Chapter 8

Conics: Set of all points in the plane in such a way that its distance from a fixed point in a constant ratio to its distance from a fixed straight line.

Or F is a fixed point in a plane and L is the line in the same plane, then the set of all points P in the plane that satisfy the condition $\frac{|PF|}{|PM|}=e$

Or Let $P(x,y)$ be any point on conic with focus F, the distance between P and F is $|PF|$ perpendicular distance of a point P from directrix $|PM|$ in all conics

$\frac{|PF|}{|PM|}=e$ where e is eccentricity of the conic

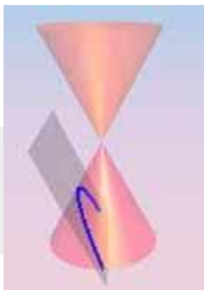
When $e<1$ Ellipse
When $e=1$ Parabola
When $e>1$ Hyperbola

The General Equation for a Conic:

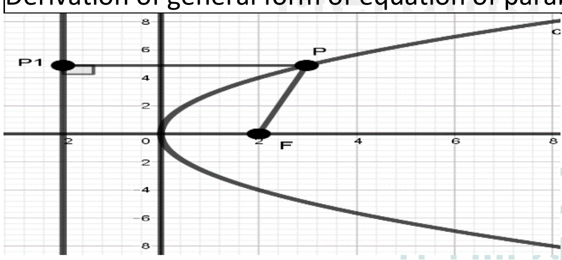
$$ax^2+hxy+by^2+2gx+2fy+c=0$$

if $h^2-ab=0$ equation of conic becomes

1). Parabola,
2). Two parallel lines
3). 1 line
4). No curve.



Derivation of general form of equation of parabola



Let the fixed point/focus $F(p,0)$ and the equation of directrix $x=-p$ parallel to the y-axis

Let $P(x,y)$ be any point on the curve & $P_1(-p,y)$ be any point on the directrix then by definition of parabola

$$\frac{|PF|}{|PP_1|}=e=1 \qquad \Rightarrow |PF|=|PP_1|$$

$$\sqrt{(x-p)^2+(y-0)^2}=\sqrt{(x-(-p))^2+(y-y)^2}$$

Squaring both sides

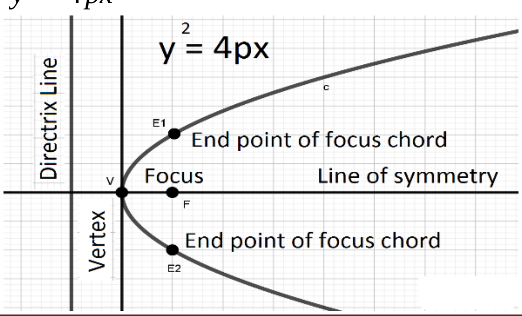
$$(x-p)^2+y^2=(x+p)^2+(0)^2$$

$$x^2+p^2-2px+y^2=x^2+p^2+2px$$

$$-2px+y^2=2px$$

$$y^2=2px+2px$$

$$y^2=4px$$



Derivation of equation of tangent on parabola

We have $y^2=4px$ Differentiating with respect to x

$$2y\frac{dy}{dx}=4p\frac{d}{dx}x$$

$$2y\frac{dy}{dx}=4p$$

$$\frac{dy}{dx}=\frac{4p}{2y}$$

Since derivative at the point $P(x_1,y_1)$ gives slope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{2p}{y_1}$$

We know that equation of line having slope and passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{2p}{y_1}(x-x_1)$$

$$y_1y-y_1^2=2px-2px_1$$

$$y_1y-4px_1=2px-2px_1 \qquad \therefore y_1^2=4px_1$$

$$y_1y=2px-2px_1+4px_1$$

$$y_1y=2px+2px_1$$

$$y_1y=2p(x+x_1)$$

Equation of normal using slope $m=\frac{2p}{y_1}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$$y-y_1=\frac{-y_1}{2p}(x-x_1)$$

Derivation of equation of tangent on parabola

We have $y^2=-4px$ Differentiating w r t x

$$2y\frac{dy}{dx}=-4p\frac{d}{dx}x$$

$$2y\frac{dy}{dx}=-4p$$

$$\frac{dy}{dx}=\frac{-4p}{2y}$$

Since derivative at the point $P(x_1,y_1)$ gives slope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-2p}{y_1}$$

We know that the equation of line having slope and passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{-2p}{y_1}(x-x_1)$$

$$y_1y-y_1^2=-2px+2px_1$$

$$y_1y-(-4px_1)=-2px+2px_1 \qquad \therefore y_1^2=-4px_1$$

$$y_1y+4px_1=-2px+2px_1$$

$$y_1y=-2px+2px_1-4px_1$$

$$y_1y=-2px-2px_1$$

$$y_1y=-2p(x+x_1)$$

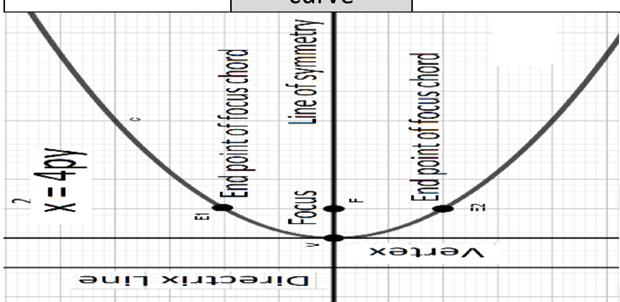
Equation of normal using slope $m=\frac{-2p}{y_1}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$$y-y_1=\frac{y_1}{2p}(x-x_1)$$

Curve along the x-axis

Left	Curve	Right
$y^2=-4px$	Parabola	$y^2=4px$
$F(-p,0)$	Focus	$F(p,0)$
$E(-p,\pm 2p)$	End points of focus chord	$E(p,\pm 2p)$
$x=p$	Equation of Directrix	$x=-p$
$y=0$	Line of symmetry	$y=0$
$(0,0)$	Vertex	$(0,0)$
$\lvert 4p\rvert$	Length of Latus Rectum	$\lvert 4p\rvert$
$x=0$	Tangent at vertex	$x=0$
$y_1y=-2p(x+x_1)$	Tangent on curve	$yy_1=2p(x+x_1)$
$y-y_1=\frac{y_1}{2p}(x-x_1)$	Normal on curve	$y-y_1=\frac{-y_1}{2p}(x-x_1)$



Derivation of equation of tangent on parabola

We have $x^2=4py$ Differentiating with respect to x

$$2x=4p\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{2x}{4p}$$
$$\frac{dy}{dx}=\frac{x}{2p}$$

Since derivative at the point $P(x_1,y_1)$ gives slope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{x_1}{2p}$$

We know that the equation of line having slope and passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{x_1}{2p}(x-x_1)$$
$$2p(y-y_1)=x_1(x-x_1)$$
$$2py-2py_1=x_1x-x_1^2$$
$$2py-2py_1=x_1x-4py_1 \qquad \therefore x_1^2=4py_1$$
$$2py-2py_1+4py_1=x_1x$$
$$2py+2py_1=x_1x$$
$$2p(y+y_1)=x_1x$$

Equation of normal using slope $m=\frac{x_1}{2p}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$$y-y_1=\frac{-2p}{x_1}(x-x_1)$$

Derivation of equation of tangent on parabola

We have $x^2=-4py$ Differentiating w.r.t x

$$2x=-4p\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{2x}{-4p}$$
$$\frac{dy}{dx}=\frac{-x}{2p}$$

Since derivative at the point $P(x_1,y_1)$ gives lope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-x_1}{2p}$$

We know that equation of line having slope & passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{-x_1}{2p}(x-x_1)$$
$$2p(y-y_1)=-x_1(x-x_1)$$
$$2py-2py_1=-x_1x+x_1^2$$
$$2py-2py_1=-x_1x-4py_1 \qquad \therefore x_1^2=-4py_1$$
$$2py-2py_1+4py_1=-x_1x$$
$$2py+2py_1=-x_1x$$
$$2py+2py_1=-x_1x$$
$$-2p(y+y_1)=x_1x$$

Equation of normal using slope $m=\frac{-x_1}{2p}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$$y-y_1=\frac{2p}{x_1}(x-x_1)$$

Curve along the y-axis

Down	Curve	Up
$x^2=-4py$	Parabola	$x^2=4py$
$F(0,-p)$	Focus	$F(0,p)$
$E(\pm 2p,-p)$	End points of focus chord	$E(\pm 2p,p)$
$y=p$	Equation of Directrix	$y=-p$
$x=0$	Line of symmetry	$x=0$
$(0,0)$	Vertex	$(0,0)$
$\lvert 4p\rvert$	Length of Latus Rectum	$\lvert 4p\rvert$
$y=0$	Tangent at vertex	$y=0$
$x_1x=-2p(y+y_1)$	Tangent on curve	$x_1x=2p(y+y_1)$
$y-y_1=\frac{2p}{x_1}(x-x_1)$	Normal on curve	$y-y_1=\frac{-2p}{x_1}(x-x_1)$

Exercise 8.1

Q1. In each case, sketch the parabola represented by equation, indicate the vertex, the focus, the end point of the focal chord (latus rectum) and the axis of symmetry

a). $x^2=2y$

Sol: we have Parabola $x^2=2y$

Comparing with the general equation $x^2=4py$

Curve is along the y-axis and Upword

$4p=2 \qquad \Rightarrow p=\frac{2}{4}=\frac{1}{2}$

Length of Latus Rectum $4p=2$

Focus $F(0,p)=F\left(0,\frac{1}{2}\right)$

End points of focal chord

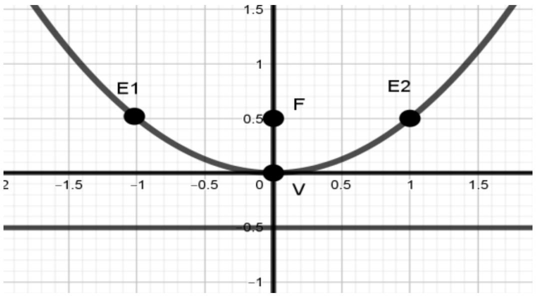
$E(\pm 2p, p) = E\left(\pm 2\left(\frac{1}{2}\right), \frac{1}{2}\right) = E\left(\pm 1, \frac{1}{2}\right)$

End points are $E_1\left(1, \frac{1}{2}\right)$ and $E_2\left(-1, \frac{1}{2}\right)$

Equation of Directrix $y = -p$

$y = -\frac{1}{2}$

Line of symmetry $x = 0$ Vertex $V(0,0)$



b). $y^2 = -3(x+1)$

Sol: We have $y^2 = -3(x+1)$

Comparing with the general equation

$(y-k)^2 = -4p(x-h)$

We get $k = 0$ and $h = -1$

Curve is along the x-axis and left

$4p = -3$

$p = \frac{-3}{4}$

Length of Latus Rectum $|4p| = |-3| = 3$

Focus $F(h+p, k) = F\left(-1 + \frac{-3}{4}, 0\right) = F\left(\frac{-7}{4}, 0\right)$

End points of focal chord

$E(-p+h, \pm 2p+k) = E\left(\frac{-3}{4} - 1, \pm 2\left(\frac{-3}{4} + 0\right)\right) = E\left(\frac{-7}{4}, \mp \frac{3}{2}\right)$

End points are $E_1\left(\frac{-7}{4}, \frac{3}{2}\right)$ and $E_2\left(\frac{-7}{4}, \frac{-3}{2}\right)$

Equation of Directrix $x = -p + h$

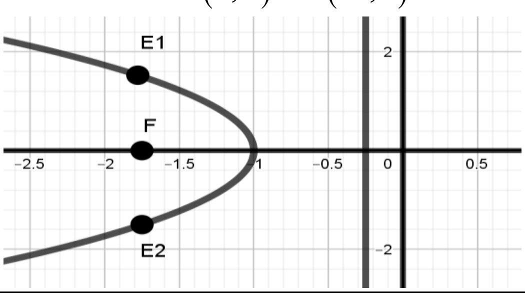
$x = -\left(\frac{-3}{4}\right) - 1$

$x = \frac{3-4}{4}$

$x = \frac{-1}{4} = -0.25$

Line of symmetry $y = k = 0$

Thus vertex is $V(h,k) = V(-1, 0)$



c). $(y-3)^2 = x$

Sol: We have $(y-3)^2 = x$

Comparing with the general equation

$(y-k)^2 = -4p(x-h)$

We get $k = 3$ and $h = 0$

Curve is along the x-axis and left

$4p = 1$

$p = \frac{1}{4}$

Length of Latus Rectum $|4p| = |1| = 1$

Focus $F(-p+h, k) = F\left(\frac{1}{4} + 0, 3\right) = F\left(\frac{1}{4}, 3\right)$

End points of focus chord

$E(-p+h, \pm 2p+k) = E\left(\frac{1}{4} - 0, \pm 2\left(\frac{1}{4}\right) + 3\right) = E\left(\frac{1}{4}, \pm \frac{1}{2} + 3\right)$

End points are $E_1\left(\frac{1}{4}, \frac{7}{2}\right)$ and $E_2\left(\frac{1}{4}, \frac{5}{2}\right)$

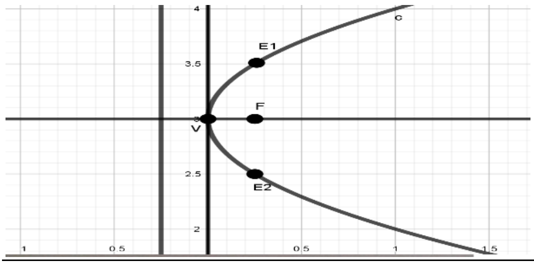
Equation of Directrix $x = -p + h$

$x = -\left(\frac{1}{4}\right)$

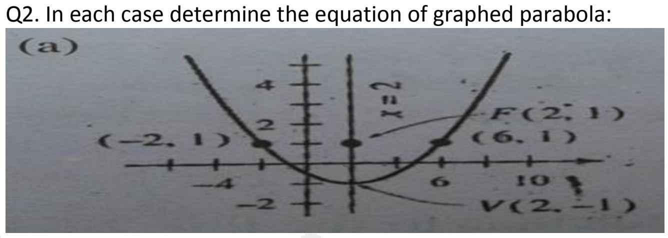
$x = \frac{-1}{4}$

Line of symmetry $y = k = 3$

Thus vertex is $V(h,k) = V(0,3)$



Q2. In each case determine the equation of graphed parabola:



Sol: We have Curve is along the y-axis and Upward

So eq of parabola $(x-h)^2 = 4p(y-k)$..........(1)

Vertex $V(h,k) = V(2,-1)$

$\Rightarrow h = 2$ $k = -1$

Focus $F(h, p+k) = F(2,1)$

$\Rightarrow h = 2$ $p + k = 1$

$p - 1 = 1$

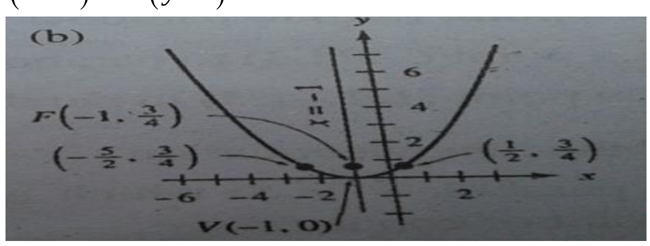
$p = 1 + 1$

$p = 2$

Putting the values of h, k & p in (1) we get

$(x-2)^2 = 4(2)(y-(-1))$

$(x-2)^2 = 8(y+1)$



Sol: We have Curve is along the y-axis and Upward

So eq of parabola $(x-h)^2 = 4p(y-k)$..........(1)

Vertex $V(h,k) = V(-1,0)$

$\Rightarrow h = -1$ $k = 0$

Focus $F(h, p+k) = F\left(-1, \frac{3}{4}\right)$

$\Rightarrow h = -1$ $p + k = \frac{3}{4}$

$p + 0 = \frac{3}{4}$

$p = \frac{3}{4}$

Putting the values of h, k & p in (1) we get

$(x-(-1))^2 = 4\left(\frac{3}{4}\right)(y-0)$

$(x+1)^2 = 3y$

Q3. In each case, write the equation of parabola through the given information:

a). Focus at $F(0,3)$ directrix $y=-3$

Sol: Focus $F(0,p)=F(0,3)$

$\Rightarrow p=3$

Since $x^2=4py$ putting the value of p we get

$x^2=4(3)y$

$x^2=12y$

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b). Focus at $F(4,0)$ directrix $x=-4$

Sol: We have $F(p,0)=F(4,0)$

$\Rightarrow p=4$

Since $y^2=4px$ putting the value of p we get

$y^2=4(4)x$

$y^2=16x$

---

c). Vertex at $V(0,0)$ x-axis is the line of symmetry, passes through $(3,6)$

Sol: X-axis i.e., $y=0$ is line of symmetry & $V(0,0)$

Then the eq of parabola $y^2=4px$................(1)

Passing through the point $(3,6)$

$6^2=4p(3)$

$36=12p$

$\frac{36}{12}=p$

$p=3$

Putting the value of p in equation (1) we get

$y^2=4(3)x$

$y^2=12x$

---

d). Vertex at $V(0,0)$ y-axis is the line of symmetry, passes through $(-12,-3)$

Sol: y-axis i.e., $x=0$ is line of symmetry and $V(0,0)$

Then the eq of parabola $x^2=4py$................(1)

Passing through the point $(-12,-3)$

$(-12)^2=4p(-3)$

$144=-12p$

$p=-12$

Putting the value of p in equation (1) we get

$x^2=4(-12)y$

$x^2=-48y$

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e). Line of symmetry is vertical, passes through $(-3,4)$, vertex at $V(5,1)$

Sol: Line of symmetry is Vertical line (line is parallel to y-axis) and $V(5,1)$

$\Rightarrow h=5 \qquad k=1$

Then eq of parabola $(x-h)^2=4p(y-k)$...........(1)

Passing through the point $(-3,4)$

$(-3-5)^2=4p(4-1)$

$(-8)^2=4p(3)$

$64=12p$

$p=\frac{64}{12}=\frac{16}{3}$

Putting the value of p in equation (1) we get

$(x-5)^2=4\left(\frac{16}{3}\right)(y-1)$

$(x-5)^2=\frac{64}{3}(y-1)$

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f). Line of symmetry is horizontal, passes through $(7,9)$ vertex at $V(3,-7)$

Sol: Line of symmetry is horizontal(parallel to X-axis) and $V(3,-7)$

$\Rightarrow h=3 \qquad k=-7$

Then eq of parabola

$(y-k)^2=4p(x-h)$................(1)

Passing through the point $(7,9)$

$(9-(-7))^2=4p(7-3)$

$(9+7)^2=4p(4)$

$16^2=16p$

$\frac{256}{16}=p \qquad \Rightarrow p=16$

Putting the value of p in equation (1) we get

$(y-(-7))^2=4(16)(x-3)$

$(y+7)^2=64(x-3)$

Q4. Find equation of set of all points with distance from $(4,3)$ that equal their distance from $(-2,1)$

Sol: Let $P(x,y)$ by any point and $A(4,3)$ $B(-2,1)$ according to condition $|PA|=|PB|$

$\sqrt{(x-4)^2+(y-3)^2}=\sqrt{(x-(-2))^2+(y-1)^2}$

Squaring both sides

$\left(\sqrt{(x-4)^2+(y-3)^2}\right)^2=\left(\sqrt{(x-(-2))^2+(y-1)^2}\right)^2$

$(x-4)^2+(y-3)^2=(x+2)^2+(y-1)^2$

$x^2-8x+16+y^2-6y+9=x^2+4x+4+y^2-2y+1$

$x^2-x^2+y^2-y^2=4x+8x-2y+6y+4+1-9-16$

$12x+4y-20=0 \qquad \div by\, 4$

$3x+y-5=0$

Q5. Find an equation of parabola whose focal chord has length 6, if it is known that parabola has focus $(4,-2)$ and its directrix is parallel to the y-axis.

Sol: Length of the focal chord = 6

$4p=6$

$p=\frac{6}{4}=\frac{3}{2}$

Directrix is parallel to y-axis then

$(y-k)^2=4p(x-h)$..................(1)

Having Focus $F(h+p,k)=F(4,-2)$

$\Rightarrow h+p=4 \qquad k=-2$

$h+\frac{3}{2}=4$

$h=4-\frac{3}{2}=\frac{8-3}{2}$

$h=\frac{5}{2}$

Putting the values of h,k & p in (1) we get

$$\left(y-(-2)\right)^2=4\left(\tfrac{3}{2}\right)\left(x-\tfrac{5}{2}\right)$$

$$(y+2)^2=6\left(x-\tfrac{5}{2}\right)$$

Q6. In each case, find the points of intersection in between the line and the parabola

a). $y^2=9x,\quad x-y+2=0$

Sol: We have $y^2=9x$..........................(1)

$x-y+2=0$..............................(2)

$\Rightarrow x+2=y$...............................(3)

Putting the value of y in equation (1) we get

$$(x+2)^2=9x$$
$$x^2+4x+4=9x$$
$$x^2+4x-9x+4=0$$
$$x^2-5x+4=0$$
$$x^2-4x-1x+4=0$$
$$x(x-4)-1(x-4)=0$$
$$(x-1)(x-4)=0$$

Either $x-1=0$ or $x-4=0$

$x=1$ $x=4$

Putting the values of x in equation (3) we get

$y=1+2$ $y=4+2$

$y=3$ $y=6$

Hence the points of intersection are $(1,3),(4,6)$

b). $y^2+3x=-8,\quad x-y+2=0$

Sol: We have $y^2+3x=-8$............................(1)

$x-y+2=0$....................................(2)

$x=y-2$..(3)

Putting the value of x in equation (1) we get

$$y^2+3(y-2)=-8$$
$$y^2+3y-6+8=0$$
$$y^2+3y+2=0$$
$$y^2+2y+y+2=0$$
$$y(y+2)+1(y+2)=0$$
$$(y+1)(y+2)=0$$

Either $y+1=0$ or $y+2=0$

$y=-1$ $y=-2$

Putting the values of y in equation (3) we get

$x=-1-2$ $x=-2-2$

$x=-3$ $x=-4$

Hence the points of intersection are $(-3,-1),(-4,-2)$

c). $x^2=2y,\quad x-y-2=0$

Sol: We have $x^2=2y$................................(1)

$x-y-2=0$.......................................(2)

$x=y+2$...(3)

Putting the value of x in equation (1) we get

$$(y+2)^2=2y$$
$$y^2+4y+4=2y$$
$$y^2+4y-2y+4=0$$
$$y^2+2y+4=0$$

Using quadratic formula $y=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

putting $y=\dfrac{-2\mp\sqrt{2^2-4(1)(4)}}{2(1)}$

$$y=\frac{-2\mp\sqrt{4-16}}{2}$$
$$y=\frac{-2\mp\sqrt{-12}}{2}$$

Here discriminant is negative i.e.,

$$b^2-4ac=-12<0$$

Therefore line and the parabola does not intersect

d). $y^2=-2(x+1),\quad x+y-2=0$

Sol: We have $y^2=-2(x+1)$.....................(1)

$x+y-2=0$..............................(2)

$x=2-y$...................................(3)

Putting the value of x in equation (1) we get

$$y^2=-2(2-y+1)$$
$$y^2=-2(3-y)$$
$$y^2=-6+2y$$
$$y^2-2y+6=0$$

Using quadratic formula $y=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

putting $y=\dfrac{-2\mp\sqrt{2^2-4(1)(6)}}{2(1)}$

$$y=\frac{-2\mp\sqrt{4-24}}{2}$$
$$y=\frac{-2\mp\sqrt{-20}}{2}$$

Here discriminant is negative i.e.,

$$b^2-4ac=-20<0$$

Therefore line and the parabola does not intersect

Q7. For what value of c

a). line $x-y+c=0$ will touch parabola $y^2=9x$

Sol: We have the parabola $y^2=9x$...........(1)

And the line $x-y+c=0$

Or $x=y-c$........................(2)

Putting the value of x in equation (1) we get

$$y^2=9(y-c)$$
$$y^2=9y-9c$$
$$y^2-9y+9c=0$$

The line and the parabola touch each other if $B^2-4AC=0$ putting the values

$(-9)^2-4(1)(9c)=0$

$81-36c=0$

$81=36c$

$\frac{81}{36}=c$

b). Line $x-y+c=0$ will touch parabola $y^2=8x$

Sol: We have the parabola $y^2=8x$...........(1)

And the line $x-y+c=0$

Or $x=y-c$.........................(2)

Putting the value of x in equation (1) we get

$y^2=8(y-c)$

$y^2=8y-8c$

$y^2-8y+8c=0$

The line and the parabola touch each other if

$B^2-4AC=0$ putting the values

$(-8)^2-4(1)(8c)=0$

$64-32c=0$

$64=32c \qquad \Rightarrow c=2$

c). line $x-y+c=0$ will touch parabola $x^2=-8y$

Sol: We have the parabola $x^2=-8y$...........(1)

And the line $x-y+c=0$

Or $y=x+c$.........................(2)

Putting the value of x in equation (1) we get

$x^2=-8(x+c)$

$x^2=-8x-8c$

$x^2+8x+8c=0$

The line and the parabola touch each other if

$B^2-4AC=0$ putting the values

$(8)^2-4(1)(8c)=0$

$64=32c$

$\Rightarrow c=2$

d). line $x-y+c=0$ will touch parabola $x^2=\frac{2}{3}y$

Sol: We have the parabola $x^2=\frac{2}{3}y$...........(1)

And the line $x-y+c=0$

Or $y=x+c$.........................(2)

Putting the value of x in equation (1) we get

$x^2=\frac{2}{3}(x+c)$

$3x^2=2x+2c$

$3x^2-2x-2c=0$

The line and the parabola touch each other if

$B^2-4AC=0$ putting the values

$(-2)^2-4(3)(-2c)=0$

$4+24c=0$

$24c=-4 \qquad \Rightarrow c=\frac{-1}{6}$

Q8. In each case, fine tangent equation & normal eq

a). at a point $(1,2)$ to the parabola $y^2=4x$

Sol: parabola $y^2=4x$ And point $(1,2)$

Comparing with $y^2=4px$

$\Rightarrow 4p=4$

$p=1$

Since Equation of tangent at the point $P(x_1,y_1)$

$yy_1=2p(x+x_1)$ Putting the values

$y(2)=2(1)(x+1)$

$2y=2(x+1)$

$2y=2x+2$

$2x-2y+2=0 \div by\, 2$

$x-y+1=0$

Since Equation of normal at the point $P(x_1,y_1)$

$y-y_1=\frac{-y_1}{2p}(x-x_1)$ Putting the values

$y-2=\frac{-2}{2(1)}(x-1)$

$y-2=-(x-1)$

$y-2=-x+1$

$x+y-2-1=0$

$x+y-3=0$

b). at a point $(3,6)$ to the parabola $y^2=12x$

Sol: parabola $y^2=12x$ And point $(3,6)$

Comparing with $y^2=4px$

$\Rightarrow 4p=12$

$p=3$

Since Equation of tangent at the point $P(x_1,y_1)$

$yy_1=2p(x+x_1)$ Putting the values

$y(6)=2(3)(x+3)$

$6y=6(x+3)$

$6y=6x+18$

$6x-6y+18=0 \qquad \div by\, 6$

$x-y+3=0$

Since Equation of normal at the point $P(x_1,y_1)$

$y-y_1=\frac{-y_1}{2p}(x-x_1)$ Putting the values

$y-6=\frac{-6}{2(3)}(x-3)$

$y-6=-(x-3)$

$y-6=-x+3$

$x+y-6-3=0$

$x+y-9=0$

c). at a point $(\frac{-3}{4},3)$ to the parabola $y^2=-12x$

Sol: parabola $y^2=-12x$ And point $(\frac{-3}{4},3)$

Comparing with $y^2=4px$

$\Rightarrow 4p=-12$

$p=-3$

Since Equation of tangent at the point $P(x_1,y_1)$

$yy_1=2p(x+x_1)$ Putting the values

$y(3)=2(-3)(x+(\frac{-3}{4}))$

$3y=-6(x-\frac{3}{4})$

$3y=-6x+\frac{9}{2}$

$6x+3y-\frac{9}{2}=0 \qquad \times by\, 2$

$12x+6y-9=0 \qquad \div by\, 3$

$4x+2y-3=0$

Since Equation of normal at the point $P(x_1, y_1)$

$y - y_1 = \frac{-y_1}{2p}(x - x_1)$ Putting the values

$y - 3 = \frac{-3}{2(-3)}\left(x - \left(\frac{-3}{4}\right)\right)$

$y - 3 = \frac{1}{2}\left(x + \frac{3}{4}\right)$

$2(y-3) = x + \frac{3}{4}$

$2y - 6 = x + \frac{3}{4}$

$x - 2y + 6 + \frac{3}{4} = 0 \qquad \times by\, 4$

$4x - 8y + 24 + 3 = 0$

$4x - 8y + 27 = 0$

---

d). at a point $\left(\frac{-1}{2}, \frac{-1}{3}\right)$ to parabola $x^2 = \frac{-3}{4}y$

Sol: parabola $x^2 = \frac{-3}{4}y$ And point $\left(\frac{-1}{2}, \frac{-1}{3}\right)$

Comparing with $y^2 = 4px$

$\Rightarrow 4p = \frac{-3}{4} \qquad \Rightarrow 2p = \frac{-3}{8}$

Since Equation of tangent at the point $P(x_1, y_1)$

$xx_1 = 2p(y + y_1)$ Putting the values

$-\frac{1}{2}x = \frac{-3}{8}\left(y - \frac{1}{3}\right) \qquad \times by 8$

$-4x = -3\left(y - \frac{1}{3}\right)$

$-4x = -3y + 1$

$4x - 3y + 1 = 0$

Since Equation of normal at the point $P(x_1, y_1)$

$x - x_1 = \frac{-x_1}{2p}(y - y_1)$ Putting the values

$x + \frac{1}{2} = \frac{-\frac{1}{2}}{\frac{3}{8}}\left(y + \frac{1}{3}\right)$

$\frac{3}{8}\left(x + \frac{1}{2}\right) = \frac{-1}{2}\left(y + \frac{1}{3}\right) \qquad \times by\ 8$

$3\left(x + \frac{1}{2}\right) = -4\left(y + \frac{1}{3}\right)$

$3x + \frac{3}{2} = -4y - \frac{4}{3}$

$3x + 4y + \frac{3}{2} + \frac{4}{3} = 0$

$3x + 4y + \frac{9+8}{6} = 0$

$3x + 4y + \frac{17}{6} = 0$

$18x + 24y + 17 = 0$

---

d). at a point $\left(\frac{1}{2}, \frac{-1}{3}\right)$ to the parabola $x^2 = \frac{-3}{4}y$

Sol: parabola $x^2 = \frac{-3}{4}y$ And point $\left(\frac{-1}{2}, \frac{-1}{3}\right)$

Comparing with $y^2 = 4px$

$\Rightarrow 4p = \frac{-3}{4} \qquad \Rightarrow 2p = \frac{-3}{8}$

Since Equation of tangent at the point $P(x_1, y_1)$

$xx_1 = 2p(y + y_1)$ Putting the values

$\frac{1}{2}x = \frac{-3}{8}\left(y - \frac{1}{3}\right) \qquad \times by 8$

$4x = -3\left(y - \frac{1}{3}\right)$

$4x = -3y + 1$

$4x + 3y - 1 = 0$

Since Equation of normal at the point $P(x_1, y_1)$

$x - x_1 = \frac{-x_1}{2p}(y - y_1)$ Putting the values

$x - \frac{1}{2} = \frac{-\frac{1}{2}}{\frac{-3}{8}}\left(y + \frac{1}{3}\right)$

$\frac{3}{8}\left(x - \frac{1}{2}\right) = \frac{1}{2}\left(y + \frac{1}{3}\right) \qquad \times by\ 8$

$3\left(x - \frac{1}{2}\right) = 4\left(y + \frac{1}{3}\right)$

$3x - \frac{3}{2} = 4y + \frac{4}{3}$

$3x - 4y - \frac{3}{2} - \frac{4}{3} = 0$

$3x - 4y - \frac{17}{6} = 0$

$18x - 24y - 17 = 0$

Q9. Find the tangent equation

a). to parabola $y^2 = 8x$ which is perpendicular to line $2y - x + 1 = 0$ Find also the point of contact

Sol: We have the parabola $y^2 = 8x$.................(1)

Comparing with $y^2 = 4px \qquad \Rightarrow 4p = 8$

$p = 2$

And the line $-x + 2y + 1 = 0$

Slope of the line $m = \frac{-a}{b}$ putting the values

$m_1 = \frac{-(-1)}{2} = \frac{1}{2}$

Slope of tangent/Perpendicular

$m_2 = \frac{-1}{m_1} = \frac{-1}{\frac{1}{2}} = -2$

Equation of tangent $y = mx + \frac{p}{m}$ putting values

$y = -2x + \frac{2}{-2}$

$y = -2x - 1$............................(2)

For point of contact Putting value of y in eq (1)

$(-2x-1)^2 = 8x$

$4x^2 + 4x + 1 = 8x$

$4x^2 + 4x - 8x + 1 = 0$

$4x^2 - 4x + 1 = 0$

$(2x)^2 - 2(2x)(1) + (1)^2 = 0$

$(2x-1)^2 = 0$

$\Rightarrow 2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$

Putting the value of x in equation (2) we get

$y = -2\left(\frac{1}{2}\right) - 1$

$y = -1 - 1$

$y = -2$

Hence the point of contact is $\left(\frac{1}{2}, -2\right)$

---

b). to parabola $y^2 = 5x$ which is parallel to the line $x + 4y + 1 = 0$ Find also the point of contact

Sol: We have the parabola $y^2 = 5x$.................(1)

Comparing with $y^2=4px$

$\Rightarrow 4p=5$

$p=\frac{5}{4}$

And the line $x+4y+1=0$

Slope of the line $m=\frac{-a}{b}$ putting values $m_1=\frac{-1}{4}$

Slope of tangent/parallel $m_1=m_2=\frac{-1}{4}$

Equation of tangent $y=mx+\frac{p}{m}$ putting values

$y=\frac{-1}{4}x+\frac{\frac{5}{4}}{\frac{-1}{4}}$

$y=\frac{-1}{4}x-5$

$y+5=\frac{-1}{4}x$

$4(y+5)=-x$

$x=-4(y+5)$.............................(2)

For point of contact Putting value of x in eq (1)

$y^2=5\{-4(y+5)\}$

$y^2=-20(y+5)$

$y^2=-20y-100$

$y^2+20y+100=0$

$(y)^2+2(y)(10)+(10)^2=0$

$(y+10)^2=0$

$\Rightarrow y+10=0$

$y=-10$

Putting the value of y in equation (2) we get

$x=-4(-10+5)$

$x=-4(-5)$

$x=20$

Hence the point of contact is $(20,-10)$

Q10. Find the tangent equation

a). to parabola $y^2=x$ which makes an angle of 45^o with the x-axis.

Sol: We have the parabola $y^2=x$.................(1)

Comparing with $y^2=4px$

$\Rightarrow 4p=1$

$p=\frac{1}{4}$

Slope $m=\tan 45^o=1$

Equation of tangent $y=mx+\frac{p}{m}$ putting values

$y=1x+\frac{\frac{1}{4}}{1}$

$y=x+\frac{1}{4}$

b). to parabola $y^2=x$ which makes an angle of 60^o with the x-axis.

Sol: We have the parabola $y^2=x$.................(1)

Comparing with $y^2=4px$ $\Rightarrow 4p=1$

$p=\frac{1}{4}$

Slope $m=\tan 60^o=\sqrt{3}$

Equation of tangent $y=mx+\frac{p}{m}$ putting values

$y=\sqrt{3}x+\frac{\frac{1}{4}}{\sqrt{3}}$

$y=\sqrt{3}x+\frac{1}{4\sqrt{3}}$

c). to parabola $y^2=x$ which makes an angle of 135^o with the x-axis.

Sol: We have the parabola $y^2=x$.................(1)

Comparing with $y^2=4px$

$\Rightarrow 4p=1$

$p=\frac{1}{4}$

Slope $m=\tan 135^o=-1$

Equation of tangent $y=mx+\frac{p}{m}$ putting values

$y=-1x+\frac{\frac{1}{4}}{-1}$ $\Rightarrow y=-x-\frac{1}{4}$

Q11. Find the tangent equation

a). to parabola $x^2=y$ which makes an angle of 45^o with the x-axis.

Sol: We have the parabola $y^2=x$.................(1)

Comparing with $y^2=4px$

$\Rightarrow 4p=1$

$p=\frac{1}{4}$

Slope $m=\tan 45^o=1$

Equation of tangent $y=mx-pm^2$ putting values

$y=1x-\frac{1}{4}(1)^2$

$y=x-\frac{1}{4}$

b). to parabola $x^2=y$ which makes an angle of 60^o with the x-axis.

Sol: We have the parabola $y^2=x$.................(1)

Comparing with $y^2=4px$

$\Rightarrow 4p=1$

$p=\frac{1}{4}$

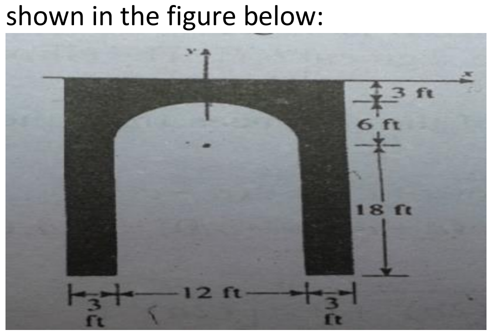
Slope $m=\tan 60^o=\frac{\sqrt{3}}{1}=\sqrt{3}$

Equation of tangent $y=mx-pm^2$ putting values

$y=\sqrt{3}x+\frac{1}{4}(\sqrt{3})^2$

$y=\sqrt{3}x+\frac{3}{4}$

Q12. Find the equation of the parabolic portion of archway, if parabolic archway has the dimensions shown in the figure below:



Sol: from figure vertex $(h,k)=(0,-3)$

Or $h=0$ $k=-3$

Point of parabola $P_1(6,-9)$ and $P_2(-6,-9)$

Eq of parabola $(x-h)^2=4p(y-k)$............(1)

Putting value of h,k and p_1 in equation (1) we get

$$(6-0)^2=4p(-9-(-3))$$
$$36=4p(-9+3)$$
$$36=4p(-6)$$
$$36=-24p$$
$$p=\frac{36}{-24}=-\frac{3}{2}$$

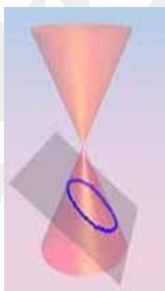
Putting the value of h,k and p in eq (1) we get

$$(x-0)^2=4\left(\tfrac{-3}{2}\right)(y-(-3))$$
$$x^2=-6(y+3)$$

Ellipse The set of all points in a plane the sum of whose distance from two fixed points/foci F_1, F_2 is constant $(2a)$

Derivation of equation of ellipse

Let we place the foci on the x-axis at the points $F_1(-c,0)\,\&\,F_2(c,0)$ so that the origin is halfway between the foci. $P(x,y)$ be any point on ellipse Let the sum of distance from the point on the ellipse to the foci be $2a$



$$|PF_1|+|PF_2|=2a$$
$$\sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$
$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

Squaring both sides

$$(x+c)^2+y^2=4a^2+(x-c)^2+y^2-2(2a)\sqrt{(x-c)^2+y^2}$$
$$(x+c)^2-(x-c)^2=4a^2-4a\sqrt{(x-c)^2+y^2}$$
$$4cx=4\left\{a^2-a\sqrt{(x-c)^2+y^2}\right\}$$
$$cx=a^2-a\sqrt{(x-c)^2+y^2}$$
$$cx-a^2=-a\sqrt{(x-c)^2+y^2}$$

Squaring again

$$\left(cx-a^2\right)^2=\left\{a\sqrt{(x-c)^2+y^2}\right\}^2$$
$$c^2x^2+a^4-2a^2cx=a^2\left\{x^2+c^2-2cx+y^2\right\}$$
$$c^2x^2+a^4-2a^2cx=a^2x^2+a^2c^2-2a^2cx+a^2y^2$$
$$a^4-a^2c^2=a^2x^2-c^2x^2+a^2y^2$$
$$a^2\left(a^2-c^2\right)=\left(a^2-c^2\right)x^2+a^2y^2$$
$$a^2b^2=b^2x^2+a^2y^2 \qquad \therefore b^2=a^2-c^2$$

Dividing both sides by a^2b^2

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Derivation of equation of tangent on Ellipse

We have $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Multiply both sides by a^2b^2

$b^2x^2+a^2y^2=a^2b^2$...........(1) Differentiating w.r.t x

$$2b^2x+2a^2y\frac{dy}{dx}=0$$
$$2a^2y\frac{dy}{dx}=-2b^2x$$
$$\frac{dy}{dx}=\frac{-2b^2x}{2a^2y} \qquad \Rightarrow \frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

Since derivative at the point $P(x_1,y_1)$ gives slope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^2x_1}{a^2y_1}$$

We know that equation of line having slope and passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{-b^2x_1}{a^2y_1}(x-x_1)$$
$$a^2y_1(y-y_1)=-b^2x_1(x-x_1)$$
$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$
$$b^2x_1x+a^2y_1y=b^2x_1^2+a^2y_1^2$$
$$b^2x_1x+a^2y_1y=a^2b^2 \qquad \therefore b^2x_1^2+a^2y_1^2=a^2b^2$$

Dividing by a^2b^2

$$\frac{x.x_1}{a^2}+\frac{y.y_1}{b^2}=1$$

Equation of normal using slope $m=\frac{-b^2x_1}{a^2y_1}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$
$$\frac{y-y_1}{x-x_1}=\frac{a^2y_1}{b^2x_1}$$

Derivation of equation of tangent on Ellipse

We have $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ Multiply both sides by a^2b^2

$b^2y^2+a^2x^2=a^2b^2$........(1) Differentiating w.r.t x

$$2b^2y\frac{dy}{dx}+2a^2x=0$$
$$2b^2y\frac{dy}{dx}=-2a^2x$$
$$\frac{dy}{dx}=\frac{-2a^2x}{2b^2y} \qquad \Rightarrow \frac{dy}{dx}=\frac{-a^2x}{b^2y}$$

Since derivative at the point $P(x_1,y_1)$ gives slope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-a^2x_1}{b^2y_1}$$

We know that equation of line having slope and passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{-a^2x_1}{b^2y_1}(x-x_1)$$
$$b^2y_1(y-y_1)=-a^2x_1(x-x_1)$$
$$b^2y_1y-b^2y_1^2=-a^2x_1x+a^2x_1^2$$
$$a^2x_1x+b^2y_1y=a^2x_1^2+b^2y_1^2$$
$$a^2x_1x+b^2y_1y=a^2b^2 \qquad \therefore b^2y_1^2+a^2x_1^2=a^2b^2$$

Dividing by a^2b^2

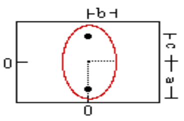
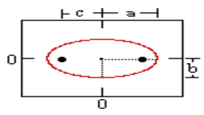
$$\frac{y.y_1}{a^2}+\frac{x.x_1}{b^2}=1$$

Equation of normal using slope $m=\frac{-b^2x_1}{a^2y_1}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$y-y_1=\frac{b^2y_1}{a^2x_1}(x-x_1)$

$\frac{y-y_1}{x-x_1}=\frac{b^2y_1}{a^2x_1}$

Major axis along Vertical	Ellipse	Major axis along Horizantal
	Curve With $c=ae$ $a>b$	
$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$	$c^2=a^2-b^2$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
$F(0,\pm c)$	Focus	$F(\pm c,0)$
$E(0,\pm a)$	End points of Major axis	$E(\pm a,0)$
$E(\pm b,0)$	End points of Minor axis	$E(0,\pm b)$
$y=\pm\frac{a}{e}$	Equation of Directrix	$x=\pm\frac{a}{e}$
$(0,0)$	Centre	$(0,0)$
$\left\|\frac{2b^2}{a}\right\|$	Length of Latus Rectum	$\left\|\frac{2b^2}{a}\right\|$
$\left(\pm\frac{b^2}{a},\pm ae\right)$	Tangent at vertex	$\left(\pm ae,\pm\frac{b^2}{a}\right)$
$x=0$	Equation of Major axis	$y=0$
$\frac{y.y_1}{a^2}+\frac{x.x_1}{b^2}=1$	Tangent on curve	$\frac{x.x_1}{a^2}+\frac{y.y_1}{b^2}=1$
$\frac{y-y_1}{x-x_1}=\frac{b^2y_1}{a^2x_1}$	Normal on curve	$\frac{y-y_1}{x-x_1}=\frac{a^2y_1}{b^2x_1}$

Exercise 8.2

Q1. In each case, sketch the ellipse represented by the equation, indicate the center, foci, end points of major axis and end point of minor axis:

a). $\frac{x^2}{9}+\frac{y^2}{4}=1$

Sol: We have $\frac{x^2}{9}+\frac{y^2}{4}=1$ Comparing $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

We get $a^2=9, b^2=4$

$\Rightarrow a=3, b=2$

Major axis is along x-axis with centre $(0,0)$

Now using $c^2=a^2-b^2$

$c^2=9-4=5$

$\Rightarrow c=\sqrt{5}$

Foci $F(\pm c,0)=F(\pm\sqrt{5},0)$

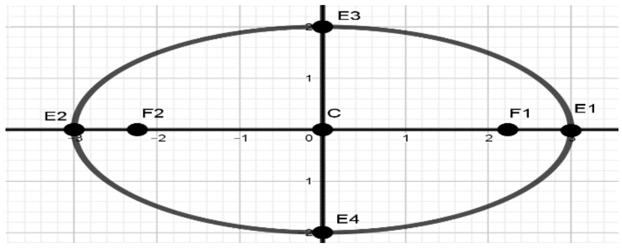
Then foci are $F_1(\sqrt{5},0), F_2(-\sqrt{5},0)$

End points of Major axis $E(\pm a,0)=E(\pm 3,0)$

End points of major axis are $E_1(3,0), E_2(-3,0)$

End points of Minor axis $E(0,\pm b)=E(0,\pm 2)$

End points of Minor axis $E_3(0,2), E_4(0,-2)$



b). $\frac{x^2}{16}+\frac{y^2}{25}=1$

Sol: We have $\frac{x^2}{16}+\frac{y^2}{25}=1$ Comparing $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

We get $a^2=25, b^2=16$

$\Rightarrow a=5, b=4$

Major axis is along y-axis with centre $(0,0)$

Now using $c^2=a^2-b^2$

$c^2=25-16=9$

$\Rightarrow c=3$

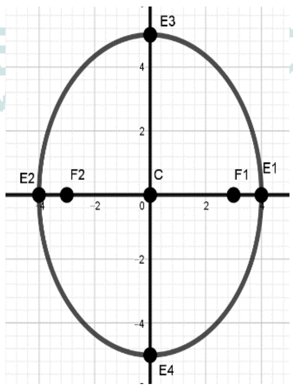
Foci $F(0,\pm c)=F(0,\pm 3)$

Then foci are $F_1(0,3), F_2(0,-3)$

End points of Minor axis $E(\pm b,0)=E(\pm 4,0)$

End points of Minor axis $E_1(4,0), E_2(-4,0)$ End points of Major axis $E(0,\pm a)=E(0,\pm 5)$

End points of major axis are $E_3(0,5), E_4(0,-5)$



c). $\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$

Sol: We have $\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$

Comparing with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

We get $a^2=16, b^2=9$ with $h=-1,\ k=2$

$\Rightarrow a=4, b=3$

Major axis is along x-axis with centre $C(-1,2)$

Now using $c^2=a^2-b^2$

$c^2=16-9=7$

$\Rightarrow c=\sqrt{7}$

Foci $F(\pm c+h,k)=F(\pm\sqrt{7}-1,2)$

Then foci are $F_1(\sqrt{7}-1,2),F_2(-\sqrt{7}-1,2)$

End points of Major axis

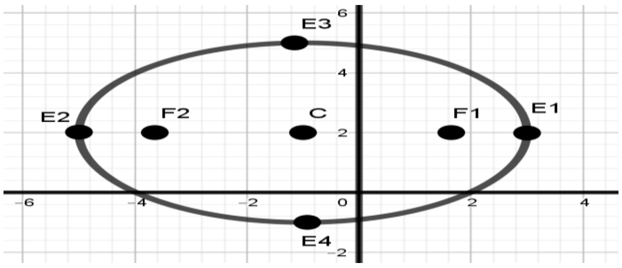
$E(h\pm a,k)=E(-1\pm 4,2)$

End points of major axis are $E_1(3,2),E_2(-5,2)$

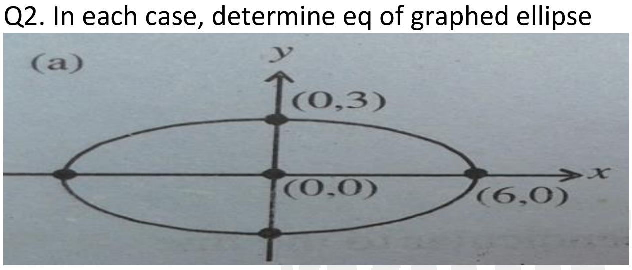
End points of Minor axis

$E(h,k\pm b)=E(-1,2\pm 3)$

End points of Minor axis $E_3(-1,5),E_4(-1,-1)$



Q2. In each case, determine eq of graphed ellipse



Sol: From the figure Centre is $(h,k)=(0,0)$

Major axis is along x-axis i.e.,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.............(1)$$

End point of major axis $E(a,0)=E(6,0)$

$$\Rightarrow a=6$$

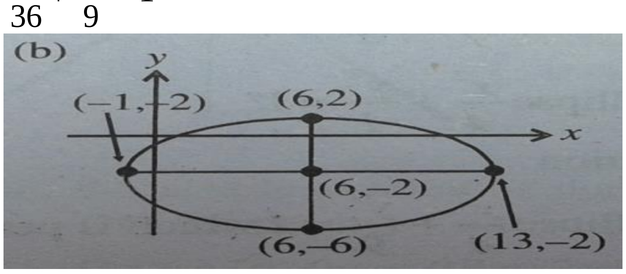
End point of minor axis $E(0,b)=E(0,3)$

$$\Rightarrow b=3$$

Putting the value of a,b in equation (1) we get

$$\frac{x^2}{6^2}+\frac{y^2}{3^2}=1$$

$$\frac{x^2}{36}+\frac{y^2}{9}=1$$



Sol: From the figure Centre is $(h,k)=(6,-2)$

$$\Rightarrow h=6,k=-2$$

Major axis is along x-axis i.e.,

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1..........................(1)$$

End point of major axis $E(a+h,k)=E(13,-2)$

$$\Rightarrow a+h=13$$

$$a+6=13$$

$$a=13-6$$

$$a=7$$

End point of minor axis $E(h,k+b)=E(6,2)$

$$\Rightarrow k+b=2$$

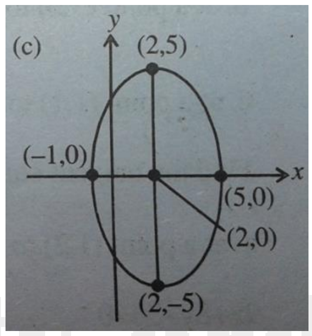
$$-2+b=2$$

$$b=2+2 \qquad \Rightarrow b=4$$

Putting the value of a,b and h,k in eq (1) we get

$$\frac{(x-6)^2}{7^2}+\frac{(y+2)^2}{4^2}=1$$

$$\frac{(x-6)^2}{49}+\frac{(y+2)^2}{16}=1$$



Sol: From the figure Centre is $(h,k)=(2,0)$

$$\Rightarrow h=2,k=0$$

Major axis is along y-axis i.e.,

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1..........................(1)$$

End point of major axis $E(h,k+a)=E(2,5)$

$$\Rightarrow k+a=5$$

$$0+a=5$$

$$a=5$$

End point of minor axis $E(h+b,k)=E(5,0)$

$$\Rightarrow h+b=5$$

$$2+b=5$$

$$b=5-2$$

$$b=3$$

Putting the value of a,b and h,k in eq (1) we get

$$\frac{(x-2)^2}{3^2}+\frac{(y-0)^2}{5^2}=1$$

$$\frac{(x-2)^2}{9}+\frac{y^2}{25}=1$$

Q3. In each case, determine the equation of ellipse through the given information

a)centre at $(-3,2)$, $a=2,b=1$ major axis is horizontal

Sol: We have centre $(h,k)=(-3,2)$

$$\Rightarrow h=-3,k=2$$

Given that $a=2,b=1$ & major axis horizontal

So using $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, Putting values

$$\frac{(x-(-3))^2}{2^2}+\frac{(y-2)^2}{1^2}=1$$
$$\frac{(x+3)^2}{4}+\frac{(y-2)^2}{1}=1$$

b). Vertices are at $(4,2)$ and $(12,2)$, $b=2$

Sol: Vertices are $(4,2)$ and $(12,2)$

Midpoint of vertices is centre $=\left(\frac{4+12}{2},\frac{2+2}{2}\right)$

Centre $(h,k)=\left(\frac{16}{2},\frac{4}{2}\right)$

Centre $(h,k)=(8,2)$

$\Rightarrow h=8, k=2$

Since vertices $(h\pm a,k)$

Take $h-a=4$
$8-a=4$
$8-4=a$
$a=4$

Given that $b=2$

So using $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, Putting values

$$\frac{(x-8)^2}{4^2}+\frac{(y-2)^2}{2^2}=1$$
$$\frac{(x-8)^2}{16}+\frac{(y-2)^2}{4}=1$$

c). A focus is at $(-3,3)$ a vertex is at $(6,3)$ length of minor axis is 6

Sol: Length of minor axis $2b=6$

$\Rightarrow b=3$

Foci $(h\pm c,k)$ take $F(h-c,k)=(-3,3)$

$\Rightarrow h-c=-3,\quad k=3$

$h+3=c$

Since vertices $(h\pm a,k)$ take Vertex

$(h+a,k)=(6,3)$

$\Rightarrow h+a=6,\quad k=3$

$a=6-h$

Now using $c^2=a^2-b^2$

$(h+3)^2=(6-h)^2-3^2$

$h^2+6h+9=36-12h+h^2-9$

$h^2-h^2+6h+12h=36-9-9$

$18h=18$

$h=1$

Put in $h+a=6$ we get

$1+a=6$

$a=6-1$

$a=5$

So using $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, Putting values

$$\frac{(x-1)^2}{5^2}+\frac{(y-3)^2}{3^2}=1$$
$$\frac{(x-1)^2}{25}+\frac{(y-3)^2}{9}=1$$

d). vertices are at $(0,8)$ and $(0,2)$, $c=\sqrt{5}$

Sol: Vertices are $(0,8)$ and $(0,2)$

Midpoint of vertices is centre $=\left(\frac{0+0}{2},\frac{8+2}{2}\right)$

Centre $(h,k)=\left(\frac{0}{2},\frac{10}{2}\right)$

Centre $(h,k)=(0,5)$

$\Rightarrow h=0, k=5$

Since vertices $(h,k\pm a)$

Take $k+a=8$
$5+a=8$
$a=8-5$
$a=3$

Now using $c^2=a^2-b^2$

$(\sqrt{5})^2=3^2-b^2$

$5=9-b^2$

$b^2=9-5$

$b^2=4$

$\Rightarrow b=2$

So using $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, Putting values

$$\frac{(x-0)^2}{2^2}+\frac{(y-5)^2}{3^2}=1$$
$$\frac{x^2}{4}+\frac{(y-5)^2}{9}=1$$

Q4. The shape of an ellipse depends on the eccentricity of the ellipse $e=\frac{c}{a}$ Determine

a). the eccentricity of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

Sol: We have the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

Comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

We get $a^2=25, b^2=16$

$\Rightarrow a=5, b=4$

Major axis is along x-axis with centre $(0,0)$

Now using $c^2=a^2-b^2$

$c^2=25-16=9$

$\Rightarrow c=3$

Now using eccentricity $e=\frac{c}{a}$ putting the values

$\Rightarrow e=\frac{3}{5}$

b). the equation of the ellipse with vertices are at $(-5,0)$ and $(5,0)$, and the eccentricity is $e=\frac{3}{5}$

Sol: vertices are $(-5,0)$ and $(5,0)$

Midpoint of vertices is centre $=\left(\frac{-5+5}{2},\frac{0+0}{2}\right)$

Centre $(h,k)=\left(\frac{0}{2},\frac{0}{2}\right)$

Centre $(h,k)=(0,0)$

$\Rightarrow h=0,k=0$

Therefore equation of ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$............(1)

Since $e=\frac{c}{a}$ Given that $e=\frac{3}{5}$ By comparing

$\Rightarrow c=3,\qquad a=5$

Now using $c^2=a^2-b^2$

$3^2=5^2-b^2$

$9=25-b^2$

$b^2=25-9$

$b^2=16$

$\Rightarrow\qquad b=4$

Putting the value of a, b in equation (1) we get

$\frac{x^2}{5^2}+\frac{y^2}{4^2}=1$

$\frac{x^2}{25}+\frac{y^2}{16}=1$

---

c). the eccentricity of the ellipse if the length of the semi-major axis is $a=4$ and the length of the semi-minor axis is $b=2$

Sol: We have $a=4$ and $b=2$

Now using $c^2=a^2-b^2$

$c^2=4^2-2^2$

$c^2=16-4$

$c^2=12$

$\sqrt{c^2}=\sqrt{4\times 3}$

$\Rightarrow c=2\sqrt{3}$

Since $e=\frac{c}{a}$ putting the values

$e=\frac{2\sqrt{3}}{4}\qquad\Rightarrow e=\frac{\sqrt{3}}{2}$

Q5. For what value of c

a). line $x-y+c=0$ will touch ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

Sol: We have ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

Or $x^2+4y^2=4$............................(1)

And the line $x-y+c=0$

$y=x+c$

Putting the value of y in equation (1) we get

$x^2+4(x+c)^2=4$

$x^2+4\left(x^2+2cx+c^2\right)=4$

$x^2+4x^2+8cx+4c^2-4=0$

$5x^2+8cx+4c^2-4=0$

Given line will tangent to ellipse if Discriminant = 0

$b^2-4ac=0$

$(8c)^2-4(5)\left(4c^2-4\right)=0$

$64c^2-20\left(4c^2-4\right)=0$

$64c^2-80c^2+80=0$

$-16c^2=-80$

$c^2=5$

$\Rightarrow c=\pm\sqrt{5}$

---

b)line $2x-y+c=0$ will touch ellipse

$\frac{x^2}{3}+\frac{y^2}{4}=1$

Sol: We have ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$

Or $4x^2+3y^2=12$............................(1)

And the line $2x-y+c=0$

$y=2x+c$

Putting the value of y in equation (1) we get

$4x^2+3(2x+c)^2=12$

$4x^2+3\left(4x^2+4cx+c^2\right)=12$

$4x^2+12x^2+12cx+3c^2-12=0$

$16x^2+12cx+3c^2-12=0$

Given line will tangent to ellipse if Discriminant = 0

$b^2-4ac=0$

$(12c)^2-4(16)\left(3c^2-12\right)=0$

$144c^2-64\left(3c^2-12\right)=0$

$144c^2-192c^2+768=0$

$-48c^2=-768$

$c^2=\frac{-768}{-48}=16$

$\Rightarrow\sqrt{c^2}=\pm\sqrt{16}$

$\Rightarrow c=\pm 4$

---

c).line $x+y+c=0$ will touch ellipse $\frac{x^2}{25}+\frac{y^2}{11}=1$

Sol: We have ellipse $\frac{x^2}{25}+\frac{y^2}{11}=1$

Or $11x^2+25y^2=275$............................(1)

And the line $x+y+c=0$

$y=-x-c$

Putting the value of y in equation (1) we get

$11x^2+25(-x-c)^2=275$

$11x^2+25\left(x^2+2cx+c^2\right)=275$

$11x^2+25x^2+50cx+25c^2-275=0$

$36x^2+50cx+25c^2-275=0$

Given line will tangent to ellipse if Discriminant = 0

$b^2-4ac=0$

$(50c)^2-4(36)\left(25c^2-275\right)=0$

$2500c^2 - 144(25c^2 - 275) = 0$

$2500c^2 - 3600c^2 + 39600 = 0$

$-1100c^2 = -39600$

$\Rightarrow c = \pm\sqrt{36}$ $\qquad \Rightarrow c = \pm 6$

Q6. In each case, find tangent eq & normal eq

a). at a point $\left(\frac{2\sqrt{5}}{3}, 2\right)$ to ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Sol: We have $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Compare with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ We get $a^2 = 9, b^2 = 4$

Equation of tangent $\frac{x.x_1}{b^2} + \frac{y.y_1}{a^2} = 1$ Putting values

$\frac{2\sqrt{5}}{3}\left(\frac{x}{4}\right) + 2\left(\frac{y}{9}\right) = 1$

$\frac{\sqrt{5}x}{3\times 2} + \frac{2y}{9} = 1$ $\qquad \times by\ 36$

$6(\sqrt{5}x) + 4(2y) = 36$

$6\sqrt{5}x + 8y = 36$

Equation of normal $\frac{b^2(x - x_1)}{x_1} = \frac{a^2(y - y_1)}{y_1}$

Putting $\frac{4\left(x - \frac{2\sqrt{5}}{3}\right)}{\frac{2\sqrt{5}}{3}} = \frac{9(y-2)}{2}$ cross multiplication

$2\times 4\left(x - \frac{2\sqrt{5}}{3}\right) = \frac{2\sqrt{5}}{3}\times 9(y-2)$

$8\left(x - \frac{2\sqrt{5}}{3}\right) = 6\sqrt{5}(y-2)$

$8x - \frac{16\sqrt{5}}{3} = 6\sqrt{5}y - 12\sqrt{5}$

$8x - 6\sqrt{5}y - \frac{16\sqrt{5}}{3} + \frac{3}{3}\times\frac{12\sqrt{5}}{1} = 0$

$8x - 6\sqrt{5}y + \frac{20\sqrt{5}}{3} = 0$

b). at a point $(0,2)$ to ellipse $\frac{x^2}{7} + \frac{y^2}{4} = 1$

Sol: We have $\frac{x^2}{7} + \frac{y^2}{4} = 1$

Compare with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ We get $a^2 = 7, b^2 = 4$

Equation of tangent $\frac{x.x_1}{a^2} + \frac{y.y_1}{b^2} = 1$ Putting values

$(0)\frac{x}{7} + (2)\frac{y}{4} = 1$

$0 + \frac{y}{2} = 1$

$y = 2$

Equation of normal $\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$

$\frac{7(x-0)}{0} = \frac{4(y-2)}{2}$

$2\times 7(x-0) = 0\times 4(y-2)$

$14x = 0$

$\Rightarrow x = 0$

c). at a point $\left(\sqrt{3}, \frac{1}{2}\right)$ to ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

Sol: We have $\frac{x^2}{4} + \frac{y^2}{1} = 1$

Compare with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ We get $a^2 = 4, b^2 = 1$

Equation of tangent $\frac{x.x_1}{a^2} + \frac{y.y_1}{b^2} = 1$ Putting values

$(\sqrt{3})\frac{x}{4} + \left(\frac{1}{2}\right)\frac{y}{1} = 1$ $\qquad \times by\ 4$

$\sqrt{3}x + 2y = 4$

Equation of normal $\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$

Putting $\frac{4(x - \sqrt{3})}{\sqrt{3}} = \frac{1(y - \frac{1}{2})}{\frac{1}{2}}$

$\frac{1}{2}\times 4(x-1) = \sqrt{3}(y - \frac{1}{2})$

$2x - 2 = \sqrt{3}y - \frac{\sqrt{3}}{2}$

$2x - \sqrt{3}y = 2 - \frac{\sqrt{3}}{2}$

Q7. Find the equation of tangent

a). to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which is perpendicular to the line $9x + 8y - 36 = 0$

Sol: We have ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ Multiply by 36

$9x^2 + 4y^2 = 36$ .... (1)

And the line $9x + 8y - 36 = 0$

So perpendicular line $8x - 9y + k = 0$

$9y = 8x + k$ $\qquad \Rightarrow y = \frac{8x+k}{9}$

Putting the value of y in equation (1)

$9x^2 + 4\left(\frac{8x+k}{9}\right)^2 = 36$

$9x^2 + 4\left(\frac{64x^2 + 16kx + k^2}{81}\right) = 36$ $\qquad \times by\ 4\times 81$

$81\times 9x^2 + 4(64x^2 + 16kx + k^2) = 81\times 36$

$729x^2 + 256x^2 + 64kx + 4k^2 = 2916$

$985x^2 + 64kx + 4k^2 - 2916 = 0$

For tangent $b^2 - 4ac = 0$

$(64k)^2 - 4(985)(4k^2 - 2916) = 0$

$4096k^2 - 15760k^2 + 11489040 = 0$

$-11664k^2 + 11489040 = 0$

$11664k^2 = 11489040$

$k^2 = 985$

$k = \pm\sqrt{985}$

Therefor equation of tangents

$8x - 9y \pm \sqrt{985} = 0$

b). to the ellipse $\frac{x^2}{7} + \frac{y^2}{4} = 1$ which is parallel to the line $6x + 21y - 14 = 0$

Sol: We have ellipse $\frac{x^2}{7}+\frac{y^2}{4}=1$ multiply by 28

$4x^2+7y^2=28$ ....(1)

Given line $6x+21y-14=0$

So the parallel line $6x+21y+k=0$

$21y=-6x-k \qquad \Rightarrow y=\frac{-6x-k}{21}$

Putting the value of y in equation (1)

$$4x^2+7\left(\frac{-6x-k}{21}\right)^2=28$$

$$4x^2+7\left(\frac{36x^2+k^2+12kx}{21\times 21}\right)=28$$

$$4x^2+\frac{36x^2+k^2+12kx}{3\times 21}=28 \qquad \times\ by\ 63$$

$$63\times 4x^2+36x^2+k^2+12k=28\times 63$$

$$252x^2+36x^2+12kx+k^2-1764=0$$

$$288x^2+12kx+k^2-1764=0$$

For tangent $b^2-4ac=0$

$$(12k)^2-4(288)(k^2-1764)=0$$

$$144k^2-4\times 2\times 144k^2+4\times 2\times 144\times 1764=0$$

$$144(1-8)k^2=-4\times 2\times 144\times 1764$$

$$-7k^2=-8\times 1764$$

$$k^2=8\times 252$$

$$k=\pm\sqrt{2016}=\pm\sqrt{16\times 126}$$

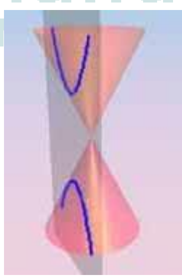
$$k=\pm 4\sqrt{126}$$

Therefor equation of tangent

$$6x+21y\pm 4\sqrt{126}=0$$

Eccentricity $e=\frac{c}{a}>1$

Hyperbola set of all points in a plane the difference of distance from two fixed points/foci F_1,F_2 is constant $(2a)$



Derivation of equation of Hyperbola

Let we place the foci on the x-axis at the points $F_1(-c,0)\ \&\ F_2(c,0)$ so that the origin is halfway between the foci. $P(x,y)$ be any point on hyperbola Let the difference of the distance from the point on the hyperbola to the foci be $2a$

$$|PF_1|-|PF_2|=2a$$

$$\sqrt{(x+c)^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Squaring both sides

$$(x+c)^2+y^2=4a^2+(x-c)^2+y^2+2(2a)\sqrt{(x-c)^2+y^2}$$

$$(x+c)^2-(x-c)^2=4a^2+4a\sqrt{(x-c)^2+y^2}$$

$$4cx=4\left\{a^2+a\sqrt{(x-c)^2+y^2}\right\}$$

$$cx=a^2+a\sqrt{(x-c)^2+y^2}$$

$$cx-a^2=a\sqrt{(x-c)^2+y^2}$$

Squaring again

$$\left\{a\sqrt{(x-c)^2+y^2}\right\}^2=(cx-a^2)^2$$

$$a^2\left\{x^2+c^2-2cx+y^2\right\}=c^2x^2+a^4-2a^2cx$$

$$a^2x^2+a^2c^2-2a^2cx+a^2y^2=c^2x^2+a^4-2a^2cx$$

$$a^2c^2-a^4=c^2x^2-a^2x^2-a^2y^2$$

$$a^2(c^2-a^2)=(c^2-a^2)x^2-a^2y^2$$

$$a^2b^2=b^2x^2-a^2y^2 \qquad \therefore b^2=c^2-a^2$$

Dividing both sides by a^2b^2

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Derivation of equation of tangent on Hyperbola

We have $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Multiply both sides by a^2b^2

$b^2x^2-a^2y^2=a^2b^2$........(1) Differentiating w.r.t x

$$2b^2x-2a^2y\frac{dy}{dx}=0$$

$$2a^2y\frac{dy}{dx}=2b^2x$$

$$\frac{dy}{dx}=\frac{2b^2x}{2a^2y}$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Since derivative at the point $P(x_1,y_1)$ gives slope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{b^2x_1}{a^2y_1}$$

We know that equation of line having slope and passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1(y-y_1)=b^2x_1(x-x_1)$$

$$a^2y_1y-a^2y_1^2=b^2x_1x-b^2x_1^2$$

$$b^2x_1x-a^2y_1y=b^2x_1^2-a^2y_1^2$$

$$b^2x_1x-a^2y_1y=a^2b^2 \qquad \therefore b^2x_1^2-a^2y_1^2=a^2b^2$$

Dividing by a^2b^2

$$\frac{x.x_1}{a^2}-\frac{y.y_1}{b^2}=1$$

Equation of normal using slope $m=\frac{b^2x_1}{a^2y_1}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$$y-y_1=\frac{-a^2y_1}{b^2x_1}(x-x_1)$$

Or $\quad \frac{y-y_1}{x-x_1}=\frac{-a^2y_1}{b^2x_1}$

Derivation of equation of tangent on Hyperbola

We have $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ Multiply both sides by a^2b^2

$b^2y^2-a^2x^2=a^2b^2$...........(1) Differentiating w.r.t x

$$2b^2y\frac{dy}{dx}-2a^2x=0$$

$$2b^2y\frac{dy}{dx}=2a^2x$$

$$\frac{dy}{dx}=\frac{2a^2x}{2b^2y}$$

$$\frac{dy}{dx}=\frac{a^2x}{b^2y}$$

Since derivative at the point $P(x_1,y_1)$ gives slope

$$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{a^2x_1}{b^2y_1}$$

We know that equation of line having slope and passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$$y-y_1=\frac{a^2x_1}{b^2y_1}(x-x_1)$$

$$b^2y_1(y-y_1)=a^2x_1(x-x_1)$$

$$b^2y_1y-b^2y_1^2=a^2x_1x-a^2x_1^2$$

$$b^2y_1y-a^2x_1x=b^2y_1^2-a^2x_1^2$$

$$b^2y_1y-a^2x_1x=a^2b^2 \qquad \therefore b^2y_1^2-a^2x_1^2=a^2b^2$$

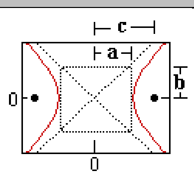
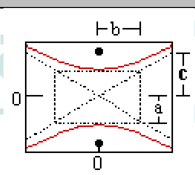
Dividing by a^2b^2

$$\frac{y.y_1}{a^2}-\frac{x.x_1}{b^2}=1$$

Equation of normal using slope $m=\frac{a^2x_1}{b^2y_1}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$y-y_1=\frac{-b^2y_1}{a^2x_1}(x-x_1)$ or $\frac{y-y_1}{x-x_1}=\frac{-b^2y_1}{a^2x_1}$

Along x-axis	Hyperbola	Aong y-axis
	Shape $c^2=a^2+b^2$	
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$c=ae$ Equation	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
$(0,0)$	Centre	$(0,0)$
$(\pm c,0)$	Foci,F,F'	$(0,\pm c)$
$x=\pm\frac{a}{e}=\frac{c}{e^2}$	Eq of Directrix, l,l'	$y=\pm\frac{a}{e}=\frac{c}{e^2}$
$(\pm a,0)$	Vertices A,A'	$(0,\pm a)$
$\lvert 2a\rvert$	Length of Transverse axis	$\lvert 2a\rvert$
$\lvert 2b\rvert$	Length of Conjugate axis	$\lvert 2b\rvert$
$\left(\pm c,\pm\frac{b^2}{a}\right)$	P,P',Q,Q' end points of Letra Recta	$\left(\pm\frac{b^2}{a},\pm c\right)$
$\left\lvert\frac{2b^2}{a}\right\rvert$	Length of Latus Rectum	$\left\lvert\frac{2b^2}{a}\right\rvert$
$x=\pm c$	Eq of Latus Rectum	$y=\pm c$
$\lvert 2c\rvert$	Distance b/w Foci	$\lvert 2c\rvert$
$\frac{x.x_1}{a^2}-\frac{y.y_1}{b^2}=1$	Tangent at (x_1,y_1)	$\frac{y_1y}{b^2}-\frac{x_1x}{a^2}=1$
$\frac{y-y_1}{x-x_1}=\frac{-a^2y_1}{b^2x_1}$	normal at (x_1,y_1)	$\frac{y-y_1}{x-x_1}=\frac{-b^2y_1}{a^2x_1}$
$y=\pm\frac{b}{a}x$	asymotote	$x=\pm\frac{a}{b}y$

Exercise 8.3

Q1. In each case, sketch the hyperbola represented by the equation, indicate the center, foci and the equations of asymptotes.

a). $\frac{x^2}{4}-\frac{y^2}{9}=1$

Sol: We have $\frac{x^2}{4}-\frac{y^2}{9}=1$

Compare with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

We get $a^2=4,\qquad b^2=9$

$\Rightarrow a=2,\qquad b=3$

Centre $(h,k)=(0,0)$

Using $c^2=a^2+b^2$

$c^2=2^2+3^2$

$c^2=4+9$

$c=\pm\sqrt{13}$

Foci $F(\pm c,0)=F(\pm\sqrt{13},0)$

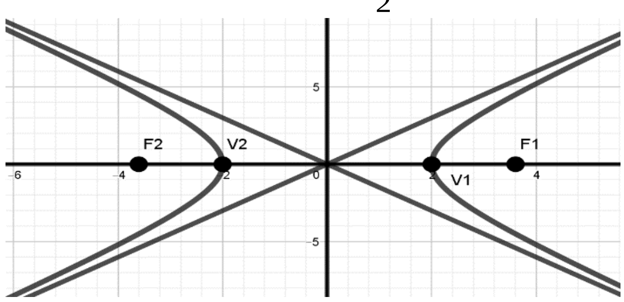
Foci are $F_1(\sqrt{13},0),F_2(-\sqrt{13},0)$

Vertices $V(\pm a,0)=V(\pm 2,0)$

Vertices are $V_1(2,0),V_2(-2,0)$

Equation of asymptotes $y=\pm\frac{b}{a}x$ putting values

$y=\pm\frac{3}{2}x$



b). $\frac{y^2}{25}-\frac{x^2}{4}=1$

Sol: We have $\frac{y^2}{25}-\frac{x^2}{4}=1$

Compare with $\frac{-x^2}{b^2}+\frac{y^2}{a^2}=1$

We get $a^2=25,\qquad b^2=4$

$\Rightarrow a=5,\qquad b=2$

Centre $(h,k)=(0,0)$

Using $c^2 = a^2 + b^2$

$$c^2 = 2^2 + 5^2$$
$$c^2 = 4 + 25$$
$$c^2 = 29$$
$$c = \pm\sqrt{29}$$

Foci $F(0, \pm c) = F(0, \pm\sqrt{29})$

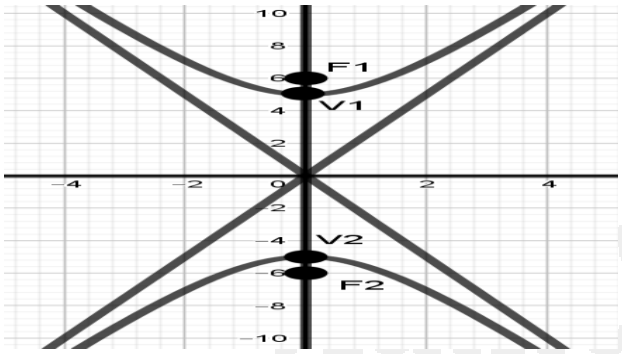
Foci are $F_1(0, \sqrt{29}), F_2(0, -\sqrt{29})$

Vertices $V(0, \pm a) = V(0, \pm 5)$

Vertices are $V_1(0,5), V_2(0,-5)$

Equation of asymptotes $y = \pm\dfrac{a}{b}x$ putting values

$$y = \pm\frac{5}{2}x$$



c). $\dfrac{(x-2)^2}{9} - \dfrac{(y-3)^2}{16} = 1$

Sol: We have $\dfrac{(x-2)^2}{9} - \dfrac{(y-3)^2}{16} = 1$

Compare with $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

We get $a^2 = 9$, $b^2 = 16$

$\Rightarrow a = 3$, $b = 4$

And $h = 2$, $k = 3$

Centre $(h,k) = (2,3)$

Using $c^2 = a^2 + b^2$

$$c^2 = 3^2 + 4^2$$
$$c^2 = 9 + 16$$
$$c^2 = 25$$
$$c = \pm 5$$

Foci $F(h \pm c, k) = F(2 \pm 5, 3)$

Foci are $F_1(7,3), F_2(-3,3)$

Vertices $V(h \pm a, k) = V(2 \pm 3, 3)$

Vertices are $V_1(5,3), V_2(-1,3)$

Eq of asymptotes $(y-k) = \pm\dfrac{b}{a}(x-h)$ putting

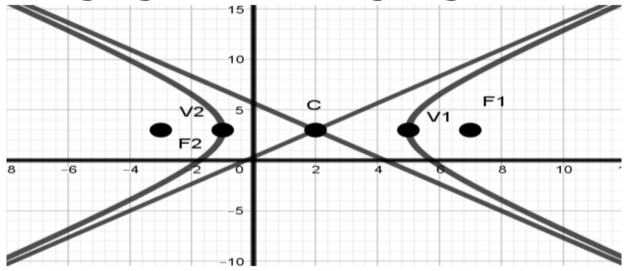
$$y - 3 = \pm\tfrac{4}{3}(x-2)$$
$$y = \pm\tfrac{4}{3}(x-2) + 3$$

Either or

$y = \dfrac{4}{3}(x-2) + 3$ $\quad$ $y = -\dfrac{4}{3}(x-2) + 3$

$y = \dfrac{4x}{3} - \dfrac{8}{3} + 3$ $\quad$ $y = \dfrac{-4x}{3} + \dfrac{8}{3} + 3$

$y = \dfrac{4x}{3} + \dfrac{1}{3}$ $\quad$ $y = \dfrac{-4x}{3} + \dfrac{17}{3}$



d). $\dfrac{(y+1)^2}{16} - \dfrac{(x+3)^2}{25} = 1$

Sol: We have $\dfrac{(y+1)^2}{16} - \dfrac{(x+3)^2}{25} = 1$

Compare with $\dfrac{-x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

We get $a^2 = 16$, $b^2 = 25$

$\Rightarrow a = 4$, $b = 5$

And $h = -3$, $k = -1$

Centre $(h,k) = (-3,-1)$

Using $c^2 = a^2 + b^2$

$$c^2 = 16 + 25$$
$$c^2 = 41$$
$$c = \pm\sqrt{41}$$

Foci $F(h, k \pm c) = F(-3, -1 \pm \sqrt{41})$

Foci are $F_1(-3, -1+\sqrt{41}), F_2(-3, -1-\sqrt{41})$

Vertices $V(h, k \pm a) = V(-3, -1 \pm 4)$

Vertices are $V_1(-3,3), V_2(-3,-5)$

Eq of asymptotes $y - k = \pm\dfrac{a}{b}(x-h)$ putting

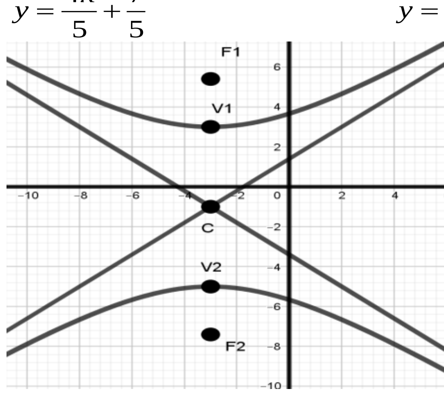
$$y + 1 = \pm\frac{4}{5}(x+3)$$

Either or

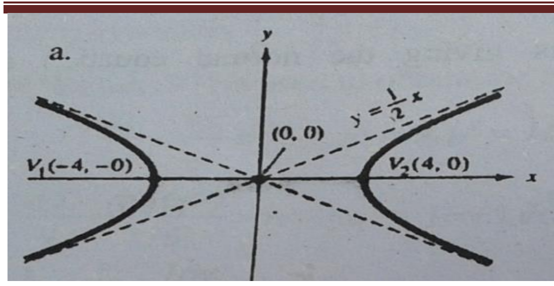
$y + 1 = \dfrac{4}{5}(x+3)$ $\quad$ $y + 1 = -\dfrac{4}{5}(x+3)$

$y = \dfrac{4x}{5} + \dfrac{12}{5} - 1$ $\quad$ $y = \dfrac{-4x}{5} - \dfrac{12}{5} - 1$

$y = \dfrac{4x}{5} + \dfrac{7}{5}$ $\quad$ $y = \dfrac{-4x}{5} - \dfrac{17}{5}$



Q2. In each case, determine equation of graphed hyperbola



Sol: curve is along x-axis so, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$........(1)

from the figure centre $(0,0)$

Vertices $V(\pm a,0)=(\pm 4,0) \quad \Rightarrow a=4$

equation of asymptote $y=\frac{1}{2}x$ or $y=\frac{2}{4}x$

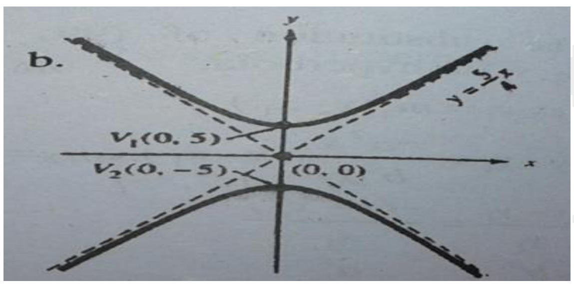
Now compare with $y=\frac{b}{a}x$ we get

$b=2$ because a must be 4

Putting the values in equation (1) we get

$$\frac{x^2}{4^2}-\frac{y^2}{2^2}=1$$

$$\frac{x^2}{16}-\frac{y^2}{4}=1$$



Sol: curve is along y-axis so, $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

from the figure centre $(0,0)$

Vertices $V(0,\pm a)=(0,\pm 5) \quad \Rightarrow a=5$

equation of asymptote $y=\frac{5}{4}x$

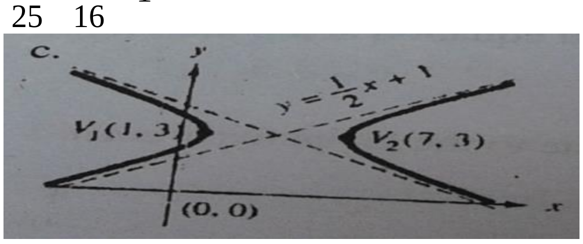
Now compare with $y=\frac{a}{b}x$ we get

$b=4$ & $a=5$

Putting the values in equation (1) we get

$$\frac{y^2}{5^2}-\frac{x^2}{4^2}=1$$

$$\frac{y^2}{25}-\frac{x^2}{16}=1$$



Sol: curve is parallel to x-axis so,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1\text{....................}(1)$$

Vertices are $(1,3),(7,3)$

Midpoint of vertices is centre

$$(h,k)=\left(\frac{1+7}{2},\frac{3+3}{2}\right)$$

$$(h,k)=\left(\frac{8}{2},\frac{6}{2}\right)$$

$$(h,k)=(4,3)$$

Vertices $V(h\pm a,k)$ are $(1,3),(7,3)$

$h-a=1,$ $\qquad h+a=7$ and $k=3$

$4-a=1$

$4-1=a$

$a=3$

Equation of asymptote $y=\frac{1}{2}x+1$..................(2)

Since $y-k=\frac{b}{a}(x-h)$ putting the values

$$y-3=\frac{b}{3}(x-4)$$

$$y=\frac{b}{3}x-\frac{4b}{3}+3\text{...........................}(3)$$

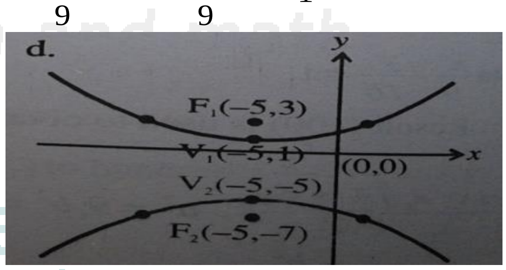
Comparing equation (2) and (3) we get

$$\frac{b}{3}=\frac{1}{2} \qquad \Rightarrow b=\frac{3}{2}$$

Putting the values in equation (1) we get

$$\frac{(x-4)^2}{3^2}-\frac{(y-3)^2}{\left(\frac{3}{2}\right)^2}=1$$

$$\frac{(x-4)^2}{9}-\frac{4(y-3)^2}{9}=1$$



Sol: curve is parallel to y-axis so,

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1\text{...............................}(1)$$

Vertices are $(-5,-5),(-5,1)$

Midpoint of vertices is centre

$$(h,k)=\left(\frac{-5-5}{2},\frac{-5+1}{2}\right)$$

$$(h,k)=\left(\frac{-10}{2},\frac{-4}{2}\right)$$

$$(h,k)=(-5,-2)$$

Vertices $V(h,k\pm a)$ are $(-5,-5),(-5,1)$

$h=-5$ and $k-a=-5,$ $\qquad k+a=1$

$-2-a=-5$

$-2+5=a$

$a=3$

From the figure Foci are $(-5,3),(-5,-7)$

Compare with $(h,k\pm c)$ we get

$h=-5$ and $k+c=3,$ $\qquad k-c=-7$

$-2+c=3$

$c=5$

Using $c^2=a^2+b^2$

$5^2 = 3^2 + b^2$

$25 = 9 + b^2$

$25 - 9 = b^2$

$b^2 = 16$

$b = 4$

Putting the values in equation (1) we get

$$\frac{(y-(-2))^2}{3^2} - \frac{(x-(-5))^2}{4^2} = 1$$

$$\frac{(y+2)^2}{9} - \frac{(x+5)^2}{16} = 1$$

Q3. In each case, write the equation of hyperbola through the given information:

a)foci are $(0,3)$ & $(0,-3)$ one of vertex is at $(0,-2)$

Sol: foci are $(0,3)$ and $(0,-3)$

So the curve/hyperbola is along y-axis

Midpoint of foci is centre

$$(h,k) = \left(\frac{0+0}{2}, \frac{3+(-3)}{2}\right)$$

$(h,k) = (0,0)$

Compare with $F(0,\pm c) = F(0,\pm 3)$ we get $c = 3$

Vertex is at $(0,-2)$

Compare with $(0,-a)$ we get $a = 2$

Using $c^2 = a^2 + b^2$

$3^2 = 2^2 + b^2$

$9 = 4 + b^2$

$9 - 4 = b^2$

$b^2 = 5$

$b = \sqrt{5}$

general equation of hyperbola along y-axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Putting the values in the

$$\frac{(y-0)^2}{2^2} - \frac{(x-0)^2}{(\sqrt{5})^2} = 1$$

$$\frac{y^2}{4} - \frac{x^2}{5} = 1$$

---

b)foci are $(\sqrt{2},0)$ & $(-\sqrt{2},0)$ one of vertex is at $(1,0)$

Sol: foci are $(\sqrt{2},0)$ and $(-\sqrt{2},0)$

So the curve/hyperbola is along x-axis

Midpoint of foci is centre

$$(h,k) = \left(\frac{\sqrt{2}+(-\sqrt{2})}{2}, \frac{0+0}{2}\right)$$

$(h,k) = (0,0)$

Comparing $F(\pm c,0) = F(\pm\sqrt{2},0)$ we get $c = \sqrt{2}$

Vertex is at $(1,0)$

Compare with $(a,0)$ we get $a = 1$

Using $c^2 = a^2 + b^2$

$(\sqrt{2})^2 = 1^2 + b^2$

$2 = 1 + b^2$

$2 - 1 = b^2$

$b^2 = 1$

$b = 1$

general equation of hyperbola along x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Putting the values in the

$$\frac{(x-0)^2}{1^2} - \frac{(y-0)^2}{1^2} = 1$$

$x^2 - y^2 = 1$

---

c)vertices are at $(5,0)$ & $(-5,0)$ one of focus is at $(-7,0)$

Sol: vertices are $(5,0)$ and $(-5,0)$

So the curve/hyperbola is along x-axis

Midpoint of vertices is centre

$$(h,k) = \left(\frac{5+(-5)}{2}, \frac{0+0}{2}\right)$$

$(h,k) = (0,0)$

Compare with $F(\pm c,0) = F(7,0)$ we get $c = 7$

Vertex is at $(5,0)$

Compare with $(a,0)$ we get $a = 5$

Using $c^2 = a^2 + b^2$

$7^2 = 5^2 + b^2$

$49 = 25 + b^2$

$49 - 25 = b^2$

$b^2 = 24$

$b = \sqrt{24}$

general equation of hyperbola along x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Putting the values in the

$$\frac{(x-0)^2}{5^2} - \frac{(y-0)^2}{(\sqrt{24})^2} = 1$$

$$\frac{x^2}{25} - \frac{y^2}{24} = 1$$

---

d). vertices are at $(3,0)$ and $(-3,0)$, and asymptotes are the lines $y = 3x$ and $y = -3x$

Sol: vertices are $(3,0)$ and $(-3,0)$

So the curve/hyperbola is along x-axis

Midpoint of vertices is centre

$$(h,k) = \left(\frac{3+(-3)}{2}, \frac{0+0}{2}\right)$$

$(h,k) = (0,0)$

Vertices are at $(\pm 3,0)$

Compare with $(\pm a,0)$ we get $a = 3$

asymptotes are the lines

$y = \pm 3x$..........................(1)

Since $y = \pm\frac{b}{a}x$ put $a = 3$ we get

$y=\pm\frac{b}{3}x$..(2)

Comparing (1) and (2) we get

$\frac{b}{3}=3 \qquad \Rightarrow b=9$

general equation of hyperbola along x-axis

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Putting the values in the

$\frac{(x-0)^2}{3^2}-\frac{(y-0)^2}{9^2}=1$

$\frac{x^2}{9}-\frac{y^2}{81}=1$

---

e). Transverse axis is the x-axis, asymptotes are the lines $y=3x$ and $y=-3x$

Sol: Transverse axis is the x-axis, so

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$...............................(1)

asymptotes are the lines

$y=\pm 3x$..........................(2)

Since $y=\pm\frac{b}{a}x$..(3)

Comparing (2) and (3) we get

$b=3 \qquad a=1$

Putting the values in equation (1) we get

$\frac{x^2}{1^2}-\frac{y^2}{3^2}=1$

$\frac{x^2}{1}-\frac{y^2}{9}=1$

---

f). Transverse axis is the y-axis, asymptotes are the lines $y=3x$ and $y=-3x$

Sol: Transverse axis is the y-axis, so

$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$...............................(1)

asymptotes are the lines

$y=\pm 3x$..........................(2)

Since $y=\pm\frac{b}{a}x$..(3)

Comparing (2) and (3) we get

$b=3 \qquad a=1$

Putting the values in equation (1) we get

$\frac{y^2}{3^2}-\frac{x^2}{1^2}=1$

$\frac{y^2}{9}-\frac{x^2}{1}=1$

---

g). Foci are at $(5,0)$ and $(-5,0)$, eccentricity is $\frac{5}{3}$

Sol: foci are $(5,0)$ and $(-5,0)$

So the curve/hyperbola is along x-axis

Midpoint of foci is centre

$(h,k)=\left(\frac{5+(-5)}{2},\frac{0+0}{2}\right)$

$(h,k)=(0,0)$

Compare with $F(\pm c,0)=F(\pm 5,0)$ we get $c=5$

eccentricity $e=\frac{5}{3}$

$\therefore c=ae$ putting the values

$5=a\left(\frac{5}{3}\right) \qquad \Rightarrow a=3$

Using $c^2=a^2+b^2$

$5^2=3^2+b^2$

$25=9+b^2$

$25-9=b^2$

$b^2=16$

$b=4$

general equation of hyperbola along x-axis

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Putting the values

$\frac{(x-0)^2}{3^2}-\frac{(y-0)^2}{4^2}=1$

$\frac{x^2}{9}-\frac{y^2}{16}=1$

---

h). vertices are at $(9,0)$ and $(-9,0)$, and asymptotes are perpendicular to each other.

Sol: vertices are $(9,0)$ and $(-9,0)$

So the curve/hyperbola is along x-axis

Midpoint of vertices is centre

$(h,k)=\left(\frac{9+(-9)}{2},\frac{0+0}{2}\right)$

$(h,k)=(0,0)$

Vertex is at $(\pm 9,0)$

Compare with $(\pm a,0)$ we get $a=9$

asymptotes $y=\pm\frac{b}{a}x$

either $y=\frac{b}{a}x$ or $y=\frac{-b}{a}x$

Slope are

$m_1=\frac{b}{a} \qquad m_2=\frac{-b}{a}$

Given asymptotes are perpendicular to each other

$m_1.m_2=-1$

$\left(\frac{b}{a}\right)\left(\frac{-b}{a}\right)=-1$

$\Rightarrow b^2=a^2$

Put $a=9$ we get

$b^2=9^2$

$b=9$

general equation of hyperbola along x-axis

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Putting the values

$\frac{(x-0)^2}{9^2}-\frac{(y-0)^2}{9^2}=1$

$\frac{x^2}{81}-\frac{y^2}{81}=1$

Q4. Determine the path of a point that moves so that the difference of its distance from a). the points $(-5,0)$ and $(5,0)$ is 8

Sol: Length of transverse axis $2a=8$

$a=4$

foci are $(5,0)$ and $(-5,0)$

So the curve/hyperbola is along x-axis
Midpoint of foci is centre

$(h,k)=\left(\frac{5+(-5)}{2},\frac{0+0}{2}\right)$

$(h,k)=(0,0)$

Compare with $F(\pm c,0)=F(\pm 5,0)$ we get $c=5$

Using $c^2=a^2+b^2$

$5^2=4^2+b^2$

$25=16+b^2$

$25-16=b^2$

$b^2=9$

$b=3$

general equation of hyperbola along x-axis

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Putting the values

$\frac{(x-0)^2}{4^2}-\frac{(y-0)^2}{3^2}=1$

$\frac{x^2}{16}-\frac{y^2}{9}=1$

b). the points $(0,-13)$ and $(0,13)$ is 10

Sol: Length of transverse axis $2a=10$

$a=5$

foci are $(0,-13)$ and $(0,13)$

So the curve/hyperbola is along y-axis
Midpoint of foci is centre

$(h,k)=\left(\frac{0+0}{2},\frac{-13+13}{2}\right)$

$(h,k)=(0,0)$

Comparing $F(0,\pm c)=F(0,\pm 13)$ we get $c=13$

Using $c^2=a^2+b^2$

$13^2=5^2+b^2$

$169=25+b^2$

$169-25=b^2$

$b^2=144$

$b=12$

general equation of hyperbola along x-axis

$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

Putting the values

$\frac{(y-0)^2}{5^2}-\frac{(x-0)^2}{12^2}=1$

$\frac{y^2}{25}-\frac{x^2}{144}=1$

Q5. Write an equation of hyperbola

a). with vertices at $(2,-2)$ and $(-4,-2)$, and that passes through point with coordinate $(5,1)$

Sol: vertices are $(2,-2)$ and $(-4,-2)$

So the curve/hyperbola is along x-axis
Midpoint of vertices is centre

$(h,k)=\left(\frac{2+(-4)}{2},\frac{-2+(-2)}{2}\right)$

$(h,k)=\left(\frac{-2}{2},\frac{-4}{2}\right)$

$(h,k)=(-1,-2)$

vertices are $(2,-2)$ and $(-4,-2)$

Compare with $(h\pm a,k)$ we get

$h-a=-4,\quad h+a=2,\quad k=-2$

$-1-a=-4$

$-1+4=a$

$a=3$

general equation of hyperbola along x-axis

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ ....(1)

Putting centre $h=-1,k=-2$ & $(5,1)$ with $a=3$

$\frac{(5-(-1))^2}{3^2}-\frac{(1-(-2))^2}{b^2}=1$

$\frac{(6)^2}{9}-\frac{(3)^2}{b^2}=1$

$\frac{36}{9}-1=\frac{9}{b^2}$

$4-1=\frac{9}{b^2}$

$3b^2=9$

$b=\sqrt{3}$

Putting the values in equation (1) we get

$\frac{(x-(-1))^2}{3^2}-\frac{(y-(-2))^2}{(\sqrt{3})^2}=1$

$\frac{(x+1)^2}{9}-\frac{(y+2)^2}{3}=1$

b). with vertices at $(-3,1)$ and $(-3,3)$, and that passes through the point with coordinate $(0,4)$

Sol: vertices are $(-3,1)$ and $(-3,3)$

So the curve/hyperbola is along y-axis
Midpoint of vertices is centre

$(h,k)=\left(\frac{-3+(-3)}{2},\frac{1+3}{2}\right)$

$(h,k)=\left(\frac{-6}{2},\frac{4}{2}\right)$

$(h,k)=(-3,2)$

vertices are $(-3,1)$ and $(-3,3)$

Compare with $(h,k\pm a)$ we get

$h=-3,\quad k+a=3,\quad k-a=1$

$2+a=3$

$a=1$

general equation of hyperbola along y-axis

$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ ....(1)

Putting centre $h=-3,k=2$ & $(0,4)$ with $a=1$

$$\frac{(4-2)^2}{1^2}-\frac{(0-(-3))^2}{b^2}=1$$

$$\frac{(2)^2}{1}-\frac{(3)^2}{b^2}=1$$

$$4-\frac{9}{b^2}=1$$

$$4-1=\frac{9}{b^2}$$

$$3b^2=9$$

$$b^2=3$$

$$b=\sqrt{3}$$

Putting the values in equation (1) we get

$$\frac{(y-2)^2}{1^2}-\frac{(x-(-3))^2}{(\sqrt{3})^2}=1$$

$$\frac{(y-2)^2}{1}-\frac{(x+3)^2}{3}=1$$

Q6. In each case, sketch the rectangular hyperbola and identify the vertices, the foci and the asymptotes:

a). $(x+1)^2-(y-2)^2=1$

Sol: We have $(x+1)^2-(y-2)^2=1$

general equation of hyperbola

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ We get

$h=-1, k=2,$ $a^2=1, b^2=1$

$a=1, b=1$

Using $c^2=a^2+b^2$

$c^2=1^2+1^2$

$c^2=1+1$

$c^2=2$

$c=\sqrt{2}$

And Centre $(h,k)=(-1,2)$

Foci $F(h\pm c,k)=F(-1\pm\sqrt{2},2)$

Foci are $F_1(-1+\sqrt{2},2), F_2(-1-\sqrt{2},2)$

Vertices $V(h\pm a,k)=V(-1\pm 1,2)$

Vertices are $V_1(0,2), V_2(-2,2)$

Eq of asymptotes $(y-k)=\pm\frac{b}{a}(x-h)$ putting

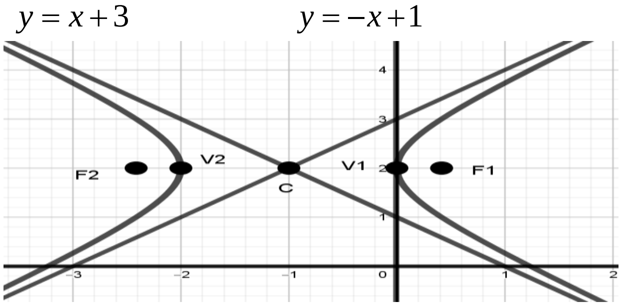
$y-2=\pm\frac{1}{1}(x-(-1))$

$y=\pm(x+1)+2$

Either or

$y=x+1+2$ $y=-x-1+2$

$y=x+3$ $y=-x+1$



b). $\frac{(x-3)^2}{4}-\frac{(y+1)^2}{4}=1$

Sol: We have $\frac{(x-3)^2}{4}-\frac{(y+1)^2}{4}=1$

general equation of hyperbola

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ We get

$h=3, k=-1,$ $a^2=4, b^2=4$

$a=2, b=2$

Using $c^2=a^2+b^2$

$c^2=2^2+2^2$

$c=\sqrt{8}=\sqrt{4\times 2}$

$c=2\sqrt{2}$

And Centre $(h,k)=(3,-1)$

Foci $F(h\pm c,k)=F(3\pm 2\sqrt{2},-1)$

Foci are $F_1(3+2\sqrt{2},-1), F_2(3-2\sqrt{2},-1)$

Vertices $V(h\pm a,k)=V(3\pm 2,-1)$

Vertices are $V_1(5,-1), V_2(1,-1)$

Eq of asymptotes $(y-k)=\pm\frac{b}{a}(x-h)$ putting

$y-(-1)=\pm\frac{2}{2}(x-3)$

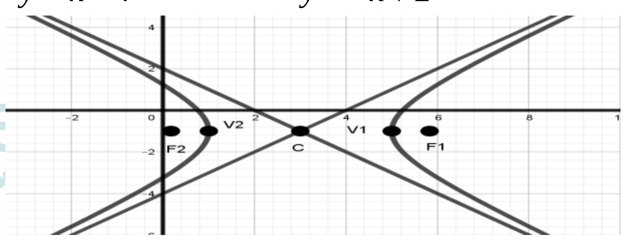
$y+1=\pm(x-3)$

$y=\pm(x-3)-1$

Either or

$y=x-3-1$ $y=-x+3-1$

$y=x-4$ $y=-x+2$



Q7. In each case, find the points of intersection between the line and the hyperbola

a). $xy=4,$ $y=x-3$

Sol: We have $xy=4$..............................(1)

$y=x-3$.............................(2)

Putting the value of y in equation (1), we get

$x(x-3)=4$

$x^2-3x-4=0$

$x^2-4x+1x-4=0$

$x(x-4)+1(x-4)=0$

$(x+1)(x-4)=0$

Either or

$x+1=0$ $x-4=0$

$x=-1$ $x=4$

Putting the values of x in equation (2) we get

$y=-1-3$ $y=4-3$

$y=-4$ $y=1$

Thus points of intersection are $(-1,-4)\&(4,1)$

b). $x^2 - y^2 = 1 \qquad y = \sqrt{3}(x-1)$

Sol: We have $x^2 - y^2 = 1$..............................(1)

$y = \sqrt{3}(x-1)$..(2)

Putting the value of y in equation (1), we get

$x^2 - \left(\sqrt{3}(x-1)\right)^2 = 1$

$x^2 - 3(x-1)^2 = 1$

$x^2 - 3(x^2 - 2x + 1) = 1$

$x^2 - 3x^2 + 6x - 3 - 1 = 0$

$-2x^2 + 6x - 4 = 0 \qquad \div \; by \; -2$

$x^2 - 3x + 2 = 0$

$x^2 - 2x - 1x + 2 = 0$

$x(x-2) - 1(x-2) = 0$

$(x-1)(x-2) = 0$

Either $x - 1 = 0$, $x = 1$ or $x - 2 = 0$, $x = 2$

Putting the values of x in equation (2) we get

$y = \sqrt{3}(1-1)$, $y = \sqrt{3}(0)$, $y = 0$

$y = \sqrt{3}(2-1)$, $y = \sqrt{3}(1)$, $y = \sqrt{3}$

Thus points of intersection are $(1,0)$ & $(2,\sqrt{3})$

c). $\frac{y^2}{1} - \frac{x^2}{4} = 1 \qquad x + 4y - 4 = 0$

Sol: We have $\frac{y^2}{1} - \frac{x^2}{4} = 1$............................(1)

$x + 4y - 4 = 0$

$x = 4 - 4y$

$x = 4(1-y)$......................(2)

Putting the value of x in equation (1), we get

$\frac{y^2}{1} - \frac{\left(4(1-y)\right)^2}{4} = 1$

$\frac{y^2}{1} - \frac{16(1-y)^2}{4} = 1$

$y^2 - 4(1-y)^2 = 1$

$y^2 - 4(1 - 2y + y^2) = 1$

$y^2 - 4 + 8y - 4y^2 - 1 = 0$

$-3y^2 + 8y - 5 = 0$

$3y^2 - 8y + 5 = 0$

$3y^2 - 3y - 5y + 5 = 0$

$3y(y-1) - 5(y-1) = 0$

$(3y-5)(y-1) = 0$

Either $3y - 5 = 0$, $y = \frac{5}{3}$ or $y - 1 = 0$, $y = 1$

Putting the values of y in equation (2) we get

$x = 4\left(1 - \frac{5}{3}\right)$

$x = 4\left(\frac{3-5}{3}\right)$

$x = 4\left(\frac{-2}{3}\right)$

$x = \frac{-8}{3}$

$x = 4(1-1)$, $x = 4(0)$, $x = 0$

Thus points of intersection are $\left(\frac{-8}{3}, \frac{5}{3}\right)$ & $(0,1)$

d). $\frac{y^2}{9} - \frac{x^2}{4} = 1 \qquad y = 3(x+1)$

Sol: We have $\frac{y^2}{9} - \frac{x^2}{4} = 1$..........................(1)

$y = 3(x+1)$...........................(2)

Putting the value of y in equation (1), we get

$\frac{\left(3(x+1)\right)^2}{9} - \frac{x^2}{4} = 1$

$\frac{9(x+1)^2}{9} - \frac{x^2}{4} = 1$

$(x+1)^2 - \frac{x^2}{4} = 1 \qquad \times \; by \; 4$

$4(x^2 + 2x + 1) - x^2 = 4$

$4x^2 + 8x + 4 - x^2 - 4 = 0$

$3x^2 + 8x = 0$

$x(3x+8) = 0$

Either $x = 0$ or $3x + 8 = 0$, $x = \frac{-8}{3}$

Putting the values of x in equation (2) we get

$y = 3(0+1)$, $y = 3(1)$, $y = 3$

$y = 3\left(\frac{-8}{3} + 1\right)$, $y = 3\left(\frac{-8+3}{3}\right)$, $y = -5$

Thus points of intersection are $(0,3)$ & $\left(\frac{-8}{3}, -5\right)$

Q8. For what value of c

a). line $y = x + c$ will touch hyperbola $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Sol: We have $\frac{x^2}{16} - \frac{y^2}{4} = 1$.........................(1)

$y = x + c$.......................................(2)

Putting the value of y in equation (1), we get

$\frac{x^2}{16} - \frac{(x+c)^2}{4} = 1 \qquad \times \; by \; 16$

$x^2 - 4(x+c)^2 = 16$

$x^2 - 4(x^2 + 2cx + c^2) = 16$

$x^2 - 4x^2 - 8cx - 4c^2 - 16 = 0$

$-3x^2 - 8cx - 4c^2 - 16 = 0$

$3x^2 + 8cx + 4c^2 + 16 = 0$

Given line will touch hyperbola if Discriminant = 0

$b^2 - 4ac = 0$ putting the values

$(8c)^2-4(3)(4c^2+16)=0$

$64c^2-12(4c^2+16)=0$

$64c^2-48c^2-192=0$

$16c^2=192$

$c^2=\frac{192}{16}=12$

$c=\pm\sqrt{4\times 3}$

$c=\pm 2\sqrt{3}$

b). line $y=-x+c$ will touch hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$

Sol: We have $\frac{x^2}{16}-\frac{y^2}{9}=1$..........................(1)

$y=-x+c$

$y=c-x$..................................(2)

Putting the value of y in equation (1), we get

$\frac{x^2}{16}-\frac{(c-x)^2}{9}=1 \qquad \times by\ 144$

$9x^2-16(c^2-2cx+x^2)=144$

$9x^2-16c^2+32cx-16x^2-144=0$

$-7x^2+32cx-16c^2-144=0$

$7x^2-32cx+16c^2+144=0$

Given line will touch hyperbola if Discriminant = 0

$b^2-4ac=0$ putting the values

$(-32c)^2-4(7)(16c^2+144)=0$

$1024c^2-28(16c^2+144)=0$

$1024c^2-448c^2-4032=0$

$576c^2=4032$

$c^2=\frac{4032}{576}=7$

$c=\pm\sqrt{7}$

c). line $y=x+c$ will touch hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$

Sol: We have $\frac{x^2}{4}-\frac{y^2}{1}=1$..........................(1)

$y=x+c$.......................................(2)

Putting the value of y in equation (1), we get

$\frac{x^2}{4}-\frac{(x+c)^2}{1}=1 \qquad \times by\ 4$

$x^2-4(x^2+2cx+c^2)=4$

$x^2-4x^2-8cx-4c^2-4=0$

$-3x^2-8cx-4c^2-4=0$

$3x^2+8cx+4c^2+4=0$

Given line will touch hyperbola if Discriminant = 0

$b^2-4ac=0$ putting the values

$(8c)^2-4(3)(4c^2+4)=0$

$64c^2-12(4c^2+4)=0$

$64c^2-48c^2-48=0$

$16c^2=48$

$c^2=3$

$c=\pm\sqrt{3}$

Q9. In each case, find equations of tangent & normal

a). at $\left(-\sqrt{13},\frac{9}{2}\right)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$

Sol: We have $\frac{x^2}{4}-\frac{y^2}{9}=1$ Point $\left(-\sqrt{13},\frac{9}{2}\right)$

Compare with general eq $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ We get

$a^2=4, \qquad b^2=9$

$\Rightarrow \quad a=2, \qquad b=3$

Since equation of tangent $\frac{x.x_1}{a^2}-\frac{y.y_1}{b^2}=1$ Putting

$\frac{-\sqrt{13}x}{4}-\frac{9y}{2(9)}=1$

$\frac{-\sqrt{13}x}{4}-\frac{y}{2}=1 \qquad \times by\ -4$

$\sqrt{13}x+2y=-4$

Since equation of normal $\frac{y-y_1}{x-x_1}=\frac{-a^2y_1}{b^2x_1}$ Putting

$\frac{y-\frac{9}{2}}{x+\sqrt{13}}=\frac{-4\left(\frac{9}{2}\right)}{9\left(-\sqrt{13}\right)}$

$\frac{y-\frac{9}{2}}{x+\sqrt{13}}=\frac{2}{\sqrt{13}}$

$\sqrt{13}\left(y-\frac{9}{2}\right)=2\left(x+\sqrt{13}\right)$

$\sqrt{13}y-\frac{9}{2}\sqrt{13}=2x+2\sqrt{13}$

$2x-\sqrt{13}y+2\sqrt{13}+\frac{9}{2}\sqrt{13}=0$

$4x-2\sqrt{13}y+4\sqrt{13}+9\sqrt{13}=0$

$4x-2\sqrt{13}y+13\sqrt{13}=0$

b). at point $\left(\frac{16}{3},5\right)$ to the hyperbola $\frac{y^2}{9}-\frac{x^2}{16}=1$

Sol: We have $\frac{y^2}{9}-\frac{x^2}{16}=1 \qquad$ Point $\left(\frac{16}{3},5\right)$

Compare with general eq $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ We get

$a^2=9, \qquad b^2=16$

$\Rightarrow \quad a=3, \qquad b=4$

Since equation of tangent $\frac{y.y_1}{a^2}-\frac{x.x_1}{b^2}=1$ Putting

$\frac{5y}{9}-\frac{\frac{16}{3}x}{16}=1$

$\frac{5y}{9}-\frac{x}{3}=1 \qquad \times by\ 9$

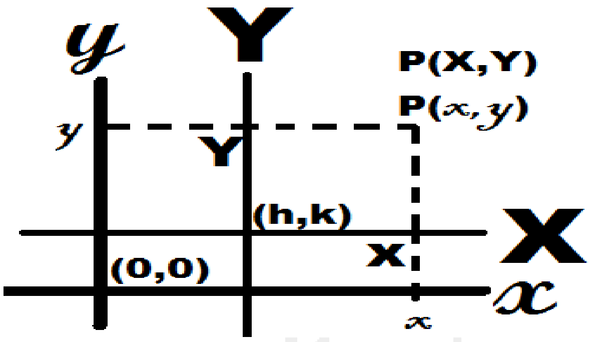
$5y-3x=9$

Since equation of normal $\dfrac{y-y_1}{x-x_1}=\dfrac{-b^2y_1}{a^2x_1}$ putting

$$\frac{y-5}{x-\frac{16}{3}}=\frac{-16(5)}{9\left(\frac{16}{3}\right)}$$
$$\frac{y-5}{x-\frac{16}{3}}=\frac{-80}{48}=\frac{-5}{3}$$
$$3(y-5)=-5\left(x-\tfrac{16}{3}\right)$$
$$3y-15=-5x+\tfrac{80}{3}$$
$$5x+3y-15-\tfrac{80}{3}=0$$
$$15x+9y-45-80=0$$
$$15x+9y-125=0$$

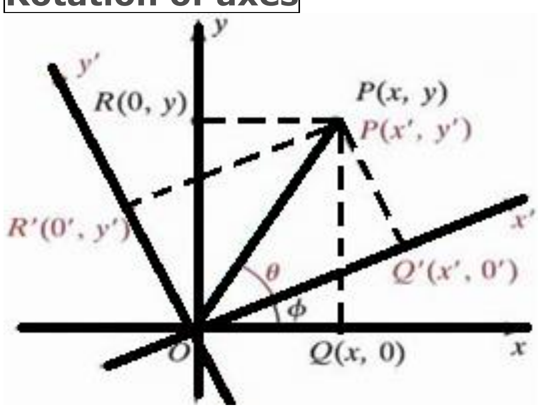
Translation of Axis

The shifting of the reference axes without turning so that each axis remains parallel to its original position.



$$x=X+h,\qquad y=Y+k$$
$$X=x-h,\qquad Y=y-k$$

Rotation of axes



$$x=x'\cos\phi-y'\sin\phi,\quad y=y'\cos\phi+x'\sin\phi$$
$$x'=y\sin\phi+x\cos\phi,\quad y'=y\cos\phi-x\sin\phi$$

Or

$$x=X\cos\phi-Y\sin\phi,\quad y=Y\cos\phi+X\sin\phi$$
$$X=y\sin\phi+x\cos\phi,\quad Y=y\cos\phi-x\sin\phi$$

Rotation and translation of axes

$$x=h+X\cos\phi-Y\sin\phi,\ y=k+Y\cos\phi+X\sin\phi$$

Exercise 8.4

Q1. Translate to parallel axes through the

a). point $(0,2)$ the equation $2x-y+2=0$

Sol: equation of line $2x-y+2=0$...............(1)

And the point $(h,k)=(0,2)$

$$\Rightarrow h=0,\qquad k=2$$

Using translation equations

$$x=X+h,\qquad y=Y+k \quad \text{putting the values}$$
$$x=X+0,\qquad y=Y+2$$
$$x=X,\qquad y=Y+2$$

Put in equation (1) we get

$$2(X)-(Y+2)+2=0$$
$$2X-Y-2+2=0$$
$$2X-Y=0$$

---

b). point $(-1,2)$ the equation

$$x^2+y^2+2x-4y+1=0$$

Sol: equation $x^2+y^2+2x-4y+1=0$.....(1)

And the point $(h,k)=(-1,2)$

$$\Rightarrow h=-1,\qquad k=2$$

Using translation equations

$$x=X+h,\qquad y=Y+k \quad \text{putting the values}$$
$$x=X+(-1),\quad y=Y+2$$
$$x=X-1,\qquad y=Y+2$$

Put in equation (1) we get

$$(X-1)^2+(Y+2)^2+2(X-1)-4(Y+2)+1=0$$
$$X^2-2X+1+Y^2+4Y+4+2X-2-4Y-8+1=0$$
$$X^2+Y^2-2X+2X+4Y-4Y+1+4-2-8+1=0$$
$$X^2+Y^2-4=0$$

---

c). point $(3,-4)$ the equation

$$x^2+2y^2-6x+16y+39=0$$

Sol: equation $x^2+2y^2-6x+16y+39=0$.....(1)

And the point $(h,k)=(3,-4)$

$$\Rightarrow h=3,\qquad k=-4$$

Using translation equations

$$x=X+h,\qquad y=Y+k \quad \text{putting the values}$$
$$x=X+3,\qquad y=Y+(-4)$$
$$x=X+3,\qquad y=Y-4$$

Put in equation (1) we get

$$(X+3)^2+2(Y-4)^2-6(X+3)+16(Y-4)+39=0$$
$$X^2+6X+9+2(Y^2-8Y+16)-6X-18+16Y-64+39=0$$
$$X^2+6X+9+2Y^2-16Y+32-6X-18+16Y-64+39=0$$
$$X^2+2Y^2+6X-6X-16Y+16Y+32+9-18-64+39=0$$
$$X^2+2Y^2-2=0$$

---

d). point $(-2,2)$ the equation

$$x^2+y^2-3xy+10x-10y+21=0$$

Sol: $x^2+y^2-3xy+10x-10y+21=0$.....(1)

And the point $(h,k)=(-2,2)$

$$\Rightarrow h=-2,\qquad k=2$$

Using translation equations

$$x=X+h,\qquad y=Y+k \quad \text{putting the values}$$
$$x=X+(-2),\quad y=Y+2$$
$$x=X-2,\qquad y=Y+2$$

Put in equation (1) we get

$(X-2)^2+(Y+2)^2-3(X-2)(Y+2)$
$+10(X-2)-10(Y+2)+21=0$

$X^2-4X+4+Y^2+4Y+4-3(XY+2X-2Y-4)$
$+10X-20-10Y-20+21=0$

$X^2+Y^2-4X+4+4Y+4-3XY-6X+6Y+12$
$+10X-10Y-20-20+21=0$

$X^2+Y^2-3XY-4X-6X+10X$
$+4Y+6Y-10Y+12+4+4-20-20+21=0$

$X^2+Y^2-3XY+1=0$

Q2. Transform to axes inclined at an angle

a). 45^o to original axes of the conic $x^2-y^2=a^2$

Sol: We have $x^2-y^2=a^2$ Angle $\phi=45^o$

Using $x=X\cos\phi-Y\sin\phi$, $y=Y\cos\phi+X\sin\phi$

$x=X\cos 45^o-Y\sin 45^o$ $y=Y\cos 45^o+X\sin 45^o$

$x=\frac{\sqrt{2}}{2}X-\frac{\sqrt{2}}{2}Y$ $y=\frac{\sqrt{2}}{2}Y+\frac{\sqrt{2}}{2}X$

$x=\frac{\sqrt{2}}{2}(X-Y)$ $y=\frac{\sqrt{2}}{2}(X+Y)$

Putting the values in the given equation, we get

$\left\{\frac{\sqrt{2}}{2}(X-Y)\right\}^2-\left\{\frac{\sqrt{2}}{2}(X+Y)\right\}^2=a^2$

$\frac{2}{4}(X-Y)^2-\frac{2}{4}(X+Y)^2=a^2$

$\frac{1}{2}(X^2-2XY+Y^2)-\frac{1}{2}(X^2+2XY+Y^2)=a^2$

$\frac{1}{2}X^2-XY+\frac{1}{2}Y^2-\frac{1}{2}X^2-XY-\frac{1}{2}Y^2=a^2$

$\frac{1}{2}X^2-\frac{1}{2}X^2-XY-XY+\frac{1}{2}Y^2-\frac{1}{2}Y^2=a^2$

$-2XY=a^2$

---

b). 90^o to the original axes of the conic $y^2=4px$

Sol: We have $y^2=4px$ Angle $\phi=90^o$

Using $x=X\cos\phi-Y\sin\phi$, $y=Y\cos\phi+X\sin\phi$

$x=X\cos 90^o-Y\sin 90^o$ $y=Y\cos 90^o+X\sin 90^o$

$x=X(0)-Y(1)$ $y=Y(0)+X(1)$

$x=-Y$ $y=X$

Putting the values in the given equation, we get

$y^2=4px$

$(X)^2=4pY$

$X^2=4pY$

---

c). 45^o to original axes of conic $x^2+y^2+4xy-1=0$

Sol: We have $x^2+y^2+4xy-1=0$ Angle $\phi=45^o$

Using $x=X\cos\phi-Y\sin\phi$, $y=Y\cos\phi+X\sin\phi$

$x=X\cos 45^o-Y\sin 45^o$ $y=Y\cos 45^o+X\sin 45^o$

$x=\frac{\sqrt{2}}{2}X-\frac{\sqrt{2}}{2}Y$ $y=\frac{\sqrt{2}}{2}Y+\frac{\sqrt{2}}{2}X$

$x=\frac{\sqrt{2}}{2}(X-Y)$ $y=\frac{\sqrt{2}}{2}(X+Y)$

Putting the values in the given equation, we get

$\left\{\frac{\sqrt{2}}{2}(X-Y)\right\}^2+\left\{\frac{\sqrt{2}}{2}(X+Y)\right\}^2$
$+4\left\{\frac{\sqrt{2}}{2}(X-Y)\right\}\left\{\frac{\sqrt{2}}{2}(X+Y)\right\}-1=0$

$\frac{2}{4}(X-Y)^2+\frac{2}{4}(X+Y)^2+4\left(\frac{2}{4}\right)(X^2-Y^2)-1=0$

$\frac{1}{2}(X^2-2XY+Y^2)+\frac{1}{2}(X^2+2XY+Y^2)$
$+2(X^2-Y^2)-1=0$ $\times$ by 2

$X^2-2XY+Y^2+X^2+2XY+Y^2+4(X^2-Y^2)-2=0$

$X^2-2XY+Y^2+X^2+2XY+Y^2+4X^2-4Y^2-2=0$

$X^2+X^2+4X^2-2XY+2XY+Y^2+Y^2-4Y^2-2=0$

$6X^2-2Y^2-2=0$ $\div$ by 2

$3X^2-Y^2-1=0$

---

d). 45^o to the original axes of the conic

$x^2-y^2-2\sqrt{2}x-10\sqrt{2}y+2=0$

Sol: We have $x^2-y^2=a^2$ Angle $\phi=45^o$

Using $x=X\cos\phi-Y\sin\phi$,
$y=Y\cos\phi+X\sin\phi$

$x=X\cos 45^o-Y\sin 45^o$

$x=\frac{\sqrt{2}}{2}X-\frac{\sqrt{2}}{2}Y$

$y=Y\cos 45^o+X\sin 45^o$

$y=\frac{\sqrt{2}}{2}Y+\frac{\sqrt{2}}{2}X$

$x=\frac{\sqrt{2}}{2}(X-Y)$ $y=\frac{\sqrt{2}}{2}(X+Y)$

Putting the values in the given equation, we get

$\left\{\frac{\sqrt{2}}{2}(X-Y)\right\}^2-\left\{\frac{\sqrt{2}}{2}(X+Y)\right\}^2-2\sqrt{2}\left\{\frac{\sqrt{2}}{2}(X-Y)\right\}$
$-10\sqrt{2}\left\{\frac{\sqrt{2}}{2}(X+Y)\right\}+2=0$

$\frac{2}{4}(X-Y)^2-\frac{2}{4}(X+Y)^2-2(X-Y)-10(X+Y)+2=0$

$\frac{1}{2}(X-Y)^2-\frac{1}{2}(X+Y)^2-2(X-Y)-10(X+Y)+2=0$

$X^2-2XY+Y^2-X^2-2XY-Y^2-4(X-Y)-20(X+Y)+4=0$

$X^2-X^2-2XY-2XY+Y^2-Y^2-4X+4Y-20X-20Y+4=0$

$-4XY-24X-16Y+4=0$ $\div$ by -4

$XY+6X+4Y-1=0$

Q3. Transform to new axes inclined at an angle

a). $\tan^{-1}\left(\frac{1}{2}\right)$ to the original axes of the conic

$14x^2+11y^2-36x+48y-4xy+41=0$ through $(1,-2)$

Sol: $14x^2+11y^2-36x+48y-4xy+41=0 \ldots(1)$

$\phi=\tan^{-1}\left(\frac{1}{2}\right)$ TOA

$\tan\phi=\frac{1}{2}$ $\tan\phi=\frac{\text{Opp}}{\text{Adj}}$

$\Rightarrow$ Opposite side = 1, adjacent side = 2

Using Pythagoras theorem

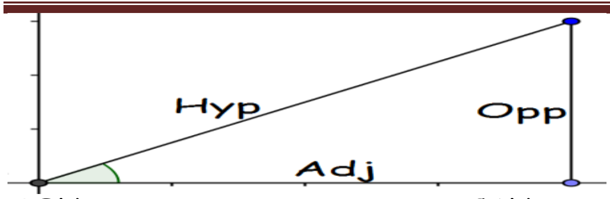
$(\text{Hyp})^2=(\text{Adj})^2+(\text{Opp})^2$

$(\text{Hyp})^2=(2)^2+(1)^2$

$(\text{Hyp})^2=4+1$

$(\text{Hyp})^2=5$

$\Rightarrow \text{Hyp}=\sqrt{5}$



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$\sin\phi = \dfrac{\text{Opp}}{\text{Hyp}}$, $\cos\phi = \dfrac{\text{adj}}{\text{Hyp}}$

$\text{Sin}\phi = \frac{1}{\sqrt{5}}$ $\text{Cos}\phi = \frac{2}{\sqrt{5}}$

using translation $(1,-2)$ rotation $\tan^{-1}\left(\frac{1}{2}\right)$ eqs

$$x = h + X\cos\phi - Y\sin\phi \,,\ y = k + Y\cos\phi + X\sin\phi$$

$$x = 1 + \frac{2}{\sqrt{5}}X - \frac{1}{\sqrt{5}}Y \quad , \quad y = -2 + \frac{2}{\sqrt{5}}Y + \frac{1}{\sqrt{5}}X$$

$$x = \frac{\sqrt{5} + 2X - Y}{\sqrt{5}} \qquad y = \frac{-2\sqrt{5} + 2Y + X}{\sqrt{5}}$$

$$x = \frac{2X - Y + \sqrt{5}}{\sqrt{5}} \qquad y = \frac{X + 2Y - 2\sqrt{5}}{\sqrt{5}}$$

Putting the values in equation (1), we get

$$14\left(\frac{2X - Y + \sqrt{5}}{\sqrt{5}}\right)^2 + 11\left(\frac{X + 2Y - 2\sqrt{5}}{\sqrt{5}}\right)^2 - 36\left(\frac{2X - Y + \sqrt{5}}{\sqrt{5}}\right)$$
$$+48\left(\frac{X + 2Y - 2\sqrt{5}}{\sqrt{5}}\right) - 4\left(\frac{2X - Y + \sqrt{5}}{\sqrt{5}}\right)\left(\frac{X + 2Y - 2\sqrt{5}}{\sqrt{5}}\right) + 41 = 0$$

$$14\left(2X - Y + \sqrt{5}\right)^2 + 11\left(X + 2Y - 2\sqrt{5}\right)^2$$
$$-36\sqrt{5}\left(2X - Y + \sqrt{5}\right) + 48\sqrt{5}\left(X + 2Y - 2\sqrt{5}\right)$$
$$-4\left(2X - Y + \sqrt{5}\right)\left(X + 2Y - 2\sqrt{5}\right) + 205 = 0$$
$$14\left(4X^2 + Y^2 + 5 - 4XY - 2\sqrt{5}Y + 4\sqrt{5}X\right)$$
$$+11\left(X^2 + 4Y^2 + 20 + 4XY - 8\sqrt{5}Y - 4\sqrt{5}X\right)$$
$$-36\sqrt{5}\left(2X - Y + \sqrt{5}\right) + 48\sqrt{5}\left(X + 2Y - 2\sqrt{5}\right)$$
$$-4\begin{pmatrix} 2X^2 + 4XY - 4\sqrt{5}X - XY - 2Y^2 + 2\sqrt{5}Y \\ +\sqrt{5}X + 2\sqrt{5}Y - 10 \end{pmatrix} + 205 = 0$$
$$56X^2 + 14Y^2 + 70 - 56XY - 28\sqrt{5}Y + 56\sqrt{5}X$$
$$+11X^2 + 44Y^2 + 220 + 44XY - 88\sqrt{5}Y - 44\sqrt{5}X$$
$$-72\sqrt{5}X + 36\sqrt{5}Y - 180 + 48\sqrt{5}X + 96\sqrt{5}Y - 480$$
$$-8X^2 - 16XY + 16\sqrt{5}X + 4XY + 8Y^2 - 8\sqrt{5}Y$$
$$-4\sqrt{5}X - 8\sqrt{5}Y + 40 + 205 = 0$$
$$56X^2 + 11X^2 - 8X^2$$
$$+14Y^2 + 44Y^2 + 8Y^2$$
$$-56XY + 44XY - 16XY + 4XY$$
$$-28\sqrt{5}Y - 88\sqrt{5}Y + 36\sqrt{5}Y + 96\sqrt{5}Y - 8\sqrt{5}Y - 8\sqrt{5}Y$$
$$+56\sqrt{5}X - 44\sqrt{5}X - 72\sqrt{5}X + 48\sqrt{5}X + 16\sqrt{5}X - 4\sqrt{5}X$$
$$+70 + 220 - 180 - 480 + 40 + 205 = 0$$
$$59X^2 + 66Y^2 - 24XY - 125 = 0$$

b). $\tan^{-1}\left(\frac{-4}{3}\right)$ to the original axes of the conic

$11x^2 + 4y^2 - 20x - 40y + 24xy - 5 = 0$ through $(2,-1)$

Sol: $11x^2 + 4y^2 - 20x - 40y + 24xy - 5 = 0 \ldots(1)$

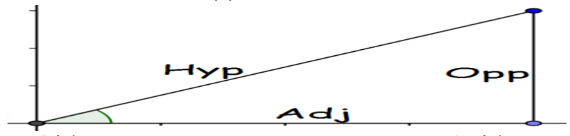
$\phi = \tan^{-1}\left(\frac{-4}{3}\right)$ TOA

$\tan\phi = \frac{-4}{3}$ $\tan\phi = \dfrac{\text{Opp}}{\text{Adj}}$

$\Rightarrow$ Opposite side = -4, adjacent side = 3

Using Pythagoras theorem

$$(\text{Hyp})^2 = (\text{Adj})^2 + (\text{Opp})^2$$
$$(\text{Hyp})^2 = (3)^2 + (-4)^2$$
$$(\text{Hyp})^2 = 9 + 16$$
$$(\text{Hyp})^2 = 25$$
$$\Rightarrow \text{Hyp} = 5$$



SOH CAH

$\sin\phi = \dfrac{\text{Opp}}{\text{Hyp}}$ $\cos\phi = \dfrac{\text{adj}}{\text{Hyp}}$

$\text{Sin}\phi = \frac{-4}{5}$ $\text{Cos}\phi = \frac{3}{5}$

using translation $(2,-1)$ rotation $\tan^{-1}\left(\frac{-4}{3}\right)$ eqs

$$x = h + X\cos\phi - Y\sin\phi \,,\ y = k + Y\cos\phi + X\sin\phi$$

$$x = 2 + \frac{3}{5}X - \frac{-4}{5}Y \qquad y = -1 + \frac{3}{5}Y + \frac{-4}{5}X$$

$$x = \frac{10 + 3X + 4Y}{5} \quad , \quad y = \frac{-5 + 3Y - 4X}{5}$$

$$x = \frac{3X + 4Y + 10}{5} \qquad y = \frac{3Y - 4X - 5}{5}$$

Putting the values in equation (1), we get

$$11\left(\frac{3X + 4Y + 10}{5}\right)^2 + 4\left(\frac{3Y - 4X - 5}{5}\right)^2 - 20\left(\frac{3X + 4Y + 10}{5}\right)$$
$$-40\left(\frac{3Y - 4X - 5}{5}\right) + 24\left(\frac{3X + 4Y + 10}{5}\right)\left(\frac{3Y - 4X - 5}{5}\right) - 5 = 0$$
$$11(3X + 4Y + 10)^2 + 4(3Y - 4X - 5)^2 - 100(3X + 4Y + 10)$$
$$-200(3Y - 4X - 5) + 24(3X + 4Y + 10)(3Y - 4X - 5) - 125 = 0$$
$$11\left(9X^2 + 16Y^2 + 100 + 24XY + 80Y + 60X\right)$$
$$+4\left(9Y^2 + 16X^2 + 25 - 24XY + 40X - 30Y\right)$$
$$-100\left(3X + 4Y + 10X\right) - 200\left(3Y - 4X - 5\right)$$
$$+24\begin{pmatrix} 9XY - 12X^2 - 15X + 12Y^2 - 16XY \\ -20Y + 30Y - 40X - 50 \end{pmatrix} - 125 = 0$$
$$99X^2 + 176Y^2 + 1100 + 264XY + 880Y + 660X$$
$$+36Y^2 + 64X^2 + 100 - 96XY + 160X - 120Y$$
$$-300X - 400Y - 1000X - 600Y + 800X + 1000$$
$$+216XY - 288X^2 - 360X + 288Y^2 - 384XY$$
$$-480Y + 720Y - 960X - 1200 - 125 = 0$$
$$99X^2 + 64X^2 - 288X^2$$
$$+176Y^2 + 36Y^2 + 288Y^2$$
$$+264XY - 96XY + 216XY - 384XY$$
$$+880Y - 120Y - 400Y - 600Y - 480Y + 720Y$$
$$+660X + 160X - 300X - 1000X + 800X - 360X - 960X$$
$$+1100 + 100 + 1000 - 1200 - 125 = 0$$
$$-125X^2 + 500Y^2 - 100X + 875 = 0 \qquad \div By - 25$$
$$5X^2 - 10Y^2 + 20X - 35 = 0$$

c). $\tan^{-1}\left(\frac{3}{4}\right)$ to the original axes of the conic

$3x^2 + 10y^2 + 6x + 52y - 24xy = 0$ through $(3,1)$

Sol: $3x^2 + 10y^2 + 6x + 52y - 24xy = 0 \ldots(1)$

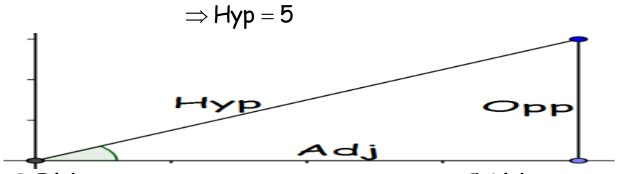
$\phi = \tan^{-1}\left(\frac{3}{4}\right)$ TOA

$\tan\phi = \frac{3}{4}$ $\tan\phi = \frac{Opp}{Adj}$

$\Rightarrow$ Opposite side = 3, adjacent side = 4

Using Pythagoras theorem

$$(Hyp)^2 = (Adj)^2 + (Opp)^2$$
$$(Hyp)^2 = (4)^2 + (3)^2$$
$$(Hyp)^2 = 16 + 9$$
$$(Hyp)^2 = 25$$
$$\Rightarrow Hyp = 5$$



SOH CAH

$\sin\phi = \frac{Opp}{Hyp}$ $\cos\phi = \frac{adj}{Hyp}$

$Sin\phi = \frac{3}{5}$ $Cos\phi = \frac{4}{5}$

using translation $(3,1)$ rotation $\tan^{-1}\left(\frac{3}{4}\right)$ equations

$x = h + X\cos\phi - Y\sin\phi$, $y = k + Y\cos\phi + X\sin\phi$

$$x = 3 + \frac{4}{5}X - \frac{3}{5}Y \qquad y = 1 + \frac{4}{5}Y + \frac{3}{5}X$$

$$x = \frac{15 + 4X - 3Y}{5}, \qquad y = \frac{5 + 4Y + 3X}{5}$$

$$x = \frac{4X - 3Y + 15}{5}, \qquad y = \frac{4Y + 3X + 5}{5}$$

Putting the values in equation (1), we get

$$3\left(\frac{4X-3Y+15}{5}\right)^2 + 10\left(\frac{4Y+3X+5}{5}\right)^2 + 6\left(\frac{4X-3Y+15}{5}\right)$$
$$+52\left(\frac{4Y+3X+5}{5}\right) - 24\left(\frac{4X-3Y+15}{5}\right)\left(\frac{4Y+3X+5}{5}\right) = 0$$

$$3(4X-3Y+15)^2 + 10(4Y+3X+5)^2 + 30(4X-3Y+15)$$
$$+260(4Y+3X+5) - 24(4X-3Y+15)(4Y+3X+5) = 0$$

$$3(16X^2 + 9Y^2 + 225 - 24XY - 90Y + 120X)$$
$$+10(16Y^2 + 9X^2 + 25 + 24XY + 30X + 40Y)$$
$$+30(4X - 3Y + 15) + 260(4Y + 3X + 5)$$
$$-24\begin{pmatrix}16XY + 12X^2 + 20X - 12Y^2 - 9XY \\ -15Y + 60Y + 45X + 75\end{pmatrix} = 0$$

$$48X^2 + 27Y^2 + 675 - 72XY - 270Y + 360X$$
$$+160Y^2 + 90X^2 + 250 + 240XY + 300X + 400Y$$
$$+120X - 90Y + 450 + 1040Y + 780X + 1300$$
$$-384XY - 288X^2 - 480X + 288Y^2 + 216XY$$
$$+360Y - 1440Y - 1080X - 1800 = 0$$

$$48X^2 + 90X^2 - 288X^2$$
$$+27Y^2 + 160Y^2 + 288Y^2$$
$$-72XY + 240XY - 384XY + 216XY$$
$$+300X + 120X + 360X + 780X - 480X - 1080X$$
$$+400Y - 270Y - 90Y + 1040Y + 360Y - 1440Y$$
$$+250 + 675 + 450 + 1300 - 1800 = 0$$

$-150X^2 + 475Y^2 + 875 = 0$ divided by - 5

$$30X^2 - 95Y^2 - 175 = 0$$
$$6X^2 - 19Y^2 - 35 = 0$$

Q4. At what angle the axes are rotated about the origin so that the transformed equation of the conic

a). $11x^2 + 4y^2 - 20x - 40y + 24xy - 5 = 0$ does not contain the term involving XY?

Sol: We have

$$11x^2 + 4y^2 - 20x - 40y + 24xy - 5 = 0$$

Compare with general equation of conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ We get

$a = 11,$ $2h = 24,$ $b = 4$

Using $\phi = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$ putting the values

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{24}{11-4}\right)$$
$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{24}{7}\right)$$
$$\phi = \tfrac{1}{2}\tan^{-1}(3.4286)$$
$$\phi = \tfrac{1}{2}(73.74^o)$$
$$\phi = 36.87^0$$

b). $5x^2 + 7y^2 + 2\sqrt{3}xy - 16 = 0$ does not contain the term involving XY?

Sol: We have $5x^2 + 7y^2 + 2\sqrt{3}xy - 16 = 0$

Compare with general equation of conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ We get

$a = 5,$ $2h = 2\sqrt{3},$ $b = 7$

Using $\phi = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$ putting the values

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{3}}{5-7}\right)$$
$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)$$
$$\phi = \tfrac{1}{2}\tan^{-1}\left(-\sqrt{3}\right)$$
$$\phi = \tfrac{-1}{2}\tan^{-1}\left(\sqrt{3}\right)$$
$$\phi = \tfrac{-1}{2}(60^o)$$
$$\phi = -30^o$$