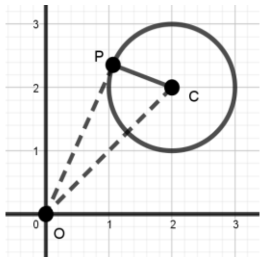
Chapter 7

Derivation of equation of circle:

Let $C(h,k)$ be a fixed point/centre and $P(x,y)$ be any point of the circle then



$OP=(x,y), OC=(h,k)$ &

$OP=OC+CP$

$CP=OP-OC$

$CP=(x,y)-(h,k)$

$CP=(x-h,y-k)$

$|CP|=\sqrt{(x-h)^2+(y-k)^2}$

$r^2=(x-h)^2+(y-k)^2 \qquad \therefore |CP|=r$

Equation of circle having centre (h,k) & radius r

$(x-h)^2+(y-k)^2=r^2$

Equation of circle having centre origin $(0,0)$ & radius r

$x^2+y^2=r^2$

Properties of the equation of circle:

1). Eq of the circle is a second degree equation in x & y.

2). Coefficients of x^2 and y^2 are same

3). There is no term containing xy

4). Radius must be positive i.e. $r\ge 0$

General form of an equation of circle

general equation of the second degree in x & y is

$ax^2+2hxy+by^2+2gx+2fy+c=0$

Dividing each term by a

$x^2+2\frac{h}{a}xy+\frac{b}{a}y^2+2\frac{g}{a}x+2\frac{f}{a}y+\frac{c}{a}=0$

For circle $h=0$, $\frac{b}{a}=1$, $g_1=\frac{g}{a}$, $f_1=\frac{f}{a}$ & $c_1=\frac{c}{a}$ gives

$x^2+y^2+2g_1x+2f_1y+c_1=0$ rearranging

$x^2+2g_1x+y^2+2f_1y+c_1=0$ Adding b s g_1^2, f_1^2

$x^2+2g_1x+g_1^2+y^2+2f_1y+f_1^2=g_1^2+f_1^2-c_1$

$(x+g_1)^2+(y+f_1)^2=g_1^2+f_1^2-c_1$

$[x-(-g_1)]^2+[y-(-f_1)]^2=\left[\sqrt{g_1^2+f_1^2-c_1}\right]^2$........(1)

Thus centre $(-g_1,-f_1)$ & radius $r=\sqrt{g_1^2+f_1^2-c_1}$

If $r>0$ then, circle is real and different from zero

If $r=0$ then, circle shrinks into a point

$(-g,-f)$ is called point circle

If $r<0$ then, circle is imaginary or virtual

Exercise 7.1

Q1. In each case, find an equation of a circle when the centre and radius are the following

a). $(0,0)$, $r=4$

Solution: We have centre $(0,0)$, $r=4$

Equation of circle having centre (h,k) & radius r

$(x-h)^2+(y-k)^2=r^2$ putting the values

$(x-0)^2+(y-0)^2=4^2$

Hence $x^2+y^2=16$ is the required eq of circle

b). $(3,2)$ $r=1$

Solution: We have centre $(3,2)$ $r=1$

Equation of circle having centre (h,k) & radius r

$(x-h)^2+(y-k)^2=r^2$ putting the values

$(x-3)^2+(y-2)^2=1^2$

Hence $(x-3)^2+(y-2)^2=1^2$ is required eq of circle

c). $(-4,-3)$ $r=4$

Solution: We have centre $(-4,-3)$ $r=4$

Equation of circle having centre (h,k) & radius r

$(x-h)^2+(y-k)^2=r^2$ putting the values

$(x-(-4))^2+(y-(-3))^2=4^2$

$(x+4)^2+(y+3)^2=4^2$

Hence $(x+4)^2+(y+3)^2=4^2$ is required eq of circle

d). $(-a,-b)$, $r=a+b$

Solution: We have centre $(-a,-b)$, $r=a+b$

Equation of circle having centre (h,k) & radius r

$(x-h)^2+(y-k)^2=r^2$ putting the values

$(x-(-a))^2+(y-(-b))^2=(a+b)^2$

$(x+a)^2+(y+b)^2=(a+b)^2$

Hence above eq is the required equation of circle

Q2. In each case, determine the equation of a circle using the given information:

a). centre $C(0,0)$, tangent line $x=-5$

Solution: centre $C(0,0)$, tangent line $x=-5$

Or $\quad x+0.y+5=0$

The tangent line passes through only one point of the circumference of the circle and

The shortest distance from the circumference/tangent line of the circle to the centre is radius, using formula

$r=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$r=\left|\frac{(1)(0)+(0)(0)+5}{\sqrt{(1)^2+(0)^2}}\right|$

$r=\left|\frac{0+0+5}{\sqrt{1+0}}\right|=5$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre & radius

$(x-0)^2+(y-0)^2=5^2$

$x^2+y^2=25$

b). $C(0,0)$, tangent to the line $y=6$

Sol: We have centre $C(0,0)$, tangent line $y=6$

Or $\quad 0.x+y-6=0$

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The tangent line passes through only one point of the circumference of the circle and
The shortest distance from the circumference/tangent line of the circle to the centre is radius, using formula

$r=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$r=\left|\frac{(0)(0)+(1)(0)-6}{\sqrt{(0)^2+(1)^2}}\right|$

$r=\left|\frac{0+0-6}{\sqrt{0+1}}\right|$

$r=\left|\frac{-6}{\sqrt{1}}\right|=6$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre and radius

$(x-0)^2+(y-0)^2=6^2$

$x^2+y^2=36$

c). $C(6,-6)$ circumference passes through origin

Sol: centre $C(6,-6)$ and point of circumference $(0,0)$

The distance from the point of the circumference of the circle to the centre is radius, using formula

$r=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ putting the values

$r=\sqrt{(6-0)^2+(-6-0)^2}$

$r=\sqrt{(6)^2+(-6)^2}$

$r=\sqrt{36+36}$

$r=\sqrt{72}$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre and radius

$(x-(6))^2+(y-(-6))^2=(\sqrt{72})^2$

$(x-6)^2+(y+6)^2=72$

d). centre $C(0,5)$ and point of circumference $(5,0)$

Sol: centre $C(0,5)$ and point of circumference $(5,0)$

The distance from the point of the circumference of the circle to the centre is radius, using formula

$r=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ putting the values

$r=\sqrt{(5-0)^2+(0-5)^2}$

$r=\sqrt{(5)^2+(-5)^2}$

$r=\sqrt{25+25}$

$r=\sqrt{50}$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre and radius

$(x-0)^2+(y-5)^2=(\sqrt{50})^2$

$x^2+(y-5)^2=50$

e). $C(-9,-6)$ circumference passes through point $(-20,8)$

Sol: centre $C(-9,-6)$ and point of circumference $(-20,8)$

The distance from the point of the circumference of the circle to the centre is radius, using formula

$r=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ putting the values

$r=\sqrt{(-20-(-9))^2+(8-(-6))^2}$

$r=\sqrt{(-20+9)^2+(8+6)^2}$

$r=\sqrt{(-11)^2+(14)^2}=\sqrt{121+196}$

$r=\sqrt{317}$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre and radius

$(x-(-9))^2+(y-(-6))^2=(\sqrt{317})^2$

$(x+9)^2+(y+6)^2=317$

f) $C(2,-8)$ circumference passes through point $(-10,-6)$

Sol: centre $C(2,-8)$ & point of circumference $(-10,-6)$

The distance from the point of the circumference of the circle to the centre is radius, using formula

$r=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ putting the values

$r=\sqrt{(-10-2)^2+(-6-(-8))^2}$

$r=\sqrt{(-12)^2+(-6+8)^2}$

$r=\sqrt{144+2^2}$

$r=\sqrt{144+4}=\sqrt{148}$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre and radius

$(x-(2))^2+(y-(-8))^2=(\sqrt{148})^2$

$(x-2)^2+(y+8)^2=148$

g). $C(-5,4)$, tangent to the x-axis

Solution: We have centre $C(-5,4)$,

Equation of x-axis = tangent line $y=0$

Or $0.x+y+0=0$

The tangent line passes through only one point of the circumference of the circle and the shortest distance from the circumference/tangent line of the circle to the centre is radius, using formula

$r=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$r=\left|\frac{(0)(-5)+(1)(4)+0}{\sqrt{(0)^2+(1)^2}}\right|$

$r=\left|\frac{0+4+0}{\sqrt{0+1}}\right|=\left|\frac{4}{\sqrt{1}}\right|=\frac{4}{1}$ $\Rightarrow r=4$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre and radius

$(x-(-5))^2+(y-(4))^2=(4)^2$

$(x+5)^2+(y-4)^2=16$

h). $C(5,3)$, tangent to the y-axis

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Solution: We have centre $C(5,3)$,

Equation of y-axis = tangent line $x=0$

Or $x+o.y+0=0$

The tangent line passes through only one point of the circumference of the circle and the shortest distance from the circumference/tangent line of the circle to the centre is radius, using formula

$r=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$$r=\left|\frac{(1)(5)+(0)(3)+0}{\sqrt{(1)^2+(0)^2}}\right|$$

$$r=\left|\frac{5+0+0}{\sqrt{1+0}}\right|=\frac{5}{1}$$

$r=5$

Now to find the equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting centre and radius

$$(x-5)^2+(y-3)^2=(5)^2$$

$$(x-5)^2+(y-3)^2=25$$

Q3. In each case, find the centre $C(-g,-f)$ and the radius $r=\sqrt{g^2+f^2-c}$ of the following:

a). $x^2+y^2-8x-6y+9=0$

Solution: We have $x^2+y^2-8x-6y+9=0$

By comparing with general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=-8$ $\quad 2f=-6$

$g=\frac{-8}{2}=-4$ $\quad f=\frac{-6}{2}=-3$ $\quad c=9$

Therefore the coordinates of centre of the circle

$$C(-g,-f)=C(-(-4),-(-3))$$

$$C(-g,-f)=C(4,3)$$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$$r=\sqrt{(-4)^2+(-3)^2-(9)}$$

$$r=\sqrt{16+9-9}=\sqrt{16}=4$$

Hence centre $C(4,3)$ and the radius $r=4$

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b). $4x^2+4y^2+16x-12y-7=0$

Solution: We have $4x^2+4y^2+16x-12y-7=0$

Dividing it by 4 to make coefficients of x^2 & y^2 equal to 1

$$\frac{4}{4}x^2+\frac{4}{4}y^2+\frac{16}{4}x-\frac{12}{4}y-\frac{7}{4}=\frac{0}{4}$$

$$x^2+y^2+4x-3y-\tfrac{7}{4}=0$$

By comparing with general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=4$ $\quad 2f=-3$ $\quad c=\frac{-7}{4}$

$g=2$ $\quad f=\frac{-3}{2}$

Therefore the coordinates of centre of the circle

$$C(-g,-f)=C\left(-(2),-\left(\tfrac{-3}{2}\right)\right)$$

$$C(-g,-f)=C\left(-2,\tfrac{3}{2}\right)$$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$$r=\sqrt{(2)^2+\left(\frac{-3}{2}\right)^2-\left(\frac{-7}{4}\right)}$$

$$r=\sqrt{4+\frac{9}{4}+\frac{7}{4}}=\sqrt{\frac{4}{4}\times\frac{4}{1}+\frac{9}{4}+\frac{7}{4}}$$

$$r=\sqrt{\frac{16+9+7}{4}}=\sqrt{\frac{32}{4}}=\sqrt{8}$$

$$r=\sqrt{4\times 2}=\sqrt{4}\sqrt{2}=2\sqrt{2}$$

Hence centre $C\left(-2,\tfrac{-3}{2}\right)$ and the radius $r=2\sqrt{2}$

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c). $x^2+y^2+4x-6y+13=0$

Solution: We have $x^2+y^2+4x-6y+13=0$

By comparing with general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=4$ $\quad 2f=-6$ $\quad c=13$

$g=2$ $\quad f=-3$

Therefore the coordinates of centre of the circle

$$C(-g,-f)=C(-(2),-(-3))$$

$$C(-g,-f)=C(-2,3)$$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$$r=\sqrt{(2)^2+(-3)^2-(13)}$$

$$r=\sqrt{4+9-13}=\sqrt{13-13}=0$$

Hence centre $C(-2,3)$ and the radius $r=0$

When radius is zero then equation of circle becomes equation of point.

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d). $x^2+y^2-x-8y+18=0$

Solution: We have $x^2+y^2-x-8y+18=0$

By comparing with general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=-1$ $\quad 2f=-8$ $\quad c=18$

$g=\frac{-1}{2}$ $\quad f=\frac{-8}{2}=-4$

Therefore the coordinates of centre of the circle

$$C(-g,-f)=C\left(-\left(\tfrac{-1}{2}\right),-(-4)\right)$$

$$C(-g,-f)=C\left(\tfrac{1}{2},4\right)$$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$$r=\sqrt{\left(\tfrac{-1}{2}\right)^2+(-4)^2-(18)}$$

$$r=\sqrt{\frac{1}{4}+16-18}=\sqrt{\frac{1}{4}-2}$$

$$r=\sqrt{\frac{1}{4}-\frac{2}{1}.\frac{4}{4}}=\sqrt{\frac{1-8}{4}}=\sqrt{\frac{-7}{4}}$$

Which is not possible

If we take $c=-18$

$r=\sqrt{\left(\frac{-1}{2}\right)^2+(-4)^2-(-18)}$

$r=\sqrt{\frac{1}{4}+16+18}=\sqrt{\frac{1}{4}+34}$

$r=\sqrt{\frac{1}{4}-\frac{34}{1}.\frac{4}{4}}=\sqrt{\frac{1+136}{4}}=\sqrt{\frac{137}{4}}=\frac{\sqrt{137}}{2}$

Hence centre $C\left(\frac{1}{2},4\right)$ and the radius $r=\frac{\sqrt{137}}{2}$

Q4. In each case, determine whether the given equation represents a circle. If no state why not, if it is, then state the coordinates of the centre and the radius

a). $x^2+y^2-8x-4y+16=0$

Solution: We have $x^2+y^2-8x-4y+16=0$

Here the coefficients of x^2 & y^2 are equal therefore the given equation represents a circle,

Now to find centre and radius

By comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=-8$ $2f=-4$

$g=\frac{-8}{2}=-4$ $f=\frac{-4}{2}=-2$ $c=16$

Therefore the coordinates of centre of the circle

$C(-g,-f)=C(-(-4),-(-2))$

$C(-g,-f)=C(4,2)$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$r=\sqrt{(-4)^2+(-2)^2-(16)}$

$r=\sqrt{16+4-16}=\sqrt{4}=2$

Hence centre $C(4,2)$ and the radius $r=2$

b). $x^2+y^2+8y+6x=0$

Solution: We have $x^2+y^2+8y+6x=0$

Here the coefficients of x^2 & y^2 are equal therefore the given equation represents a circle,

Now to find centre and radius

By comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=6$ $2f=8$

$g=\frac{6}{2}=3$ $f=\frac{8}{2}=4$ $c=0$

Therefore the coordinates of centre of the circle

$C(-g,-f)=C(-3,-4)$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$r=\sqrt{(3)^2+(4)^2-(0)}$

$r=\sqrt{9+16-0}=\sqrt{25}=5$

Hence centre $C(-3,-4)$ and the radius $r=5$

c). $x^2+y^2-y-2=0$

Solution: We have $x^2+y^2-y-2=0$

Here the coefficients of x^2 & y^2 are equal therefore the given equation represents a circle,

Now to find centre and radius

By comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=0$ $2f=-1$

$g=\frac{0}{2}=0$ $f=\frac{-1}{2}$ $c=-2$

Therefore the coordinates of centre of the circle

$C(-g,-f)=C\left(-(0),-\left(\frac{-1}{2}\right)\right)$

$C(-g,-f)=C\left(0,\frac{1}{2}\right)$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$r=\sqrt{(0)^2+\left(\frac{-1}{2}\right)^2-(-2)}$

$r=\sqrt{0+\frac{1}{4}+2}=\sqrt{\frac{1}{4}+\frac{2}{1}\times\frac{4}{1}}=\sqrt{\frac{1+8}{4}}$

$r=\sqrt{\frac{9}{4}}=\frac{3}{2}$

Hence centre $C\left(0,\frac{1}{2}\right)$ and the radius $r=\frac{3}{2}$

d). $x^2+y^2+5x=0$

Solution: We have $x^2+y^2+5x=0$

Here the coefficients of x^2 & y^2 are equal therefore the given equation represents a circle,

Now to find centre and radius

By comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=5$ $2f=0$

$g=\frac{5}{2}$ $f=0$ $c=0$

Therefore the coordinates of centre of the circle

$C(-g,-f)=C\left(-\left(\frac{5}{2}\right),-(0)\right)$

$C(-g,-f)=C\left(\frac{5}{2},0\right)$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$r=\sqrt{\left(\frac{5}{2}\right)^2+(0)^2-(0)}=\sqrt{\left(\frac{5}{2}\right)^2}$

$r=\frac{5}{2}$

Hence centre $C\left(\frac{5}{2},0\right)$ and the radius $r=\frac{5}{2}$

e). $3x^2+3y^2+6x-6y=0$

Solution: We have $3x^2+3y^2+6x-6y=0$

Here the coefficients of x^2 & y^2 are equal therefore the given equation represents a circle, To make coefficients of x^2 & y^2 are equal 1 Dividing it by 3

$\frac{3}{3}x^2+\frac{3}{3}y^2+\frac{6}{3}x-\frac{6}{3}y=\frac{0}{3}$

$x^2+y^2+2x-2y=0$

Now to find centre and radius

By comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

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$2g=2$ $\quad 2f=-2$

$g=\frac{2}{2}=1$ $\quad f=\frac{-2}{2}=-1$ $\quad c=0$

Therefore the coordinates of centre of the circle

$C(-g,-f)=C(-(1),-(-1))$

$C(-g,-f)=C(-1,1)$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$r=\sqrt{(1)^2+(-1)^2-(0)}$

$r=\sqrt{1+1-0}=\sqrt{2}$

Hence centre $C(-1,1)$ and the radius $r=\sqrt{2}$

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f). $2x^2+2y^2-8x+12y+8=0$

Solution: We have $2x^2+2y^2-8x+12y+8=0$

Here the coefficients of x^2 & y^2 are equal therefore the given equation represents a circle,To make coefficients of x^2 & y^2 are equal 1 Dividing it by 2

$\frac{2}{2}x^2+\frac{2}{2}y^2-\frac{8}{2}x+\frac{12}{2}y+\frac{8}{2}=\frac{0}{2}$

$x^2+y^2-4x+6y+4=0$

Now to find centre and radius

By comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ we get

$2g=-4$ $\quad 2f=6$ $\quad c=4$

$g=\frac{-4}{2}=-2$ $\quad f=\frac{6}{2}=3$

Therefore the coordinates of centre of the circle

$C(-g,-f)=C(-(-2),-(3))$

$C(-g,-f)=C(2,-3)$

And the radius $r=\sqrt{g^2+f^2-c}$ Putting values

$r=\sqrt{(-2)^2+(3)^2-(4)}$

$r=\sqrt{4+9-4}=\sqrt{9}=3$

Hence centre $C(2,-3)$ and the radius $r=3$

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g). $x^2+2y^2-2x-2y=0$

Solution: We have $x^2+2y^2-2x-2y=0$

Here the coefficients of x^2 & y^2 are not equal therefore the given equation not represents a circle.

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h). $3x^2+2y^2+3x+2y=0$

Solution: We have $3x^2+2y^2+3x+2y=0$

Here the coefficients of x^2 & y^2 are not equal therefore the given equation not represents a circle.

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i). $x^2+y^2+25=0$

Solution: We have $x^2+y^2+25=0$

Here the coefficients of x^2 & y^2 are equal

So $x^2+y^2=-25$

Now to find centre and radius, By comparing with general equation of circle with centre at origin $(0,0)$

$x^2+y^2=r^2$ we get $r^2=-25$

Which is not possible because $r^2\geq 0$

The given equation not represents a circle.

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j). $x^2+y^2+16=0$

Solution: We have $x^2+y^2+16=0$

Here the coefficients of x^2 & y^2 are equal

So $x^2+y^2=-16$

Now to find centre & radius, By comparing with general equation of circle with centre at origin $(0,0)$

$x^2+y^2=r^2$ we get $r^2=-16$

Which is not possible because $r^2\geq 0$

The given equation not represents a circle.

Q5. In each case, find an equation of a circle which passes through the three points

a). $(-3,0),(5,4),(6,-3)$

Sol: The points of circle $(-3,0),(5,4),(6,-3)$

The given points are satisfies the equation of circle

$x^2+y^2+2gx+2fy+c=0$......................(1)

At $(-3,0)$ equation (1) becomes

$(-3)^2+(0)^2+2g(-3)+2f(0)+c=0$

$9+0-6g+0+c=0$

$9-6g+c=0$

$c=6g-9$ ....(2)

At $(5,4)$ equation (1) becomes

$(5)^2+(4)^2+2g(5)+2f(4)+c=0$

$25+16+10g+8f+c=0$

$41+10g+8f+c=0$.....................(3)

At $(6,-3)$ equation (1) becomes

$(6)^2+(-3)^2+2g(6)+2f(-3)+c=0$

$36+9+12g-6f+c=0$

$45+12g-6f+c=0$..................(4)

Putting the value of c in equation (3)

$41+10g+8f+6g-9=0$

$16g+8f+32=0$ $\quad \div by\ 8$

$2g+f+4=0$...................(5)

Putting the value of c in equation (4)

$45+12g-6f+6g-9=0$

$18g-6f+36=0$ $\quad \div by\ 6$

$3g-f+6=0$....................(6)

Adding equation (5) and (6) we get

$2g+f+4=0$

$\underline{3g-f+6=0}$

$5g \quad +10=0$

$5g=-10$

$g=\frac{-10}{5}=-2$

Putting the value of g in equation (5) we get

$2(-2)+f+4=0$

$-4+f+4=0$

$f=0$

Putting value of f and g in equation (2) we have

$c=6(-2)-9$

$c=-12-9$

$c=-21$

Putting value of f,g and c in equation (1) we have

$x^2+y^2+2(-2)x+2(0)y+(-21)=0$

$x^2+y^2-4x-21=0$

Hence $x^2+y^2-4x-21=0$ is the required equation of circle which is passing through given points.

b). $(7,-1),(5,3),(-4,6)$

Sol: The points of circle $(7,-1),(5,3),(-4,6)$

The given points are satisfies the equation of circle

$x^2+y^2+2gx+2fy+c=0$.......................(1)

At $(7,-1)$ equation (1) becomes

$(7)^2+(-1)^2+2g(7)+2f(-1)+c=0$

$49+1+14g-2f+c=0$

$50+14g-2f+c=0$

$c=2f-14g-50$ ....(2)

At $(5,3)$ equation (1) becomes

$(5)^2+(3)^2+2g(5)+2f(3)+c=0$

$25+9+10g+6f+c=0$

$34+10g+6f+c=0$......................(3)

At $(-4,6)$ equation (1) becomes

$(-4)^2+(6)^2+2g(-4)+2f(6)+c=0$

$16+36-8g+12f+c=0$

$52-8g+12f+c=0$..................(4)

Putting the value of c in equation (3)

$34+10g+6f+2f-14g-50=0$

$10g-14g+6f+2f-50+34=0$

$-4g+8f-16=0 \quad \div by\ 4$

$-g+2f-4=0$...................(5)

Putting the value of c in equation (4)

$52-8g+12f+2f-14g-50=0$

$-8g-14g+12f+2f-50+52=0$

$-22g+14f+2=0 \quad \div by\ -2$

$11g-7f-1=0$.....................(6)

Multiply eq (5) with 11 and adding in eq (6) we get

$-11g+22f-44=0$

$11g-7f-1=0$

$15f-45=0$

$15f=45$

$f=\frac{45}{15}=3$

Putting the value of f in equation (5) we get

$-g+2(3)-4=0$

$-g+6-4=0$

$-g+2=0$

$g=2$

Putting value of f and g in equation (2) we have

$c=2(3)-14(2)-50$

$c=6-28-50$

$c=-72$

Putting value of f,g and c in equation (1) we have

$x^2+y^2+2(2)x+2(3)y+(-72)=0$

$x^2+y^2+4x+6y-72=0$

Hence $x^2+y^2+4x+6y-72=0$ is the required equation of circle which is passing through given points.

c). $(1,2),(3,-4),(5,-6)$

Sol: The points of circle $(1,2),(3,-4),(5,-6)$

The given points are satisfies the equation of circle

$x^2+y^2+2gx+2fy+c=0$.......................(1)

At $(1,2)$ equation (1) becomes

$(1)^2+(2)^2+2g(1)+2f(2)+c=0$

$1+4+2g+4f+c=0$

$5+2g+4f+c=0$

$c=-2g-4f-5$ ....(2)

At $(3,-4)$ equation (1) becomes

$(3)^2+(-4)^2+2g(3)+2f(-4)+c=0$

$9+16+6g-8f+c=0$

$25+6g-8f+c=0$......................(3)

At $(5,-6)$ equation (1) becomes

$(5)^2+(-6)^2+2g(5)+2f(-6)+c=0$

$25+36+10g-12f+c=0$

$61+10g-12f+c=0$..................(4)

Putting the value of c in equation (3)

$25+6g-8f-2g-4f-5=0$

$6g-2g-8f-4f-5+25=0$

$4g-12f+20=0 \quad \div by\ 4$

$g-3f+5=0$...................(5)

Putting the value of c in equation (4)

$61+10g-12f-2g-4f-5=0$

$10g-2g-12f-4f-5+61=0$

$8g-16f+56=0 \quad \div by 8$

$g-2f+7=0$....................(6)

Subtracting equation (6) from (5) we get

$g-3f+5=0$

$\pm g\mp 2f\pm 7=0$

$-f-2=0$

$f=-2$

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Putting the value of f in equation (5) we get

$g-3(-2)+5=0$

$g+6+5=0$

$g=-11$

Putting value of f and g in equation (2) we have

$c=-2(-11)-4(-2)-5$

$c=22+8-5$

$c=30-5$

$c=25$

Putting value of f, g and c in equation (1) we have

$x^2+y^2+2(-11)x+2(-2)y+(25)=0$

$x^2+y^2-22x-4f+25=0$

Hence $x^2+y^2-22x-4y+25=0$ is the required equation of circle which is passing through given points.

d). $(-3,4),(-2,0),(1,5)$

Sol: The points of circle $(-3,4),(-2,0),(1,5)$

The given points are satisfies the equation of circle

$x^2+y^2+2gx+2fy+c=0$.......................(1)

At $(-2,0)$ equation (1) becomes

$(-2)^2+(0)^2+2g(-2)+2f(0)+c=0$

$4+0-4g+0+c=0$

$4-4g+c=0$

$c=4g-4$......................(2)

At $(-3,4)$ equation (1) becomes

$(-3)^2+(4)^2+2g(-3)+2f(4)+c=0$

$9+16-6g+8f+c=0$

$25-6g+8f+c=0$.........................(3)

At $(1,5)$ equation (1) becomes

$(1)^2+(5)^2+2g(1)+2f(5)+c=0$

$1+25+2g+10f+c=0$

$26+2g+10f+c=0$...................(4)

Putting the value of c in equation (3)

$25-6g+8f+4g-4=0$

$-6g+4g+8f-4+25=0$

$-2g+8f+21=0$..................(5)

Putting the value of c in equation (4)

$26+2g+10f+4g-4=0$

$2g+4g+10f-4+26=0$

$6g+10f+22=0$....................(6)

Multiply eq (5) by 3 and add in eq (6) we get

$6g+10f+22=0$

$\underline{-6g+24f+63=0}$

$+34f+85=0$

$34f=-85$

$f=\frac{-85}{34}=\frac{-5}{2}$

Putting the value of f in equation (5) we get

$-2g+8\left(\frac{-5}{2}\right)+21=0$

$-2g+4(-5)+21=0$

$-2g-20+21=0$

$-2g=-1 \qquad \Rightarrow g=\frac{1}{2}$

Putting value of f and g in equation (2) we have

$c=4\left(\frac{1}{2}\right)-4$

$c=2-4$

$c=-2$

Putting value of f, g and c in equation (1) we have

$x^2+y^2+2\left(\frac{1}{2}\right)x+2\left(\frac{-5}{2}\right)y+(-2)=0$

$x^2+y^2+x-5y-2=0$

Hence $x^2+y^2+x-5y-2=0$ is required equation of circle which is passing through the given points.

Q6. In each case, find an equation of circle which

a). contains the point $(2,6),(6,4)$ and has its centre on the line $3x+2y-1=0$

Sol: The points of circle $A=(2,6), B=(6,4)$ And equation of line $3x+2y-1=0$ Passes through centre (h,k) i.e.,

$3h+2k-1=0$..........................(1)

Distance from the centre to point of circle is radius i.e., $|CA|=|CB|$ putting the coordinates

$\sqrt{(h-2)^2+(k-6)^2}=\sqrt{(h-6)^2+(k-4)^2}$

Squaring both sides we get

$\left(\sqrt{(h-2)^2+(k-6)^2}\right)^2=\left(\sqrt{(h-6)^2+(k-4)^2}\right)^2$

$(h-2)^2+(k-6)^2=(h-6)^2+(k-4)^2$

$(h)^2-2(h)(2)+(2)^2+(k)^2-2(k)(6)+(6)^2=(h)^2$
$-2(h)(6)+(6)^2+(k)^2-2(k)(4)+(4)^2$

$h^2-4h+4+k^2-12k+36=h^2-12h+36+k^2-8k+16$

$-4h+12h-12k+8k+36+4-36-16=h^2-h^2+k^2-k^2$

$8h-4k-12=0 \qquad \div by\ 2$

$4h-2k-6=0$.................................(2)

adding equation (1) and equation (2) we get

$4h-2k-6=0$

$\underline{3h+2k-1=0}$

$7h \quad -7=0$

$7h=7$

$h=1$

Putting the value of h in equation (1), we get

$3(1)+2k-1=0$

$3+2k-1=0$

$2k+2=0$

$2k=-2$

$k=-1$

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Therefore the centre $C(h,k)=C(1,-1)$

Now to find radius $r=|CA|$

$r=\sqrt{(1-2)^2+(-1-6)^2}$

$r=\sqrt{(-1)^2+(-7)^2}=\sqrt{1+49}$

$r=\sqrt{50}$

Hence equation of circle

$(x-1)^2+(y-(-1))^2=(\sqrt{50})^2$

$(x-1)^2+(y+1)^2=50$

$(x)^2-2(x)(1)+(1)^2+(y)^2+2(y)(1)+(1)^2=50$

$x^2-2x+1+y^2+2y+1-50=0$

$x^2+y^2-2x+2y+1+1-50=0$

$x^2+y^2-2x+2y-48=0$

---

b). contains the point $(4,1),(6,5)$ and has its centre on the line $4x+y-16=0$

Solution: The points of circle $A=(4,1), B=(6,5)$

And equation of line $4x+y-16=0$ Passes through centre (h,k) i.e.,

$4h+k-16=0$.........................(1)

Distance from the centre to point of circle is radius i.e., $|CA|=|CB|$ putting the coordinates

$\sqrt{(h-4)^2+(k-1)^2}=\sqrt{(h-6)^2+(k-5)^2}$

Squaring both sides we get

$\left(\sqrt{(h-4)^2+(k-1)^2}\right)^2=\left(\sqrt{(h-6)^2+(k-5)^2}\right)^2$

$(h-4)^2+(k-1)^2=(h-6)^2+(k-5)^2$

$(h)^2-2(h)(4)+(4)^2+(k)^2-2(k)(1)+(1)^2=(h)^2$

$-2(h)(6)+(6)^2+(k)^2-2(k)(5)+(5)^2$

$h^2-8h+16+k^2-2k+1=h^2-12h+36+k^2-10k+25$

$-8h+12h-2k+10k+16+1-36-25=h^2-h^2+k^2-k^2$

$4h+8k-44=0$..................................(2)

Subtracting equation (1) from equation (2) we get

$4h+8k-44=0$

$\underline{\pm 4h\pm k\mp 16=0}$

$7k-28=0$

$7k=28$

$k=\frac{28}{7}=4$

Putting the value of k in equation (1), we get

$4h+4-16=0$

$4h-12=0$

$4h=12$

$h=\frac{12}{4}=3$

Therefore the centre $C(h,k)=C(3,4)$

Now to find radius $r=|CA|$

$r=\sqrt{(3-4)^2+(4-1)^2}$

$r=\sqrt{(-1)^2+(3)^2}=\sqrt{1+9}$

$r=\sqrt{10}$

Hence the equation of circle

$(x-3)^2+(y-4)^2=(\sqrt{10})^2$

$(x)^2-2(x)(3)+(3)^2+(y)^2-2(y)(4)+(4)^2=10$

$x^2-6x+9+y^2-8y+16-10=0$

$x^2+y^2-6x-8y+9+16-10=0$

$x^2+y^2-6x-8y+15=0$

---

c). contains the point $A=(1,-2), B=(4,3)$ and has its centre on the line $3x+4y-7=0$

Sol: points of circle $A=(1,-2), B=(4,3)$ And eq of line $3x+4y-7=0$ Passes through centre (h,k) i.e., $3h+4k-7=0$.........................(1)

Distance from centre to the point of circle is radius i.e., $|CA|=|CB|$ putting the coordinates

$\sqrt{(h-1)^2+(k-(-2))^2}=\sqrt{(h-4)^2+(k-3)^2}$

Squaring both sides we get

$\left(\sqrt{(h-1)^2+(k+2)^2}\right)^2=\left(\sqrt{(h-4)^2+(k-3)^2}\right)^2$

$(h-1)^2+(k+2)^2=(h-4)^2+(k-3)^2$

$(h)^2-2(h)(1)+(1)^2+(k)^2+2(k)(2)+(2)^2=(h)^2$

$-2(h)(4)+(4)^2+(k)^2-2(k)(3)+(3)^2$

$h^2-2h+1+k^2+4k+4=h^2-8h+16+k^2-6k+9$

$-2h+8h+4k+6k+1+4-16-9=h^2-h^2+k^2-k^2$

$6h+10k-20=0 \qquad \div by\ 2$

$3h+5k-10=0$..................................(2)

subtracting equation (1) from equation (2) we get

$3h+5k-10=0$

$\underline{\pm 3h\pm 4k\mp 7=0}$

$k-3=0 \qquad \Rightarrow k=3$

Putting the value of k in equation (1), we get

$3h+4(3)-7=0$

$3h+12-7=0$

$3h+5=0$

$3h=-5 \qquad \Rightarrow h=\frac{-5}{3}$

Therefore the centre $C(h,k)=C\left(\frac{-5}{3},3\right)$

Now to find radius $r=|CA|$

$r=\sqrt{\left(\frac{-5}{3}-1\right)^2+(3-(-2))^2}$

$r=\sqrt{\left(\frac{-5-3}{3}\right)^2+(3+2)^2}$

$r=\sqrt{\left(\frac{-8}{3}\right)^2+(5)^2}=\sqrt{\frac{64}{9}+25}$

$r=\sqrt{\frac{64}{9}+\frac{25}{1}\times\frac{9}{9}}=\sqrt{\frac{94+225}{9}}$

$r=\sqrt{\frac{289}{9}}$

Hence the equation of circle

$\left(x-\left(\frac{-5}{3}\right)\right)^2+(y-3)^2=\left(\sqrt{\frac{289}{9}}\right)^2$

$\left(x+\frac{5}{3}\right)^2+(y-3)^2=\frac{289}{9}$

$(x)^2+2(x)\left(\frac{5}{3}\right)+\left(\frac{5}{3}\right)^2+(y)^2-2(y)(3)+(3)^2=\frac{289}{9}$

$x^2+\frac{10}{3}x+\frac{25}{9}+y^2-6y+9-\frac{289}{9}=0$

$x^2+y^2+\frac{10}{3}x-6y+\frac{25}{9}+9-\frac{289}{9}=0$

Multiply each term by 9 we get

$9\left(x^2+y^2\right)+9\left(\frac{10}{3}\right)x-9\times6y+9\left(\frac{25}{9}\right)$

$+9\times9-9\left(\frac{289}{9}\right)=9\times0$

$9\left(x^2+y^2\right)+3(10)x-54y+25+81-289=0$

$9\left(x^2+y^2\right)+30x-54y+25+81-289=0$

$9\left(x^2+y^2\right)+30x-54y-183=0$

Book answer is wrong I have checked with geogebra

d). contains point $(0,3),(4,1)$ and has its centre on x-axis

Sol: The points of circle $(0,3),(4,1)$ And equation of x-axis $y=0$ Passes through centre (h,k) i.e., (h,k) i.e., $k=0$ therefore the centre $C(h,0)$

Distance from centre to the point of circle is radius i.e., $|CA|=|CB|$ putting the coordinates

$\sqrt{(h-0)^2+(0-3)^2}=\sqrt{(h-4)^2+(0-1)^2}$

Squaring both sides we get

$\left(\sqrt{(h-0)^2+(0-3)^2}\right)^2=\left(\sqrt{(h-4)^2+(0-1)^2}\right)^2$

$(h)^2+(-3)^2=(h-4)^2+(-1)^2$

$h^2+9=(h)^2-2(h)(4)+(4)^2+1$

$h^2+9=h^2-8h+16+1$

$8h-16-1+9=h^2-h^2$

$8h-17+9=0$

$8h-8=0$

$8h=8$

$h=1$...............................(2)

Therefore the centre $C(h,k)=C(1,0)$

Now to find radius $r=|CA|$

$r=\sqrt{(1-0)^2+(0-3)^2}$

$r=\sqrt{(1)^2+(-3)^2}=\sqrt{1+9}$ $\Rightarrow r=\sqrt{10}$

Hence the equation of circle

$(x-1)^2+(y-0)^2=\left(\sqrt{10}\right)^2$

$(x-1)^2+(y)^2=10$

$(x)^2-2(x)(1)+(1)^2+y^2=10$

$x^2-2x+1+y^2-10=0$

$x^2+y^2-2x+1-10=0$

$x^2+y^2-2x-9=0$

Q7.

Sol: The points of circle $A=(0,0),B=(0,3)$

The slope of tangent line $4x-5y=0$ is $m=\frac{-a}{b}$

putting the values of a & b we get $m=\frac{-4}{-5}=\frac{4}{5}$

The perpendicular line to the tangent line which passes through the point of contact $(0,0)$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the values

$y-0=\frac{-5}{4}(x-0)$

$y=\frac{-5}{4}x$

$4y=-5x$

$5x+4y=0$

Thus $5x+4y=0$ is perpendicular to tangent will passes through centre i.e.,

$5h+4k=0$................(1)

Distance from centre to the point of circle is radius i.e., $|CA|=|CB|$ putting the coordinates

$\sqrt{(h-0)^2+(k-0)^2}=\sqrt{(h-0)^2+(k-3)^2}$

Squaring both sides we get

$\left(\sqrt{(h)^2+(k)^2}\right)^2=\left(\sqrt{(h)^2+(k-3)^2}\right)^2$

$h^2+k^2=h^2+(k-3)^2$

$h^2+k^2=h^2+(k)^2-2(k)(3)+(3)^2$

$h^2+k^2=h^2+k^2-6k+9$

$6k=h^2-h^2+k^2-k^2+9$

$6k=9$ $\Rightarrow k=\frac{3}{2}$..................................(2)

Putting the value of k in equation (1), we get

$5h+4\left(\frac{3}{2}\right)=0$

$5h+2(3)=0$

$5h+6=0$

$5h=-6$ $\Rightarrow h=\frac{-6}{5}$

Therefore the centre $C(h,k)=C\left(\frac{-6}{5},\frac{3}{2}\right)$

Now to find radius $r=|CA|$

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$$r=\sqrt{\left(\frac{-6}{5}-0\right)^2+\left(\frac{3}{2}-0\right)^2}$$

$$r=\sqrt{\left(\frac{-6}{5}\right)^2+\left(\frac{3}{2}\right)^2}=\sqrt{\frac{36}{25}+\frac{9}{4}}$$

$$r=\sqrt{\frac{4}{4}\times\frac{36}{25}+\frac{9}{4}\times\frac{25}{25}}=\sqrt{\frac{144+225}{100}}$$

$$r=\sqrt{\frac{369}{100}}$$

Hence the equation of circle

$$\left(x-\left(\frac{-6}{5}\right)\right)^2+\left(y-\left(\frac{3}{2}\right)\right)^2=\left(\sqrt{\frac{369}{100}}\right)^2$$

$$\left(x+\frac{6}{5}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{369}{100}$$

$$(x)^2+2(x)\left(\frac{6}{5}\right)+\left(\frac{6}{5}\right)^2+(y)^2-2(y)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2=\frac{369}{100}$$

$$x^2+\frac{12}{5}x+\frac{36}{25}+y^2-3y+\frac{9}{4}-\frac{369}{100}=0$$

$$x^2+y^2+\frac{12}{5}x-3y+\frac{36}{25}+\frac{9}{4}-\frac{369}{100}=0$$

multiply by 100 we have

$$100\left(x^2+y^2\right)+100\times\frac{12}{5}x-100\times 3y+100\times\frac{36}{25}$$
$$+100\times\frac{9}{4}-100\times\frac{369}{100}=100\times 0$$

$$100\left(x^2+y^2\right)+20(12)x-300y+4(36)+25(9)-369=0$$

$$100\left(x^2+y^2\right)+240x-300y+144+225-369=0$$

$$100\left(x^2+y^2\right)+240x-300y=0$$

divided by 20

$$5\left(x^2+y^2\right)+12x-15y=0$$

2nd method

Q7. Find an eq of a circle which passes through points a). $(0,0),(0,3)$ & line $4x-5y=0$ is tangent to it at (0,0)

Sol: The points of circle $A=(0,0), B=(0,3)$

The given points are satisfies the equation of circle

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots\ldots(1)$$

At $(0,0)$ equation (1) becomes

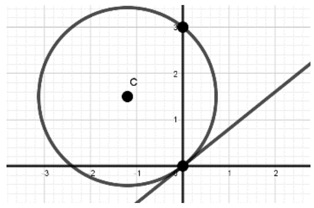
$$(0)^2+(0)^2+2g(0)+2f(0)+c=0$$
$$0+0+0+0+c=0$$
$$c=0$$

At $(0,3)$ using $c=0$ in equation (1) we get

$$(0)^2+(3)^2+2g(0)+2f(3)+0=0$$
$$0+9+0+6f+0=0$$
$$9+6f=0$$
$$6f=-9$$
$$f=\frac{-9}{6}=\frac{-3}{2}$$

The slope of tangent line $4x-5y=0$ is $m_1=\frac{-a}{b}$

putting the values of a & b we get $m_1=\frac{4}{5}$



Slope of the line segment AC using formula

$m_2=\frac{y_2-y_1}{x_2-x_1}$ putting the coordinates of A and C

$$m_2=\frac{0-(-f)}{0-(-g)}=\frac{f}{g}$$

Since line CA is perpendicular to tangent line,

So $m_1.m_2=-1$ putting the values

$$\frac{4}{5}.\frac{f}{g}=-1$$
$$4f=-5g$$

putting the value of f

$$4\left(\tfrac{-3}{2}\right)=-5g \qquad \Rightarrow g=\frac{6}{5}$$
$$-6=-5g$$

Putting values of g, f and c in equation (1) we get

$$x^2+y^2+2\left(\tfrac{6}{5}\right)x+2\left(\tfrac{-3}{2}\right)y+0=0$$
$$x^2+y^2+\frac{12}{5}x-3y=0$$
$$5\left(x^2+y^2\right)+12x-15y=0$$

---

b). $(0,-1),(3,0)$ and the line $3x+y=9$ is tangent to it at $(3,0)$

Sol: The points of circle $A=(0,-1), B=(3,0)$

The given points are satisfies the equation of circle

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots\ldots(1)$$

At $A=(0,-1)$ equation (1) becomes

$$(0)^2+(-1)^2+2g(0)+2f(-1)+c=0$$
$$0+1+0-2f+c=0$$
$$c=2f-1\ldots\ldots\ldots\ldots(2)$$

At $B=(3,0)$ in equation (1) we get

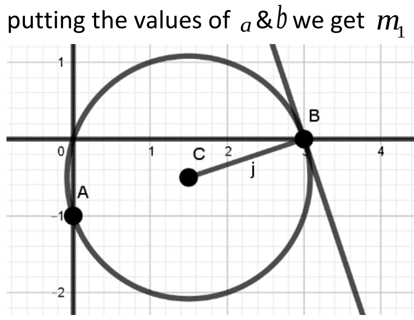
$$(3)^2+(0)^2+2g(3)+2f(0)+c=0$$
$$9+0+6g+0+c=0$$
$$9+6g+c=0$$

Putting the value of c

$$9+6g+2f-1=0$$
$$6g+2f+8=0 \qquad \div by\ 2$$
$$3g+f+4=0\ldots\ldots\ldots\ldots(3)$$

The slope of tangent line $3x+y=9$ is $m_1=\frac{-a}{b}$

putting the values of a & b we get $m_1=-3$



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Slope of the line segment AC using formula

$m_2=\frac{y_2-y_1}{x_2-x_1}$ putting the coordinates of A and C

$$m_2=\frac{0-(-f)}{3-(-g)}=\frac{f}{3+g}$$

Since line CA is perpendicular to tangent line,

So $m_1.m_2=-1$ putting the values

$$-3.\frac{f}{3+g}=-1$$

$$3f=3+g$$

$$g-3f+3=0 \qquad \times by\,3$$

$$3g-9f+9=0\ldots\ldots\ldots\ldots(4)$$

Subtracting equation (4) from Equation (3) we get

$$3g+f+4=0$$

$$\pm3g\mp9f\pm9=0$$

$$10f-5=0$$

$$10f=5$$

$$f=\tfrac{5}{10}=\tfrac{1}{2}$$

putting the value of f in equation (3) we get

$$3g+\tfrac{1}{2}+4=0$$

$$3g=-\tfrac{2}{2}\times\tfrac{4}{1}-\tfrac{1}{2}$$

$$3g=\tfrac{-8-1}{2}=\tfrac{-9}{2}$$

$$g=\tfrac{-3}{2}$$

putting the value of f in equation (2) we get

$$c=2\left(\tfrac{1}{2}\right)-1$$

$$c=1-1$$

$$c=0$$

Putting values of g, f and c in equation (1) we get

$$x^2+y^2+2\left(\frac{-3}{2}\right)x+2\left(\frac{1}{2}\right)y+0=0$$

$$x^2+y^2-3x+y=0$$

c). $(0,1),(3,-1)$ and the line $2x+2y-2=0$ is tangent to it at $(0,1)$

Sol: The points of circle $A=(0,1), B=(3,-1)$

The given points are satisfies the equation of circle

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots\ldots(1)$$

At $A=(0,1)$ equation (1) becomes

$$(0)^2+(1)^2+2g(0)+2f(1)+c=0$$

$$0+1+0+2f+c=0$$

$$c=-2f-1\ldots\ldots\ldots\ldots(2)$$

At $B=(3,-1)$ in equation (1) we get

$$(3)^2+(-1)^2+2g(3)+2f(-1)+c=0$$

$$9+1+6g-2f+c=0$$

$$10+6g-2f+c=0$$

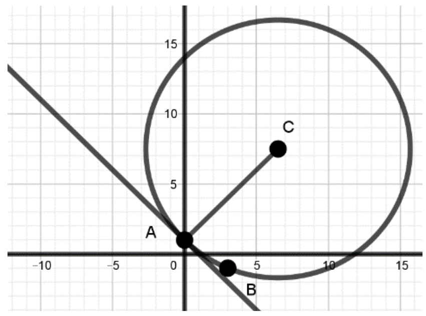
Putting the value of c

$$10+6g-2f-2f-1=0$$

$$6g-4f+9=0\ldots\ldots\ldots\ldots(3)$$

slope of tangent line $2x+2y-2=0$ is $m_1=\frac{-a}{b}$

putting the values of a & b we get $m_1=\frac{-2}{2}=-1$



Slope of the line segment AC using formula

$m_2=\frac{y_2-y_1}{x_2-x_1}$ putting the coordinates of A and C

$$m_2=\frac{1-(-f)}{0-(-g)}=\frac{1+f}{g}$$

Since line CA is perpendicular to tangent line,

So $m_1.m_2=-1$ putting the values

$$-1.\frac{1+f}{g}=-1$$

$$1+f=g$$

$$g-f-1=0 \qquad \times by\,4$$

$$4g-4f-4=0\ldots\ldots\ldots\ldots(4)$$

Subtracting equation (3) from Equation (4) we get

$$6g-4f+9=0$$

$$\pm4g\mp4f\mp4=0$$

$$2g \quad +13=0$$

$$2g=-13$$

$$g=\tfrac{-13}{2}$$

putting the value of g in equation (3) we get

$$6\left(\frac{-13}{2}\right)-4f+9=0$$

$$3(-13)+9=4f$$

$$4f=-39+9$$

$$f=\frac{-30}{4}=\frac{-15}{2}$$

putting the value of f in equation (2) we get

$$c=-2\left(\frac{-15}{2}\right)-1$$

$$c=15-1$$

$$c=14$$

Putting values of g, f and c in equation (1) we get

$$x^2+y^2+2\left(\frac{-13}{2}\right)x+2\left(\frac{-15}{2}\right)y+14=0$$

$$x^2+y^2-13x-15y+14=0$$

d). $(0,4),(2,6)$ & line $x+y-4=0$ is tangent to it at (0,4)

Sol: The points of circle $A=(0,4), B=(2,6)$

The given points are satisfies the equation of circle

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots\ldots(1)$$

At $A=(0,4)$ equation (1) becomes

$$(0)^2+(4)^2+2g(0)+2f(4)+c=0$$

$$0+16+0+8f+c=0$$

$$c=-8f-16\ldots\ldots\ldots\ldots(2)$$

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At $B=(2,6)$ in equation (1) we get

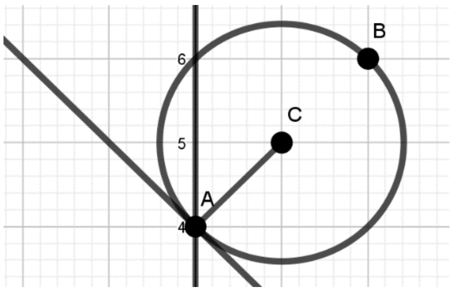
$$(2)^2+(6)^2+2g(2)+2f(6)+c=0$$
$$4+36+4g+12f+c=0$$
$$40+4g+12f+c=0$$

Putting the value of c

$$40+4g+12f-8f-16=0$$
$$4g+4f+24=0 \qquad \div by\,4$$
$$g+f+6=0 \ldots\ldots\ldots\ldots(3)$$

slope of tangent line $x+y-4=0$ is $m_1=\frac{-a}{b}$

putting the values of a & b we get $m_1=-1$



Slope of the line segment AC using formula

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$$m_2=\frac{4-(-f)}{0-(-g)}=\frac{4+f}{g}$$

Since line CA is perpendicular to tangent line,

So $m_1.m_2=-1$ putting the values

$$-1.\frac{4+f}{g}=-1$$
$$4+f=g$$
$$g-f-4=0 \ldots\ldots\ldots\ldots(4)$$

adding equation (3) and Equation (4) we get

$$g+f+6=0$$
$$g-f-4=0$$
$$2g \qquad -2=0$$
$$2g=-2 \qquad \Rightarrow g=-1$$

putting the value of g in equation (3) we get

$$(-1)+f+6=0$$
$$f+5=0 \qquad \Rightarrow f=-5$$

putting the value of f in equation (2) we get

$$c=-8(-5)-16$$
$$c=40-16$$
$$c=24$$

Putting values of g, f and c in equation (1) we get

$$x^2+y^2+2(-1)x+2(-5)y+24=0$$
$$x^2+y^2-2x-10y+24=0$$

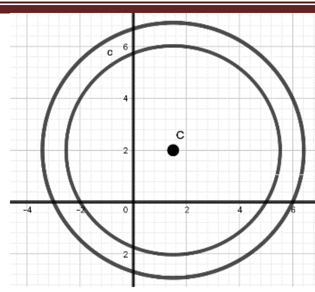
Q8.

Solution: We have $x^2+y^2-3x-4y-10=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$$2g=-3 \qquad 2f=-4$$
$$g=\tfrac{-3}{2} \qquad f=-2$$



Since concentric circles have the same centre, so

$$C(-g,-f)=C\left(-\left(\frac{-3}{2}\right),-(-2)\right)$$
$$C(-g,-f)=C\left(\frac{3}{2},2\right)$$

To find the radius of the required circle use

$C=\left(\frac{3}{2},2\right)$ and the point $A=(-3,0)$

$$r=|CA|=\sqrt{\left(-3-\frac{3}{2}\right)^2+(2-0)^2}$$
$$r=\sqrt{\left(\frac{-6-3}{2}\right)^2+(2)^2}=\sqrt{\left(\frac{-9}{2}\right)^2+4}$$
$$r=\sqrt{\frac{81}{4}+\frac{4}{1}\times\frac{4}{4}}=\sqrt{\frac{81+16}{4}}=\sqrt{\frac{97}{4}}$$

To find required equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting the values

$$\left(x-\frac{3}{2}\right)^2+(y-2)^2=\left(\sqrt{\frac{97}{4}}\right)^2$$
$$\left(x-\frac{3}{2}\right)^2+(y-2)^2=\frac{97}{4}$$

Q9. Find an equation of a circle that concentric to circle

a). $2x^2+2y^2+16x-7y=0$ & is tangent to y-axis.

Solution: We have $2x^2+2y^2+16x-7y=0$

Divided by 2 $x^2+y^2+8x-\frac{7}{2}y=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$$2g=8 \qquad 2f=\tfrac{-7}{2}$$
$$g=\tfrac{8}{2}=4 \qquad f=\tfrac{-7}{4}$$

Since concentric circles have the same centre, so

$$C(-g,-f)=C\left(-(4),-\left(\tfrac{-7}{4}\right)\right)$$
$$C(-g,-f)=C\left(-4,\tfrac{7}{4}\right)$$

To find the radius of the required circle use

$C=\left(-4,\frac{7}{4}\right)$ and tangent to y-axis i.e., $x=0$

Use shortest distance from the point to the line

$r=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$$r=\left|\frac{(1)(-4)+(0)\left(\frac{7}{4}\right)+0}{\sqrt{1^2+0^2}}\right|=\left|\frac{-4+0+0}{\sqrt{1}}\right|$$
$$r=\left|\frac{-4}{1}\right|=4$$

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To find required equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting the values

$(x-(-4))^2+(y-\frac{7}{4})^2=(4)^2$

$(x+4)^2+(y-\frac{7}{4})^2=16$

b). $x^2+y^2-8x+4=0$ and is tangent to the line $x+2y+6=0$

Solution: We have $x^2+y^2-8x+4=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=-8$ $\quad 2f=0$

$g=-4$ $\quad f=0$

Since concentric circles have the same centre, so

$C(-g,-f)=C(-(-4),-0)$

$C(-g,-f)=C(4,0)$

To find the radius of the required circle use $C(4,0)$ and tangent line $x+2y+6=0$

Use shortest distance from the point to the line

$r=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$r=\left|\frac{(1)(4)+(2)(0)+6}{\sqrt{1^2+2^2}}\right|=\left|\frac{4+0+6}{\sqrt{1+4}}\right|$

$r=\left|\frac{10}{\sqrt{5}}\right|=\frac{10}{\sqrt{5}}$

To find required equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting the values

$(x-4)^2+(y-0)^2=\left(\frac{10}{\sqrt{5}}\right)^2$

$(x-4)^2+y^2=\frac{100}{5}$

$(x-4)^2+y^2=20$

c). $x^2+y^2+6x-10y+33=0$ and is touching x-axis

Solution: We have $x^2+y^2+6x-10y+33=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=6$ $\quad 2f=-10$

$g=\frac{6}{2}=3$ $\quad f=\frac{-10}{2}=-5$

Since concentric circles have the same centre, so

$C(-g,-f)=C(-(3),-(-5))$

$C(-g,-f)=C(-3,5)$

To find the radius of the required circle use $C(-3,5)$ and tangent to x axis i.e., $y=0$

Use shortest distance from the point to the line

$r=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$r=\left|\frac{(0)(-3)+(1)(5)+0}{\sqrt{0^2+1^2}}\right|=\left|\frac{0+5+0}{\sqrt{1}}\right|=\left|\frac{5}{1}\right|$

$r=5$

To find required equation of circle using formula

$(x-h)^2+(y-k)^2=r^2$ putting the values

$(x-(-3))^2+(y-5)^2=(5)^2$

$(x+3)^2+(y-5)^2=25$

Q10. In each case, find an equation of a circle which passes through origin whose intercepts on the coordinate axes are

a). 3 and 4

Solution: We have the points origin $(0,0)$

x-intercept $(3,0)$ and y-intercept $(0,4)$

since the general equation of circle

$x^2+y^2+2gx+2fy+c=0$........................(1)

put origin $(0,0)$ in equation (1), we get

$(0)^2+(0)^2+2g(0)+2f(0)+c=0$

$0+0+0+0+c=0$ $\quad \Rightarrow c=0$

Put x-intercept $(3,0)$ using $c=0$ in eq (1), we get

$(3)^2+(0)^2+2g(3)+2f(0)+0=0$

$9+0+6g+0+0=0$

$6g=-9$

$g=\frac{-9}{6}=\frac{-3}{2}$

Put y-intercept $(0,4)$ using $c=0$ in eq (1), we get

$(0)^2+(4)^2+2g(0)+2f(4)+0=0$

$0+16+0+8f+0=0$

$8f=-16$

$f=\frac{-16}{8}=-2$

Putting values of g, f and c in equation (1) we get

$x^2+y^2+2(\frac{-3}{2})x+2(-2)y+0=0$

$x^2+y^2-3x-4y=0$

b). 2 and 4

Solution: We have the points origin $(0,0)$

x-intercept $(2,0)$ and y-intercept $(0,4)$

since the general equation of circle

$x^2+y^2+2gx+2fy+c=0$........................(1)

put origin $(0,0)$ in equation (1), we get

$(0)^2+(0)^2+2g(0)+2f(0)+c=0$

$0+0+0+0+c=0$ $\quad \Rightarrow c=0$

Put x-intercept $(2,0)$ using $c=0$ in eq (1), we get

$(2)^2+(0)^2+2g(2)+2f(0)+0=0$

$4+0+4g+0+0=0$

$4g=-4$ $\quad \Rightarrow g=-1$

Put y-intercept $(0,4)$ using $c=0$ in eq (1), we get

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$(0)^2+(4)^2+2g(0)+2f(4)+0=0$

$0+16+0+8f+0=0$

$8f=-16$

$f=\frac{-16}{8}=-2$

Putting values of g, f and c in equation (1) we get

$x^2+y^2+2(-1)x+2(-2)y+0=0$

$x^2+y^2-2x-4y=0$

Derivation of tangent equation to the circle:

Since equation of circle $x^2+y^2=a^2$

Differentiating with respect to x

$2x+2y\frac{dy}{dx}=0$

$2y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\frac{-x}{y}$

Since derivative at the point $P(x_1,y_1)$ gives slope

$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-x_1}{y_1}$

We know that equation of line having slope & passing through point $P(x_1,y_1)$ gives equation of tangent

$y-y_1=m(x-x_1)$ putting the slope

$y-y_1=\frac{-x_1}{y_1}(x-x_1)$

$y_1(y-y_1)=-x_1(x-x_1)$

$y_1y-y_1^2=-x_1x+x_1^2$

$x_1x+y_1y=x_1^2+y_1^2$

$x_1x+y_1y=a^2 \qquad \therefore a^2=x_1^2+y_1^2$

Equation of normal using slope $m=\frac{-x_1}{y_1}$

$y-y_1=\frac{-1}{m}(x-x_1)$ putting the slope

$y-y_1=\frac{y_1}{x_1}(x-x_1)$

$\frac{y-y_1}{y_1}=\frac{x-x_1}{x_1}$

$\frac{y}{y_1}-\frac{y_1}{y_1}=\frac{x}{x_1}-\frac{x_1}{x_1}$

$\frac{y}{y_1}-1=\frac{x}{x_1}-1$

$\frac{y}{y_1}=\frac{x}{x_1}$

When centre	Origin i.e., $(0,0)$
Equation of circle	$x^2+y^2=a^2$
Equation of tangent	$xx_1+yy_1=a^2$
Equation of normal	$xy_1-yx_1=0$ or $\frac{x}{x_1}=\frac{y}{y_1}$

When centre	$(h,k)=(-g,-f)$
Equation of circle	$x^2+y^2+2gx+2fy+c=0$
Equation of tangent	$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$
Equation of normal	$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}$

Point of intersection b/w circle with centre origin & line

$y=mx+c$......................(1)

$x^2+y^2=a^2$...................(2)

Putting the value of y in equation (2) we get

$x^2+(mx+c)^2=a^2$

$x^2+(mx)^2+2(mx)(c)+(c)^2-a^2=0$

$x^2+m^2x^2+2mc\,x+c^2-a^2=0$

$(1+m^2)x^2+2mc\,x+c^2-a^2=0$

$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ putting the values

$x=\frac{-2mc\pm\sqrt{(2mc)^2-4(1+m^2)(c^2-a^2)}}{2(1+m^2)}$

When Discriminant >0 then the line and the circle intersect at two different points

$4m^2c^2-4(1+m^2)(c^2-a^2)>0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$4m^2c^2-4(1+m^2)(c^2-a^2)=0$

When Discriminant < 0 then the line and the circle will not intersect

$4m^2c^2-4(1+m^2)(c^2-a^2)<0$

Point of intersection b/w circle with centre is (-g,-f) & line

$y=mx+c$..(1)

$x^2+y^2+2g\,x+2f\,y+c=0$..................(2)

Putting the value of y in equation (2) we get

$x^2+(mx+c)^2\,2g\,x+2f(mx+c)+c=0$

$x^2+(mx)^2+2(mx)(c)+(c)^2+2g\,x+2\,fmx+2\,fc+c=0$

$x^2+m^2x^2+2mc\,x+2g\,x+2\,fmx+c^2+2\,fc+c=0$

$(1+m^2)x^2+2(mc+g+fm)x+c^2+2\,fc+c=0$

$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ putting the values

$x=\frac{-2(mc+g+fm)\pm\sqrt{4(mc+g+fm)^2-4(1+m^2)(c^2+2fc+c)}}{2(1+m^2)}$

When Discriminant >0 then the line and the circle intersect at two different points

$4(mc+g+fm)^2-4(1+m^2)(c^2+2fc+c)>0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$4(mc+g+fm)^2-4(1+m^2)(c^2+2fc+c)=0$

When Discriminant < 0 then the line and the circle will not intersect

$4(mc+g+fm)^2-4(1+m^2)(c^2+2fc+c)<0$

Condition for line to touch a circle

Equation of line

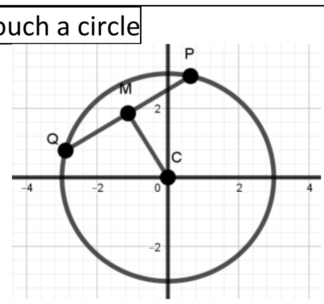
$y=mx+c$

Equation of circle

$x^2+y^2=a^2$

Coordinates of P

$P(x_1,y_1)$



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Coordinates of Q $Q(x_2, y_2)$ M is midpoint of PQ $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Perpendicular distance from Centre of circle $O(0,0)$ to line $mx - y + c = 0$ at M is

$$|OM| = \frac{|m(0)-(0)+c|}{\sqrt{m^2+(-1)^2}} = \frac{c}{\sqrt{m^2+1}}$$

radius $|OP| = a = |OQ|$ using Pythagoras theorem

$$|OP|^2 = |OM|^2 + |MP|^2$$

Or $|MP|^2 = |OP|^2 - |OM|^2$ putting the values

$$|MP|^2 = a^2 - \frac{c^2}{m^2+1} = \frac{a^2(m^2+1)-c^2}{m^2+1}$$ Taking square root we get $|MP| = \sqrt{\frac{a^2(m^2+1)-c^2}{m^2+1}}$

The secant line PQ is 2 times of MP therefore

$$|PQ| = 2|MP| = 2\sqrt{\frac{a^2(m^2+1)-c^2}{m^2+1}}$$

When distance between PQ becomes zero the secant line convert to tangent line i.e.

$$2\sqrt{\frac{a^2(m^2+1)-c^2}{m^2+1}} = 0$$

$$\Rightarrow \sqrt{a^2(m^2+1)-c^2} = 0$$

$\Rightarrow a^2(m^2+1)-c^2 = 0$ After simplifying

$c = a\sqrt{m^2+1}$ then equation of tangent becomes

$y = mx + a\sqrt{m^2+1}$ to the circle $x^2+y^2=a^2$

Point of contact of tangent line and circle

Let $y = mx + a\sqrt{m^2+1}$ is tangent to the circle $x^2+y^2=a^2$ is identical to $x_1x + y_1y = a^2$

The coefficients of line terms of

$y = mx + a\sqrt{m^2+1}$ compared with

$y_1y = -x_1x + a^2$ we get

$$\frac{x_1}{-m} = \frac{y_1}{1} = \frac{a^2}{a\sqrt{m^2+1}}$$ by transitive property

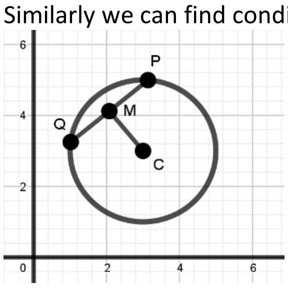
Either $\frac{x_1}{-m} = \frac{a^2}{a\sqrt{m^2+1}}$ or $\frac{y_1}{1} = \frac{a^2}{a\sqrt{m^2+1}}$

$x_1 = \frac{-am}{\sqrt{m^2+1}}$ $y_1 = \frac{a}{\sqrt{m^2+1}}$

Thus, point of contact is

$$(x_1, y_1) = \left(\frac{-am}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$$

Similarly we can find condition and point of contact for



Equation of line $y = mx + c$

Equation of circle

$$x^2+y^2+2gx+2fy+c=0$$

Coordinates of $P(x_1, y_1)$ & Coordinates of $Q(x_2, y_2)$

M is midpoint of PQ $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Exercise 7.2

Q1 In each case, find an equation tangent & normal

a). at a point $(1,2)$ to the circle $x^2+y^2=5$

Solution: The equation of circle $x^2+y^2=5$

To find the equation of tangent at point $(1,2)$

Using formula $x_1x + y_1y = r^2$ putting the point

$(1)x + (2)y = 5$

$x + 2y = 5$

To find the equation of normal at point $(1,2)$

Using formula $\frac{x}{x_1} = \frac{y}{y_1}$ putting the point

$\frac{x}{1} = \frac{y}{2}$

$2x = y$

Hence Equation of tangent $x + 2y = 5$

Equation of normal $2x = y$

b). at a point $(-1,3)$ to the circle

$x^2+y^2+6x-y-1=0$

Solution: The equation of circle

$x^2+y^2+6x-y-1=0$

By comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g = 6$ $2f = -1$

$g = \frac{6}{2} = 3$ $f = \frac{-1}{2}$ $c = -1$

To find equation of tangent at point $(-1,3)$

Using formula $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$

putting $(-1)x + (3)y + 3(x-1) - \frac{1}{2}(y+3) - 1 = 0$

$-x + 3y + 3x - 3 - \frac{1}{2}y - \frac{3}{2} - 1 = 0$

$-x + 3x + 3y - \frac{1}{2}y - \frac{3}{2} - 3 - 1 = 0$

$2x + 3y - \frac{1}{2}y - \frac{3}{2} - 4 = 0$ $\times by\ 2$

$4x + 6y - y - 3 - 8 = 0$

$4x + 5y - 11 = 0$

To find the equation of normal at point $(1,2)$

Using formula $\frac{x-x_1}{x_1+g} = \frac{y-y_1}{y_1+f}$ putting the point

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$$\frac{x-(-1)}{-1+3}=\frac{y-3}{3-\frac{1}{2}} \qquad \therefore 3-\frac{1}{2}$$

$$\frac{x+1}{2}=\frac{y-3}{\frac{5}{2}} \qquad \frac{2}{2}.\frac{3}{1}-\frac{1}{2}=\frac{6-1}{2}=\frac{5}{2}$$

By cross multiplication

$\frac{5}{2}(x+1)=2(y-3)$

$5(x+1)=4(y-3)$

$5x+5=4y-12$

$5x-4y+5+12=0$

$5x-4y+17=0$

Hence Equation of tangent $4x+5y-11=0$

Equation of normal $5x-4y+17=0$

Q2 In each case, find an equation tangent and normal

a). at a point $(2\cos 45^o, 2\sin 45^o)$ to circle $x^2+y^2=4$

Sol: The equation of circle $x^2+y^2=4$ The point

$(2\cos 45^o, 2\sin 45^o)=\left(2\left(\frac{\sqrt{2}}{2}\right),2\left(\frac{\sqrt{2}}{2}\right)\right)$

$(2\cos 45^o, 2\sin 45^o)=(\sqrt{2},\sqrt{2})$

To find the equation of tangent at point $(\sqrt{2},\sqrt{2})$

Using formula $x_1x+y_1y=r^2$ putting the point

$\sqrt{2}x+\sqrt{2}y=4 \qquad \div by\ \sqrt{2}$

$x+y=\frac{4}{\sqrt{2}}$

$x+y=\frac{4}{\sqrt{2}}.\frac{\sqrt{2}}{\sqrt{2}}$

$x+y=\frac{4\sqrt{2}}{2}$

$x+y=2\sqrt{2}$

To find the equation of normal at point $(\sqrt{2},\sqrt{2})$

Using formula $\frac{x}{x_1}=\frac{y}{y_1}$ putting the point

$\frac{x}{\sqrt{2}}=\frac{y}{\sqrt{2}} \qquad \times by\ \sqrt{2}$

$x=y$

Hence Equation of tangent $x+y=2\sqrt{2}$

Equation of normal $x=y$

---

b). at a point $(\cos 30^o, \sin 30^o)$ to circle $x^2+y^2=1$

Sol: The equation of circle $x^2+y^2=1$ point

$(\cos 30^o, \sin 30^o)=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

To find the equation of tangent at point $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

Using formula $x_1x+y_1y=r^2$ putting the point

$\frac{\sqrt{3}}{2}x+\frac{1}{2}y=1 \qquad \times by\ 2$

$\sqrt{3}x+y=2$

To find the equation of normal at point $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

Using formula $\frac{x}{x_1}=\frac{y}{y_1}$ putting the point

$\frac{x}{\frac{\sqrt{3}}{2}}=\frac{y}{\frac{1}{2}}$

$\frac{1}{2}x=\frac{\sqrt{3}}{2}y \qquad \times by\ 2$

$x=\sqrt{3}y$

Hence Equation of tangent $\sqrt{3}x+y=2$

Equation of normal $x=\sqrt{3}y$

Q3. Find the condition that

a). the line $x+y+n=0$ touches to the circle $x^2+y^2=9$

Solution: the circle $x^2+y^2=9$ And the line

$x+y+n=0$

$y=-x-n \qquad \Rightarrow \begin{cases} m=-1, \\ c=-n \end{cases}$

condition for the circle and the line to touch

Discriminant $=4m^2c^2-4(1+m^2)(c^2-a^2)=0$

$4(-1)^2(-n)^2-4(1+(-1)^2)((-n)^2-9)=0$

$4(1)(n^2)-4(1+1)(n^2-9)=0$

$4n^2-8n^2+72=0$

$-4n^2+72=0$

$-4(n^2-18)=0$

$n^2-18=0$

2nd method

Q3. Find the condition that

a). the line $x+y+n=0$ touches to the circle $x^2+y^2=9$

Sol: the circle $x^2+y^2=9$..................(1) And the line

$x+y+n=0$..............................(2)

$y=-x-n$

Putting the value of y in equation (1) we get

$x^2+(-x-n)^2=9$

$x^2+(-x)^2+2(-x)(-n)+(-n)^2=9$

$x^2+x^2+2nx+n^2-9=0$

$2x^2+2nx+n^2-9=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$(2n)^2-4(2)(n^2-9)=0$

$4n^2-8(n^2-9)=0$

$4n^2-8n^2+72=0$

$-4n^2+72=0$

$-4(n^2-18)=0$

$n^2-18=0$

---

b). the line $2x+2y+n=0$ touches to the circle $x^2+y^2=81$

Sol: the circle $x^2+y^2=81$................(1) And the line

$2x+2y+n=0$..............................(2)

$2y=-2x-n$

$y=-x-\frac{n}{2}$

Putting the value of y in equation (1) we get

$x^2+\left(-x-\frac{n}{2}\right)^2=81$

$x^2+(-x)^2+2(-x)\left(\frac{-n}{2}\right)+\left(\frac{-n}{2}\right)^2=81$

$x^2+x^2+nx+\frac{n^2}{4}-81=0$

$2x^2+nx+\frac{n^2}{4}-81=0 \qquad \times by\ 4$

$8x^2+4nx+n^2-324=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$(4n)^2-4(8)(n^2-324)=0$

$16n^2-32(n^2-324)=0$

$16n^2-32n^2+10368=0$

$-16n^2+10368=0$

$-16(n^2-648)=0$

$n^2-648=0$

Q4.

Sol: the circle $x^2+y^2=9$..................(1) And the line

$x+y+n=0$..............................(2)

$y=-x-n$

Putting the value of y in equation (1) we get

$x^2+(-x-n)^2=9$

$x^2+(-x)^2+2(-x)(-n)+(-n)^2=9$

$x^2+x^2+2nx+n^2-9=0$

$2x^2+2nx+n^2-9=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$(2n)^2-4(2)(n^2-9)=0$

$4n^2-8(n^2-9)=0$

$4n^2-8n^2+72=0$

$-4n^2+72=0$

$-4(n^2-18)=0$

$n^2-18=0$

$n^2=18$

$\sqrt{n^2}=\sqrt{18}=\sqrt{9\times 2}$

$n=\pm 3\sqrt{2}$

b). line $2x+2y+n=0$ touches to circle $x^2+y^2=81$

Sol: the circle $x^2+y^2=81$................(1) And the line

$2x+2y+n=0$..............................(2)

$2y=-2x-n$

$y=-x-\frac{n}{2}$

Putting the value of y in equation (1) we get

$x^2+\left(-x-\frac{n}{2}\right)^2=81$

$x^2+(-x)^2+2(-x)\left(\frac{-n}{2}\right)+\left(\frac{-n}{2}\right)^2=81$

$x^2+x^2+nx+\frac{n^2}{4}-81=0$

$2x^2+nx+\frac{n^2}{4}-81=0 \qquad \times by\ 4$

$8x^2+4nx+n^2-324=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$(4n)^2-4(8)(n^2-324)=0$

$16n^2-32(n^2-324)=0$

$16n^2-32n^2+10368=0$

$-16n^2+10368=0$

$-16(n^2-648)=0$

$n^2-648=0$

$n^2=648$

$\sqrt{n^2}=\sqrt{648}=\sqrt{36\times 9\times 2}$

$n=\pm 6\times 3\sqrt{2}$

$n=\pm 18\sqrt{2}$

Q5. For what value of c

a). the line $y=mx+c$ touches circle $x^2+y^2=a^2$

solution: the line and the circle

$y=mx+c$......................(1)

$x^2+y^2=a^2$..................(2)

Putting the value of y in equation (2) we get

$x^2+(mx+c)^2=a^2$

$x^2+(mx)^2+2(mx)(c)+(c)^2-a^2=0$

$x^2+m^2x^2+2mc\,x+c^2-a^2=0$

$(1+m^2)x^2+2mc\,x+c^2-a^2=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$(2mc)^2-4(1+m^2)(c^2-a^2)=0$

$4m^2c^2=4(1+m^2)(c^2-a^2) \qquad \div by\ 4$

$m^2c^2=(1+m^2)(c^2-a^2)$

$m^2c^2=c^2-a^2+m^2c^2-m^2a^2$

$m^2c^2-m^2c^2+m^2a^2+a^2=c^2$

$c^2=a^2+a^2m^2$

$c^2=a^2(1+m^2)$

$\sqrt{c^2}=\sqrt{a^2(1+m^2)}$

$c=\pm a\sqrt{1+m^2}$

b). the line $y=-x+c$ touches circle $x^2+y^2=9$

Sol: the circle $x^2+y^2=9$..................(1) And the line

$y=-x+c$..............................(2)

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Putting the value of y in equation (1) we get

$x^2+(-x+c)^2=9$

$x^2+(-x)^2+2(-x)(c)+(c)^2=9$

$x^2+x^2-2cx+c^2-9=0$

$2x^2-2cx+c^2-9=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$(-2c)^2-4(2)(c^2-9)=0$

$4c^2-8(c^2-9)=0$

$4c^2-8c^2+72=0$

$-4c^2+72=0$

$-4(c^2-18)=0$

$c^2-18=0$

$c^2=18$

$\sqrt{c^2}=\sqrt{18}=\sqrt{9\times 2}$

$c=\pm 3\sqrt{2}$

Q6. Find condition at which line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$

Sol: the circle $x^2+y^2+2gx+2fy+c=0$.........(1)

And the line $lx+my+n=0$............................(2)

$my=-lx-n$

$y=\frac{-lx-n}{m}$

Putting the value of y in equation (1) we get

$x^2+\left(\frac{-lx-n}{m}\right)^2+2gx+2f\left(\frac{-lx-n}{m}\right)+c=0$

$x^2+\frac{(-lx-n)^2}{m^2}+2gx+2f\left(\frac{-lx-n}{m}\right)+c=0 \quad \times\ by\ m^2$

$m^2x^2+(-lx-n)^2+2m^2gx+2fm(-lx-n)+cm^2=0$

$m^2x^2+l^2x^2+2nlx+n^2+2m^2gx-2lfmx-2fmn+cm^2=0$

$m^2x^2+l^2x^2+2nlx+2m^2gx-2lfmx+n^2-2fmn+cm^2=0$

$(m^2+l^2)x^2+2(nl+m^2g-lfm)x+(n^2-2fmn+cm^2)=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$4(nl+m^2g-lfm)^2-4(m^2+l^2)(n^2-2fmn+cm^2)=0$

$4(nl+m^2g-lfm)^2=4(m^2+l^2)(n^2-2fmn+cm^2)$

$(nl+m^2g-lfm)^2=(m^2+l^2)(n^2-2fmn+cm^2)$

$n^2l^2+m^4g^2+l^2f^2m^2+2m^2ngl-2m^3lfg-2l^2fmn$

$\quad =m^2n^2-2fm^3n+cm^4+n^2l^2-2l^2fmn+cl^2m^2$

$n^2l^2-n^2l^2-2l^2fmn+2l^2fmn$

$\quad +m^4g^2+l^2f^2m^2+2m^2ngl-2m^3lfg$

$\quad =m^2n^2-2fm^3n+cm^4+cl^2m^2 \qquad \div\ by\ m^2$

$m^2g^2+l^2f^2+2ngl-2mlfg$

$\quad =n^2-2fmn+cm^2+cl^2$

$cl^2-l^2f^2+2mlfg+cm^2-m^2g^2-2fmn-2ngl+n^2=0$

$l^2(c-f^2)+2mlfg+m^2(c-g^2)-2n(fm-gl)+n^2=0$

Q7. For what value of n

a). the line $3x+4y+n=0$ touching the circle $x^2+y^2-4x-6y-12=0$

Solution: the circle

$x^2+y^2-4x-6y-12=0$..............(1) And the line

$3x+4y+n=0$..................(2)

$4y=-3x-n$

$y=\frac{-3x-n}{4}$

Putting the value of y in equation (1) we get

$x^2+\left(\frac{-3x-n}{4}\right)^2-4x-6\left(\frac{-3x-n}{4}\right)-12=0$

$x^2+\frac{9x^2+6nx+n^2}{16}-4x-\frac{-18x-6n}{4}-12=0 \quad \times by16$

$16x^2+9x^2+6nx+n^2-64x-4(-18x-6n)-192=0$

$25x^2+6nx-64x+72x+24n+n^2-192=0$

$25x^2+6nx+8x+n^2+24n-192=0$

$25x^2+2(3n+4)x+n^2+24n-192=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$4(3n+4)^2-4(25)(n^2+24n-192)=0$

$(9n^2+24n+16)=25(n^2+24n-192)$

$9n^2+24n+16=25n^2+600n-4800$

$25n^2-9n^2+600n-24n-4800-16=0$

$16n^2+576n-4816=0$

$n^2+36n-301=0$

$n^2+43n-7n-301=0$

$n(n+43)-7(n+43)=0$

$(n-7)(n+43)=0$

Either $n-7=0$, $n=7$ or $n+43=0$, $n=-43$

b). the line $x-2y+n=0$ touching the circle $x^2+y^2+3x+6y-5=0$

Sol: the circle $x^2+y^2+3x+6y-5=0$......(1) And the line

$x-2y+n=0$..............................(2)

$x=2y-n$

Putting the value of y in equation (1) we get

$(2y-n)^2+y^2+3(2y-n)+6y-5=0$

$(2y)^2-2(2y)(n)+(n)^2+y^2+6y-3n+6y-5=0$

$4y^2-4ny+n^2+y^2+6y-3n+6y-5=0$

$4y^2+y^2-4ny+6y+6y+n^2-3n-5=0$

$5y^2-4ny+12y+n^2-3n-5=0$

$5y^2-4(n-3)y+n^2-3n-5=0$

When Discriminant = 0 then the line and the circle intersect at only one points/touches only

$B^2-4AC=0$ putting the values

$16(n-3)^2-4(5)(n^2-3n-5)=0$

$16(n^2-6n+3^2)=4(5)(n^2-3n-5)$

$4(n^2-6n+9)=5(n^2-3n-5)$

$4n^2-24n+36=5n^2-15n-25$

$5n^2-4n^2-15n+24n-25-36=0$

$n^2+9n-61=0$

$\therefore n=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ putting the values

$n=\dfrac{-9\pm\sqrt{(9)^2-4(1)(-61)}}{2(1)}$

$n=\dfrac{-9\pm\sqrt{81+244}}{2}$

$n=\dfrac{-9\pm\sqrt{325}}{2}=\dfrac{-9\pm\sqrt{25\times13}}{2}$

$n=\dfrac{-9\pm5\sqrt{13}}{2}$

Either $n=\dfrac{-9+5\sqrt{13}}{2}$ or $n=\dfrac{-9-5\sqrt{13}}{2}$

$n=4.513878$ $n=-13.513878$

Q8.

Sol: We have the circle $x^2+y^2=9$

$x^2+y^2-9=0$........................(1) and the line

$3x+4y+3=0$.............................(2)

Let $P(x,y)$ be any point

Length of tangent = perpendicular distance

$\sqrt{x^2+y^2-9}=\left|\dfrac{3x+4y+3}{\sqrt{3^2+4^2}}\right|$

$\sqrt{x^2+y^2-9}=\dfrac{3x+4y+3}{\sqrt{9+16}}$

$\sqrt{x^2+y^2-9}=\dfrac{3x+4y+3}{\sqrt{25}}$

$\sqrt{x^2+y^2-9}=\dfrac{3x+4y+3}{5}$

$5\sqrt{x^2+y^2-9}=3x+4y+3$

Squaring on both sides

$\left(5\sqrt{x^2+y^2-9}\right)^2=(3x+4y+3)^2$

$25\left(\sqrt{x^2+y^2-9}\right)^2=(3x+4y+3)^2$

$25(x^2+y^2-9)=(3x+4y+3)^2$

Q9. The length of the tangent from (f,g) to the circle $x^2+y^2=6$ is twice the length of tangent to the circle $x^2+y^2+3x+3y=0$ Prove that $f^2+g^2+4f+4g+2=0$

Sol: We have the circle $x^2+y^2=6$

$x^2+y^2-6=0$...........................(1)

another circle

$x^2+y^2+3x+3y=0$...................(2)

Let $P(f,g)$ be any point

Length of tangent = 2 x length of tangent

$\sqrt{f^2+g^2-6}=2\sqrt{f^2+g^2+3f+3g}$

Squaring on both sides

$\left(\sqrt{f^2+g^2-6}\right)^2=\left(2\sqrt{f^2+g^2+3f+3g}\right)^2$

$f^2+g^2-6=4(f^2+g^2+3f+3g)$

$f^2+g^2-6=4f^2+4g^2+12f+12g$

$4f^2-f^2+4g^2-g^2+12f+12g+6=0$

$3f^2+3g^2+12f+12g+6=0$ $\div by 3$

$f^2+g^2+4f+4g+2=0$

Q10. Find the equations of tangents to the circle $x^2+y^2=25$ which are parallel to the straight line $3x+4y+3=0$

Solution: parallel to the line $3x+4y+3=0$ is

$3x+4y+k=0$.........................(1)

and equation of circle with centre is origin $(0,0)$ is

$x^2+y^2=25$......................(2)

centre $=(0,0)$ and radius $=5$

perpendicular distance from the line to centre = radius

$\left|\dfrac{3(0)+4(0)+k}{\sqrt{3^2+4^2}}\right|=5$

$\left|\dfrac{0+0+k}{\sqrt{9+16}}\right|=5$

$\left|\dfrac{k}{\sqrt{25}}\right|=5$

$\dfrac{k}{\pm5}=5$

$k=\pm25$

Therefore equation (1) becomes

$3x+4y\pm25=0$

Hence equation of tangent to the circle

Either $3x+4y+25=0$ or $3x+4y-25=0$

Q11a). Prove that the line $x=8$ and $y=7$ touch the circle $x^2+y^2-6x-4y-12=0$ Find also point of contact

Solution: To check the line $x=8$ touches the circle

$x^2+y^2-6x-4y-12=0$.......................(1)

Put $x=8$ in equation (1) we get

$(8)^2+y^2-6(8)-4y-12=0$

$64+y^2-48-4y-12=0$

$y^2-4y+64-48-12=0$

$y^2-4y+4=0$...............................(2)

When Discriminant = 0 then the line touches the circle

$B^2-4AC=0$ putting the values

$B^2-4AC=(-4)^2-4(1)(4)$

$B^2-4AC=16-16$

$B^2-4AC=0$

Hence the line $x=8$ touches the given circle.
For the point of contact solving equation (2)

$y^2-4y+4=0$

$(y)^2-2(y)(2)+(2)^2=0$

$(y-2)^2=0$

$\Rightarrow y-2=0$

$y=2$

Therefore the point of contact is $(8,2)$

To check the line $y=7$ touches the circle

$x^2+y^2-6x-4y-12=0$........................(1)

Put $y=7$ in equation (1) we get

$x^2+(7)^2-6x-4(7)-12=0$

$x^2+49-6x-28-12=0$

$x^2-6x+49-28-12=0$

$x^2-6x+9=0$..............................(3)

When Discriminant = 0 then line touches the circle

$B^2-4AC=0$ putting the values

$B^2-4AC=(-6)^2-4(1)(9)$

$B^2-4AC=36-36$

$B^2-4AC=0$

Hence the line $y=7$ touches the given circle.
For the point of contact solving equation (3)

$x^2-6x+9=0$

$(x)^2-2(x)(3)+(3)^2=0$

$(x-3)^2=0$

$\Rightarrow x-3=0$

$x=3$

Therefore the point of contact is $(3,7)$

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b).
Solution: To check the line $x+y-1=0$ or $y=1-x$ touches the circle

$x^2+y^2-4x-2y+3=0$........................(1)

Put $y=1-x$ in equation (1) we get

$x^2+(1-x)^2-4x-2(1-x)+3=0$

$x^2+1-2x+x^2-4x-2+2x+3=0$

$x^2+x^2-2x-4x+2x-2+1+3=0$

$2x^2-4x+2=0$

When Discriminant = 0 then line touches the circle

$B^2-4AC=0$ putting the values

$B^2-4AC=(-4)^2-4(2)(2)$

$B^2-4AC=16-16$

$B^2-4AC=0$

Hence the line $x+y-1=0$ touches the circle
Now to find the point of intersection

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ putting the values

$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(2)}}{2(2)}$

$x=\frac{4\pm\sqrt{16-16}}{4}=\frac{4\pm\sqrt{0}}{4}=\frac{4}{4}$

$x=1$

Put the value of x in the line $x+y-1=0$

$1+y-1=0$

$y=0$

Hence the point of contact $(1,0)$

To check the line $x-y+1=0$ or $y=1+x$ touches the circle

$x^2+y^2-4x-2y+3=0$........................(1)

Put $y=1+x$ in equation (1) we get

$x^2+(1+x)^2-4x-2(1+x)+3=0$

$x^2+1+2x+x^2-4x-2-2x+3=0$

$x^2+x^2+2x-4x-2x-2+1+3=0$

$2x^2-4x+2=0$

When Discriminant = 0 then line touches the circle

$B^2-4AC=0$ putting the values

$B^2-4AC=(-4)^2-4(2)(2)$

$B^2-4AC=16-16$

$B^2-4AC=0$

Hence the line $x-y+1=0$ touches the circle
Now to find the point of intersection

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ putting the values

$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(2)}}{2(2)}$

$x=\frac{4\pm\sqrt{16-16}}{4}=\frac{4\pm\sqrt{0}}{4}=\frac{4}{4}$

$x=1$

Put the value of x in the line $x-y+1=0$

$1-y+1=0$

$1+1=y$

$y=2$

Hence the point of contact $(1,2)$

Q12. Find equation of tangents to the circle $x^2+y^2=2$, which make an angle of 45^o with the x-axis.

Solution: equation of circle $x^2+y^2=2$

slope $m=\tan 45^o=1$ and radius

$a^2=2 \Rightarrow a=\sqrt{2}$

Since the equation of tangent

$y=mx\pm a\sqrt{1+m^2}$ putting the values

$y=1x\pm\sqrt{2}\sqrt{1+1^2}$

$y=x\pm\sqrt{2}\sqrt{1+1}$

$y=x\pm\sqrt{2}\sqrt{2}$

$y=x\pm 2$

Q13. Find the equation of tangents drawn from the point $P(4,3)$ to the circle $x^2+y^2=9$

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Solution: the equation of circle $x^2+y^2=9$

Here $a^2=9 \qquad \Rightarrow a=3$

The number of tangents

$m^2\left(x_1^2-a^2\right)-2mx_1y_1+\left(y_1^2-a^2\right)=0$

Putting $P(4,3)$ and $a=3$ we get

$m^2\left(4^2-3^2\right)-2m(4)(3)+\left(3^2-3^2\right)=0$

$m^2(16-9)-24m+0=0$

$7m^2-24m=0$

$m(7m-24)=0$

Either $m=0$ or $7m-24=0$

$7m=24$

$m=\frac{24}{7}$

The equation of line having slope and point

$y=mx+c$....................(1) putting the values

If $m=0$ put in equation (1) we get

$y=0x+c$

$y=c$ ....(2) to find c put $(4,3)$

$3=c \qquad \Rightarrow c=3$ so

$y=3$ is eq of tangent which passing through $(4,3)$

The equation of line having slope and point

$y=mx+c$....................(1) putting the values

If $m=\frac{24}{7}$ put in equation (1) we get

$y=\frac{24}{7}x+c$ ....(2)to find c put $(4,3)$

$3=\frac{24}{7}(4)+c \qquad \Rightarrow c=\frac{-75}{7}$ so

$c=3-\frac{96}{7}=\frac{21-96}{7}$

$y=\frac{24}{7}x-\frac{75}{7}$ is eq of tangent, passing through $(4,3)$

Perpendicular from the centre of a circle on a chord bisect the chord

Equation of circle

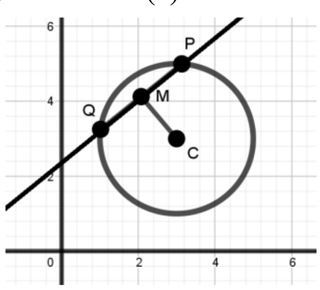
$x^2+y^2+2gx+2fy+c=0$.....(1)

Equation of line $y=mx+c$

Coordinates of $P(x_1,y_1)$ &

Coordinates of $Q(x_2,y_2)$

Which passing through a circle



$x_1^2+y_1^2+2gx_1+2fy_1+c=0$........(2)

$x_2^2+y_2^2+2gx_2+2fy_2+c=0$........(3)

Subtracting equation (3) from equation (2)

$\left(x_2^2-x_1^2\right)+\left(y_2^2-y_1^2\right)+2gx_2-2gx_1+2fy_2-2fy_1=0$

$\left(x_2^2-x_1^2\right)+2g\left(x_2-x_1\right)+\left(y_2^2-y_1^2\right)+2f\left(y_2-y_1\right)=0$

$\left(x_2-x_1\right)\left(x_2+x_1+2g\right)+\left(y_2-y_1\right)\left(y_2+y_1+2f\right)=0$

or $m_1=\dfrac{y_2-y_1}{x_2-x_1}=-\dfrac{x_2+x_1+2g}{y_2+y_1+2f}$

M is midpoint of PQ $M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ & centre

$C(-g,-h)$ Therefore the slope of CM

$m_2=\dfrac{\frac{y_1+y_2}{2}+f}{\frac{x_1+x_2}{2}+g}=\dfrac{y_1+y_2+2f}{x_1+x_2+2g}$

To check CM is perpendicular to PQ so

$m_1.m_2=-\dfrac{x_1+x_2+2g}{y_1+y_2+2f}\times\dfrac{y_1+y_2+2f}{x_1+x_2+2g}$

$m_1.m_2=-1$

i.e. CM is perpendicular bisector to PQ

Congruent chords of a circle are equidistant from centre

Let circle eq $x^2+y^2+2gx+2fy+c=0$.....(1) with

centre $C(-g,-f)$

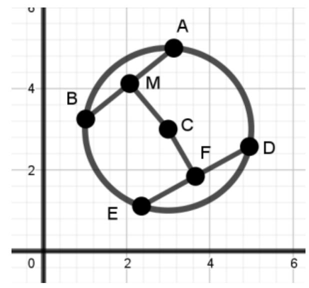
Coordinates of A, B, C and D are

$A(x_1,y_1)$,

$B(x_2,y_2)$

$C(x_3,y_3)$

$D(x_4,y_4)$



Given that $|AB|=|DE| \qquad \Rightarrow |AB|^2=|DE|^2$

We have to show that

$|CM|^2=|CN|^2 \Rightarrow |CM|=|CN|$

M is midpoint of AB $\quad M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

N is midpoint of DE $\quad N\left(\frac{x_3+x_4}{2},\frac{y_3+y_4}{2}\right)$

Measure of central angle of a minor arc is double of angle subtend by the corresponding major arc

Let equation circle $x^2+y^2=a^2$.....(1)

with centre $C(0,0)$

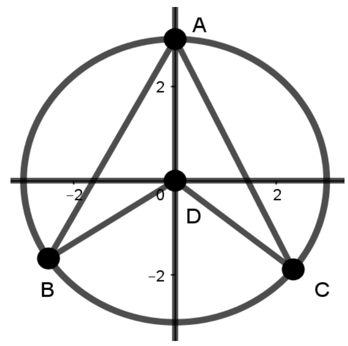
Coordinates of A, B, and C are $A(0,a)$

$B(-x_1,-y_1)$

$C(x_1,-y_1)$

We have to show that

$\angle BOC=2\angle BAC$



Let $\angle BOC=\theta$ and $\angle BAC=2\theta$

Slope of BA $m_1=\dfrac{a+y_1}{0+x_1}=\dfrac{a+y_1}{x_1}$

Slope of AC $m_2=\dfrac{-y_1-a}{x_1-0}=\dfrac{-(a+y_1)}{x_1}$

Then, angle $\tan\angle BAC=\dfrac{m_1-m_2}{1+m_1m_2}$ putting values

$\tan\angle BAC=\dfrac{\frac{a+y_1}{x_1}+\frac{a+y_1}{x_1}}{1-\left(\frac{a+y_1}{x_1}\right)\left(\frac{a+y_1}{x_1}\right)}$

$=\dfrac{2(a+y_1)}{x_1}\div\left[1-\left(\dfrac{a+y_1}{x_1}\right)^2\right]$

$\tan\theta=\dfrac{2(a+y_1)}{x_1}\div\left[\dfrac{x_1^2-(a+y_1)^2}{x_1^2}\right]$

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$$\tan\theta = \frac{2(a+y_1)}{x_1} \times \frac{x_1^2}{x_1^2-(a+y_1)^2}$$

$$\tan\theta = \frac{2x_1(a+y_1)}{x_1^2-a^2-y_1^2-2ay_1}$$

$$\tan\theta = \frac{2x_1(a+y_1)}{x_1^2-x_1^2-y_1^2-y_1^2-2ay_1} \qquad \therefore x_1^2+y_1^2=a^2$$

$$\tan\theta = \frac{2x_1(a+y_1)}{-2y_1^2-2ay_1} = \frac{2x_1(a+y_1)}{-2y_1(y_1+a)}$$

$$\tan\theta = -\frac{x_1}{y_1}$$

Slope of BO $m_3 = \dfrac{0+y_1}{0+x_1} = \dfrac{y_1}{x_1}$

Slope of CO $m_4 = \dfrac{0+y_1}{0-x_1} = \dfrac{-y_1}{x_1}$

Then, angle $\tan\angle BOC = \dfrac{m_3-m_4}{1+m_3m_4}$ putting values

$$\tan\angle BOC = \frac{\frac{y_1}{x_1}+\frac{y_1}{x_1}}{1+\left(\frac{y_1}{x_1}\right)\left(\frac{-y_1}{x_1}\right)} = \frac{\frac{2y_1}{x_1}}{1-\frac{y_1^2}{x_1^2}}$$

$$\tan 2\theta = \frac{2y_1}{x_1} \div \frac{x_1^2-y_1^2}{x_1^2} = \frac{2y_1}{x_1} \times \frac{x_1^2}{x_1^2-y_1^2}$$

$$\tan 2\theta = \frac{2x_1y_1}{x_1^2-y_1^2}$$

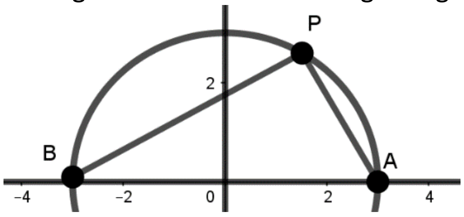
Now using trigonometric identity

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ putting the values

$$\tan 2\theta = \frac{\frac{-2x_1}{y_1}}{1-\frac{x_1^2}{y_1^2}} = \frac{-2x_1}{y_1} \div \frac{y_1^2-x_1^2}{y_1^2}$$

$$\tan 2\theta = \frac{2x_1y_1}{x_1^2-y_1^2}$$

An angle in a semi-circle is a right angle



Let the circle equation be $x^2+y^2=a^2$

And the coordinates of A,B and P are

$A(a,0)$ $B(-a,0)$ and $P(x_1,y_1)$

Slope of AP $m_1 = \dfrac{y_1-0}{x_1-a} = \dfrac{y_1}{x_1-a}$

Slope of PB $m_2 = \dfrac{y_1-0}{x_1+a} = \dfrac{y_1}{x_1+a}$

Now $m_1.m_2 = \dfrac{y_1}{x_1-a}.\dfrac{y_1}{x_1+a}$

$$m_1.m_2 = \frac{y_1^2}{x_1^2-a^2} = \frac{y_1^2}{x_1^2-x_1^2-y_1^2} \qquad \therefore x_1^2+y_1^2=a^2$$

$$m_1.m_2 = \frac{y_1^2}{-y_1^2} = -1$$

Thus the line AP and PB are perpendicular

Or $m\angle APB = 90^o$

Perpendicular at the outer end of radial segment is tangent to the circle

Let circle equation

$x^2+y^2+2gx+2fy+c=0.....(1)$ with centre

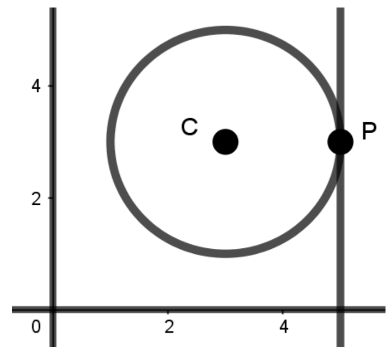
$C(-g,-f)$

Coordinates of P

$P(x_1,y_1)$

Slope of CP is

$$m_1 = \frac{y_1+f}{x_1+g}$$



Equation of tangent to the circle

$$x_1x+y_1y+g(x+x_1)+f(y+y_1)+c=0$$

$$x_1x+y_1y+gx+gx_1+fy+fy_1+c=0$$

$$x_1x+gx+y_1y+fy+gx_1+fy_1+c=0$$

$$(x_1+g)x+(y_1+f)y+gx_1+fy_1+c=0$$

Slope of tangent equation $m_2=-\frac{a}{b}$ putting values

$m_2 = -\dfrac{x_1+g}{y_1+f}$ now

$$m_1.m_2 = \left(\frac{y_1+f}{x_1+g}\right)\left(-\frac{x_1+g}{y_1+f}\right)$$

$$m_1.m_2 = -1$$

Thus, the perpendicular at the outer end of a circle is tangent to the circle

Exercise 7.3

Q1. If $A(2,2)$ and $B(3,1)$ are the end points of the chord AB of the circle

$x^2+y^2-4x-2y+4=0$ then show that

a). line from the center of the circle is perpendicular to AB, also bisects the chord AB

Sol: end points of the chord $A(2,2)$ and $B(3,1)$

Let D is the midpoint of AB $= \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

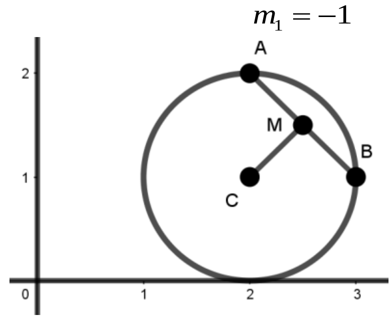
$$D = \left(\tfrac{2+3}{2}, \tfrac{2+1}{2}\right)$$

$$D = \left(\tfrac{5}{2}, \tfrac{3}{2}\right)$$

Slope of the line AB $m_1 = \dfrac{y_2-y_1}{x_2-x_1}$

$$m_1 = \frac{1-2}{3-2} = \frac{-1}{1}$$

$$m_1 = -1$$



Equation of circle $x^2+y^2-4x-2y+4=0$

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Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=-4$ $\quad$ $2f=-2$

$g=\frac{-4}{2}=-2$ $\quad$ $f=\frac{-2}{2}=-1$

Thus center

$C(-g,-f)=C(-(-2),-(-1))=C(2,1)$

Slope of the line CD $m_2=\frac{y_2-y_1}{x_2-x_1}$

$$m_2=\frac{\frac{3}{2}-1}{\frac{5}{2}-2}=\frac{3-2}{2}\div\frac{5-4}{2}$$

$$m_2=\frac{1}{2}\div\frac{1}{2}=1$$

Now

$$m_1.m_2=(-1)(1)$$

$$m_1.m_2=-1$$

Thus line CD is perpendicular bisector of AB

---

b). line from the center of the circle to the mid-point of chord AB is perpendicular to the chord AB

Sol: end points of the chord $A(2,2)$ and $B(3,1)$

Let D is the midpoint of AB$=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$$D=\left(\frac{2+3}{2},\frac{2+1}{2}\right)$$

$$D=\left(\frac{5}{2},\frac{3}{2}\right)$$

Equation of circle $x^2+y^2-4x-2y+4=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=-4$ $\quad$ $2f=-2$

$g=\frac{-4}{2}=-2$ $\quad$ $f=\frac{-2}{2}=-1$

Thus center

$C(-g,-f)=C(-(-2),-(-1))=C(2,1)$

Slope of the line CD $m_2=\frac{y_2-y_1}{x_2-x_1}$

$$m_2=\frac{\frac{3}{2}-1}{\frac{5}{2}-2}=\frac{3-2}{2}\div\frac{5-4}{2}$$

$$m_2=\frac{1}{2}\div\frac{1}{2} \qquad \Rightarrow m_2=1$$

The line from centre to the midpoint of chord

$y-y_1=m_2(x-x_1)$

$y-\frac{3}{2}=1\left(x-\frac{5}{2}\right)$

$y-\frac{3}{2}=x-\frac{5}{2}$

$$x-y+\frac{3}{2}-\frac{5}{2}=0$$

$$x-y+\frac{3-5}{2}=0$$

$$x-y-1=0$$

Slope of the line AB $m_1=\frac{y_2-y_1}{x_2-x_1}$

$$m_1=\frac{1-2}{3-2}$$

$$m_1=\frac{-1}{1} \qquad \Rightarrow m_1=-1$$

The line AB with slope $m_1=-1$ and point $A(2,2)$

$y-y_1=m_1(x-x_1)$

$y-2=-1(x-2)$

$y-2=-x+2$

$x+y-2-2=0$

$x+y-4=0$

Thus the line CD $x-y-1=0$

And the line AB $x+y-4=0$

Are perpendicular to each other

Hence line from the center of the circle to the mid-point of the chord AB is perpendicular to chord AB

---

c). the perpendicular bisector CD of the chord AB passes through the center of the given circle.

Sol: end points of the chord $A(2,2)$ and $B(3,1)$

Slope of the line AB $m_1=\frac{y_2-y_1}{x_2-x_1}$

$$m_1=\frac{1-2}{3-2}$$

$$m_1=\frac{-1}{1}$$

$$m_1=-1$$

Let m_2 is the slope perpendicular to AB then

$$m_1.m_2=-1$$

$$(-1)m_2=-1$$

$$m_2=\frac{-1}{-1}=1$$

Let D is the midpoint of AB$=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$$D=\left(\frac{2+3}{2},\frac{2+1}{2}\right)$$

$$D=\left(\frac{5}{2},\frac{3}{2}\right)$$

Thus line perpendicular bisector {midpoint} to chord AB

$$y-y_1=m_2(x-x_1)$$

$$y-\frac{3}{2}=1\left(x-\frac{5}{2}\right)$$

$$y-\frac{3}{2}=x-\frac{5}{2}$$

$$x-y+\frac{3}{2}-\frac{5}{2}=0$$

$$x-y+\frac{3-5}{2}=0$$

$$x-y-1=0$$

Equation of circle $x^2+y^2-4x-2y+4=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=-4$ $\quad$ $2f=-2$

$g=\frac{-4}{2}=-2$ $\quad$ $f=\frac{-2}{2}=-1$

Thus center

$C(-g,-f)=C(-(-2),-(-1))=C(2,1)$

To show that line $x-y-1=0$ is perpendicular bisector of chord AB is passes through centre $C(2,1)$

$$x-y-1=0$$

$$2-1-1=0$$

$$0=0$$

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Hence perpendicular bisector CD of the chord AB passes through the center of the given circle

Q2. If $A(0,0)$ and $B(0,5)$ are the end points of chord AB of the circle $x^2+y^2+4x-5y=0$ then show that

a). the line from the center of the circle is perpendicular to AB, also bisects the chord AB

Sol: end points of the chord $A(0,0)$ and $B(0,5)$

Let D is the midpoint of AB $=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$$D=\left(\tfrac{0+0}{2},\tfrac{0+5}{2}\right)$$

$$D=\left(0,\tfrac{5}{2}\right)$$

Slope of the line AB $m_1=\dfrac{y_2-y_1}{x_2-x_1}$

$$m_1=\tfrac{5-0}{0-0}$$

$$m_1=\tfrac{5}{0}=\tfrac{1}{0}$$

Equation of circle $x^2+y^2+4x-5y=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=4$ $\quad 2f=-5$

$g=\frac{4}{2}=2$ $\quad f=\frac{-5}{2}$

Thus center

$C(-g,-f)=C\left(-(2),-\left(\tfrac{5}{2}\right)\right)=C\left(-2,\tfrac{5}{2}\right)$

Slope of the line CD $m_2=\dfrac{y_2-y_1}{x_2-x_1}$

$$m_2=\frac{\frac{5}{2}-\frac{5}{2}}{-2-0}$$

$$m_2=\tfrac{0}{-2}$$

Take $\quad m_2=\dfrac{0}{-1}$

Now $\quad m_1.m_2=\left(\dfrac{1}{0}\right)\left(\dfrac{0}{-1}\right)$

$$m_1.m_2=-1$$

Thus line CD is perpendicular bisector of AB

b). The line from the center of the circle to the mid-point of the chord AB is perpendicular to chord AB

Sol: end points of the chord $A(0,0)$ and $B(0,5)$

Let D is the midpoint of AB $=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$$D=\left(\tfrac{0+0}{2},\tfrac{0+5}{2}\right)$$

$$D=\left(0,\tfrac{5}{2}\right)$$

Equation of circle $x^2+y^2+4x-5y=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=4$ $\quad 2f=-5$

$g=\frac{4}{2}=2$ $\quad f=\frac{-5}{2}$

Thus center

$C(-g,-f)=C\left(-(2),-\left(\tfrac{5}{2}\right)\right)=C\left(-2,\tfrac{5}{2}\right)$

Slope of the line CD $m_2=\dfrac{y_2-y_1}{x_2-x_1}$

$$m_2=\frac{\frac{5}{2}-\frac{5}{2}}{-2-0}$$

$$m_2=\frac{0}{-2}$$

$$m_2=0$$

line from the centre to the midpoint point of AB

$y-y_1=m_2(x-x_1)$ Putting the values

$y-\frac{5}{2}=0(x-0)$

$y-\frac{5}{2}=0$

$y=\frac{5}{2}$

Slope of the line AB $m_1=\dfrac{y_2-y_1}{x_2-x_1}$

$$m_1=\tfrac{5-0}{0-0}$$

$$m_1=\tfrac{5}{0}=\tfrac{1}{0}$$

The line AB with slope $m_1=\dfrac{1}{0}$ and point $A(0,0)$

$y-y_1=m_1(x-x_1)$ Putting the values

$y-0=\dfrac{1}{0}(x-0)$

$(0)y=x$

$x=0$

Thus the line CD $y=\frac{5}{2}$

And the line AB $x=0$

Are perpendicular to each other

Hence line from the center of the circle to the mid-point of the chord AB is perpendicular to chord AB

c). the perpendicular bisector CD of the chord AB passes through the center of the given circle.

Sol: end points of the chord $A(0,0)$ and $B(0,5)$

Slope of the line AB $m_1=\dfrac{y_2-y_1}{x_2-x_1}$

$$m_1=\tfrac{5-0}{0-0}$$

$$m_1=\tfrac{5}{0}$$

Let m_2 is the slope perpendicular to AB then

$$m_1.m_2=-1$$

$$\left(\frac{5}{0}\right)m_2=-1$$

$$5m_2=0$$

$$m_2=0$$

Let D is the midpoint of AB $=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$$D=\left(\tfrac{0+0}{2},\tfrac{0+5}{2}\right)$$

$$D=\left(0,\tfrac{5}{2}\right)$$

Thus line perpendicular bisector{midpoint} to chord AB

$$y-y_1=m_2(x-x_1)$$

$$y-\tfrac{5}{2}=0(x-0)$$

$$y-\tfrac{5}{2}=0$$

$$y=\tfrac{5}{2}$$

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Equation of circle $x^2+y^2+4x-5y=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=4$ $\quad$ $2f=-5$

$g=\frac{4}{2}=2$ $\quad$ $f=\frac{-5}{2}$

Thus center

$C(-g,-f)=C\left(-(2),-\left(\frac{5}{2}\right)\right)=C\left(-2,\frac{5}{2}\right)$

To show that line $y=\frac{5}{2}$ is perpendicular bisector of the chord AB is passes through centre $C\left(-2,\frac{5}{2}\right)$

$y=\frac{5}{2}$ Putting centre $\frac{5}{2}=\frac{5}{2}$

Hence perpendicular bisector CD of the chord AB passes through the center of the given circle

---

Q3. Show that the chords AB and DE are equidistant from the center $C(0,0)$ of the circle

a). $x^2+y^2=4$ the coordinates of end points of the two chords AB and DE are $A(-2,0),B(0,2)$ $D(0,2)$ and $E(2,0)$

Solution: Equation of the circle $x^2+y^2=4$

Hence center of the circle is origin $(0,0)$

End points of the chord are $A(-2,0),B(0,2)$

$y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$ putting the values

$y-0=\dfrac{2-0}{0-(-2)}(x-(-2))$

$y=\dfrac{2}{2}(x+2)$

$y=x+2$

$x-y+2=0$

Shortest distance d_1 from centre of circle to chord AB is

$d_1=\left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$d_1=\left|\dfrac{(1)(0)+(-1)(0)+3}{\sqrt{1^2+(-1)^2}}\right|=\left|\dfrac{0+0+2}{\sqrt{1+1}}\right|$

$d_1=\left|\dfrac{2}{\sqrt{2}}\right|=\dfrac{2}{\sqrt{2}}$

End points of the chord are $D(0,2),E(2,0)$

$y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$ putting the values

$y-0=\frac{0-2}{2-0}(x-2)$

$y=\frac{-2}{2}(x-2)$

$y=-1(x-2)$

$y=-x+2$

$x+y-2=0$

Shortest distance d_2 from centre of circle to chord CD is

$d_2=\left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$d_2=\left|\dfrac{(1)(0)+(1)(0)-2}{\sqrt{1^2+(1)^2}}\right|=\left|\dfrac{0+0-2}{\sqrt{1+1}}\right|$

$d_2=\left|\dfrac{-2}{\sqrt{2}}\right|=\dfrac{2}{\sqrt{2}}$

Hence the distance from the centre of circle to the both chords AB and CD is same.

---

b). $x^2+y^2=16$ the coordinates of the end points of the two chords AB and DE are $A(-4,0),B(0,4)$ $D(0,4)$ and $E(4,0)$

Solution: Equation of the circle $x^2+y^2=16$

Hence center of the circle is origin $(0,0)$

End points of the chord are $A(-4,0),B(0,4)$

$y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$ putting the values

$y-0=\dfrac{4-0}{0-(-4)}(x-(-4))$

$y=\dfrac{4}{4}(x+4)$

$y=x+4$

$x-y+4=0$

Shortest distance d_1 from centre of circle to chord AB is

$d_1=\left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$d_1=\left|\dfrac{(1)(0)+(-1)(0)+4}{\sqrt{1^2+(-1)^2}}\right|=\left|\dfrac{0+0+4}{\sqrt{1+1}}\right|$

$d_1=\left|\dfrac{4}{\sqrt{2}}\right|=\dfrac{4}{\sqrt{2}}$

End points of the chord are $D(0,4),E(4,0)$

$y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$ putting the values

$y-0=\frac{0-4}{4-0}(x-4)$

$y=\frac{-4}{4}(x-4)$

$y=-1(x-4)$

$y=-x+4$

$x+y-4=0$

Shortest distance d_2 from centre of circle to chord CD is

$d_2=\left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$d_2=\left|\dfrac{(1)(0)+(1)(0)-4}{\sqrt{1^2+(1)^2}}\right|=\left|\dfrac{0+0-4}{\sqrt{1+1}}\right|$

$d_2=\left|\dfrac{-4}{\sqrt{2}}\right|=\dfrac{4}{\sqrt{2}}$

Hence the distance from the centre of circle to the both chords AB and CD is same.

---

c).

Solution: Equation of the circle $x^2+y^2=9$

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Hence center of the circle is origin $(0,0)$

End points of the chord are $A(-3,0), B(0,3)$

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ putting the values

$y-0=\frac{3-0}{0-(-3)}(x-(-3))$

$y=\frac{3}{3}(x+3)$

$y=x+3$

$x-y+3=0$

Shortest distance d_1 from centre of circle to chord AB is

$d_1=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$d_1=\left|\frac{(1)(0)+(-1)(0)+3}{\sqrt{1^2+(-1)^2}}\right|=\left|\frac{0+0+3}{\sqrt{1+1}}\right|$

$d_1=\left|\frac{3}{\sqrt{2}}\right|=\frac{3}{\sqrt{2}}$

End points of the chord are $D(0,3), E(3,0)$

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ putting the values

$y-0=\frac{0-3}{3-0}(x-3)$

$y=\frac{-3}{3}(x-3)$

$y=-1(x-3)$

$y=-x+3$

$x+y-3=0$

Shortest distance d_2 from centre of circle to chord CD is

$d_2=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ putting the values

$d_2=\left|\frac{(1)(0)+(1)(0)-3}{\sqrt{1^2+(1)^2}}\right|=\left|\frac{0+0-3}{\sqrt{1+1}}\right|$

$d_2=\left|\frac{-3}{\sqrt{2}}\right|=\frac{3}{\sqrt{2}}$

Hence the distance from the centre of circle to the both chords AB and CD is same.

Q4. Show that angle subtended by minor arc AB of circle a). $x^2+y^2=9$ is two times the angle subtended in the major arc. The coordinates of the minor arc AB are $A(2,\sqrt{5}), B(2,-\sqrt{5})$

Solution: circle $x^2+y^2=9$ with centre O(0,0)

For x-intercepts put $y=0$ in equation (1), we get

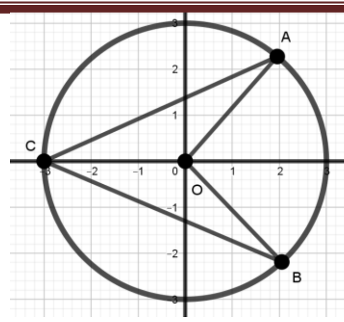
$x^2+0^2=9$

$x^2=9$ *taking square root*

$x=\pm 3$

So x-intercepts are $C(-3,0), D(3,0)$

Take a point on major arc C(-3,0)



slope of OA

$m_1=\frac{\sqrt{5}-0}{2-0}$

$m_1=\frac{\sqrt{5}}{2}$

slope of OB

$m_2=\frac{-\sqrt{5}-0}{2-0}$

$m_2=\frac{-\sqrt{5}}{2}$

slope of CA

$m_3=\frac{\sqrt{5}-0}{2-(-3)}=\frac{\sqrt{5}}{2+3}$

$m_3=\frac{\sqrt{5}}{5}=\frac{1}{\sqrt{5}}$

Let m_4 is a slope of BC

$m_4=\frac{-\sqrt{5}-0}{2-(-3)}=\frac{-\sqrt{5}}{2+3}$

$m_4=\frac{-\sqrt{5}}{5}=\frac{-1}{\sqrt{5}}$

$\tan\angle BOA=\frac{m_2-m_1}{1+m_2m_1}$ putting the values

$\tan\angle BOA=\frac{\frac{-\sqrt{5}}{2}-\frac{\sqrt{5}}{2}}{1+\left(\frac{-\sqrt{5}}{2}\right)\left(\frac{\sqrt{5}}{2}\right)}$

$\tan\angle BOA=\frac{\frac{-2\sqrt{5}}{2}}{1-\frac{5}{4}}=\frac{-\sqrt{5}}{\frac{4-5}{4}}$

$\tan\angle BOA=\frac{-\sqrt{5}}{\frac{-1}{4}}$

$\tan\angle BOA=4\sqrt{5}$

$\tan\angle BCA=\frac{m_4-m_3}{1+m_4m_3}$ putting the values

$\tan\angle BCA=\frac{\frac{-1}{\sqrt{5}}-\frac{1}{\sqrt{5}}}{1+\left(\frac{-1}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)}$

$\tan\angle BCA=\frac{\frac{-2}{\sqrt{5}}}{1-\frac{1}{5}}=\frac{\frac{-2}{\sqrt{5}}}{\frac{5-1}{5}}$

$\tan\angle BCA=\frac{\frac{-2}{\sqrt{5}}}{\frac{4}{5}}=\frac{-2}{\sqrt{5}}\times\frac{5}{4}$

$\tan\angle BCA=\frac{-\sqrt{5}}{2}$

According to condition

$m\angle BOA=2m\angle BCA$

$\tan(\angle BOA)=\tan(2\angle BCA)$

$\tan(\angle BOA)=\frac{2\tan(\angle BCA)}{1-\tan^2(\angle BCA)}$

By using formula

$\tan(\angle BOA)=\frac{2\left(\frac{-\sqrt{5}}{2}\right)}{1-\left(\frac{-\sqrt{5}}{2}\right)^2}=\frac{-\sqrt{5}}{1-\frac{5}{4}}$

$4\sqrt{5}=\frac{-\sqrt{5}}{\frac{4-5}{4}}=\frac{-\sqrt{5}}{\frac{-1}{4}}$

$4\sqrt{5}=4\sqrt{5}$

This satisfied the given condition

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b). $x^2+y^2=4$ is two times the angle subtended in the major arc. The coordinates of the minor arc AB are $A\left(1,\sqrt{3}\right), B\left(1,-\sqrt{3}\right)$

Solution: circle $x^2+y^2=4$ with centre O(0,0)

For x-intercepts put $y=0$ in equation (1), we get

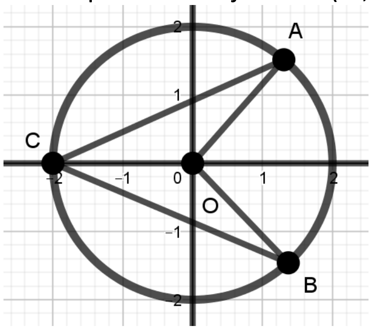
$x^2+0^2=4$

$x^2=4 \qquad \textit{taking square root}$

$x=\pm 2$

So x-intercepts are $C\left(-2,0\right), \mathrm{D}\left(2,0\right)$

Take a point on major arc C(-2,0)



Let m_1 is a slope of OA

$$m_1=\frac{\sqrt{3}-0}{1-0}=\frac{\sqrt{3}}{1}=\sqrt{3}$$

Let m_2 is a slope of OB

$$m_2=\frac{-\sqrt{3}-0}{1-0}=\frac{-\sqrt{3}}{1}=-\sqrt{3}$$

Let m_3 is a slope of CA

$$m_3=\frac{\sqrt{3}-0}{1-\left(-2\right)}=\frac{\sqrt{3}}{1+2}$$

$$m_3=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}$$

Let m_4 is a slope of BC

$$m_4=\frac{-\sqrt{3}-0}{1-\left(-2\right)}=\frac{-\sqrt{3}}{1+2}$$

$$m_4=\frac{-\sqrt{3}}{3}=\frac{-1}{\sqrt{3}}$$

$\tan\angle BOA=\dfrac{m_2-m_1}{1+m_2m_1}$ putting the values

$$\tan\angle BOA=\frac{-\sqrt{3}-\sqrt{3}}{1+\left(-\sqrt{3}\right)\left(\sqrt{3}\right)}$$

$$\tan\angle BOA=\frac{-2\sqrt{3}}{1-3}=\frac{-2\sqrt{3}}{-2}$$

$$\tan\angle BOA=\sqrt{3}$$

$\tan\angle BCA=\dfrac{m_4-m_3}{1+m_4m_3}$ putting the values

$$\tan\angle BCA=\frac{\frac{-1}{\sqrt{3}}-\frac{1}{\sqrt{3}}}{1+\left(\frac{-1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)}$$

$$\tan\angle BCA=\frac{\frac{-2}{\sqrt{3}}}{1-\frac{1}{3}}=\frac{\frac{-2}{\sqrt{3}}}{\frac{3-1}{3}}$$

$$\tan\angle BCA=\frac{\frac{-2}{\sqrt{3}}}{\frac{2}{3}}=\frac{-2}{\sqrt{3}}\times\frac{3}{2}$$

$$\tan\angle BCA=-\sqrt{3}$$

According to condition

$$m\angle BOA=2m\angle BCA$$

$$\tan\left(\angle BOA\right)=\tan\left(2\angle BCA\right)$$

$$\tan\left(\angle BOA\right)=\frac{2\tan\left(\angle BCA\right)}{1-\tan^2\left(\angle BCA\right)}$$

By using formula

$$\tan\left(\angle BOA\right)=\frac{2\left(-\sqrt{3}\right)}{1-\left(-\sqrt{3}\right)^2}=\frac{-2\sqrt{3}}{1-3}$$

$$\sqrt{3}=\frac{-2\sqrt{3}}{-2}$$

$$\sqrt{3}=\sqrt{3}$$

This satisfied the given condition

---

c). $x^2+y^2=16$ is two times the angle subtended in the major arc. The coordinates of the minor arc AB are $A\left(3,\sqrt{7}\right), B\left(3,-\sqrt{7}\right)$

Solution: circle $x^2+y^2=16$ with centre O(0,0)

For x-intercepts put $y=0$ in equation (1), we get

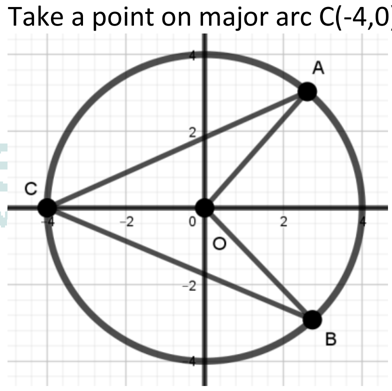
$x^2+0^2=16$

$x^2=16 \qquad \textit{taking square root}$

$x=\pm 4$

So x-intercepts are $C\left(-4,0\right), \mathrm{D}\left(4,0\right)$

Take a point on major arc C(-4,0)



Let m_1 is a slope of OA

$$m_1=\frac{\sqrt{7}-0}{3-0}=\frac{\sqrt{7}}{3}$$

Let m_2 is a slope of OB

$$m_2=\frac{-\sqrt{7}-0}{3-0}=\frac{-\sqrt{7}}{3}$$

Let m_3 is a slope of CA

$$m_3=\frac{\sqrt{7}-0}{3-\left(-4\right)}=\frac{\sqrt{7}}{3+4}$$

$$m_3=\frac{\sqrt{7}}{7}=\frac{1}{\sqrt{7}}$$

Let m_4 is a slope of BC

$$m_4=\frac{-\sqrt{7}-0}{3-\left(-4\right)}=\frac{-\sqrt{7}}{3+4}$$

$$m_4=\frac{-\sqrt{7}}{7}=\frac{-1}{\sqrt{7}}$$

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$\tan \angle BOA = \dfrac{m_2 - m_1}{1 + m_2 m_1}$ putting the values

$\tan \angle BOA = \dfrac{\frac{-\sqrt{7}}{3} - \frac{\sqrt{7}}{3}}{1 + \left(\frac{-\sqrt{7}}{3}\right)\left(\frac{\sqrt{7}}{3}\right)}$

$\tan \angle BOA = \dfrac{\frac{-2\sqrt{7}}{3}}{1 - \frac{7}{9}} = \dfrac{\frac{-2\sqrt{7}}{3}}{\frac{9-7}{9}}$

$\tan \angle BOA = \dfrac{\frac{-2\sqrt{7}}{3}}{\frac{2}{9}} = \dfrac{-2\sqrt{7}}{3} \times \dfrac{9}{2}$

$\tan \angle BOA = -3\sqrt{7}$

$\tan \angle BCA = \dfrac{m_4 - m_3}{1 + m_4 m_3}$ putting the values

$\tan \angle BCA = \dfrac{\frac{-1}{\sqrt{7}} - \frac{1}{\sqrt{7}}}{1 + \left(\frac{-1}{\sqrt{7}}\right)\left(\frac{1}{\sqrt{7}}\right)}$

$\tan \angle BCA = \dfrac{\frac{-2}{\sqrt{7}}}{1 - \frac{1}{7}} = \dfrac{\frac{-2}{\sqrt{7}}}{\frac{7-1}{7}}$

$\tan \angle BCA = \dfrac{\frac{-2}{\sqrt{7}}}{\frac{6}{7}} = \dfrac{-2}{\sqrt{7}} \times \dfrac{7}{6}$

$\tan \angle BCA = \dfrac{-\sqrt{7}}{3}$

According to condition

$m\angle BOA = 2m\angle BCA$

$\tan(\angle BOA) = \tan(2\angle BCA)$

$\tan(\angle BOA) = \dfrac{2\tan(\angle BCA)}{1 - \tan^2(\angle BCA)}$

By using formula

$\tan(\angle BOA) = \dfrac{2\left(\frac{-\sqrt{7}}{3}\right)}{1 - \left(\frac{-\sqrt{7}}{3}\right)^2} = \dfrac{\frac{-2\sqrt{7}}{3}}{1 - \frac{7}{9}}$

$-3\sqrt{7} = \dfrac{\frac{-2\sqrt{7}}{3}}{\frac{9-7}{9}} = \dfrac{-2\sqrt{7}}{3} \times \dfrac{9}{2}$

$-3\sqrt{7} = -3\sqrt{7}$

This satisfied the given condition

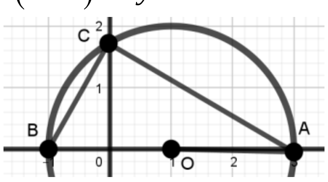
Q5. Show that angle in the semicircle of the circle

a). $(x-h)^2 + y^2 = a^2,\ h = 1, a = 2$ is a right angle.

Sol: We have $(x-h)^2 + y^2 = a^2,\ h = 1, a = 2$

Then equation of the circle becomes

$(x-1)^2 + y^2 = 2^2$.....................(1)



For x-intercepts put $y = 0$ in equation (1), we get

$(x-1)^2 + 0^2 = 2^2$

$(x-1)^2 = 2^2$ *taking square root*

$x - 1 = \pm 2$

$x = 1 \pm 2$

Either or

$x = 1 + 2$ $x = 1 - 2$

$x = 3$ $x = -1$

So x-intercepts are $A(3,0), B(-1,0)$

For y-intercepts put $x = 0$ in equation (1), we get

$(0-1)^2 + y^2 = 2^2$

$(0-1)^2 + y^2 = 2^2$

$1 + y^2 = 4$

$y^2 = 4 - 1$

$y^2 = 3$ *taking square root*

$y = \pm\sqrt{3}$

Take only one condition $y = \sqrt{3}$

So y-intercepts are $C(0, \sqrt{3})$

Let m_1 is a slope of AC

$m_1 = \dfrac{\sqrt{3} - 0}{0 - 3} = \dfrac{\sqrt{3}}{-3} = \dfrac{-1}{\sqrt{3}}$

Let m_2 is a slope of BC

$m_2 = \dfrac{\sqrt{3} - 0}{0 - (-1)} = \dfrac{\sqrt{3}}{+1} = \sqrt{3}$

For the right angle between AC and BC

$m_1.m_2 = \left(\frac{-1}{\sqrt{3}}\right)\left(\sqrt{3}\right)$

$m_1.m_2 = -1$

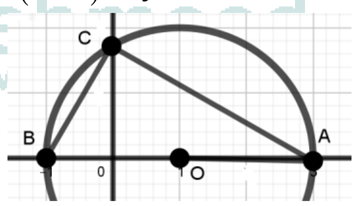
Hence angle in the semi circle is right

b). $(x-h)^2 + y^2 = a^2,\ h = 3, a = 4$ is a right angle.

Sol: We have $(x-h)^2 + y^2 = a^2,\ h = 3, a = 4$

Then equation of the circle becomes

$(x-3)^2 + y^2 = 4^2$.....................(1)



For x-intercepts put $y = 0$ in equation (1), we get

$(x-3)^2 + 0^2 = 4^2$

$(x-3)^2 = 4^2$ *taking square root*

$x - 3 = \pm 4$

$x = 3 \pm 4$

Either or

$x = 3 + 4$ $x = 3 - 4$

$x = 7$ $x = -1$

So x-intercepts are $A(7,0), B(-1,0)$

For y-intercepts put $x = 0$ in equation (1), we get

$(0-3)^2 + y^2 = 4^2$

$(-3)^2 + y^2 = 4^2$

$9 + y^2 = 16$

$y^2 = 16 - 9$

$y^2 = 7$ *taking square root*

$y = \pm\sqrt{7}$

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Take only one condition $y=\sqrt{7}$

So y-intercepts are $C\left(0,\sqrt{7}\right)$

Let m_1 is a slope of AC

$$m_1=\frac{\sqrt{7}-0}{0-7}=\frac{\sqrt{7}}{-7}=\frac{-1}{\sqrt{7}}$$

Let m_2 is a slope of BC

$$m_2=\frac{\sqrt{7}-0}{0-(-1)}=\frac{\sqrt{7}}{+1}=\sqrt{7}$$

For the right angle between AC and BC

$$m_1.m_2=\left(\frac{-1}{\sqrt{7}}\right)\left(\sqrt{7}\right)$$

$$m_1.m_2=-1$$

Hence angle in the semi circle is right

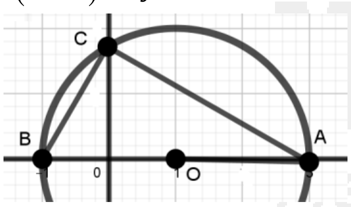
c). $\left(x-h\right)^2+y^2=a^2,\ h=2,a=3$ is a right angle

Solution: We have $\left(x-h\right)^2+y^2=a^2,$

$h=2,a=3$

Then equation of the circle becomes

$$\left(x-2\right)^2+y^2=3^2 ....(1)$$



For x-intercepts put $y=0$ in equation (1), we get

$$\left(x-2\right)^2+0^2=3^2$$

$$\left(x-2\right)^2=3^2 \qquad \textit{taking square root}$$

$$x-2=\pm3$$

$$x=2\pm3$$

Either or

$x=2+3$ $\qquad$ $x=2-3$

$x=5$ $\qquad$ $x=-1$

So x-intercepts are $A\left(5,0\right),B\left(-1,0\right)$

For y-intercepts put $x=0$ in equation (1), we get

$$\left(0-2\right)^2+y^2=3^2$$

$$\left(0-2\right)^2+y^2=3^2$$

$$4+y^2=9$$

$$y^2=9-4$$

$$y^2=5 \qquad \textit{taking square root}$$

$$y=\pm\sqrt{5}$$

Take only one condition $y=\sqrt{5}$

So y-intercepts are $C\left(0,\sqrt{5}\right)$

Let m_1 is a slope of AC

$$m_1=\frac{\sqrt{5}-0}{0-5}=\frac{\sqrt{5}}{-5}=\frac{-1}{\sqrt{5}}$$

Let m_2 is a slope of BC

$$m_2=\frac{\sqrt{5}-0}{0-(-1)}=\frac{\sqrt{5}}{+1}=\sqrt{5}$$

For the right angle between AC and BC

$$m_1.m_2=\left(\frac{-1}{\sqrt{5}}\right)\left(\sqrt{5}\right)$$

$$m_1.m_2=-1$$

Hence angle in the semi-circle is right

Q6. Show that perpendicular at the outer endpoint

a). $P\left(1,5\right)$ of the radial segment is tangent to the circle $x^2+y^2+x-5y-2=0$

Solution: the Circle $x^2+y^2+x-5y-2=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=1$ $\qquad$ $2f=-5$

$g=\frac{1}{2}$ $\qquad$ $f=\frac{-5}{2}$

Thus center

$$C\left(-g,-f\right)=C\left(-\left(\tfrac{1}{2}\right),-\left(\tfrac{-5}{2}\right)\right)=C\left(\tfrac{-1}{2},\tfrac{5}{2}\right)$$

Equation of tangent at $P\left(1,5\right)$ of given circle

Using formula $xx_1+yy_1+g\left(x+x_1\right)+f\left(y+y_1\right)+c=0$

putting

$$(1)x+(5)y+\frac{1}{2}(x+1)-\frac{5}{2}(y+5)-2=0$$

$$x+5y+\frac{1}{2}(x+1)-\frac{5}{2}(y+5)-2=0 \qquad \times by\ 2$$

$$2x+10y+\left(x+1\right)-5\left(y+5\right)-4=0$$

$$2x+10y+x+1-5y-25-4=0$$

$$2x+x+10y-5y+1-25-4=0$$

$$3x+5y-28=0$$

Slope of line $m_1=\frac{-a}{b}$ putting the values $m_1=\frac{-3}{5}$

Slope of the radial segment CP $m_2=\frac{y_2-y_1}{x_2-x_1}$

$$m_2=\frac{\frac{5}{2}-5}{\frac{-1}{2}-1}=\frac{5-10}{2}\div\frac{-1-2}{2}$$

$$m_2=\tfrac{-5}{2}\times\tfrac{2}{-3} \qquad \Rightarrow m_2=\tfrac{5}{3}$$

Slope of radial segment x slope of tangent

$$m_1.m_2=\left(\tfrac{-3}{5}\right)\left(\tfrac{5}{3}\right)$$

$$m_1.m_2=-1$$

Hence Perpendicular at the outer endpoint of radial segment is tangent to the circle

b). $P\left(5,-6\right)$ of the radial segment is tangent to the circle $x^2+y^2-22x-4y+25=0$

Solution: the Circle $x^2+y^2-22x-4y+25=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=-22$ $\qquad$ $2f=-4$

$g=\frac{-22}{2}=-11$ $\qquad$ $f=\frac{-4}{2}=-2$

Thus center

$C(-g,-f)=C(-(-11),-(-2))=C(11,2)$

Equation of tangent at $P(5,10)$ of given circle

Using formula $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

putting

$(5)x+(-6)y-11(x+5)-2(y-6)+25=0$

$5x-6y-11x-55-2y+12+25=0$

$5x-11x-6y-2y-55+12+25=0$

$-6x-8y-18=0$

Slope of line $m_1=\frac{-a}{b}$ putting values

$m_1=\frac{-(-6)}{-8}=-\frac{3}{4}$

Slope of the radial segment CP $m_2=\frac{y_2-y_1}{x_2-x_1}$

$$m_2=\frac{-6-2}{5-11}=\frac{-8}{-6}$$

$$m_2=\frac{4}{3}$$

Slope of radial segment x slope of tangent

$m_1.m_2=\left(-\frac{3}{4}\right)\left(\frac{4}{3}\right)$

$m_1.m_2=-1$

Hence Perpendicular at the outer endpoint of radial segment is tangent to the circle

---

c). $P(0,0)$ of the radial segment is tangent to the circle $x^2+y^2-ax-by=0$

Solution: the Circle $x^2+y^2-ax-by=0$

Comparing with the general equation of circle

$x^2+y^2+2gx+2fy+c=0$ We get

$2g=-a$ $\qquad$ $2f=-b$

$g=\frac{-a}{2}$ $\qquad$ $f=\frac{-b}{2}$

Thus center

$C(-g,-f)=C\left(-\left(\frac{-a}{2}\right),-\left(\frac{-b}{2}\right)\right)=C\left(\frac{a}{2},\frac{b}{2}\right)$

Equation of tangent at $P(0,0)$ of given circle

Using formula $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

putting

$(0)x+(0)y-\frac{a}{2}(x+0)-\frac{b}{2}(y+0)=0$

$0+0-\frac{a}{2}(x)-\frac{b}{2}(y)=0$

$-\frac{a}{2}x-\frac{b}{2}y=0$ $\qquad$ $\times by(-2)$

$ax+by=0$

Slope of the line $m_1=\frac{-a}{b}$ putting the values

$m_1=\frac{-a}{b}$

Slope of the radial segment CP $m_2=\frac{y_2-y_1}{x_2-x_1}$

$$m_2=\frac{\frac{b}{2}-0}{\frac{a}{2}-0}$$

$$m_2=\frac{\frac{b}{2}}{\frac{a}{2}}$$

$$m_2=\frac{b}{a}$$

Slope of radial segment x slope of tangent

$m_1.m_2=\left(\frac{-a}{b}\right)\left(\frac{b}{a}\right)$

$m_1.m_2=-1$

Hence Perpendicular at the outer endpoint of radial segment is tangent to the circle

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Merging man and math
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