Chapter 5

Integration formulae	Differentiation formulae
$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$
$\int \frac{dx}{x} = \ln\lvert x\rvert + C$	$\frac{d}{dx}(\ln x + C) = \frac{1}{x}$
$\int \cos kx\, dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k} + C\right) = \cos kx$
$\int \sin kx\, dx = \frac{-\cos kx}{k} + C$	$\frac{d}{dx}\left(\frac{-\cos kx}{k} + C\right) = \sin kx$
$\int \sec^2 xdx = \tan x + C$	$\frac{d}{dx}(\tan x + C) = \sec^2 x$
$\int -\cos ec^2 xdx = \cot x + C$	$\frac{d}{dx}(\cot x + C) = -\cos ec^2 x$
$\int \frac{\sin x}{\cos x} dx = -\ln\lvert\cos x\rvert + c$	$\frac{d}{dx}(-\ln\cos x + C) = \tan x$
$\int \cot xdx = \ln\lvert\sin x\rvert + C$	$\frac{d}{dx}(\ln\sin x + C) = \cot x$
$\int e^{mx} dx = \frac{e^{mx}}{m} + C$	$\frac{d}{dx}\left(\frac{e^{mx}}{m} + C\right) = e^{mx}$
$\int a^x dx = \frac{a^x}{\ln a} + C$	$\frac{d}{dx}\left(\frac{a^x}{\ln a} + C\right) = a^x$
$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$	$\frac{d}{dx}\left(\tan^{-1} x + C\right) = \frac{1}{1+x^2}$
$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2+x^2}$
$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$	$\frac{d}{dx}\left(\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert\right) = \frac{1}{a^2-x^2}$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	$\frac{d}{dx}\left(\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert\right) = \frac{1}{x^2-a^2}$
$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$	$\frac{d}{dx}\left(\sin^{-1} x + C\right) = \frac{1}{\sqrt{1-x^2}}$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$	$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a} + C\right) = \frac{1}{\sqrt{a^2-x^2}}$
$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert$	$\frac{d}{dx}\left(\ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert\right) = \frac{1}{\sqrt{x^2 \pm a^2}}$

Constant Rule $\int kdx = k\int dx + C$

Constant multiple Rule $\int kf(x)dx = k\int f(x)dx + C$

Sum Rule $\int\left[f(x) + g(x)\right]dx = \int f(x)dx + \int g(x)dx$

Power Rule $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Logarithmic Rule $\int \frac{1}{x} dx = \ln\lvert x\rvert + C$

Exercise 5.1

Q1. Evaluate the following indefinite integral and check the result through differentiation:

a). $\int\left(x^4 + 3x^3 - 7\right)dx$

Sol: Given $\int\left(x^4 + 3x^3 - 7\right)dx$

$$\int\left(x^4 + 3x^3 - 7\right)dx = \int x^4 dx + 3\int x^3 dx - 7\int dx$$

$$\int\left(x^4 + 3x^3 - 7\right)dx = \frac{x^{4+1}}{4+1} + \frac{3x^{3+1}}{3+1} - 7x + C$$

$$\int\left(x^4 + 3x^3 - 7\right)dx = \frac{x^5}{5} + \frac{3x^4}{4} - 7x + C$$

For checking the result, Let

$y = \frac{x^5}{5} + \frac{3x^4}{4} - 7x + C$ Differentiating w. r. t "x"

$$\frac{d}{dx}y = \frac{1}{5}\frac{d}{dx}x^5 + \frac{3}{4}\frac{d}{dx}x^4 - 7\frac{d}{dx}x + \frac{d}{dx}C$$

$$\frac{dy}{dx} = \frac{1}{5}\left(5x^{5-1}\right)\frac{d}{dx}x + \frac{3}{4}\left(4x^{4-1}\right)\frac{d}{dx}x - 7 + 0$$

$$\frac{dy}{dx} = x^4 + 3x^3 - 7$$

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b). $\int \frac{1}{x^3} dx$

Sol: Given $\int \frac{1}{x^3} dx = \int x^{-3} dx$

$$\int \frac{1}{x^3} dx = \frac{x^{-3+1}}{-3+1} + C$$

$$\int \frac{1}{x^3} dx = \frac{x^{-2}}{-2} + C$$

$$\int \frac{1}{x^3} dx = \frac{-1}{2x^2} + C$$

For checking the result, Let

$y = \frac{-1}{2x^2} + C = \frac{-1}{2}x^{-2} + C$ Differentiating w. r. t "x"

$$\frac{dy}{dx} = \frac{-1}{2}\frac{d}{dx}x^{-2} + \frac{d}{dx}C$$

$$\frac{dy}{dx} = \frac{-1}{2}(-2)x^{-2-1} + 0$$

$$\frac{dy}{dx} = x^{-3} = \frac{1}{x^3}$$

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c). $\int\left(2x^4 + x^{\frac{-2}{3}} - x^{\frac{-5}{3}}\right)dx$

Sol: Given $\int\left(2x^4 + x^{\frac{-2}{3}} - x^{\frac{-5}{3}}\right)dx$

$$\int\left(2x^4 + x^{\frac{-2}{3}} - x^{\frac{-5}{3}}\right)dx = 2\int x^4 dx + \int x^{\frac{-2}{3}} dx - \int x^{\frac{-5}{3}} dx$$

$$\int\left(2x^4 + x^{\frac{-2}{3}} - x^{\frac{-5}{3}}\right)dx = \frac{2x^{4+1}}{4+1} + \frac{x^{\frac{-2}{3}+1}}{\frac{-2}{3}+1} - \frac{x^{\frac{-5}{3}+1}}{\frac{-5}{3}+1} + C$$

$$\int\left(2x^4 + x^{\frac{-2}{3}} - x^{\frac{-5}{3}}\right)dx = \frac{2x^5}{5} + \frac{x^{\frac{1}{3}}}{\frac{1}{3}} - \frac{x^{\frac{-2}{3}}}{\frac{-2}{3}} + C$$

$$\int\left(2x^4 + x^{\frac{-2}{3}} - x^{\frac{-5}{3}}\right)dx = \frac{2x^5}{5} + 3x^{\frac{1}{3}} - \frac{3x^{\frac{-2}{3}}}{-2} + C$$

$$\int\left(2x^4 + x^{\frac{-2}{3}} - x^{\frac{-5}{3}}\right)dx = \frac{2}{5}x^5 + 3x^{\frac{1}{3}} + \frac{3}{2}x^{\frac{-2}{3}} + C$$

For checking the result, Let

$y = \frac{2}{5}x^5 + 3x^{\frac{1}{3}} + \frac{3}{2}x^{\frac{-2}{3}} + C$ Differentiating w. r. t "x"

$$\frac{dy}{dx} = \frac{2}{5}\frac{d}{dx}x^5 + 3\frac{d}{dx}x^{\frac{1}{3}} + \frac{3}{2}\frac{d}{dx}x^{\frac{-2}{3}} + \frac{d}{dx}C$$

$$\frac{dy}{dx}=\frac{2}{5}(5)x^{5-1}+\frac{3}{3}x^{\frac{1}{3}-1}+\frac{3}{2}\left(\frac{-2}{3}\right)x^{\frac{-2}{3}-1}+0$$

$$\frac{dy}{dx}=2x^4+x^{\frac{-2}{3}}-x^{\frac{-5}{3}}$$

d). $\int(1+3t)t^3dt$

Sol: Given $\int(1+3t)t^3dt$

$$\int(1+3t)t^3dt=\int\left(t^3+3t^4\right)dt$$

$$\int(1+3t)t^3dt=\int t^3dt+3\int t^4dt$$

$$\int(1+3t)t^3dt=\frac{t^{3+1}}{3+1}+3\frac{t^{4+1}}{4+1}+C$$

$$\int(1+3t)t^3dt=\frac{1}{4}t^4+\frac{3}{5}t^5+C$$

For checking the result, Let

$y=\frac{1}{4}t^4+\frac{3}{5}t^5+C$ Differentiating w. r. t "x"

$$\frac{dy}{dt}=\frac{1}{4}\frac{d}{dt}t^4+\frac{3}{5}\frac{d}{dt}t^5+\frac{d}{dt}C$$

$$\frac{dy}{dt}=\frac{1}{4}\left(4t^{4-1}\right)\frac{d}{dt}t+\frac{3}{5}\left(5t^{5-1}\right)\frac{d}{dt}t+0$$

$$\frac{dy}{dt}=t^3+3t^4$$

$$\frac{dy}{dt}=(1+3t)t^3$$

e). $\int\left(t^2-1\right)^2dt$

Sol: Given $\int\left(t^2-1\right)^2dt$

$$\int\left(t^2-1\right)^2dt=\int\left\{\left(t^2\right)^2-2\left(t^2\right)(1)+(1)^2\right\}dt$$

$$\int\left(t^2-1\right)^2dt=\int\left\{t^4-2t^2+1\right\}dt$$

$$\int\left(t^2-1\right)^2dt=\int t^4dt-2\int t^2dt+\int 1dt$$

$$\int\left(t^2-1\right)^2dt=\frac{t^{4+1}}{4+1}-2\frac{t^{2+1}}{2+1}+t+C$$

$$\int\left(t^2-1\right)^2dt=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C$$

For checking the result, Let

$y=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C$ Differentiating w. r. t "x"

$$\frac{dy}{dt}=\frac{1}{5}\frac{d}{dt}t^5-\frac{2}{3}\frac{d}{dt}t^3+\frac{d}{dt}t+\frac{d}{dt}C$$

$$\frac{dy}{dt}=\frac{1}{5}\left(5t^{5-1}\right)\frac{d}{dt}t-\frac{2}{3}\left(3t^{3-1}\right)\frac{d}{dt}t+1+0$$

$$\tfrac{dy}{dt}=t^4-2t^2+1$$

$$\tfrac{dy}{dx}=\left(t^2\right)^2-2\left(t^2\right)(1)+(1)^2$$

$$\tfrac{dy}{dx}=\left(t^2-1\right)^2$$

f). $\int\frac{x^3+1}{x^3}dx$

Sol: Given $\int\frac{x^3+1}{x^3}dx$

$$\int\frac{x^3+1}{x^3}dx=\int\left(\frac{x^3}{x^3}+\frac{1}{x^3}\right)dx$$

$$\int\frac{x^3+1}{x^3}dx=\int\left(1+x^{-3}\right)dx$$

$$\int\frac{x^3+1}{x^3}dx=\int 1dx+\int x^{-3}dx$$

$$\int\frac{x^3+1}{x^3}dx=x+\frac{x^{-3+1}}{-3+1}+C$$

$$\int\frac{x^3+1}{x^3}dx=x+\frac{x^{-2}}{-2}+C$$

$$\int\frac{x^3+1}{x^3}dx=x-\frac{1}{2}x^{-2}+C$$

For checking the result, Let

$y=x-\frac{1}{2}x^{-2}+C$ Differentiating w. r. t "x"

$$\frac{dy}{dx}=\frac{d}{dx}x-\frac{1}{2}\frac{d}{dx}x^{-2}+\frac{d}{dx}C$$

$$\frac{dy}{dx}=1-\frac{1}{2}(-2)x^{-2-1}+0$$

$$\tfrac{dy}{dx}=1+x^{-3}$$

$$\frac{dy}{dx}=1+\frac{1}{x^3}=\frac{x^3+1}{x^3}$$

g). $\int z^2\sqrt{z}dz$

Sol: Given $\int z^2\sqrt{z}dz$

$$\int z^2\sqrt{z}dz=\int z^{2+\frac{1}{2}}dz$$

$$\int z^2\sqrt{z}dz=\int z^{\frac{5}{2}}dz$$

$$\int z^2\sqrt{z}dz=\frac{z^{\frac{5}{2}+1}}{\frac{5}{2}+1}+c$$

$$\int z^2\sqrt{z}dz=\frac{z^{\frac{7}{2}}}{\frac{7}{2}}+c$$

$$\int z^2\sqrt{z}dz=\frac{2}{7}z^{\frac{7}{2}}+c$$

For checking the result, Let

$y=\frac{2}{7}z^{\frac{7}{2}}+c$ Differentiating w. r. t "x"

$$\frac{dy}{dz}=\frac{2}{7}\frac{d}{dz}z^{\frac{7}{2}}+\frac{d}{dz}c$$

$$\frac{dy}{dz}=\frac{2}{7}\times\frac{7}{2}z^{\frac{7}{2}-1}+0$$

$$\tfrac{dy}{dz}=z^{\frac{5}{2}}\qquad\Rightarrow\tfrac{dy}{dz}=z^2.z^{\frac{1}{2}}=z^2.\sqrt{z}$$

Q2. Evaluate the following indefinite integrals by method of substitution:

a). $\int(3x+4)^8dx$

Sol: Given $\int(3x+4)^8dx$

Let $u=3x+4$ Differentiating w. r. t "x"

$$\frac{du}{dx}=3\frac{d}{dx}x+\frac{d}{dx}4$$

$$\frac{du}{dx}=3(1)+0$$
$$\frac{du}{dx}=3$$
$$du=3xdx$$
$$\frac{du}{3}=dx$$

Substituting values of u and du in given integral

$$\int (3x+4)^8\,dx=\int u^8\frac{du}{3}$$
$$\int (3x+4)^8\,dx=\frac{1}{3}\int u^8du$$
$$\int (3x+4)^8\,dx=\frac{1}{3}\frac{u^{8+1}}{8+1}+C$$
$$\int (3x+4)^8\,dx=\frac{1}{3}.\frac{1}{9}u^9+C$$
$$\int (3x+4)^8\,dx=\frac{1}{27}u^9+C$$

Putting the values of u

$$\int (3x+4)^8\,dx=\frac{1}{27}(3x+4)^9+C$$

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b). $\int 3x^2\left(x^3-4\right)dx$

Sol: Given $\int 3x^2\left(x^3-4\right)dx$

Let $u=x^3-4$ Differentiating w. r. t "x"

$$\frac{du}{dx}=\frac{d}{dx}x^3-\frac{d}{dx}4$$
$$\frac{du}{dx}=3x^{3-1}\frac{d}{dx}x-0$$
$$\frac{du}{dx}=3x^2$$
$$du=3x^2dx$$

Substituting values of u and du in given integral

$$\int 3x^2\left(x^3-4\right)dx=\int \left(x^3-4\right)3x^2dx$$
$$\int 3x^2\left(x^3-4\right)dx=\int udu$$
$$\int 3x^2\left(x^3-4\right)dx=\frac{u^2}{2}+C$$
$$\int 3x^2\left(x^3-4\right)dx=\frac{1}{2}u^2+C$$

Putting the values of u

$$\int 3x^2\left(x^3-4\right)dx=\frac{1}{2}\left(x^3-4\right)^2+C$$

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c). $\int \left(3x^2+7\right)\left(x^3+7x\right)^8dx$

Sol: Given $\int \left(3x^2+7\right)\left(x^3+7x\right)^8dx$

Let $u=x^3+7x$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}x^3+7\tfrac{d}{dx}x$$
$$\tfrac{d}{dx}u=3x^2\tfrac{d}{dx}x+7$$
$$\frac{du}{dx}=3x^2+7$$
$$du=\left(3x^2+7\right)dx$$

Substituting values of u and du in given integral

$$\int \left(3x^2+7\right)\left(x^3+7x\right)^8dx=\int \left(x^3+7x\right)^8\left(3x^2+7\right)dx$$
$$\int \left(3x^2+7\right)\left(x^3+7x\right)^8dx=\int u^8du$$
$$\int \left(3x^2+7\right)\left(x^3+7x\right)^8dx=\frac{u^{8+1}}{8+1}+C$$
$$\int \left(3x^2+7\right)\left(x^3+7x\right)^8dx=\frac{1}{9}u^9+C$$

Putting the values of u

$$\int \left(3x^2+7\right)\left(x^3+7x\right)^8dx=\frac{1}{9}\left(x^3+7x\right)^9+C$$

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d). $\int \frac{3x^2-7}{\left(x^3-7x\right)^4}dx$

Sol: Given $\int \frac{3x^2-7}{\left(x^3-7x\right)^4}dx$

Let $u=x^3-7x$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}x^3-7\tfrac{d}{dx}x$$
$$\tfrac{d}{dx}u=3x^2\tfrac{d}{dx}x-7$$
$$\frac{du}{dx}=3x^2-7$$
$$du=\left(3x^2-7\right)dx$$

Substituting values of u and du in given integral

$$\int \frac{3x^2-7}{\left(x^3-7x\right)^4}dx=\int \frac{\left(3x^2-7\right)dx}{\left(x^3-7x\right)^4}$$
$$\int \frac{3x^2-7}{\left(x^3-7x\right)^4}dx=\int \frac{du}{u^4}=\int u^{-4}du$$
$$\int \frac{3x^2-7}{\left(x^3-7x\right)^4}dx=\frac{u^{-4+1}}{-4+1}+C$$
$$\int \frac{3x^2-7}{\left(x^3-7x\right)^4}dx=\frac{u^{-3}}{-3}+C=\frac{1}{-3u^3}+C$$

Putting the values of u

$$\int \frac{3x^2-7}{\left(x^3-7x\right)^4}dx=\frac{-1}{3\left(x^3-7x\right)^3}+C$$

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e). $\int \frac{x+3x^2}{\sqrt{x}}dx$

Sol: Given $\int \frac{x+3x^2}{\sqrt{x}}dx$

Let $u=\sqrt{x}=x^{\frac{1}{2}}$ $\Rightarrow u^2=x$

Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}x^{\frac{1}{2}}$$
$$\tfrac{d}{dx}u=\tfrac{1}{2}x^{\frac{1}{2}-1}\tfrac{d}{dx}x$$
$$\tfrac{d}{dx}u=\tfrac{1}{2}x^{\frac{-1}{2}}$$
$$\frac{du}{dx}=\frac{1}{2x^{\frac{1}{2}}}$$
$$2du=\frac{1}{\sqrt{x}}dx$$

Substituting values of u and du in given integral

$$\int \frac{x+3x^2}{\sqrt{x}}dx=\int \left(x+3x^2\right)\frac{dx}{\sqrt{x}}$$

$$\int \frac{x+3x^2}{\sqrt{x}}dx=\int \left(u^2+3u^4\right)2du$$

$$\int \frac{x+3x^2}{\sqrt{x}}dx=2\int \left(u^2+3u^4\right)du$$

$$\int \frac{x+3x^2}{\sqrt{x}}dx=2\int u^2du+2\times 3\int u^4du$$

$$\int \frac{x+3x^2}{\sqrt{x}}dx=2\frac{u^{2+1}}{2+1}+2\times 3\frac{u^{4+1}}{4+1}+C$$

$$\int \frac{x+3x^2}{\sqrt{x}}dx=\frac{2}{3}u^3+\frac{6}{5}u^5+C$$

Putting the values of u

$$\int \frac{x+3x^2}{\sqrt{x}}dx=\frac{2}{3}\sqrt{x^3}+\frac{6}{5}\sqrt{x^5}+C$$

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f). $\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx$

Sol: Given $\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx$

Let $u=x^2+2x+2$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}x^2+2\tfrac{d}{dx}x+\tfrac{d}{dx}2$$

$$\tfrac{d}{dx}u=2x^{2-1}\tfrac{d}{dx}x+2+0$$

$$\tfrac{d}{dx}u=2x+2$$

$$\frac{du}{dx}=2\left(x+1\right)$$

$$\frac{du}{2}=\left(x+1\right)dx$$

Substituting values of u and du in given integral

$$\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx=\int \frac{1}{\left(x^2+2x+2\right)^2}\frac{\left(x+1\right)dx}{1}$$

$$\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx=\int \frac{1}{u^2}\frac{du}{2}$$

$$\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx=\frac{1}{2}\int u^{-2}du$$

$$\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx=\frac{1}{2}\frac{u^{-2+1}}{-2+1}+C$$

$$\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx=\frac{1}{2}\frac{u^{-1}}{-1}+C$$

$$\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx=\frac{-1}{2}\frac{1}{u}+C$$

Putting the values of u

$$\int \frac{x+1}{\left(x^2+2x+2\right)^2}dx=\frac{-1}{2\left(x^2+2x+2\right)}+C$$

Q3. Evaluate the following indefinite integrals by method of substitution:

a). $\int 6e^{6t}dt$

Sol: Given $\int 6e^{6t}dt$

Let $u=6t$ Differentiating w. r. t "x"

$$\frac{du}{dt}=6\frac{d}{dt}t$$

$$\frac{du}{dt}=6$$

$$du=6dt$$

Substituting values of u and du in given integral

$$\int 6e^{6t}dt=\int e^{6t}\left(6dt\right)$$

$$\int 6e^{6t}dt=\int e^{u}du$$

$$\int 6e^{6t}dt=e^{u}+C$$

Putting the values of u

$$\int 6e^{6t}dt=e^{6t}+C$$

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b). $\int xe^{\left(5x^2+1\right)}dx$

Sol: Given $\int xe^{\left(5x^2+1\right)}dx$

Let $u=5x^2+1$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=5\tfrac{d}{dx}x^2+\tfrac{d}{dx}1$$

$$\tfrac{d}{dx}u=5\left(2x\right)\tfrac{d}{dx}x+0$$

$$\frac{du}{dx}=10x$$

$$\frac{du}{10}=xdx$$

Substituting values of u and du in given integral

$$\int xe^{\left(5x^2+1\right)}dx=\int e^{\left(5x^2+1\right)}xdx$$

$$\int xe^{\left(5x^2+1\right)}dx=\int e^{u}\frac{du}{10}$$

$$\int xe^{\left(5x^2+1\right)}dx=\frac{1}{10}\int e^{u}du$$

$$\int xe^{\left(5x^2+1\right)}dx=\frac{1}{10}e^{u}+C$$

Putting the values of u

$$\int xe^{\left(5x^2+1\right)}dx=\frac{1}{10}e^{\left(5x^2+1\right)}+C$$

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c). $\int \left(x^2-2\right)e^{\left(x^2-6x+4\right)}dx$

Sol: Given $\int \left(x^2-2\right)e^{\left(x^3-6x+4\right)}dx$

Let $u=x^3-6x+4$

Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}x^3-6\tfrac{d}{dx}x+\tfrac{d}{dx}4$$

$$\tfrac{d}{dx}u=3x^2\tfrac{d}{dx}x-6\left(1\right)+0$$

$$\tfrac{d}{dx}u=3x^2-6$$

$$\frac{du}{dx}=3\left(x^2-2\right)$$

$$\frac{du}{3}=\left(x^2-2\right)dx$$

Substituting values of u and du in given integral

$$\int \left(x^2-2\right)e^{\left(x^3-6x+4\right)}dx=\int e^{\left(x^3-6x+4\right)}\left(x^2-2\right)dx$$

$$\int \left(x^2-2\right)e^{\left(x^3-6x+4\right)}dx=\int e^u\frac{du}{3}$$

$$\int \left(x^2-2\right)e^{\left(x^3-6x+4\right)}dx=\frac{1}{3}\int e^u du$$

$$\int \left(x^2-2\right)e^{\left(x^3-6x+4\right)}dx=\frac{1}{3}e^u+C$$

Putting the values of u

$$\int \left(x^2-2\right)e^{\left(x^3-6x+4\right)}dx=\frac{1}{3}e^{\left(x^3-6x+4\right)}+C$$

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d). $\int 8^{7-3x^2}\left(-6x\right)dx$

Sol: Given $\int 8^{7-3x^2}\left(-6x\right)dx$

Let $u=7-3x^2$ Differentiating w. r. t “x”

$$\tfrac{d}{dx}u=\tfrac{d}{dx}7-3\tfrac{d}{dx}x^2$$

$$\tfrac{d}{dx}u=0-3\left(2x\right)\tfrac{d}{dx}x$$

$$\frac{du}{dx}=-6x$$

$$du=\left(-6x\right)dx$$

Substituting values of u and du in given integral

$$\int 8^{7-3x^2}\left(-6x\right)dx=\int 8^u du$$

$$\int 8^{7-3x^2}\left(-6x\right)dx=\frac{8^u}{\ln 8}+C$$

Putting the values of u

$$\int 8^{7-3x^2}\left(-6x\right)dx=\frac{8^{7-3x^2}}{\ln 8}+C$$

Q4. Find the equation of the particular curve that has a slope $4x^3+6x^2$ at a point $\left(1,0\right)$

Sol: Given the slope $m=\dfrac{dy}{dx}=4x^3+6x^2$

$$\frac{dy}{dx}=4x^3+6x^2$$

$$dy=\left(4x^3+6x^2\right)dx$$

Now integrating both sides

$$\int dy=\int\left(4x^3+6x^2\right)dx$$

$$y=4\int x^3dx+6\int x^2dx$$

$$y=4\frac{x^{3+1}}{3+1}+6\frac{x^{2+1}}{2+1}+C$$

$$y=\frac{4}{4}x^4+\frac{6}{3}x^3+C$$

$$y=x^4+2x^3+C\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Equation of the particular curve that passes through the point $\left(1,0\right)$ i.e., $x=1, y=0$ we get

$$0=\left(1\right)^4+2\left(1\right)^3+C$$

$$0=\left(1\right)+2\left(1\right)+C$$

$$0=1+2+C$$

$$0=3+C$$

$$C=-3$$

Putting value of C in equation (1) we get required

$$y=x^4+2x^3-3$$

Q5. A certain curve has a slope $x\left(2x^2-1\right)^2$ that passes through the point $\left(3,3\right)$ what is the equation of the specific curve?

Sol: Given the slope $m=\dfrac{dy}{dx}=x\left(2x^2-1\right)^2$

$$dy=\left(2x^2-1\right)^2xdx$$

Now integrating both sides

$$\int dy=\int\left(2x^2-1\right)^2xdx$$

Let $u=2x^2-1$

Differentiating w. r. t “x”

$$\tfrac{d}{dx}u=2\tfrac{d}{dx}x^2-\tfrac{d}{dx}1$$

$$\tfrac{d}{dx}u=2\left(2x\right)\tfrac{d}{dx}x-0$$

$$\frac{du}{dx}=4x$$

$$\frac{du}{4}=xdx$$

Substituting the values of u and du in the integral

$$\int dy=\int\left(2x^2-1\right)^2xdx$$

$$\int dy=\int u^2\frac{du}{4}$$

$$\int dy=\frac{1}{4}\int u^2du$$

$$y=\frac{1}{4}\frac{u^{2+1}}{2+1}+C$$

$$y=\frac{1}{4}\frac{u^3}{3}+C$$

$$y=\frac{1}{12}u^3+C$$

Putting the values of u

$$y=\frac{1}{12}\left(2x^2-1\right)^3+C$$

Equation of the particular curve that passes through the point $\left(3,3\right)$ i.e., $x=3, y=3$ we get

$$3=\frac{1}{12}\left(2\left(3\right)^2-1\right)^3+C$$

$$3\times 12=\left(2\left(9\right)-1\right)^3+12C$$

$$36=\left(18-1\right)^3+12C$$

$$36=17^3+12C$$

$$12C=36-4913$$

$$12C=-4877 \qquad \Rightarrow C=\tfrac{-4877}{12}=-406.41\bar{6}$$

Putting value of C in equation (1) we get required eq

$$y=\frac{1}{12}\left(2x^2-1\right)^3-406.41\bar{6}$$

Q6. A certain curve has a slope $x\sqrt{2x^2-1}$ that passes through the point $\left(3,3\right)$ what is the equation of the specific curve?

Sol: Given the slope $m=\dfrac{dy}{dx}=x\sqrt{2x^2-1}$

$$dy = \left(2x^2 - 1\right)^{\frac{1}{2}} xdx$$

Now integrating both sides

$$\int dy = \int \left(2x^2 - 1\right)^{\frac{1}{2}} xdx$$

Let $u = 2x^2 - 1$

Differentiating w. r. t "x"

$$\tfrac{d}{dx}u = 2\tfrac{d}{dx}x^2 - \tfrac{d}{dx}1$$

$$\tfrac{d}{dx}u = 2\left(2x\right)\tfrac{d}{dx}x - 0$$

$$\frac{du}{dx} = 4x$$

$$\frac{du}{4} = xdx$$

Substituting the values of u and du in the integral

$$\int dy = \int \left(2x^2 - 1\right)^{\frac{1}{2}} xdx$$

$$\int dy = \int u^{\frac{1}{2}} \frac{du}{4}$$

$$\int dy = \tfrac{1}{4}\int u^{\frac{1}{2}} du$$

$$y = \frac{1}{4}\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$y = \frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$y = \tfrac{1}{6}u^{\frac{3}{2}} + C$$

Putting the values of u

$$y = \frac{1}{6}\left(2x^2 - 1\right)^{\frac{3}{2}} + C$$

Equation of the particular curve that passes through the point $(3,3)$ i.e., $x = 3, y = 3$ we get

$$3 = \tfrac{1}{6}\left(2\left(3\right)^2 - 1\right)^{\frac{3}{2}} + C$$

$$3 \times 6 = \left(2\left(9\right) - 1\right)^{\frac{3}{2}} + 6C$$

$$18 = \left(18 - 1\right)^{\frac{3}{2}} + 6C$$

$$18 = 17^{\frac{3}{2}} + 6C$$

$$6C = 18 - 70.09$$

$$C = \tfrac{-52.09}{6} = -8.681\bar{6}$$

Putting value of C in equation (1) we get required

$$y = \tfrac{1}{6}\left(2x^2 - 1\right)^{\frac{3}{2}} - 8.681\bar{6}$$

Q7. For an average person, the rate of change of weight W (in pounds) with respect to height h (in inches) is approximately by $\dfrac{dW}{dh} = 0.0015h^2$

Sol: Given the slope $m = \frac{dW}{dh} = 0.0015h^2$

$dW = 0.0015h^2dh$ Now integrating both sides

$$\int dW = \int 0.0015h^2dh$$

$$W = 0.0015\frac{h^{2+1}}{2+1} + C$$

$$W = \frac{0.0015}{3}h^3 + C$$

$$W = 0.0005h^3 + C.............(1)$$

a). Find $W\left(h\right)$ if the weight is w=108 pounds in response of height h=60 inches

Sol: Since $W = 108$ pounds and $h = 60$ inches

$$108 = 0.0005\left(60\right)^3 + C$$

$$108 = 0.0005\left(60\right)^3 + C$$

$$C = 108 - 0.0005\left(216000\right)$$

$$C = 108 - 108$$

$$C = 0$$

Putting the value of C in equation (1) we get the required equation

$$W = 0.0005h^3 + 0$$

$$W = 0.0005h^3................(2)$$

---

b). Find the weight W of a person who is 5 feet 10 inches tall.

Sol: Given $h = 5\ feet\ 10\ inches$

$$h = 5 \times 12 + 10\ inches$$

$$h = 60 + 10\ inches$$

$$h = 70\ inches$$

Putting the value of h in equation (2) we get

$$W = 0.0005\left(70\right)^3$$

$$W = 0.0005\left(343000\right)$$

$$W = 171.5\ \ pounds$$

Q8. The rate of growth of the population N(t) of a newly incorporated city t years after incorporation is estimated to be $\frac{dN}{dt} = 400 + 600\sqrt{t},\ \ 0 \le t \le 9$ if the population was 5,000 at time of incorporation, find the population 9 years later.

Sol: Given the slope $\frac{dN}{dt} = 400 + 600\sqrt{t},$

$$dN = \left(400 + 600\sqrt{t}\right)dt$$

Now integrating both sides

$$\int dN = \int \left(400 + 600\sqrt{t}\right)dt$$

$$N = 400\int dt + 600\int t^{\frac{1}{2}}dt$$

$$N = 400t + 600\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$N = 400t + 600\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$N = 400t + \tfrac{2}{3}\left(600\ t^{\frac{3}{2}}\right) + C$$

$$N = 400t + 400\ t^{\frac{3}{2}} + C............(1)$$

When population was 5,000 at the time of incorporation i.e., $N = 5000, \quad t = 0$

$$5000 = 400\left(0\right) + 400\left(0\right)^{\frac{3}{2}} + C$$

$$5000 = 0 + 400\left(0\right) + C$$

$$5000 = 0 + 0 + C$$

$$C = 5000$$

Putting value of C in eq (1) we get required eq

$$N = 400t + 400\ t^{\frac{3}{2}} + 5000$$

Now to find the population 9 years later

$$N = 400(9) + 400(9)^{\frac{3}{2}} + 5000$$

$$N = 3600 + 400(3)^3 + 5000$$

$$N = 3600 + 400(27) + 5000$$

$$N = 19400$$

Few formulae

- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \tan x dx = -\ln|\cos x| + C$
- $\int \cot x dx = \ln|\sin x| + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \cos ec^2 x dx = -\cot x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \cos ecx \cot x dx = -\cos ecx + C$
- $\int \sec x\,dx = \ln|\sec x + \tan x|$
- $\int \cos ecx\,dx = \ln|\cos ecx - \cot x|$
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
- $\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
- $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right|$
- $\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
- $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\sqrt{a^2 - x^2} \pm \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
- $\int \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} - a\ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|$

Exercise 5.2

Q1. Evaluate the following indefinite integrals by method of substitution:

a). $\int \sin^4 x \cos x dx$

Sol: Given $\int \sin^4 x \cos x dx$

Let $u = \sin x$, Differentiating w. r. t "x"

$$\tfrac{d}{dx}u = \tfrac{d}{dx}\sin x$$

$$\tfrac{d}{dx}u = \cos x \tfrac{d}{dx}x$$

$$du = \cos x\,dx$$

Substituting values of u and du in given integral

$$\int \sin^4 x \cos x dx = \int u^4 du$$

$$\int \sin^4 x \cos x dx = \frac{u^{4+1}}{4+1} + C$$

$$\int \sin^4 x \cos x dx = \frac{u^5}{5} + C$$

$$\int \sin^4 x \cos x dx = \frac{1}{5}u^5 + C$$

Putting the values of u

$$\int \sin^4 x \cos x dx = \frac{1}{5}\sin^5 x + C$$

---

b). $\int \frac{\cos x \ln(\sin x)}{\sin x}dx$

Sol: Given $\int \frac{\cos x \ln(\sin x)}{\sin x}dx$

Let $u = \ln(\sin x)$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u = \tfrac{d}{dx}\ln(\sin x)$$

$$\frac{du}{dx} = \frac{1}{\sin x}\frac{d}{dx}\sin x$$

$$\frac{du}{dx} = \frac{1}{\sin x}\cos x \frac{d}{dx}x$$

$$du = \frac{\cos x}{\sin x}dx$$

Substituting values of u and du in given integral

$$\int \frac{\cos x \ln(\sin x)}{\sin x}dx = \int \ln(\sin x)\frac{\cos x}{\sin x}dx$$

$$\int \frac{\cos x \ln(\sin x)}{\sin x}dx = \int u\,du$$

$$\int \frac{\cos x \ln(\sin x)}{\sin x}dx = \frac{u^2}{2} + C$$

$$\int \frac{\cos x \ln(\sin x)}{\sin x}dx = \frac{1}{2}u^2 + C$$

Putting the values of u

$$\int \frac{\cos x \ln(\sin x)}{\sin x}dx = \frac{1}{2}\{\ln(\sin x)\}^2 + C$$

---

c). $\int e^x \sin e^x dx$

Sol: Given $\int e^x \sin e^x dx$

Let $u = e^x$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u = \tfrac{d}{dx}e^x$$

$$\tfrac{d}{dx}u = e^x \tfrac{d}{dx}x$$

$$\frac{du}{dx} = e^x$$

$$du = e^x dx$$

Substituting values of u and du in given integral

$$\int e^x \sin e^x dx = \int \sin(e^x)\ \ e^x dx$$

$$\int e^x \sin e^x dx = \int \sin u\ \ du$$

$$\int e^x \sin e^x dx = -\cos u + C$$

Putting the values of u

$$\int e^x \sin e^x dx = -\cos(e^x) + C$$

---

d). $\int (t+3)\cos(t+3)^2\,dt$

Sol: Given $\int (t+3)\cos(t+3)^2\,dt$

Let $u = (t+3)^2$ Differentiating w. r. t "x"

$\frac{d}{dt}u=\frac{d}{dt}(t+3)^2$

$\frac{d}{dt}u=2(t+3)\frac{d}{dt}(t+3)$

$\frac{d}{dt}u=2(t+3)\left(\frac{d}{dt}t+\frac{d}{dt}3\right)$

$\frac{d}{dt}u=2(t+3)(1+0)$

$\frac{du}{dt}=2(t+3)$

$\frac{du}{2}=(t+3)dt$

Substituting values of u and du in given integral

$\int (t+3)\cos(t+3)^2\,dt=\int \cos(t+3)^2\quad(t+3)dt$

$\int (t+3)\cos(t+3)^2\,dt=\int \cos u\frac{du}{2}$

$\int (t+3)\cos(t+3)^2\,dt=\frac{1}{2}\int \cos u\ du$

$\int (t+3)\cos(t+3)^2\,dt=\frac{1}{2}\sin u+C$

Putting the values of u

$\int (t+3)\cos(t+3)^2\,dt=\frac{1}{2}\sin(t+3)^2+C$

Q2. Evaluate the following indefinite integrals by method of substitution:

a). $\int \tan 2x\sec 2x\,dx$

Sol: Given $\int \tan 2x\sec 2x\,dx$

Let $u=\sec 2x$ Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}\sec 2x$

$\frac{d}{dx}u=\sec 2x\tan 2x\frac{d}{dx}(2x)$

$\frac{du}{dx}=2\sec 2x\tan 2x$

$\frac{du}{2}=\sec 2x\tan 2x$

Substituting values of u and du in given integral

$\int \tan 2x\sec 2x\,dx=\int\frac{du}{2}$

$\int \tan 2x\sec 2x\,dx=\frac{1}{2}\int du$

$\int \tan 2x\sec 2x\,dx=\frac{1}{2}u+C$

Putting the values of u

$\int \tan 2x\sec 2x\,dx=\frac{1}{2}\sec 2x+C$

---

b). $\int 4\sec^2 4x\,dx$

Sol: Given $\int 4\sec^2 4x\,dx$

Let $u=\tan 4x$ Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}\tan 4x$

$\frac{d}{dx}u=\sec^2(4x)\frac{d}{dx}(4x)$

$\frac{d}{dx}u=4\sec^2(4x)\frac{d}{dx}x$

$\frac{d}{dx}u=4\sec^2(4x)$

$\frac{du}{4}=\sec^2(4x)dx$

Substituting values of u and du in given integral

$\int 4\sec^2 4x\,dx=4\int \sec^2 4x\,dx$

$\int 4\sec^2 4x\,dx=4\int\frac{du}{4}$

$\int 4\sec^2 4x\,dx=\frac{4}{4}\int du$

$\int 4\sec^2 4x\,dx=u+c$

Putting the values of u

$\int 4\sec^2 4x\,dx=\tan(4x)+C$

---

c). $\int \tan x\sec^2 x\,dx$

Sol: Given $\int \tan x\sec^2 x\,dx$

Let $u=\tan x$ Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}\tan x$

$\frac{d}{dx}u=\sec^2 x\frac{d}{dx}x$

$\frac{du}{dx}=\sec^2 x$

$du=\sec^2 x\ dx$

Substituting values of u and du in given integral

$\int \tan x\sec^2 x\,dx=\int u du$

$\int \tan x\sec^2 x\,dx=\frac{u^2}{2}+C$

$\int \tan x\sec^2 x\,dx=\frac{1}{2}u^2+C$

Putting the values of u

$\int \tan x\sec^2 x\,dx=\frac{1}{2}(\tan x)^2+C$

$\int \tan x\sec^2 x\,dx=\frac{1}{2}\tan^2 x+C$

---

d). $\int (\tan 3x+\sec 3x)dx$

Sol: Given $\int (\tan 3x+\sec 3x)dx$

$\int (\tan 3x+\sec 3x)dx=\int\tan 3x\ dx+\int\sec 3x\ dx$

$\int (\tan 3x+\sec 3x)dx=I_1+I_2$

Where $I_1=\int\tan 3x\ dx$ and $I_2=\int\sec 3x\ dx$

Take $I_1=\int\tan 3x\ dx$

$I_1=\int\frac{\sin 3x}{\cos 3x}dx$

Let $u=\cos 3x$ Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}\cos 3x$

$\frac{d}{dx}u=-\sin 3x\frac{d}{dx}(3x)$

$\frac{d}{dx}u=-\sin 3x\left(3\frac{d}{dx}x\right)$

$\frac{du}{dx}=-3\sin 3x$

$\frac{du}{-3}=\sin 3x\ dx$

Substituting values of u and du in the integral I_1

$$I_1=\int\frac{\sin 3x}{\cos 3x}dx=\int\frac{1}{\cos 3x}\sin 3x\,dx$$

$$I_1=\int\frac{\sin 3x}{\cos 3x}dx=\int\frac{1}{u}\frac{-du}{3}$$

$$I_1=\int\frac{\sin 3x}{\cos 3x}dx=\frac{-1}{3}\int\frac{du}{u}=\frac{-1}{3}\ln|u|+C_1$$

$$I_1=\int\frac{\sin 3x}{\cos 3x}dx=\frac{1}{3}\ln\left|u^{-1}\right|+C_1=\frac{1}{3}\ln\left|\frac{1}{u}\right|+C_1$$

Putting the values of u

$$I_1=\int\frac{\sin 3x}{\cos 3x}dx=\frac{1}{3}\ln\left|\frac{1}{\cos 3x}\right|+C_1$$

$$I_1=\int\frac{\sin 3x}{\cos 3x}dx=\frac{1}{3}\ln|\sec 3x|+C_1$$

Now to solve integral I_2

$$I_2=\int\sec 3x\ dx=\int\sec 3x\frac{(\sec 3x+\tan 3x)}{\sec 3x+\tan 3x}dx$$

$$_2=\int\sec 3x\ dx=\int\frac{\sec^2 3x+\sec 3x\tan 3x}{\sec 3x+\tan 3x}dx$$

Let $u=\sec 3x+\tan 3x$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}\sec 3x+\tfrac{d}{dx}\tan 3x$$

$$\tfrac{d}{dx}u=\sec 3x\tan 3x\tfrac{d}{dx}(3x)+\sec^2 3x\tfrac{d}{dx}(3x)$$

$$\tfrac{d}{dx}u=3\sec 3x\tan 3x+3\sec^2 3x$$

$$\frac{du}{dx}=3\left(\sec 3x\tan 3x+\sec^2 3x\right)$$

$$\frac{du}{3}=\left(\sec 3x\tan 3x+\sec^2 3x\right)dx$$

Substituting values of u and du in given integral

$$I_2=\int\sec 3x\ dx=\int\frac{\sec^2 3x+\sec 3x\tan 3x}{\sec 3x+\tan 3x}dx$$

$$I_2=\int\sec 3x\ dx=\int\frac{1}{u}\frac{du}{3}$$

$$I_2=\int\sec 3x\ dx=\frac{1}{3}\int\frac{du}{u}$$

$$I_2=\int\sec 3x\ dx=\frac{1}{3}\ln|u|+C_2$$

Putting the values of u

$$I_2=\int\sec 3x\ dx=\frac{1}{3}\ln|\sec 3x+\tan 3x|+C_2$$

Therefore the given integral will be

$$\int(\tan 3x+\sec 3x)dx=\int\tan 3x\ dx+\int\sec 3x\ dx$$

$$\int(\tan 3x+\sec 3x)dx=\frac{1}{3}\ln|\sec 3x|+C_1+\frac{1}{3}\ln|\sec 3x+\tan 3x|+C_2$$

$$\int(\tan 3x+\sec 3x)dx=\frac{1}{3}\ln|\sec 3x|+\frac{1}{3}\ln|\sec 3x+\tan 3x|+C_2+C_1$$

$$\int(\tan 3x+\sec 3x)dx=\frac{1}{3}\ln|\sec 3x|+\frac{1}{3}\ln|\sec 3x+\tan 3x|+C$$

where $C=C_2+C_1$

---

e). $\int\frac{\cos^2 x}{\cos eex}dx$

Sol: Given $\int\frac{\cos^2 x}{\cos eex}dx=\int\cos^2 x\sin xdx$

Let $u=\cos x$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}\cos x$$

$$\tfrac{d}{dx}u=-\sin x\tfrac{d}{dx}x$$

$$\frac{du}{dx}=-\sin x$$

$$-du=\sin x\ dx$$

Substituting values of u and du in given integral

$$\int\frac{\cos^2 x}{\cos eex}dx=\int\cos^2 x\sin xdx$$

$$\int\frac{\cos^2 x}{\cos eex}dx=\int u^2(-du)$$

$$\int\frac{\cos^2 x}{\cos eex}dx=-\int u^2du$$

$$\int\frac{\cos^2 x}{\cos eex}dx=-\frac{u^{2+1}}{2+1}+C$$

$$\int\frac{\cos^2 x}{\cos eex}dx=-\frac{u^3}{3}+C$$

$$\int\frac{\cos^2 x}{\cos eex}dx=-\frac{1}{3}u^3+C$$

Putting the values of u

$$\int\frac{\cos^2 x}{\cos eex}dx=-\frac{1}{3}\cos^3 x+C$$

---

f). $\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx$

Sol: Given $\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx$

Let $u=\sqrt{x}=x^{\frac{1}{2}}$

Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}x^{\frac{1}{2}}$$

$$\tfrac{d}{dx}u=\tfrac{1}{2}x^{\frac{1}{2}-1}\tfrac{d}{dx}x$$

$$\tfrac{d}{dx}u=\tfrac{1}{2}x^{\frac{-1}{2}}$$

$$\frac{du}{dx}=\frac{1}{2x^{\frac{1}{2}}}$$

$$2du=\frac{dx}{\sqrt{x}}$$

Substituting values of u and du in given integral

$$\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx=\int\cot\sqrt{x}\ \frac{dx}{\sqrt{x}}$$

$$\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx=\int\cot u\ (2du)$$

$$\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx=2\int\frac{\cos u\ du}{\sin u}$$

$$\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx=2\ln|\sin u|+C \qquad \therefore\int\frac{dx}{x}=\ln|x|$$

Putting the values of u

$$\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx=2\ln\left|\sin\sqrt{x}\right|+C$$

---

g). $\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$

Sol: Given $\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$

Let $u=\cos x+\sin x$ Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}\cos x+\frac{d}{dx}\sin x$

$\frac{d}{dx}u=-\sin x\frac{d}{dx}x+\cos x\frac{d}{dx}x$

$\frac{d}{dx}u=-\sin x+\cos x$

$$\frac{du}{dx}=-\left(\sin x-\cos x\right)$$

$$-du=\left(\sin x-\cos x\right)dx$$

Substituting values of u and du in given integral

$$\int\frac{\sin x-\cos x}{\sin x+\cos x}dx=\int\frac{-du}{u}$$

$$\int\frac{\sin x-\cos x}{\sin x+\cos x}dx=-\int\frac{du}{u}$$

$$\int\frac{\sin x-\cos x}{\sin x+\cos x}dx=-\ln\left|u\right|+C$$

Putting the values of u

$$\int\frac{\sin x-\cos x}{\sin x+\cos x}dx=-\ln\left|\sin x+\cos x\right|+C$$

h). $\displaystyle\int\frac{\sin x}{3+2\cos x}dx$

Sol: Given $\displaystyle\int\frac{\sin x}{3+2\cos x}dx$

Let $u=3+2\cos x$ Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}3+2\frac{d}{dx}\cos x$

$\frac{d}{dx}u=0+2\left(-\sin x\right)\frac{d}{dx}x$

$$\frac{du}{dx}=-2\sin x$$

$$\frac{du}{-2}=\sin x\ dx$$

Substituting values of u and du in given integral

$$\int\frac{\sin x}{3+2\cos x}dx=\frac{-1}{2}\int\frac{1}{3+2\cos x}\frac{\sin x\,dx}{1}$$

$$\int\frac{\sin x}{3+2\cos x}dx=\frac{-1}{2}\int\frac{du}{u}$$

$$\int\frac{\sin x}{3+2\cos x}dx=\frac{-1}{2}\int\frac{du}{u}$$

$$\int\frac{\sin x}{3+2\cos x}dx=\frac{-1}{2}\ln\left|u\right|+C$$

Putting the values of u

$$\int\frac{\sin x}{3+2\cos x}dx=\frac{-1}{2}\ln\left|3+2\cos x\right|+C$$

Q3. Use substitution and table to evaluate the following indefinite integrals

a). $\displaystyle\int\frac{dx}{x^2+16}$

Sol: Given $\displaystyle\int\frac{dx}{x^2+16}$

Let $x=4\tan\theta$ Differentiating w r t θ

$\frac{d}{d\theta}x=4\frac{d}{d\theta}\tan\theta$

$$\frac{dx}{d\theta}=4\sec^2\theta\frac{d}{d\theta}\theta$$

$$dx=4\sec^2\theta d\theta$$

Substituting values of x and dx in given integral

$$\int\frac{dx}{x^2+16}=\int\frac{4\sec^2\theta d\theta}{\left(4\tan\theta\right)^2+16}$$

$$\int\frac{dx}{x^2+16}=\int\frac{4\sec^2\theta d\theta}{16\tan^2\theta+16}$$

$$\int\frac{dx}{x^2+16}=\frac{4}{16}\int\frac{\sec^2\theta d\theta}{\tan^2\theta+1}$$

$$\int\frac{dx}{x^2+16}=\frac{1}{4}\int\frac{\sec^2\theta d\theta}{\sec^2\theta}\qquad\therefore\tan^2\theta+1=\sec^2\theta$$

$$\int\frac{dx}{x^2+16}=\frac{1}{4}\int d\theta\qquad\therefore 4\tan\theta=x$$

$$\int\frac{dx}{x^2+16}=\frac{1}{4}\theta+C\qquad\Rightarrow\tan\theta=\frac{x}{4}$$

Putting the values of θ

$$\int\frac{dx}{x^2+16}=\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C\qquad\Rightarrow\theta=\tan^{-1}\left(\frac{x}{4}\right)$$

b). $\displaystyle\int\frac{\sin x}{\cos^2 x+1}dx$

Sol: Given $\displaystyle\int\frac{\sin x}{\cos^2 x+1}dx$

Let $\cos x=\tan\theta$ Differentiating w r t θ

$\frac{d}{d\theta}\cos x=\frac{d}{d\theta}\tan\theta$

$\frac{d}{dx}\cos x\frac{dx}{d\theta}=\sec^2\theta\frac{d}{d\theta}\theta$

$$-\sin x\frac{dx}{d\theta}=\sec^2\theta$$

$$\sin x\ dx=-\sec^2\theta\,d\theta$$

Putting values of $\cos x$ and $\sin x\,dx$ in the integral

$$\int\frac{\sin x}{\cos^2 x+1}dx=\int\frac{-\sec^2\theta d\theta}{\tan^2\theta+1}\quad\therefore\tan^2\theta+1=\sec^2\theta$$

$$\int\frac{\sin x}{\cos^2 x+1}dx=-\int\frac{\sec^2\theta d\theta}{\sec^2\theta}\qquad\therefore\tan\theta=\cos x$$

$$\int\frac{\sin x}{\cos^2 x+1}dx=-\int d\theta\qquad\theta=\tan^{-1}\left(\cos x\right)$$

$$\int\frac{\sin x}{\cos^2 x+1}dx=-\theta+C$$

Putting the values of θ

$$\int\frac{\sin x}{\cos^2 x+1}dx=-\tan^{-1}\left(\cos x\right)+C$$

c). $\displaystyle\int\frac{dx}{\sqrt{5-2x^2}}$

Sol: Given $\displaystyle\int\frac{dx}{\sqrt{5-2x^2}}$

Let $x=\sqrt{\frac{5}{2}}\sin\theta$ Differentiating w r t θ

$$\frac{d}{d\theta}x=\sqrt{\frac{5}{2}}\frac{d}{d\theta}\sin\theta$$

$$\frac{dx}{d\theta}=\sqrt{\frac{5}{2}}\cos\theta\frac{d}{d\theta}\theta$$

$$dx=\sqrt{\tfrac{5}{2}}\cos\theta d\theta$$

Substituting values of x and dx in given integral

$$\int \frac{dx}{\sqrt{5-2x^2}} = \int \frac{\sqrt{\frac{5}{2}}\cos\theta d\theta}{\sqrt{5-2\left(\sqrt{\frac{5}{2}}\sin\theta\right)^2}}$$

$$\int \frac{dx}{\sqrt{5-2x^2}} = \sqrt{\frac{5}{2}}\int \frac{\cos\theta d\theta}{\sqrt{5-2\left(\frac{5}{2}\right)\sin^2\theta}}$$

$$\int \frac{dx}{\sqrt{5-2x^2}} = \sqrt{\frac{5}{2}}\int \frac{\cos\theta d\theta}{\sqrt{5-5\sin^2\theta}} \qquad \therefore \sin^2\theta+\cos^2\theta=1$$

$$\int \frac{dx}{\sqrt{5-2x^2}} = \sqrt{\frac{5}{2}}\int \frac{\cos\theta d\theta}{\sqrt{5}\sqrt{1-\sin^2\theta}} \qquad \Rightarrow \cos^2\theta=1-\sin^2\theta$$

$$\int \frac{dx}{\sqrt{5-2x^2}} = \sqrt{\frac{5}{2}}\frac{1}{\sqrt{5}}\int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}}$$

$$\int \frac{dx}{\sqrt{5-2x^2}} = \sqrt{\frac{5}{2}}\frac{1}{\sqrt{5}}\int \frac{\cos\theta d\theta}{\cos\theta} \qquad \therefore \sqrt{\tfrac{5}{2}}\sin\theta = x$$

$$\int \frac{dx}{\sqrt{5-2x^2}} = \frac{1}{\sqrt{2}}\int d\theta \qquad \sin\theta = \sqrt{\frac{2}{5}}x$$

$$\int \frac{dx}{\sqrt{5-2x^2}} = \frac{1}{\sqrt{2}}\theta + C \qquad \theta = \sin^{-1}\left(\sqrt{\frac{2}{5}}x\right)$$

Putting the values of θ

$$\int \frac{dx}{\sqrt{5-2x^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{5}}x\right)+C$$

d). $\int \frac{dx}{\sqrt{e^{2x}-4}}$

Sol: Given $\int \frac{dx}{\sqrt{e^{2x}-4}}$

Let $e^x = 2\sec\theta$, Differentiating w r t θ

$$\tfrac{d}{dx}e^x\tfrac{dx}{d\theta} = 2\tfrac{d}{d\theta}\sec\theta$$

$$e^x\tfrac{d}{dx}x\tfrac{dx}{d\theta} = 2\sec\theta\tan\theta\tfrac{d}{d\theta}\theta$$

$$e^x\tfrac{dx}{d\theta} = 2\sec\theta\tan\theta \qquad \therefore e^x = 2\sec\theta$$

$$2\sec\theta\frac{dx}{d\theta} = 2\sec\theta\tan\theta$$

$$dx = \frac{2\sec\theta}{2\sec\theta}\tan\theta d\theta$$

$$dx = \tan\theta d\theta$$

Substituting values of x and dx in given integral

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \int \frac{\tan\theta d\theta}{\sqrt{4\sec^2\theta-4}} \qquad \therefore \sec^2\theta = \tan^2\theta-1$$

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \frac{1}{\sqrt{4}}\int \frac{\tan\theta d\theta}{\sqrt{\sec^2\theta-1}} \qquad \sec^2\theta-1=\tan^2\theta$$

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \frac{1}{2}\int \frac{\tan\theta d\theta}{\sqrt{\tan^2\theta}}$$

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \frac{1}{2}\int \frac{\tan\theta d\theta}{\tan\theta} \qquad \therefore 2\sec\theta = e^x$$

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \frac{1}{2}\int d\theta \qquad \sec\theta = \frac{e^x}{2}$$

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \frac{1}{2}\theta + C \qquad \theta = \sec^{-1}\left(\frac{e^x}{2}\right)$$

Putting the values of θ

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \frac{1}{2}\sec^{-1}\left(\frac{e^x}{2}\right)+C$$

e). $\int \frac{2x+5}{x^2+4x+5}dx$

Sol: Given $\int \frac{2x+5}{x^2+4x+5}dx$

$$\int \frac{2x+4+1}{x^2+4x+5}dx = \int\left(\frac{2x+4}{x^2+4x+5}+\frac{1}{x^2+4x+5}\right)dx$$

$$\int \frac{2x+4+1}{x^2+4x+5}dx = \int\frac{2x+4}{x^2+4x+5}dx+\int\frac{1}{x^2+4x+5}dx$$

$\int \frac{2x+4+1}{x^2+4x+5}dx = I_1+I_2$ Where

$$I_1 = \int\frac{2x+4}{x^2+4x+5}dx \;\; \& \;\; I_2 = \int\frac{1}{x^2+4x+5}dx$$

First to solve the integral I_1

$$I_1 = \int\frac{2x+4}{x^2+4x+5}dx$$

Let $u = x^2+4x+5$ Differentiating w. r. t “x”

$$\tfrac{d}{dx}u = \tfrac{d}{dx}x^2+4\tfrac{d}{dx}x+\tfrac{d}{dx}5$$

$$\tfrac{d}{dx}u = 2x\tfrac{d}{dx}x+4+0$$

$$\frac{du}{dx} = 2x+4$$

$$du = (2x+4)dx$$

Substituting values of u and du in the integral I_1

$$I_1 = \int\frac{2x+4}{x^2+4x+5}dx = \int\frac{du}{u} = \ln|u|+C_1$$

Putting the values of x

$$I_1 = \int\frac{2x+4}{x^2+4x+5}dx = \ln\left|x^2+4x+5\right|+C_1$$

Now to solve the integral I_2

$$I_2 = \int\frac{1}{x^2+4x+5}dx = \int\frac{1}{x^2+4x+4+1}dx$$

$$I_2 = \int\frac{1}{x^2+2(x)(2)+(2)^2+1}dx = \int\frac{1}{(x+2)^2+1}dx$$

Let $x+2=\tan\theta$ Differentiating w r t θ

$$\tfrac{d}{d\theta}x+\tfrac{d}{d\theta}2 = \tfrac{d}{d\theta}\tan\theta$$

$$\tfrac{d}{d\theta}x+0 = \sec^2\theta\tfrac{d}{d\theta}\theta$$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$dx = \sec^2\theta\, d\theta$$

Substituting values of x and dx in given integral

$$I_2 = \int\frac{1}{x^2+4x+5}dx = \int\frac{1}{(x+2)^2+1}dx$$

$$I_2 = \int\frac{1}{x^2+4x+5}dx = \int\frac{\sec^2\theta\, d\theta}{\tan^2\theta+1} \qquad \therefore \tan^2\theta+1=\sec^2\theta$$

$$I_2 = \int\frac{1}{x^2+4x+5}dx = \int\frac{\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$I_2 = \int\frac{1}{x^2+4x+5}dx = \int d\theta \qquad \therefore \tan\theta = x+2$$

$$I_2 = \int\frac{1}{x^2+4x+5}dx = \theta+C_2 \qquad \theta = \tan^{-1}(x+2)$$

Putting the values of θ

$$I_2=\int\frac{1}{x^2+4x+5}dx=\tan^{-1}\left(x+2\right)+C_2$$

Putting the solutions of the integrals I_1 & I_2

$$\int\frac{2x+4+1}{x^2+4x+5}dx=I_1+I_2$$

$$\int\frac{2x+4+1}{x^2+4x+5}dx=\ln\left|x^2+4x+5\right|+C_1+\tan^{-1}\left(x+2\right)+C_2$$

$$\int\frac{2x+4+1}{x^2+4x+5}dx=\ln\left|x^2+4x+5\right|+\tan^{-1}\left(x+2\right)+C_1+C_2$$

$$\int\frac{2x+4+1}{x^2+4x+5}dx=\ln\left|x^2+4x+5\right|+\tan^{-1}\left(x+2\right)+C$$

where $C=C_1+C_2$

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f). $\displaystyle\int\frac{2+x}{\sqrt{4-2x-x^2}}dx$

Sol: Given $\displaystyle\int\frac{2+x}{\sqrt{4-2x-x^2}}dx$

$$\int\frac{1+1+x}{\sqrt{4-2x-x^2}}dx=\int\frac{1}{\sqrt{4-2x-x^2}}dx+\int\frac{1+x}{\sqrt{4-2x-x^2}}dx$$

$$\int\frac{2+x}{\sqrt{4-2x-x^2}}dx=I_1+I_2 \text{ where}$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx \;\;\&\;\; I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx$$

First to solve the integral I_1

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{1}{\sqrt{4+1-1-2x-x^2}}dx$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{1}{\sqrt{5-\left(1+2x+x^2\right)}}dx$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{1}{\sqrt{5-\left(1+x\right)^2}}dx$$

Let $1+x=\sqrt{5}\sin\theta$, Differentiating w r t θ

$$\tfrac{d}{d\theta}1+\tfrac{d}{d\theta}x=\sqrt{5}\tfrac{d}{d\theta}\sin\theta$$

$$0+\frac{dx}{d\theta}=\sqrt{5}\cos\theta\tfrac{d}{d\theta}\theta$$

$$dx=\sqrt{5}\cos\theta\,d\theta$$

Putting the values of $1+x$ and dx in the integral I_1

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{1}{\sqrt{5-\left(1+x\right)^2}}dx$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{\sqrt{5}\cos\theta d\theta}{\sqrt{5-\left(\sqrt{5}\sin\theta\right)^2}}$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{\sqrt{5}\cos\theta d\theta}{\sqrt{5-5\sin^2\theta}}$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{\sqrt{5}\cos\theta d\theta}{\sqrt{5}\sqrt{1-\sin^2\theta}}$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}}$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}}$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int\frac{\cos\theta d\theta}{\cos\theta}\qquad\therefore\sqrt{5}\sin\theta=1+x$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\int d\theta\qquad\sin\theta=\frac{1+x}{\sqrt{5}}$$

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\theta+C_1\qquad\theta=\sin^{-1}\left(\frac{1+x}{\sqrt{5}}\right)$$

Putting the values of θ

$$I_1=\int\frac{1}{\sqrt{4-2x-x^2}}dx=\sin^{-1}\left(\frac{1+x}{\sqrt{5}}\right)+C_1$$

Now to solve the integral I_2

$$I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx$$

Let $u=4-2x-x^2$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}4-2\tfrac{d}{dx}x-\tfrac{d}{dx}x^2$$

$$\tfrac{d}{dx}u=0-2-2x\tfrac{d}{dx}x$$

$$\frac{du}{dx}=-2-2x$$

$$\frac{du}{dx}=-2\left(1+x\right)$$

$$\frac{du}{-2}=\left(1+x\right)dx$$

Substituting values of u and du in given integral

$$I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx=\int\frac{1}{\sqrt{u}}\frac{du}{-2}$$

$$I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx=\frac{-1}{2}\int u^{\frac{-1}{2}}du$$

$$I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx=\frac{-1}{2}\frac{u^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}+C_2$$

$$I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx=\frac{-1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+C_2$$

$$I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx=-\sqrt{u}+C_2$$

Putting the values of u

$$I_2=\int\frac{1+x}{\sqrt{4-2x-x^2}}dx=-\sqrt{4-2x-x^2}+C_2$$

Putting the solutions of the integrals I_1 & I_2

$$\int\frac{2+x}{\sqrt{4-2x-x^2}}dx=I_1+I_2$$

$$\int\frac{2+x}{\sqrt{4-2x-x^2}}dx=\sin^{-1}\left(\frac{1+x}{\sqrt{5}}\right)+C_1-\sqrt{4-2x-x^2}+C_2$$

$$\int\frac{2+x}{\sqrt{4-2x-x^2}}dx=\sin^{-1}\left(\frac{1+x}{\sqrt{5}}\right)-\sqrt{4-2x-x^2}+C_1+C_2$$

$$\int\frac{2+x}{\sqrt{4-2x-x^2}}dx=\sin^{-1}\left(\frac{1+x}{\sqrt{5}}\right)-\sqrt{4-2x-x^2}+C$$

where $C=C_1+C_2$

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g). $\displaystyle\int\frac{dx}{x\sqrt{7x^2-5}}$

Sol: Given $\displaystyle\int\frac{dx}{x\sqrt{7x^2-5}}$

Let $x=\sqrt{\frac{5}{7}}\cos ec\theta$ Differentiating w r t θ

$$\frac{d}{d\theta}x=\sqrt{\frac{5}{7}}\frac{d}{d\theta}\cos ec\theta$$

$$\frac{dx}{d\theta}=-\sqrt{\frac{5}{7}}\cos ec\theta\cot\theta\frac{d}{d\theta}\theta$$

$$dx=-\sqrt{\tfrac{5}{7}}\cos ec\theta\cot\theta d\theta$$

Substituting values of x and dx in given integral

$$\int\frac{dx}{x\sqrt{7x^2-5}}=\int\frac{-\sqrt{\tfrac{5}{7}}\cos ce\theta\cot\theta d\theta}{\sqrt{\tfrac{5}{7}}\cos ce\theta\sqrt{7\left(\sqrt{\tfrac{5}{7}}\cos ce\theta\right)^2-5}}$$

$$\int\frac{dx}{x\sqrt{7x^2-5}}=-\int\frac{\cot\theta d\theta}{\sqrt{7\left(\tfrac{5}{7}\cos ce^2\theta\right)-5}} \qquad \therefore \cos ce^2\theta=\cot^2\theta+1$$

$$\int\frac{dx}{x\sqrt{7x^2-5}}=-\int\frac{\cot\theta d\theta}{\sqrt{5\cos ce^2\theta-5}} \qquad \cos ce^2\theta-1=\cot^2\theta$$

$$\int\frac{dx}{x\sqrt{7x^2-5}}=\frac{-1}{\sqrt{5}}\int\frac{\cot\theta d\theta}{\sqrt{\cos ce^2\theta-1}}$$

$$\int\frac{dx}{x\sqrt{7x^2-5}}=\frac{-1}{\sqrt{5}}\int\frac{\cot\theta d\theta}{\sqrt{\cot^2\theta}} \qquad \sqrt{\tfrac{5}{7}}\cos ce\theta=x$$

$$\int\frac{dx}{x\sqrt{7x^2-5}}=\frac{-1}{\sqrt{5}}\int\frac{\cot\theta d\theta}{\cot\theta} \qquad \cos ce\theta=\sqrt{\frac{7}{5}}x$$

$$\int\frac{dx}{x\sqrt{7x^2-5}}=\frac{-1}{\sqrt{5}}\int d\theta \qquad \theta=\cos ec^{-1}\left(\sqrt{\frac{7}{5}}x\right)$$

$$\int\frac{dx}{x\sqrt{7x^2-5}}=\frac{-1}{\sqrt{5}}\theta+C$$

Putting the values of θ

$$\int\frac{dx}{x\sqrt{7x^2-5}}=\frac{-1}{\sqrt{5}}\cos ec^{-1}\left(\sqrt{\frac{7}{5}}x\right)+C$$

Rules for Partial Fractions

Partial Fraction:

If we spilt a single rational fraction into two or more rational fraction with simpler denominator, this procedure of splitting is called partial fraction.

Decomposition of rational fraction into Partial fraction:

The decomposition of rational fraction $\frac{P(x)}{Q(x)}$ into partial fraction as follows

1). If possible then Factorize the Q(x).

Case1: When Q(x) has non repeated linear factors

$$\frac{x}{(x+a)(x+b)}=\frac{A}{x+a}+\frac{B}{x+b}$$

Case 2: When Q(x) has repeated linear factors.

$$\frac{Bx}{(x+a)^n}=\frac{A_1}{(x+a)^1}+\frac{A_2}{(x+a)^2}+\cdots+\frac{A_n}{(x+a)^n}$$

Case 3: When Q(x) has non repeated irreducible quadratic factors.

$$\frac{Bx+C}{(ax^2+bx+c)(dx^2+ex+f)}=\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(dx^2+ex+f)}$$

Case 4: When Q(x) has repeated irreducible quadratic factors.

$$\frac{Ax+B}{(ax^2+bx+c)^n}=\frac{A_1x+B_1}{(ax^2+bx+c)^1}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Exercise 5.3

Q1. Decomposing the following rational functions into partial fractions:

a). $\frac{1}{x(x-3)}$

Sol: Given $\frac{1}{x(x-3)}$

Given fraction is proper and has two linear factors, so

$$\frac{1}{x(x-3)}=\frac{A}{x}+\frac{B}{x-3}\text{..................}(1)$$

Multiply each term by $x(x-3)$ we have

$$x(x-3)\frac{1}{x(x-3)}=x(x-3)\frac{A}{x}+x(x-3)\frac{B}{x-3}$$

$$1=A(x-3)+Bx$$

$$1=Ax-3A+Bx$$

$$1=Ax+Bx-3A$$

$$1=(A+B)x-3A$$

Comparing the coefficients of x and constant

$-3A=1$ $\qquad$ $A+B=0$

$A=\frac{-1}{3}$ $\qquad$ $\frac{-1}{3}+B=0$

$B=\frac{1}{3}$

Putting the values of A and B in equation(1) we get

$$\frac{1}{x(x-1)}=\frac{\frac{-1}{3}}{x}+\frac{\frac{1}{3}}{x-1}$$

$$\frac{1}{x(x-1)}=\frac{-1}{3x}+\frac{1}{3(x-1)}$$

Now integrating

$$\int\frac{1}{x(x-1)}dx=\frac{-1}{3}\int\frac{1}{x}dx+\frac{1}{3}\int\frac{1}{x-1}dx$$

$$\int\frac{1}{x(x-1)}dx=\frac{-1}{3}\ln x+\frac{1}{3}\ln(x-1)+c$$

$$\int\frac{1}{x(x-1)}dx=\frac{1}{3}\ln\frac{x-1}{x}+c$$

---

b). $\frac{3x^2+2x-1}{x(x+1)}$

Sol: Given $\frac{3x^2+2x-1}{x(x+1)}$

$$\frac{3x^2+2x-1}{x(x+1)}=\frac{3x^2+3x-x-1}{x(x+1)}$$

$$\frac{3x^2+2x-1}{x(x+1)}=\frac{3x(x+1)-1(x+1)}{x(x+1)}$$

$$\frac{3x^2+2x-1}{x(x+1)}=\frac{(3x-1)(x+1)}{x(x+1)}$$

$$\frac{3x^2+2x-1}{x(x+1)}=\frac{3x-1}{x}$$

$$\frac{3x^2+2x-1}{x(x+1)}=\frac{3x}{x}-\frac{1}{x}$$

$$\frac{3x^2+2x-1}{x(x+1)}=3-\frac{1}{x}$$

Now integrating

$$\int\frac{3x^2+2x-1}{x(x+1)}dx=\int\left(3-\frac{1}{x}\right)dx$$

$$\int\frac{3x^2+2x-1}{x(x+1)}dx=3x-\ln x+c$$

---

c). $\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}$

Sol: Given $\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}$

fraction is proper & has two linear repeating factors,

$$\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}..........(1)$$

Multiply each term by $x^2(x+1)^2$ we have

$$x^2(x+1)^2\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}=x^2(x+1)^2\frac{A}{x}+x^2(x+1)^2\frac{B}{x^2}$$
$$+x^2(x+1)^2\frac{C}{x+1}+x^2(x+1)^2\frac{D}{(x+1)^2}$$

$$4x^3+4x^2+x-1=Ax(x+1)^2+B(x+1)^2$$
$$+Cx^2(x+1)+Dx^2................(2)$$

$$4x^3+4x^2+x-1=Ax(x^2+2x+1)+B(x^2+2x+1)$$
$$+C(x^3+x^2)+Dx^2$$

$$4x^3+4x^2+x-1=A(x^3+2x^2+x)+B(x^2+2x+1)$$
$$+C(x^3+x^2)+Dx^2$$

$$4x^3+4x^2+x-1=Ax^3+2Ax^2+Ax+Bx^2+2Bx+B$$
$$+Cx^3+Cx^2+Dx^2$$

Rearranging

$$4x^3+4x^2+x-1=Ax^3+Cx^3$$
$$+2Ax^2+Bx^2+Cx^2+Dx^2+Ax+2Bx+B$$

$$4x^3+4x^2+x-1=(A+C)x^3$$
$$+(2A+B+C+D)x^2+(A+2B)x+B$$

Comparing the coefficients

$Cons\tan t$

$B=-1$

coefficients of x^3

$A+C=4$

$3+C=4$

$C=4-3$

$C=1$

coefficients of x

$A+2B=1$

$A+2(-1)=1$

$A-2=1$

$A=1+2$

$A=3$

coefficients of x^2

$2A+B+C+D=4$

$2(3)+(-1)+1+D=4$

$6-1+1+D=4$

$6+D=4$

$D=4-6$

$D=-2$

Putting values of A,B,C and D in equation (1) we get

$$\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}=\frac{3}{x}+\frac{-1}{x^2}+\frac{1}{x+1}+\frac{-2}{(x+1)^2}$$

$$\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}=\frac{3}{x}-\frac{1}{x^2}+\frac{1}{x+1}-\frac{2}{(x+1)^2}$$

$$\int\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}dx=3\int\frac{dx}{x}-\int x^{-2}dx+\int\frac{dx}{x+1}-2\int(x+1)^{-2}dx$$

$$\int\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}dx=3\ln x+\frac{1}{x}+\ln(x+1)+\frac{2}{x+1}+c$$

---

d). $\frac{1}{x^3-1}$

Sol: Given $\frac{1}{x^3-1}=\frac{1}{x^3-1^3}$

$$\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}$$

Given fraction is proper and has two factors, one is linear other is quadratic factor, so

$$\frac{1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.............(1)$$

Multiply each term by x^3-1 we have

$$(x^3-1)\frac{1}{x^3-1}=(x^3-1)\frac{A}{x-1}+(x^3-1)\frac{Bx+C}{x^2+x+1}$$

$$1=A(x^2+x+1)+(Bx+C)(x-1).................(2)$$

Put $x=1$ in equation (2) we get

$$1=A(1^2+1+1)+(B.1+C)(1-1)$$

$$1=A(1+1+1)+(B+C)(0)$$

$$1=3A \qquad \Rightarrow A=\tfrac{1}{3}$$

Taking equation (2)

$$1=Ax^2+Ax+A+Bx^2-Bx+Cx-C$$

$$1=Ax^2+Bx^2+Ax-Bx+Cx+A-C$$

$$1=(A+B)x^2+(A-B+C)x+A-C$$

Comparing the coefficients

constant

$A-C=1$

$\tfrac{1}{3}-C=1$

$\tfrac{1}{3}-1=C$

$C=\tfrac{-2}{3}$

coefficients of x^2

$A+B=0$

$B=-A$

$B=\frac{-1}{3}$

Putting values of A,B and C in equation (1) we get

$$\frac{1}{x^3-1}=\frac{\frac{1}{3}}{x-1}+\frac{\frac{-1}{3}x+\frac{-2}{3}}{x^2+x+1}$$

$$\frac{1}{x^3-1}=\frac{1}{3(x-1)}+\frac{-x-2}{3(x^2+x+1)}$$

$$\frac{1}{x^3-1}=\frac{1}{3(x-1)}+\frac{-(x+2)}{3(x^2+x+1)}$$

$$\frac{1}{x^3-1}=\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}$$

Now integrating

$$\int\frac{dx}{x^3-1}=\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3\times2}\int\frac{2(x+2)dx}{x^2+x+1}$$

$$\int\frac{dx}{x^3-1}=\frac{\ln(x-1)}{3}-\frac{1}{6}\int\frac{(2x+1+3)dx}{x^2+x+1}$$

$$\int\frac{dx}{x^3-1}=\frac{\ln(x-1)}{3}-\frac{1}{6}\int\frac{(2x+1)dx}{x^2+x+1}$$

$$-\frac{1}{6}\int\frac{3dx}{x^2+2(x)\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2+1-\left(\tfrac{1}{2}\right)^2}$$

$$\int\frac{dx}{x^3-1}=\frac{\ln(x-1)}{3}-\frac{1}{6}\ln\left(x^2+x+1\right)$$

$$-\frac{1}{6}\int\frac{3dx}{\left(x+\tfrac{1}{2}\right)^2+\left(\tfrac{\sqrt{3}}{2}\right)^2}$$

$$\int\frac{dx}{x^3-1}=\frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1}-\frac{1}{6\left(\tfrac{\sqrt{3}}{2}\right)^2}\int\frac{3dx}{\left(\frac{x+\tfrac{1}{2}}{\tfrac{\sqrt{3}}{2}}\right)^2+1^2}$$

$$\int\frac{dx}{x^3-1}=\frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1}-\frac{1}{6\left(\tfrac{3}{4}\right)\frac{1}{\tfrac{\sqrt{3}}{2}}}\tan^{-1}\left(\frac{x+\tfrac{1}{2}}{\tfrac{\sqrt{3}}{2}}\right)$$

$$\int\frac{dx}{x^3-1}=\frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1}-\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x+\tfrac{1}{2}}{\tfrac{\sqrt{3}}{2}}\right)$$

---

e). $\dfrac{x^4-x^2+2}{x^2(x-1)}$

Sol: Given $\dfrac{x^4-x^2+2}{x^2(x-1)}=\dfrac{x^4-x^2+2}{x^3-x^2}$

Given fraction is an improper, so we have to divide it

$$\begin{array}{r|llll} & & & & x+1\\ x^3-x^2 & x^4 & & -x^2 & +2\\ & \pm x^4 & \mp x^3 & & \\ \hline & & x^3 & -x^2 & +2\\ & & \pm x^3 & \mp x^2 & \\ \hline & & & & 2\end{array}$$

Therefore the given improper fraction can be written as the sum of polynomials and proper fraction

$$\frac{x^4-x^2+2}{x^2(x-1)}=x+1+\frac{2}{x^2(x-1)}\ldots\ldots\ldots\ldots(1)$$

Take $\dfrac{2}{x^2(x-1)}$ which is proper, so we decompose into partial fractions according to factors of denominator

$$\frac{2}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Multiply each term by $x^2(x-1)$ we have

$$x^2(x-1)\frac{2}{x^2(x-1)}=x^2(x-1)\frac{A}{x}$$

$$+x^2(x-1)\frac{B}{x^2}+x^2(x-1)\frac{C}{x-1}$$

$$2=Ax(x-1)+B(x-1)+Cx^2$$

$$2=A\left(x^2-x\right)+B(x-1)+Cx^2$$

$$2=Ax^2-Ax+Bx-B+Cx^2$$

$$2=Ax^2+Cx^2+Bx-Ax-B$$

$$2=(A+C)x^2+(B-A)x-B$$

Comparing the coefficients

$Cons\tan t$	$coefficients\ of\ x$	$coefficients\ of\ x^2$
$-B=2$	$B-A=0$	$A+C=0$
$B=-2$	$B=A$	$C=-A$
	$A=-2$	$C=-(-2)$
		$C=2$

Putting values of A,B and C in equation (2) we get

$$\frac{2}{x^2(x-1)}=\frac{-2}{x}+\frac{-2}{x^2}+\frac{2}{x-1}$$

$$\frac{2}{x^2(x-1)}=-\frac{2}{x}-\frac{2}{x^2}+\frac{2}{x-1}$$

Therefore equation (1) becomes

$$\frac{x^4-x^2+2}{x^2(x-1)}=x+1-\frac{2}{x}-\frac{2}{x^2}+\frac{2}{x-1}$$

Now Integrating

$$\int\frac{x^4-x^2+2}{x^2(x-1)}dx=\int\left(x+1-\frac{2}{x}-\frac{2}{x^2}+\frac{2}{x-1}\right)dx$$

$$=\int xdx+\int dx-2\int\frac{dx}{x}-2\int x^{-2}dx+2\int\frac{dx}{x-1}$$

$$\int\frac{x^4-x^2+2}{x^2(x-1)}dx=\frac{x^2}{2}+x-2\ln(x)+\frac{2}{x}+2\ln(x-1)+c$$

Q2. Evaluate the following indefinite integrals through partial fractions de-composition:

a). $\displaystyle\int\frac{dx}{x^2-1}$

Sol: Given $\displaystyle\int\frac{dx}{x^2-1}$ For decomposition of the fraction into partial fractions Take fraction only which is in the given integral

$$\frac{1}{x^2-1}=\frac{1}{x^2-1^2}$$

$$\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}$$

Fraction is proper and has two linear factors, so

$$\frac{1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}\ldots\ldots\ldots\ldots\ldots(1)$$

Multiply each term by x^2-1 we have

$$\left(x^2-1\right)\frac{1}{x^2-1}=\left(x^2-1\right)\frac{A}{x+1}+\left(x^2-1\right)\frac{B}{x-1}$$

$$1=A(x-1)+B(x+1)\ldots\ldots\ldots.(2)$$

Put $x=1$ in equation (2) we get

$$1=A(1-1)+B(1+1)$$

$$1=A(0)+B(2)$$

$$1=2B \qquad \Rightarrow B=\tfrac{1}{2}$$

Put $x=-1$ in equation (2) we get

$$1=A(-1-1)+B(-1+1)$$

$$1=A(-2)+B(0)$$

$$1=-2A \qquad \Rightarrow A=\tfrac{-1}{2}$$

Putting the values of A and B in equation(1) we get

$$\frac{1}{x^2-1}=\frac{\tfrac{-1}{2}}{x+1}+\frac{\tfrac{1}{2}}{x-1}$$

$$\frac{1}{x^2-1}=\frac{-1}{2(x+1)}+\frac{1}{2(x-1)}$$

Substituting partial fraction into the given integral

$$\int\frac{1}{x^2-1}dx=\frac{-1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx \qquad \because \int\frac{du}{u}=\ln|u|$$

$$\int\frac{1}{x^2-1}dx=\frac{-1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C$$

$$\int\frac{1}{x^2-1}dx=\frac{1}{2}\{-\ln|x+1|+\ln|x-1|\}+C$$

$$\int\frac{1}{x^2-1}dx=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

b). $\int\frac{3x+5}{x^2+2x-3}dx$

Sol: Given $\int\frac{3x+5}{x^2+2x-3}dx$

For decomposition of the fraction into partial fractions
Take fraction only which is in the given integral

$$\frac{3x+5}{x^2+2x-3}=\frac{3x+5}{x^2+3x-1x-3}$$

$$\frac{3x+5}{x^2+2x-3}=\frac{3x+5}{x(x+3)-1(x+3)}$$

$$\frac{3x+5}{x^2+2x-3}=\frac{3x+5}{(x-1)(x+3)}$$

Fraction is proper and has two linear factors, so

$$\frac{3x+5}{x^2+2x-3}=\frac{A}{x-1}+\frac{B}{x+3}\ldots\ldots\ldots\ldots\ldots(1)$$

Multiply each term by x^2+2x-3 we have

$$\left(x^2+2x-3\right)\frac{3x+5}{x^2+2x-3}=\left(x^2+2x-3\right)\frac{A}{x-1}$$
$$+\left(x^2+2x-3\right)\frac{B}{x+3}$$

$$3x+5=A(x+3)+B(x-1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Put $x=1$ in equation (2) we get

$$3(1)+5=A(1+3)+B(1-1)$$
$$3+5=A(4)+B(0)$$
$$8=4A$$
$$A=\tfrac{8}{4}=2$$

Put $x=-3$ in equation (2) we get

$$3(-3)+5=A(-3+3)+B(-3-1)$$
$$-9+5=A(0)+B(-4)$$
$$-4=-4B \qquad \Rightarrow B=1$$

Putting the values of A and B in equation(1) we get

$$\frac{3x+5}{x^2+2x-3}=\frac{2}{x-1}+\frac{1}{x+3}$$

Substituting partial fraction into the given integral

$$\int\frac{3x+5}{x^2+2x-3}dx=2\int\frac{1}{x-1}dx+\int\frac{1}{x+3}dx$$

$$\int\frac{3x+5}{x^2+2x-3}dx=2\ln|x-1|+\ln|x+3|+C$$

$$\int\frac{3x+5}{x^2+2x-3}dx=\ln\left|(x-1)^2\right|+\ln|x+3|+C$$

$$\int\frac{3x+5}{x^2+2x-3}dx=\ln\left|(x+3)(x-1)^2\right|+C$$

c). $\int\frac{-x-3}{2x^2-x-1}dx$

Sol: Given $\int\frac{-x-3}{2x^2-x-1}dx$

For decomposition of the fraction into partial fractions
Take fraction only which is in the given integral

$$\frac{-x-3}{2x^2-x-1}=\frac{-x-3}{2x^2-2x+x-1}$$

$$\frac{-x-3}{2x^2-x-1}=\frac{-x-3}{2x(x-1)+1(x-1)}$$

$$\frac{-x-3}{2x^2-x-1}=\frac{-x-3}{(2x+1)(x-1)}$$

Fraction is proper and has two linear factors, so

$$\frac{-x-3}{2x^2-x-1}=\frac{A}{2x+1}+\frac{B}{x-1}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Multiply each term by $2x^2-x-1$ we have

$$\left(2x^2-x-1\right)\frac{-x-3}{2x^2-x-1}=\left(2x^2-x-1\right)\frac{A}{2x+1}$$
$$+\left(2x^2-x-1\right)\frac{B}{x-1}$$

$$-x-3=A(x-1)+B(2x+1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Put $x=1$ in equation (2) we get

$$-1-3=A(1-1)+B\big(2(1)+1\big)$$
$$-4=A(0)+B(2+1)$$
$$-4=0+3B \qquad \Rightarrow B=\tfrac{-4}{3}$$

Put $x=\frac{-1}{2}$ in equation (2) we get

$$-\left(\tfrac{-1}{2}\right)-3=A\left(\tfrac{-1}{2}-1\right)+B\left(2\left(\tfrac{-1}{2}\right)+1\right)$$
$$\tfrac{1}{2}-3=A\left(\tfrac{-3}{2}\right)+B(-1+1)$$
$$\tfrac{1-6}{2}=\tfrac{-3}{2}A+B(0)$$
$$\tfrac{-5}{2}=\tfrac{-3}{2}A \qquad \Rightarrow A=\tfrac{5}{3}$$

Putting the values of A and B in equation(1) we get

$$\frac{-x-3}{2x^2-x-1}=\frac{\frac{5}{3}}{2x+1}+\frac{\frac{-4}{3}}{x-1}$$

$$\frac{-x-3}{2x^2-x-1}=\frac{5}{3(2x+1)}+\frac{-4}{3(x-1)}$$

$$\frac{-x-3}{2x^2-x-1}=\frac{5}{3(2x+1)}-\frac{4}{3(x-1)}$$

Substituting partial fraction into the given integral

$$\int\frac{-x-3}{2x^2-x-1}dx=\frac{5}{3}\int\frac{1}{2x+1}dx-\frac{4}{3}\int\frac{1}{x-1}dx$$

$$\int\frac{-x-3}{2x^2-x-1}dx=\frac{5}{3\times2}\int\frac{2}{2x+1}dx-\frac{4}{3}\int\frac{1}{x-1}dx$$

$$\int\frac{-x-3}{2x^2-x-1}dx=\frac{5}{6}\ln|2x+1|-\frac{4}{3}\ln|x-1|+C$$

$$\int\frac{-x-3}{2x^2-x-1}dx=\ln\left|(2x+1)^{\frac{5}{6}}\right|-\ln\left|(x-1)^{\frac{4}{3}}\right|+C$$

$$\int\frac{-x-3}{2x^2-x-1}dx=\ln\left|\frac{(2x+1)^{\frac{5}{6}}}{(x-1)^{\frac{4}{3}}}\right|+C$$

d). $\int\frac{x^2-1}{x^2-2x-15}dx$

Sol: Given $\int\frac{x^2-1}{x^2-2x-15}dx$

For decomposition of fraction into partial fractions Take fraction only which is in given integral $\dfrac{x^2-1}{x^2-2x-15}$

Fraction is an improper, so we have to divide it

$$\begin{array}{r|l} & \quad\quad\quad\quad 1 \\ x^2-2x-15 & x^2 \quad\quad -1 \\ & \underline{\pm x^2 \mp 2x \mp 15} \\ & \quad\quad 2x+14 \end{array}$$

Therefore the given improper fraction can be written as the sum of polynomials and proper fraction

$$\frac{x^2-1}{x^2-2x-15}=1+\frac{2x+14}{x^2-2x-15}\text{.................}(1)$$

Take $\dfrac{2x+14}{x^2-2x-15}$ which is proper, so we decompose into partial fractions according to factors of denominator

$$\frac{2x+14}{x^2-2x-15}=\frac{2x+14}{x^2-5x+3x-15}$$

$$\frac{2x+14}{x^2-2x-15}=\frac{2x+14}{x(x-5)+3(x-5)}$$

$$\frac{2x+14}{x^2-2x-15}=\frac{2x+14}{(x+3)(x-5)}$$

Fraction is proper and has two linear factors, so

$$\frac{2x+14}{x^2-2x-15}=\frac{A}{x+3}+\frac{B}{x-5}\text{.................}(2)$$

Multiply each term by $x^2-2x-15$ we have

$$\left(x^2-2x-15\right)\frac{2x+14}{x^2-2x-15}=\left(x^2-2x-15\right)\frac{A}{x+3}$$

$$+\left(x^2-2x-15\right)\frac{B}{x-5}$$

$$2x+14=A(x-5)+B(x+3)\text{..........................}(3)$$

Put $x=5$ in equation (3) we get

$$2(5)+14=A(5-5)+B(5+3)$$

$$10+14=A(0)+B(8)$$

$$24=0+8B$$

$$8B=24 \qquad\qquad \Rightarrow B=\tfrac{24}{8}=3$$

Put $x=-3$ in equation (3) we get

$$2(-3)+14=A(-3-5)+B(-3+3)$$

$$-6+14=A(-8)+B(0)$$

$$8=-8A+0$$

$$-8A=8 \qquad\qquad \Rightarrow A=\tfrac{8}{-8}=-1$$

Putting the values of A and B in equation (2) we get

$$\frac{2x+14}{x^2-2x-15}=\frac{-1}{x+3}+\frac{3}{x-5}$$

Putting into equation (1)

$$\frac{x^2-1}{x^2-2x-15}=1+\frac{-1}{x+3}+\frac{3}{x-5}$$

$$\frac{x^2-1}{x^2-2x-15}=1-\frac{1}{x+3}+\frac{3}{x-5}$$

Substituting partial fraction into the given integral

$$\int\frac{x^2-1}{x^2-2x-15}dx=\int dx-\int\frac{1}{x+3}dx+3\int\frac{1}{x-5}dx$$

$$\int\frac{x^2-1}{x^2-2x-15}dx=x-\ln|x+3|+3\ln|x-5|+C$$

$$\int\frac{x^2-1}{x^2-2x-15}dx=x-\ln|x+3|+\ln\left|(x-5)^3\right|+C$$

$$\int\frac{x^2-1}{x^2-2x-15}dx=x+\ln\left|\frac{(x-5)^3}{x+3}\right|+C$$

---

e). $\displaystyle\int\frac{x^2}{(x+1)^3}dx$

Sol: Given $\displaystyle\int\frac{x^2}{(x+1)^3}dx$

For decomposition of fraction into partial fractions Take fraction only which is in given integral $\dfrac{x^2}{(x+1)^3}$

Fraction is proper and has linear repeating factors, so

$$\frac{x^2}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}\text{................}(1)$$

Multiply each term by $(x+1)^3$ we have

$$(x+1)^3\frac{x^2}{(x+1)^3}=(x+1)^3\frac{A}{x+1}$$

$$+(x+1)^3\frac{B}{(x+1)^2}+(x+1)^3\frac{C}{(x+1)^3}$$

$$x^2=A(x+1)^2+B(x+1)+C$$

$$x^2=A\left(x^2+2x+1\right)+B(x+1)+C$$

$$x^2=Ax^2+2Ax+A+Bx+B+C$$

$$x^2=Ax^2+2Ax+Bx+A+B+C$$

$$x^2=Ax^2+(2A+B)x+A+B+C$$

Comparing the coefficients

$coefficients\ of\ x^2$

$A=1$

$coefficients\ of\ x$

$2A+B=0$

$2(1)+B=0$

$B=-2$

$constant$

$A+B+C=0$

$(1)+(-2)+C=0$

$1-2+C=0$

$-1+C=0$

$C=1$

Putting the values of A and B in equation (1) we get

$$\frac{x^2}{(x+1)^3}=\frac{1}{x+1}+\frac{-2}{(x+1)^2}+\frac{1}{(x+1)^3}$$

$$\frac{x^2}{(x+1)^3}=\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}$$

Substituting partial fraction into the given integral

$$\int\frac{x^2}{(x+1)^3}dx=\int\frac{1}{x+1}dx-2\int\frac{1}{(x+1)^2}dx+\int\frac{1}{(x+1)^3}dx$$

$$\int\frac{x^2}{(x+1)^3}dx=\int\frac{1}{x+1}dx-2\int(x+1)^{-2}dx+\int(x+1)^{-3}dx$$

$$\int\frac{x^2}{(x+1)^3}dx=\ln|x+1|-2\frac{(x+1)^{-2+1}}{-2+1}+\frac{(x+1)^{-3+1}}{-3+1}+C$$

$$\int\frac{x^2}{(x+1)^3}dx=\ln|x+1|-2\frac{(x+1)^{-1}}{-1}+\frac{(x+1)^{-2}}{-2}+C$$

$$\int\frac{x^2}{(x+1)^3}dx=\ln|x+1|+\frac{2}{(x+1)}-\frac{1}{2(x+1)^2}+C$$

f). $\int \frac{x}{(x+1)(x^2+1)}dx$

Sol: Given $\int \frac{x}{(x+1)(x^2+1)}dx$

For decomposition of fraction into partial fractions Take fraction only which is in given integral $\frac{x}{(x+1)(x^2+1)}$

Fraction is proper and has one linear factor and other quadratic factor, so

$$\frac{x}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}\text{.........................}(1)$$

Multiply each term by $(x+1)(x^2+1)$ we have

$$(x+1)(x^2+1)\frac{x}{(x+1)(x^2+1)}=(x+1)(x^2+1)\frac{A}{x+1}$$
$$+(x+1)(x^2+1)\frac{Bx+C}{x^2+1}$$
$$x=A(x^2+1)+(Bx+C)(x+1)\text{........................}(2)$$
$$x=Ax^2+A+Bx^2+Bx+Cx+C$$
$$x=Ax^2+Bx^2+Bx+Cx+A+C$$
$$x=(A+B)x^2+(B+C)x+A+C\text{.....................}(3)$$

Put $x=-1$ in equation (2) we get

$$-1=A\left((-1)^2+1\right)+\left(B(-1)+C\right)(-1+1)$$
$$-1=A(1+1)+(-B+C)(0)$$
$$-1=A(2)+0 \qquad \Rightarrow A=\tfrac{-1}{2}$$

Comparing the coefficients of equation (3)

constant	*coefficients of x*
$A+C=0$	$B+C=1$
$C=-A$	$B=1-C$
$C=-\left(\tfrac{-1}{2}\right)=\tfrac{1}{2}$	$B=1-\tfrac{1}{2}=\tfrac{1}{2}$

Putting the values of A,B and C in equation(1) we get

$$\frac{x}{(x+1)(x^2+1)}=\frac{\frac{-1}{2}}{x+1}+\frac{\frac{1}{2}x+\frac{1}{2}}{x^2+1}$$
$$\frac{x}{(x+1)(x^2+1)}=\frac{-1}{2(x+1)}+\frac{x+1}{2(x^2+1)}$$

Substituting partial fraction into the given integral

$$\int\frac{x}{(x+1)(x^2+1)}dx=\frac{-1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{x+1}{x^2+1}dx$$
$$\int\frac{x}{(x+1)(x^2+1)}dx=\frac{-1}{2}\int\frac{1}{x+1}dx$$
$$+\frac{1}{2}\int\left(\frac{x}{x^2+1}+\frac{1}{x^2+1}\right)dx$$
$$\int\frac{x}{(x+1)(x^2+1)}dx=\frac{-1}{2}\ln|x+1|$$
$$+\frac{1}{2\times2}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx$$
$$\int\frac{x}{(x+1)(x^2+1)}dx=\frac{-1}{2}\ln|x+1|$$
$$+\frac{1}{4}\ln|x^2+1|+\frac{1}{2}\tan^{-1}(x)+C$$
$$\int\frac{x}{(x+1)(x^2+1)}dx=\ln\left|\frac{(x^2+1)^{\frac{1}{4}}}{(x+1)^{\frac{1}{2}}}\right|+\frac{1}{2}\tan^{-1}(x)+C$$

---

g). $\int \frac{x^2+2}{(x^2+1)^2}dx$

Sol: Given $\int \frac{x^2+2}{(x^2+1)^2}dx$

For decomposition of fraction into partial fractions Take fraction only, which is in given integral $\frac{x^2+2}{(x^2+1)^2}$

Fraction is proper and has quadratic repeating factors

$$\frac{x^2+2}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}\text{....................}(1)$$

Multiply each term by $(x^2+1)^2$ we have

$$(x^2+1)^2\frac{x^2+2}{(x^2+1)^2}=(x^2+1)^2\frac{Ax+B}{x^2+1}+(x^2+1)^2\frac{Cx+D}{(x^2+1)^2}$$
$$x^2+2=(Ax+B)(x^2+1)+Cx+D$$
$$x^2+2=Ax^3+Ax+Bx^2+B+Cx+D$$
$$x^2+2=Ax^3+Bx^2+Ax+Cx+B+D$$
$$x^2+2=Ax^3+Bx^2+(A+C)x+B+D\text{.........................}(2)$$

Comparing the coefficients

coefficients of x^3	*coefficients of x^2*
$A=0$	$B=1$
coefficients of x	*constant*
$A+C=0$	$B+D=2$
$0+C=0$	$1+D=2$
$C=0$	$D=2-1$
	$D=1$

Putting values of A,B,C and D in equation(1) we get

$$\frac{x^2+2}{(x^2+1)^2}=\frac{0x+1}{x^2+1}+\frac{0x+1}{(x^2+1)^2}$$
$$\frac{x^2+2}{(x^2+1)^2}=\frac{1}{x^2+1}+\frac{1}{(x^2+1)^2}$$

Substituting partial fraction into the given integral

$$\int\frac{x^2+2}{(x^2+1)^2}dx=\int\frac{1}{x^2+1}dx+\int\frac{1}{(x^2+1)^2}dx$$
$$\int\frac{x^2+2}{(x^2+1)^2}dx=\tan^{-1}(x)+\int\frac{1}{(x^2+1)^2}dx\text{.........}(1)$$

Take $I=\int\frac{1}{(x^2+1)^2}dx$

Let $x=\tan\theta \qquad \Rightarrow \theta=\tan^{-1}(x)$

$$\tfrac{dx}{d\theta}=\sec^2\theta$$

$$dx=\sec^2\theta d\theta$$

Putting the values

$$I=\int\frac{dx}{\left(x^2+1\right)}=\int\frac{\sec^2\theta d\theta}{\left[\tan^2\theta+1\right]^2}$$

$$I=\int\frac{\sec^2\theta d\theta}{\sec^4\theta}$$

$$I=\int\frac{d\theta}{\sec^2\theta}$$

$$I=\int\cos^2\theta d\theta$$

$$I=\int\frac{1+\cos 2\theta}{2}d\theta$$

$$I=\frac{1}{2}\int d\theta+\frac{1}{2}I\int\cos2\theta d\theta$$

$$I=\frac{1}{2}\theta+\frac{1}{2}\frac{\sin 2\theta}{2}+C$$

Putting the value of θ

$$I=\frac{1}{2}\tan^{-1}\left(x\right)+\frac{1}{4}\sin\left[2\tan^{-1}\left(x\right)\right]+C$$

Putting the value of I in equation (1) we get

$$\int\frac{x^2+2}{\left(x^2+1\right)^2}dx=\tan^{-1}\left(x\right)+\frac{1}{2}\tan^{-1}\left(x\right)+\frac{1}{4}\sin\left[2\tan^{-1}\left(x\right)\right]+C$$

$$\int\frac{x^2+2}{\left(x^2+1\right)^2}dx=\frac{3}{2}\tan^{-1}\left(x\right)+\frac{1}{4}\sin\left[2\tan^{-1}\left(x\right)\right]+C$$

---

h). $\displaystyle\int\frac{x^4+1}{x^4-1}dx$

Sol: Given $\displaystyle\int\frac{x^4+1}{x^4-1}dx$

Given fraction is an improper, so we have to divide it

$$\begin{array}{r|l} & \quad 1 \\ x^4-1 & \;x^4+1 \\ & \underline{\pm x^4\mp 1} \\ & \quad 2 \end{array}$$

Therefore the given improper fraction can be written as the sum of polynomials and proper fraction

$$\frac{x^4+1}{x^4-1}=1+\frac{2}{x^4-1}\text{.........................}(1)$$

Take $\dfrac{2}{x^4-1}$ which is proper, so we decompose into partial fractions according to factors of denominator

$$\frac{2}{x^4-1}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}+\frac{D}{x-1}\text{...............}(2)$$

Multiply each term by x^4-1 we have

$$\left(x^4-1\right)\frac{2}{x^4-1}=\left(x^4-1\right)\frac{Ax+B}{x^2+1}$$
$$+\left(x^4-1\right)\frac{C}{x+1}+\left(x^4-1\right)\frac{D}{x-1}$$

$$2=\left(Ax+B\right)\left(x^2-1\right)+C\left(x^2+1\right)\left(x-1\right)$$
$$+D\left(x^2+1\right)\left(x+1\right)\text{.......}(3)$$

$$2=Ax^3-Ax+Bx^2-B+C\left(x^3-x^2+x-1\right)$$
$$+D\left(x^3+x^2+x+1\right)$$

$$2=Ax^3-Ax+Bx^2-B+Cx^3-Cx^2+Cx-C$$
$$+Dx^3+Dx^2+Dx+D$$

$$2=Ax^3+Cx^3+Dx^3+Bx^2+Dx^2-Cx^2$$
$$+Cx-Ax+Dx+D-B-C$$

$$2=\left(A+C+D\right)x^3+\left(B+D-C\right)x^2$$
$$+\left(C-A+D\right)x+D-B-C\text{...........}(3)$$

Put $x=1$ in equation (2) we get

$$2=\Big(A\left(1\right)+B\Big)\Big(\left(1\right)^2-1\Big)+C\Big(\left(1\right)^2+1\Big)\Big(\left(1\right)-1\Big)$$
$$+D\Big(\left(1\right)^2+1\Big)\Big(\left(1\right)+1\Big)$$

$$2=\left(A+B\right)\left(1-1\right)+C\left(1+1\right)\left(1-1\right)+D\left(1+1\right)\left(1+1\right)$$

$$2=\left(A+B\right)\left(0\right)+C\left(2\right)\left(0\right)+D\left(2\right)\left(2\right)$$

$$2=0+0+4D$$

$$4D=2$$

$$D=\frac{2}{4}=\frac{1}{2}$$

Put $x=-1$ in equation (2) we get

$$2=\Big(A\left(-1\right)+B\Big)\Big(\left(-1\right)^2-1\Big)+C\Big(\left(-1\right)^2+1\Big)\Big(\left(-1\right)-1\Big)$$
$$+D\Big(\left(-1\right)^2+1\Big)\Big(\left(-1\right)+1\Big)$$

$$2=\left(-A+B\right)\left(1-1\right)+C\left(1+1\right)\left(-1-1\right)$$
$$+D\left(1+1\right)\left(-1+1\right)$$

$$2=\left(-A+B\right)\left(0\right)+C\left(2\right)\left(-2\right)+D\left(2\right)\left(0\right)$$

$$2=0-4C+0$$

$$-4C=2$$

$$C=\frac{2}{-4}=\frac{-1}{2}$$

Comparing the coefficients of equations (3)

$coefficients\ of\ x^3$	$coefficients\ of\ x^3$
$A+C+D=0$	$B+D-C=0$
$A-\frac{1}{2}+\frac{1}{2}=0$	$B+\frac{1}{2}-\left(\frac{-1}{2}\right)=0$
$A=0$	$B+\frac{1}{2}+\frac{1}{2}=0$
	$B=-1$

Putting values of A,B,C and D in equation (2) we get

$$\frac{2}{x^4-1}=\frac{\left(0\right)x+\left(-1\right)}{x^2+1}+\frac{\frac{-1}{2}}{x+1}+\frac{\frac{1}{2}}{x-1}$$

$$\frac{2}{x^4-1}=\frac{-1}{x^2+1}-\frac{1}{2\left(x+1\right)}+\frac{1}{2\left(x-1\right)}$$

Therefore equation (1) becomes

$$\frac{x^4+1}{x^4-1}=1+\frac{2}{x^4-1}\text{.........................}(1)$$

$$\frac{x^4+1}{x^4-1}=1-\frac{1}{x^2+1}-\frac{1}{2\left(x+1\right)}+\frac{1}{2\left(x-1\right)}$$

Substituting partial fraction into the given integral

$$\int\frac{x^4+1}{x^4-1}dx=\int\left(1-\frac{1}{x^2+1}-\frac{1}{2\left(x+1\right)}+\frac{1}{2\left(x-1\right)}\right)dx$$

$$\int \frac{x^4+1}{x^4-1}dx = \int dx - \int \frac{1}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$\int \frac{x^4+1}{x^4-1}dx = x - \tan^{-1}(x) - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$\int \frac{x^4+1}{x^4-1}dx = x - \tan^{-1}(x) + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

Q3. Use integration by parts to evaluate following integrals:

a). $\int xe^x dx$

Sol: Given $\int xe^x dx$

$$\int xe^x dx = x\int e^x dx - \int \tfrac{d}{dx}x\left(\int e^x dx\right)dx$$

$$\int xe^x dx = xe^x - \int 1.e^x dx$$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$\int xe^x dx = xe^x - e^x + C$$

$$\int xe^x dx = e^x(x-1) + C$$

b). $\int x\sin x\ dx$

Sol: Given $\int x\sin x\ dx$

$$\int x\sin x\ dx = x\int \sin x\,dx - \int \tfrac{d}{dx}x\left(\int \sin x\,dx\right)dx$$

$$\int x\sin x\ dx = -x\cos x - \int 1.(-\cos x)\,dx$$

$$\int x\sin x\ dx = -x\cos x + \int \cos x\,dx$$

$$\int x\sin x\ dx = -x\cos x + \sin x + C$$

c). $\int \tan^{-1} x\ dx$

Sol: Given $\int \tan^{-1} x\ dx$

$$\int \tan^{-1} x\ dx = \int 1.\tan^{-1} x\ dx$$

$$\int \tan^{-1} x\ dx = \tan^{-1} x\int 1.dx - \int \tfrac{d}{dx}\tan^{-1} x\left(\int 1.dx\right)dx$$

$$\int \tan^{-1} x\ dx = x\tan^{-1} x - \int \frac{1}{1+x^2}.xdx$$

$$\int \tan^{-1} x\ dx = x\tan^{-1} x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$$

$$\int \tan^{-1} x\ dx = x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + C$$

d). $\int \sin^{-1} x\ dx$

Sol: Given $\int \sin^{-1} x\ dx$

$$\int \sin^{-1} x\ dx = \int 1.\sin^{-1} x\ dx$$

$$\int \sin^{-1} x\ dx = \sin^{-1} x\int 1.\ dx - \int\left(\tfrac{d}{dx}\sin^{-1} x\right)\left(\int dx\right)dx$$

$$\int \sin^{-1} x\ dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$\int \sin^{-1} x\ dx = x\sin^{-1} x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx$$

$$\int \sin^{-1} x\ dx = x\sin^{-1} x + \frac{1}{2}\int \left(1-x^2\right)^{\frac{-1}{2}}(-2xdx)$$

$$\int \sin^{-1} x\ dx = x\sin^{-1} x + \frac{1}{2}\frac{\left(1-x^2\right)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C$$

$$\int \sin^{-1} x\ dx = x\sin^{-1} x + \frac{1}{2}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\int \sin^{-1} x\ dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$

e). $\int x^2(x-3)^{11}dx$

Sol: Given $\int x^2(x-3)^{11}dx$

$$I = x^2\int (x-3)^{11}dx - \int\left(\tfrac{d}{dx}x^2\right)\left(\int (x-3)^{11}dx\right)dx$$

$$I = x^2\frac{(x-3)^{11+1}}{11+1} - \int (2x)\left(\frac{(x-3)^{11+1}}{11+1}\right)dx$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \int (2x)\left(\frac{(x-3)^{12}}{12}\right)dx$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \frac{1}{6}\int x(x-3)^{12}dx$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \frac{1}{6}\left[x\int (x-3)^{12}dx - \int \tfrac{d}{dx}x\left(\int (x-3)^{12}dx\right)dx\right]$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \frac{1}{6}\left[x\frac{(x-3)^{12+1}}{12+1} - \int \frac{(x-3)^{12+1}}{12+1}dx\right]$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \frac{1}{6}\left[x\frac{(x-3)^{13}}{13} - \frac{1}{13}\int (x-3)^{13}dx\right]$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \frac{1}{6}\left[x\frac{(x-3)^{13}}{13} - \frac{1}{13}\frac{(x-3)^{13+1}}{13+1}\right] + C$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \frac{1}{6}\left[x\frac{(x-3)^{13}}{13} - \frac{1}{13}\frac{(x-3)^{14}}{14}\right] + C$$

$$I = \frac{1}{12}x^2(x-3)^{12} - \frac{1}{78}x(x-3)^{13} + \frac{1}{1092}(x-3)^{14} + C$$

f). $\int e^x\cos x\ dx$

Solution: Let $I = \int e^x\cos x\ dx$

$$I = e^x\int \cos x\,dx - \int\left(\tfrac{d}{dx}e^x\right)\left(\int \cos x\,dx\right)dx$$

$$I = e^x\sin x - \int e^x\sin x\,dx$$

$$I = e^x\sin x - \left[e^x\int \sin x\,dx - \int\left(\tfrac{d}{dx}e^x\right)\left(\int \sin x\,dx\right)dx\right]$$

$$I = e^x\sin x - \left[-e^x\cos x - \int -e^x\cos x\,dx\right]$$

$$I = e^x\sin x + e^x\cos x - \int e^x\cos x\,dx + C$$

$$I = e^x(\sin x + \cos x) - I + C$$

$$I + I = e^x(\sin x + \cos x) + C$$

$$2I = e^x(\sin x + \cos x) + C$$

$$I = \frac{e^x}{2}(\sin x + \cos x) + C$$

g). $\int (x+\sin x)^2 dx$

Sol: Given $\int (x+\sin x)^2 dx$

$$\int (x+\sin x)^2 dx = \int \left(x^2 + \sin^2 x + 2x\sin x\right)dx$$

$$\int(x+\sin x)^2dx=\int x^2dx+\int\sin^2 x\,dx+2\int x\sin x\,dx$$

$$I=\frac{x^{2+1}}{2+1}+\int\frac{1-\cos 2x}{2}dx+2\left[x\int\sin x\,dx-\int\tfrac{d}{dx}x\left(\int\sin x\,dx\right)dx\right]$$

$$I=\frac{x^3}{3}+\int\left(\frac{1}{2}-\frac{\cos 2x}{2}\right)dx+2\left[-x\cos x-\int-\cos x\,dx\right]$$

$$I=\frac{x^3}{3}+\frac{1}{2}\int dx-\frac{1}{2}\int\cos 2x\,dx+2\left[-x\cos x+\int\cos x\,dx\right]$$

$$I=\frac{x^3}{3}+\frac{1}{2}x-\frac{1}{2}\frac{\sin 2x}{\frac{d}{dx}2x}+2\left[-x\cos x+\sin x\right]+C$$

$$I=\frac{x^3}{3}+\frac{1}{2}x-\frac{1}{2}\frac{\sin 2x}{2}-2x\cos x+2\sin x+C$$

$$I=\frac{x^3}{3}+\frac{1}{2}x-\frac{1}{4}\sin 2x-2x\cos x+2\sin x+C$$

---

h). $\int e^{2x}\sqrt{1-e^x}\ dx$

Sol: Given $\int e^{2x}\sqrt{1-e^x}\ dx$

$$\int e^{2x}\sqrt{1-e^x}\ dx=\int e^x.\left(1-e^x\right)^{\frac{1}{2}}e^xdx$$

$$I=e^x\int\left(1-e^x\right)^{\frac{1}{2}}e^xdx-\int\tfrac{d}{dx}e^x\left(\int\left(1-e^x\right)^{\frac{1}{2}}e^xdx\right)dx$$

$$I=-e^x\int\left(1-e^x\right)^{\frac{1}{2}}\left(-e^x\right)dx-\int e^x\left(-\int\left(1-e^x\right)^{\frac{1}{2}}\left(-e^x\right)dx\right)dx$$

$$I=-e^x\frac{\left(1-e^x\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\int-e^x\frac{\left(1-e^x\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}dx$$

$$I=-e^x\frac{\left(1-e^x\right)^{\frac{3}{2}}}{\frac{3}{2}}-\int-e^x\frac{\left(1-e^x\right)^{\frac{3}{2}}}{\frac{3}{2}}dx$$

$$I=-\frac{2}{3}e^x\left(1-e^x\right)^{\frac{3}{2}}-\frac{2}{3}\int\left(1-e^x\right)^{\frac{3}{2}}\left(-e^x\right)dx$$

$$I=-\frac{2}{3}e^x\left(1-e^x\right)^{\frac{3}{2}}-\frac{2}{3}\frac{\left(1-e^x\right)^{\frac{3}{2}+1}}{\frac{3}{2}+1}+C$$

$$I=-\frac{2}{3}e^x\left(1-e^x\right)^{\frac{3}{2}}-\frac{2}{3}\frac{\left(1-e^x\right)^{\frac{3}{2}+1}}{\frac{5}{2}}+C$$

$$I=-\frac{2}{3}e^x\left(1-e^x\right)^{\frac{3}{2}}-\frac{4}{15}\left(1-e^x\right)^{\frac{3}{2}+1}+C$$

$$I=\left(1-e^x\right)^{\frac{3}{2}}\left\{-\frac{5}{5}.\frac{2}{3}e^x-\frac{4}{15}\left(1-e^x\right)\right\}+C$$

$$I=\frac{2}{15}\left(1-e^x\right)^{\frac{3}{2}}\left\{-5e^x-2\left(1-e^x\right)\right\}+C$$

$$I=\frac{2}{15}\left(1-e^x\right)^{\frac{3}{2}}\left\{-5e^x-2+2e^x\right\}+C$$

$$I=\frac{2}{15}\left(1-e^x\right)^{\frac{3}{2}}\left\{-3e^x-2\right\}+C$$

---

i). $\int x\sin x\ \cos x\ dx$

Sol: Given $\int x\sin x\ \cos x\ dx$

$$\int x\sin x\cos x\,dx=\frac{1}{2}\int x\left(2\sin x\cos x\right)dx$$

$$\int x\sin x\cos x\,dx=\frac{1}{2}\int x\sin 2x\,dx$$

$$I=\frac{1}{2}\left\{x\int\sin 2x\,dx-\int\tfrac{d}{dx}x\left(\int\sin 2x\,dx\right)dx\right\}$$

$$I=\frac{1}{2}\left\{x\frac{-\cos 2x}{\frac{d}{dx}2x}-\int 1\frac{-\cos 2x}{\frac{d}{dx}2x}dx\right\}$$

$$I=\frac{1}{2}\left\{-\frac{x\cos 2x}{2}+\frac{1}{2}\int\cos 2xdx\right\}$$

$$I=\frac{1}{2}\left\{-\frac{x\cos 2x}{2}+\frac{1}{2}\frac{\sin 2x}{\frac{d}{dx}2x}\right\}+C$$

$$I=\frac{1}{2}\left\{-\frac{x\cos 2x}{2}+\frac{1}{2}\frac{\sin 2x}{2}\right\}+C$$

$$I=-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$$

---

j). $\int x^2\ln x\ dx$

Sol: Given $\int x^2\ln x\ dx$

$$\int x^2\ln x\ dx=\ln x\int x^2\ dx-\int\tfrac{d}{dx}\ln x\left(\int x^2\ dx\right)dx$$

$$I=\ln x\frac{x^{2+1}}{2+1}-\int\frac{1}{x}\frac{x^{2+1}}{2+1}dx$$

$$I=\frac{x^3\ln x}{3}-\frac{1}{3}\int x^2dx$$

$$I=\frac{x^3\ln x}{3}-\frac{1}{3}\frac{x^{2+1}}{2+1}+C$$

$$I=\frac{x^3\ln x}{3}-\frac{1}{3}\frac{x^3}{3}+C$$

$$I=\frac{x^3\ln x}{3}-\frac{x^3}{9}+C$$

Q4. Integrate the following integrals by parts integrations through appropriate substitution:

a). $\int\frac{\ln x\sin\left(\ln x\right)}{x}dx$

Sol: Given $\int\frac{\ln x\sin\left(\ln x\right)}{x}dx$

Let $u=\ln x$, Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}\ln x$

$\frac{d}{dx}u=\frac{1}{x}$

$du=\frac{dx}{x}$

Substituting values of u and du in given integral

$$\int\frac{\ln x\sin\left(\ln x\right)}{x}dx=\int\ln x.\sin\left(\ln x\right)\frac{dx}{x}$$

$$I=\int u\sin u\,du$$

$$I=u\int\sin u\,du-\int\tfrac{d}{du}u\left(\int\sin u\,du\right)du$$

$$I=u\left(-\cos u\right)-\int 1\left(-\cos u\right)du$$

$$I=-u\cos u+\int\cos u\,du$$

$$I=-u\cos u+\sin u+C$$

Putting the values of u

$$I=-\ln x\cos\left(\ln x\right)+\sin\left(\ln x\right)+C$$

---

b). $\int\left[\sin 2x\ \ln\left(\cos x\right)\right]dx$

Sol: Given

Let $u=\cos x$, Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}\cos x$

$\frac{du}{dx}=-\sin x$

$du=-\sin xdx$

Substituting values of u and du in given integral

$$\int \left[\sin 2x\,\ln\left(\cos x\right)\right]dx=\int 2\sin x\cos x\ln\left(\cos x\right)dx$$

$$I=-2\int \cos x.\ln\left(\cos x\right)\left(-\sin x\right)dx$$

$$I=-2\int u.\ln u\,du$$

$$I=-2\left\{\ln u\int u\,du-\int \tfrac{d}{du}\ln u\left(\int u\,du\right)du\right\}$$

$$I=-2\left\{\frac{u^2}{2}\ln u-\int\frac{1}{u}\frac{u^2}{2}du\right\}$$

$$I=-u^2\ln u+\int u\,du$$

$$I=-u^2\ln u+\frac{u^2}{2}+C$$

Putting the values of u

$$I=-\cos^2 x\ln\left|\cos x\right|+\frac{\cos^2 x}{2}+C$$

c). $\int e^{2x}\sin e^x dx$

Sol: Given $\int e^{2x}\sin e^x dx$

Let $u=e^x$, Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}e^x$$

$$\frac{du}{dx}=e^x \qquad \Rightarrow du=e^x dx$$

Substituting values of u and du in the given integral

$$\int e^{2x}\sin e^x dx=\int e^x\sin e^x\left(e^x dx\right)$$

$$\int e^{2x}\sin e^x dx=\int u\sin u\,du$$

$$\int e^{2x}\sin e^x dx=u\int\sin u\,du-\int\tfrac{d}{du}u\left(\int\sin u\,du\right)du$$

$$\int e^{2x}\sin e^x dx=-u\cos u-\int 1\left(-\cos u\right)du$$

$$\int e^{2x}\sin e^x dx=-u\cos u+\int\cos u\,du$$

$$\int e^{2x}\sin e^x dx=-u\cos u+\sin u+C$$

Putting the values of u

$$\int e^{2x}\sin e^x dx=-e^x\cos e^x+\sin e^x+C$$

Q5. rate at which the body eliminates a drug (in milliliters per hour) is given by $\frac{d}{dt}R(t)=\frac{60t}{(t+1)^2(t+2)}$ Where t is the number of hours since the drug was administered. If $R(0)=0$ is the current drug elimination, how much of the drug is eliminated during the first hour after it was administered? The fourth hour, after it was administered?

Sol: Given $\frac{d}{dt}R(t)=\frac{60t}{(t+1)^2(t+2)}$.........(1)

Take $\frac{60t}{(t+1)^2(t+2)}$ using partial fraction Since denominator has linear repeating and linear factor so

$$\frac{60t}{(t+1)^2(t+2)}=\frac{A}{t+1}+\frac{B}{(t+1)^2}+\frac{C}{t+2}\text{...............}(2)$$

Multiply each term by $(t+1)^2(t+2)$ we get

$$60t=A(t+1)(t+2)+B(t+2)+C(t+1)^2\text{...........}(3)$$

Put $t=-1$ in equation (3) we obtained

$$60(-1)=A(-1+1)(-1+2)+B(-1+2)+C(-1+1)^2$$

$$-60=A(0)(1)+B(1)+C(0)^2$$

$$-60=0+B(1)+0$$

$$B=-60$$

Put $t=-2$ in equation (3) we obtained

$$60(-2)=A(-2+1)(-2+2)+B(-2+2)+C(-2+1)^2$$

$$-120=A(-1)(0)+B(0)+C(-1)^2$$

$$-120=0+0+C$$

$$C=-120$$

Choose $t=0$ & using $B=-60$ & $C=-120$ in (3)

$$60(0)=A(0+1)(0+2)+(-60)(0+2)+(-120)(0+1)^2$$

$$0=A(1)(2)-60(2)-120(1)$$

$$0=2A-120-120$$

$$0=2A-240$$

$$2A=240 \qquad \Rightarrow A=\frac{240}{2}=120$$

Putting the value of A,B and C in equation (2)

$$\frac{60t}{(t+1)^2(t+2)}=\frac{120}{t+1}+\frac{-60}{(t+1)^2}+\frac{-120}{t+2}$$

$$\frac{60t}{(t+1)^2(t+2)}=\frac{120}{t+1}-\frac{60}{(t+1)^2}-\frac{120}{t+2}$$

Therefore equation (1) becomes

$$\frac{d}{dt}R(t)=\frac{60t}{(t+1)^2(t+2)}$$

$$\frac{d}{dt}R(t)=\frac{120}{t+1}-\frac{60}{(t+1)^2}-\frac{120}{t+2}$$

$$dR(t)=\left\{\frac{120}{t+1}-\frac{60}{(t+1)^2}-\frac{120}{t+2}\right\}dt$$

Now integrating both sides

$$\int dR(t)=\int\left\{\frac{120}{t+1}-\frac{60}{(t+1)^2}-\frac{120}{t+2}\right\}dt$$

$$R(t)=120\int\frac{1}{t+1}dt-60\int\frac{1}{(t+1)^2}dt-120\int\frac{1}{t+2}dt$$

$$R(t)=120\ln|t+1|-60\frac{(t+1)^{-2+1}}{-2+1}-120\ln|t+2|+C$$

$$R(t)=120\ln|t+1|-120\ln|t+2|-60\frac{(t+1)^{-1}}{-1}+C$$

$$R(t)=120\ln\left|\frac{t+1}{t+2}\right|+\frac{60}{t+1}+C\text{..................}(4)$$

When $R(0)=0$

$$0=R(0)=120\ln\left|\frac{0+1}{0+2}\right|+\frac{60}{0+1}+C$$

$$0=120\ln\left|\frac{1}{2}\right|+\frac{60}{1}+C$$

$$0=120(-0.6931)+60+C$$

$$0=-83.18+60+C$$
$$0=-23.18+C$$
$$C=23.18$$

Therefore equation (4) becomes

$$R(t)=120\ln\left|\frac{t+1}{t+2}\right|+\frac{60}{t+1}+23.18$$

Drug is eliminated during the first hour i.e., $t=1$

$$R(1)=120\ln\left|\frac{1+1}{1+2}\right|+\frac{60}{1+1}+23.18$$
$$R(1)=120\ln\left|\frac{2}{3}\right|+\frac{60}{2}+23.18$$
$$R(1)=120(-0.4055)+30+23.18$$
$$R(1)=-48.66+53.18$$
$$R(1)=4.52$$

Drug is eliminated during the first hour i.e., $t=4$ only

Take $R(4)-R(3)$

$$R(4)=120\ln\left|\frac{4+1}{4+2}\right|+\frac{60}{4+1}+23.18$$
$$R(4)=120\ln\left|\frac{5}{6}\right|+\frac{60}{5}+23.18$$
$$R(4)=120(-0.1823)+12+23.18$$
$$R(4)=-21.88+35.18$$
$$R(4)=13.3$$

Now $R(3)=120\ln\left|\frac{3+1}{3+2}\right|+\frac{60}{3+1}+23.18$

$$R(3)=120\ln\left|\frac{4}{6}\right|+\frac{60}{4}+23.18$$
$$R(3)=120(-0.4055)+15+23.18$$
$$R(3)=-48.66+38.18$$
$$R(3)=-10.48$$

Therefore

$$R(4)-R(3)=13.3-(-10.48)=13.3+10.48$$
$$R(4)-R(3)=23.78$$

Q6. Rate of change of the voting population of a city with respect to time t (in years) is estimated to be $\frac{dN}{dt}=\frac{100t}{\left(1+t^2\right)^2}$ where $N(t)$ is in thousands. If $N(0)$ is current voting population, then how much will this population increasing during next 3 years?

Sol: Given $\frac{dN}{dt}=\frac{100t}{\left(1+t^2\right)^2}$

$$dN=\frac{100t}{\left(1+t^2\right)^2}dt$$

Integrating both sides

$$\int dN=\int\frac{100t}{\left(1+t^2\right)^2}dt$$
$$N=50\int\frac{2t}{\left(1+t^2\right)^2}dt$$
$$N=50\int\left(1+t^2\right)^{-2}2tdt$$
$$N=50\frac{\left(1+t^2\right)^{-2+1}}{-2+1}+C$$
$$N=50\frac{\left(1+t^2\right)^{-1}}{-1}+C$$
$$N=-\frac{50}{1+t^2}+C$$

The population increasing during the next 3 years i.e., $N(3)-N(0)$

$$N(0)=-\frac{50}{1+0^2}+C$$
$$N(0)=-50+C$$

Now $N(3)=-\frac{50}{1+3^2}+C$

$$N(3)=-\frac{50}{1+9}+C$$
$$N(3)=-\frac{50}{10}+C \qquad \Rightarrow N(3)=-5+C$$

Therefore

$$N(3)-N(0)=(-5+C)-(-50+C)$$
$$N(3)-N(0)=-5+C+50-C$$
$$N(3)-N(0)=45\ \ thousands$$

Q7. An oil tanker aground on a reef is losing oil and producing an oil slick that is radiating outward at a rate approximated by $\frac{dr}{dt}=\frac{100}{\sqrt{t^2+9}},t\geq 0$ where t is the radius (in feet) of the circular slick after t minutes. Find the radius of slick after 4 minutes if the radius is $r=0$ when $t=0$

Sol: Given $\frac{dr}{dt}=\frac{100}{\sqrt{t^2+9}},t\geq 0$

$$dr=\frac{100}{\sqrt{t^2+9}}dt$$

Integrating both sides

$$\int dr=\int\frac{100}{\sqrt{t^2+9}}dt \qquad \text{Let } t=3\tan\theta$$
$$r=\int\frac{100\times 3\sec^2\theta d\theta}{\sqrt{9\tan\theta^2+9}} \qquad dt=3\sec^2\theta d\theta$$
$$r=\int\frac{100\times 3\sec^2\theta d\theta}{3\sqrt{\tan\theta^2+1}} \qquad \tfrac{t}{3}=\tan\theta$$
$$r=\int\frac{100\sec^2\theta d\theta}{\sqrt{\sec^2\theta}}$$
$$\left(\tfrac{t}{3}\right)^2+1=\tan^2\theta+1$$
$$r=100\int\sec\theta d\theta \qquad \frac{t^2+9}{9}=\sec^2\theta$$

$r=100\ln\left|\sec\theta+\tan\theta\right|+c \qquad \frac{\sqrt{t^2+9}}{3}=\sec\theta$

Putting the value of $\tan\theta$ and $\sec\theta$

$$r=100\ln\left|\frac{\sqrt{t^2+9}}{3}+\frac{t}{3}\right|+c \text{ ....(1)}$$

When radius is $r=0$ when $t=0$

$$0=100\ln\left|\frac{\sqrt{0^2+9}}{3}+\frac{0}{3}\right|+c$$

$$0=100\ln(1)+c$$

$$c=0$$

Therefore equation (1) becomes

$$r=100\ln\left|\frac{\sqrt{t^2+9}}{3}+\frac{t}{3}\right|+0$$

When $t=4$ minutes we have to find r

$$r=100\ln\left|\frac{\sqrt{4^2+9}}{3}+\frac{4}{3}\right|$$

$$r=100\ln\left|\frac{5}{3}+\frac{4}{3}\right|=100\ln|3|$$

$$r=109.86122\approx 110 \ ft$$

Q8. After a person takes a pill, the drug contained in the pill is assimilated into the bloodstream. The rate of assimilation t minutes after taking the pill is $\frac{dr}{dt}=te^{-0.2t}, \quad r(0)=0$ Find total amount of drug that is assimilated into the bloodstream during the first 10 minutes after the pill is taken.

Sol: Given $\frac{dr}{dt}=te^{-0.2t}$,

$$dr=te^{-0.2t}dt$$

Integrating both sides

$$\int dr=\int te^{-0.2t}dt$$

$$r=t\int e^{-0.2t}dt-\int\tfrac{d}{dt}t\left(\int e^{-0.2t}dt\right)dt$$

$$r=t\frac{e^{-0.2t}}{\frac{d}{dt}(-0.2t)}-\int 1.\frac{e^{-0.2t}}{\frac{d}{dt}(-0.2t)}dt$$

$$r=t\frac{e^{-0.2t}}{-0.2}-\int\frac{e^{-0.2t}}{-0.2}dt$$

$$r=\frac{1}{-0.2}te^{-0.2t}+\frac{1}{0.2}\int e^{-0.2t}dt$$

$$r=\frac{10}{-2}te^{-0.2t}+\frac{10}{2}\frac{e^{-0.2t}}{\frac{d}{dt}(-0.2t)}+C$$

$$r=-5te^{-0.2t}+5\frac{1}{-0.2}e^{-0.2t}+C$$

$$r=-5te^{-0.2t}-5\frac{10}{2}e^{-0.2t}+C$$

$$r=-5te^{-0.2t}-5(5)e^{-0.2t}+C$$

$$r=-5te^{-0.2t}-25e^{-0.2t}+C\text{............}(1)$$

When $r(0)=0$

$$0=r(0)=-5(0)e^{-0.2(0)}-25e^{-0.2(0)}+C$$

$$0=0-25(1)+C$$

$$0=-25+C$$

$$C=25$$

Therefore equation (1) becomes

$$r=-5te^{-0.2t}-25e^{-0.2t}+25$$

drug is assimilated into the bloodstream during the first 10 minutes after the pill is taken i.e., $t=10$

$$r(10)=-5(10)e^{-0.2(10)}-25e^{-0.2(10)}+25$$

$$r(10)=-50e^{-2}-25e^{-2}+25$$

$$r(10)=(-50-25)e^{-2}+25$$

$$r(10)=(-75)(0.1353)+25$$

$$r(10)=-10.15+25$$

$$r(10)=14.85$$

Interval $[a,b]$, number of subinterval n

The width of each sub interval $\Delta x=\frac{b-a}{n}$

Definite integral:

If $f(x)$ is continuous on $[a,b]$ and $[a,b]$ is divided into each subintervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n]$$

whose right end points are $x_1,x_2,x_3,\ldots,x_n$ then the definite integral of $f(x)$ from $x=a$ to $x=b$ is

$$\int_{x=a}^{x=b}f(x)dx=\lim_{n\to\infty}\frac{b-a}{n}\{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\}$$

When n is finite number so we take

$$\int_{x=a}^{x=b}f(x)dx=\frac{b-a}{n}\{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\}$$

$$\int_{x=a}^{x=b}f(x)dx=\Delta x\{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\}$$

Exercise 5.4

Q1. In each case, determine approximate area of the region bounded by $f(x)=2x+1, x=a$ & $x=b$ for n subintervals:

a). $n=2 \quad ,a=0 \quad b=2$

Solution: $f(x)=2x+1$, which is continuous

And $a=0$ $b=2$ with $n=2$ therefore subintervals are $[x_0,x_1],[x_1,x_2]$

Right end points of each subinterval x_1,x_2

Width of each subinterval $\Delta x=\frac{2-0}{2}=\frac{2}{2}=1$

Since $x_i=x_o+i.\Delta x$

At $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x \qquad f(x_1)=2(x_1)+1$

$x_1=0+1(1) \qquad f(1)=2(1)+1$

$x_1=0+1 \qquad f(1)=2+1$

$x_1=1 \qquad f(1)=3$

At $i=2$ corresponding value of function

$x_2 = x_o + 2.\Delta x \qquad f(x_2) = 2(x_2)+1$

$x_2 = 0 + 2(1) \qquad f(2) = 2(2)+1$

$x_2 = 0 + 2 \qquad f(2) = 4+1$

$x_2 = 2 = b \qquad f(2) = 5$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int_{x=a}^{x=b} f(x)dx = \Delta x\{f(x_1)+f(x_2)\}$$

Substituting the values

$$\int_{a=0}^{b=2} (2x+1)dx = 1\{3+5\}$$

$$\int_{a=0}^{b=2} (2x+1)dx = 1\{8\} = 8$$

---

b). $n = 4 \quad ,a = 0 \ \ b = 2$

Solution: $f(x) = 2x+1$, which is continuous

And $a = 0 \quad b = 2$ with $n = 4$ therefore subintervals are $[x_0,x_1],[x_1,x_2],[x_2,x_3],[x_3,x_4]$

Right end points of each subinterval x_1,x_2,x_3,x_4

Width of each subinterval

$\Delta x = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$ Since $x_i = x_o + i.\Delta x$

At $i=1$ corresponding value of function

$x_1 = x_o + 1.\Delta x \qquad f(x_1) = 2(x_1)+1$

$x_1 = 0+1(0.5) \qquad f(0.5) = 2(0.5)+1$

$x_1 = 0+0.5 \qquad f(0.5) = 1+1$

$x_1 = 0.5 \qquad f(0.5) = 2$

At $i=2$ corresponding value of function

$x_2 = x_o + 2.\Delta x \qquad f(x_2) = 2(x_2)+1$

$x_2 = 0+2(0.5) \qquad f(1) = 2(1)+1$

$x_2 = 0+1 \qquad f(1) = 2+1$

$x_2 = 1 \qquad f(1) = 3$

At $i=3$ corresponding value of function

$x_3 = x_o + 3.\Delta x \qquad f(x_3) = 2(x_3)+1$

$x_3 = 0+3(0.5) \qquad f(1.5) = 2(1.5)+1$

$x_3 = 0+1.5 \qquad f(1.5) = 3+1$

$x_1 = 1.5 \qquad f(1.5) = 4$

At $i=4$ corresponding value of function

$x_4 = x_o + 4.\Delta x \qquad f(x_2) = 2(x_2)+1$

$x_4 = 0+4(0.5) \qquad f(2) = 2(2)+1$

$x_4 = 0+2 \qquad f(2) = 4+1$

$x_4 = 2 = B \qquad f(2) = 5$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int_{x=a}^{x=b} f(x)dx = \Delta x\{f(x_1)+f(x_2)+f(x_3)+f(x_4)\}$$

Substituting the values

$$\int_{a=0}^{b=2} (2x+1)dx = 0.5\{2+3+4+5\}$$

$$\int_{a=0}^{b=2} (2x+1)dx = 0.5\{14\} = 7$$

---

c). $n = 8 \quad ,a = 0 \ \ b = 2$

Sol: Given $f(x) = 2x+1$, which is continuous & $a = 0, b = 2$ with $n = 8$ therefore subintervals are $[x_0,x_1],[x_1,x_2],[x_2,x_3],[x_3,x_4],[x_4,x_5],[x_5,x_6],[x_6,x_7],[x_7,x_8]$

Right end points of each subinterval $x_1,x_2,x_3,x_4,$ x_5,x_6,x_7,x_8 Width of each subinterval

$\Delta x = \frac{2-0}{8} = \frac{2}{8} = \frac{1}{4} = 0.25$ Since $x_i = x_o + i.\Delta x$

At $i=1$ corresponding value of function

$x_1 = x_o + 1.\Delta x \qquad f(x_1) = 2(x_1)+1$

$x_1 = 0+1(0.25) \qquad f(0.25) = 2(0.25)+1$

$x_1 = 0+0.25 \qquad f(0.25) = 0.5+1$

$x_1 = 0.25 \qquad f(0.25) = 1.5$

At $i=2$ corresponding value of function

$x_2 = x_o + 2.\Delta x \qquad f(x_2) = 2(x_2)+1$

$x_2 = 0+2(0.25) \qquad f(0.5) = 2(0.5)+1$

$x_2 = 0+0.5 \qquad f(0.5) = 1+1$

$x_2 = 0.5 \qquad f(0.5) = 2$

At $i=3$ corresponding value of function

$x_3 = x_o + 3.\Delta x \qquad f(x_3) = 2(x_3)+1$

$x_3 = 0+3(0.25) \qquad f(0.75) = 2(0.75)+1$

$x_3 = 0+0.75 \qquad f(0.75) = 1.5+1$

$x_3 = 0.75 \qquad f(0.75) = 2.5$

At $i=4$ corresponding value of function

$x_4 = x_o + 4.\Delta x \qquad f(x_4) = 2(x_4)+1$

$x_4 = 0+4(0.25) \qquad f(1) = 2(1)+1$

$x_4 = 0+1 \qquad f(1) = 2+1$

$x_4 = 1 \qquad f(1) = 3$

At $i=5$ corresponding value of function

$x_5 = x_o + 5.\Delta x \qquad f(x_5) = 2(x_5)+1$

$x_5 = 0+5(0.25) \qquad f(1.25) = 2(1.25)+1$

$x_5 = 0+1.25 \qquad f(1.25) = 2.50+1$

$x_5 = 1.25 \qquad f(1.25) = 3.5$

At $i=6$ corresponding value of function

$x_6 = x_o + 6.\Delta x \qquad f(x_6) = 2(x_6)+1$

$x_6 = 0+6(0.25) \qquad f(1.5) = 2(1.5)+1$

$x_6 = 0+1.5 \qquad f(1.5) = 3+1$

$x_6 = 1.5 \qquad f(1.5) = 4$

At $i=7$ corresponding value of function

$x_7 = x_o + 7.\Delta x \qquad f(x\ ) = 2(x_7)+1$

$x_7 = 0+7(0.25) \qquad f(1.75) = 2(1.75)+1$

$x_7 = 0+1.75 \qquad f(1.75) = 3.5+1$

$x_7 = 1.75 \qquad f(1.75) = 4.5$

At $i=8$ corresponding value of function

$x_8 = x_0 + 8.\Delta x \qquad f(x_8) = 2(x_8)+1$

$x_8 = 0+8(0.25) \qquad f(2) = 2(2)+1$

$x_8 = 0+2 \qquad f(2) = 4+1$

$x_8 = 2 = b \qquad f(2) = 5$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int_{x=a}^{x=b} f(x)dx = \Delta x\{f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7)+f(x_8)\}$$

Substituting the values

$$\int_{a=0}^{b=2} (2x+1)dx = 0.25\{1.5+2+2.5+3+3.5+4+4.5+5\}$$

$$\int_{a=0}^{b=2} (2x+1)dx = 0.25\{26\} = 6.5$$

---

d). $n=2 \quad ,a=1 \quad b=5$

Sol: Given $f(x)=2x+1$, which is continuous

& $a=1, b=5$ with $n=2$ therefore subintervals are $[x_0,x_1],[x_1,x_2]$ Right end points of each subinterval x_1,x_2 Width of each subinterval

$\Delta x = \frac{5-1}{2} = \frac{4}{2} = 2$ Since $x_i = x_o + i.\Delta x$

At $i=1$ corresponding value of function

$x_1 = x_o + 1.\Delta x$ $\quad f(x_1) = 2(x_1)+1$

$x_1 = 1+1(2)$ $\quad f(3) = 2(3)+1$

$x_1 = 1+2$ $\quad f(3) = 6+1$

$x_1 = 3$ $\quad f(3) = 7$

At $i=2$ corresponding value of function

$x_2 = x_o + 2.\Delta x$ $\quad f(x_2) = 2(x_2)+1$

$x_2 = 1+2(2)$ $\quad f(5) = 2(5)+1$

$x_2 = 1+4$ $\quad f(5) = 10+1$

$x_2 = 5 = b$ $\quad f(5) = 11$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int_{x=a}^{x=b} f(x)dx = \Delta x\{f(x_1)+f(x_2)\}$$

Substituting the values $\int_{a=1}^{b=5} (2x+1)dx = 2\{7+11\}$

$$\int_{a=1}^{b=5} (2x+1)dx = 2\{18\} = 36$$

Q2. In each case, determine the approximate area of the region bounded by $f(x)=x^2+1, x=a$ and $x=b$ for n subintervals:

a). $n=2 \quad ,a=0 \quad b=2$

Solution: Since $f(x)=x^2+1$, which is continuous

And $a=0 \quad b=2$ with $n=2$ therefore subintervals are $[x_0,x_1],[x_1,x_2]$ Right end points of each subinterval x_1,x_2 Width of each subinterval

$\Delta x = \frac{2-0}{2} = \frac{2}{2} = 1$ Since $x_i = x_o + i.\Delta x$

At $i=1$ corresponding value of function

$x_1 = x_o + 1.\Delta x$ $\quad f(x_1) = (x_1)^2+1$

$x_1 = 0+1(1)$ $\quad f(1) = (1)^2+1$

$x_1 = 0+1$ $\quad f(1) = 1+1$

$x_1 = 1$ $\quad f(1) = 2$

At $i=2$ corresponding value of function

$x_2 = x_o + 2.\Delta x$ $\quad f(x_2) = (x_2)^2+1$

$x_2 = 0+2(1)$ $\quad f(2) = (2)^2+1$

$x_2 = 0+2$ $\quad f(2) = 4+1$

$x_2 = 2 = b$ $\quad f(2) = 5$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int_{x=a}^{x=b} f(x)dx = \Delta x\{f(x_1)+f(x_2)\}$$

Substituting the values $\int_{a=0}^{b=2} (x^2+1)dx = 1\{2+5\}$

$$\int_{a=0}^{b=2} (x^2+1)dx = 1\{7\} = 7$$

---

b). $n=4 \quad ,a=0 \quad b=2$

Sol: Given $f(x)=x^2+1$, which is continuous

And $a=0 \quad b=2$ with $n=4$ therefore subintervals are $[x_0,x_1],[x_1,x_2],[x_2,x_3],[x_3,x_4]$

Right end points of each subinterval x_1,x_2,x_3,x_4

Width of each subinterval $\Delta x = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$

Since $x_i = x_o + i.\Delta x$

At $i=1$ corresponding value of function

$x_1 = x_o + 1.\Delta x$ $\quad f(x_1) = (x_1)^2+1$

$x_1 = 0+1(0.5)$ $\quad f(0.5) = (0.5)^2+1$

$x_1 = 0+0.5$ $\quad f(0.5) = 0.25+1$

$x_1 = 0.5$ $\quad f(0.5) = 1.25$

At $i=2$ corresponding value of function

$x_2 = x_o + 2.\Delta x$ $\quad f(x_2) = (x_2)^2+1$

$x_2 = 0+2(0.5)$ $\quad f(1) = (1)^2+1$

$x_2 = 0+1$ $\quad f(1) = 1+1$

$x_2 = 1$ $\quad f(1) = 2$

At $i=3$ corresponding value of function

$x_3 = x_o + 3.\Delta x$ $\quad f(x_3) = (x_3)+1$

$x_3 = 0+3(0.5)$ $\quad f(1.5) = (1.5)^2+1$

$x_3 = 0+1.5$ $\quad f(1.5) = 2.25+1$

$x_1 = 1.5$ $\quad f(1.5) = 3.25$

At $i=4$ corresponding value of function

$x_4 = x_o + 4.\Delta x$ $\quad f(x_2) = (x_2)^2+1$

$x_4 = 0+4(0.5)$ $\quad f(2) = (2)^2+1$

$x_4 = 0+2$ $\quad f(2) = 4+1$

$x_4 = 2 = B$ $\quad f(2) = 5$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int_{x=a}^{x=b} f(x)dx = \Delta x\{f(x_1)+f(x_2)+f(x_3)+f(x_4)\}$$

Putting $\int\limits_{a=0}^{b=2}\left(x^2+1\right)dx=0.5\left\{1.25+2+3.25+5\right\}$

$$\int\limits_{a=0}^{b=2}\left(x^2+1\right)dx=0.5\left\{11.5\right\}=5.75$$

---

c). $n=8 \quad ,a=0 \quad b=2$

Sol: Given $f\left(x\right)=x^2+1$, which is continuous

And $a=0,b=2$ with $n=8$ therefore subintervals are $\left[x_0,x_1\right],\left[x_1,x_2\right],\left[x_2,x_3\right],\left[x_3,x_4\right],\left[x_4,x_5\right],\left[x_5,x_6\right],\left[x_6,x_7\right],\left[x_7,x_8\right]$

Right end points of each subinterval $x_1,x_2,x_3,x_4,$ x_5,x_6,x_7,x_8 Width of each subinterval

$\Delta x=\dfrac{2-0}{8}=\dfrac{2}{8}=\dfrac{1}{4}=0.25$ Since $x_i=x_o+i.\Delta x$

At $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$ $\quad f\left(x_1\right)=\left(x_1\right)^2+1$
$x_1=0+1\left(0.25\right)$ $\quad f\left(0.25\right)=\left(0.25\right)^2+1$
$x_1=0+0.25$ $\quad f\left(0.25\right)=0.0625+1$
$x_1=0.25$ $\quad f\left(0.25\right)=1.0625$

At $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$ $\quad f\left(x_2\right)=\left(x_2\right)^2+1$
$x_2=0+2\left(0.25\right)$ $\quad f\left(0.5\right)=\left(0.5\right)^2+1$
$x_2=0+0.5$ $\quad f\left(0.5\right)=0.25+1$
$x_2=0.5$ $\quad f\left(0.5\right)=1.25$

At $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x$ $\quad f\left(x_3\right)=\left(x_3\right)^2+1$
$x_3=0+3\left(0.25\right)$ $\quad f\left(0.75\right)=\left(0.75\right)^2+1$
$x_3=0+0.75$ $\quad f\left(0.75\right)=0.5625+1$
$x_3=0.75$ $\quad f\left(0.75\right)=1.5625$

At $i=4$ corresponding value of function

$x_4=x_o+4.\Delta x$ $\quad f\left(x_4\right)=\left(x_4\right)^2+1$
$x_4=0+4\left(0.25\right)$ $\quad f\left(1\right)=\left(1\right)^2+1$
$x_4=0+1$ $\quad f\left(1\right)=1+1$
$x_4=1$ $\quad f\left(1\right)=2$

At $i=5$ corresponding value of function

$x_5=x_o+5.\Delta x$ $\quad f\left(x_5\right)=\left(x_5\right)^2+1$
$x_5=0+5\left(0.25\right)$ $\quad f\left(1.25\right)=\left(1.25\right)^2+1$
$x_5=0+1.25$ $\quad f\left(1.25\right)=1.5625+1$
$x_5=1.25$ $\quad f\left(1.25\right)=2.5625$

At $i=6$ corresponding value of function

$x_6=x_o+6.\Delta x$ $\quad f\left(x_6\right)=\left(x_6\right)^2+1$
$x_6=0+6\left(0.25\right)$ $\quad f\left(1.5\right)=\left(1.5\right)^2+1$
$x_6=0+1.5$ $\quad f\left(1.5\right)=2.25+1$
$x_6=1.5$ $\quad f\left(1.5\right)=3.25$

At $i=7$ corresponding value of function

$x_7=x_o+7.\Delta x$ $\quad f\left(x\ \right)=\left(x_7\right)^2+1$
$x_7=0+7\left(0.25\right)$ $\quad f\left(1.75\right)=\left(1.75\right)^2+1$
$x_7=0+1.75$ $\quad f\left(1.75\right)=3.0625+1$
$x_7=1.75$ $\quad f\left(1.75\right)=4.0625$

At $i=8$ corresponding value of function

$x_8=x_0+8.\Delta x$ $\quad f\left(x_8\right)=\left(x_8\right)^2+1$
$x_8=0+8\left(0.25\right)$ $\quad f\left(2\right)=\left(2\right)^2+1$
$x_8=0+2$ $\quad f\left(2\right)=4+1$
$x_8=2=b$ $\quad f\left(2\right)=5$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int\limits_{x=a}^{x=b}f\left(x\right)dx=\Delta x\left\{f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+f\left(x_4\right)+f\left(x_5\right)+f\left(x_6\right)+f\left(x_7\right)+f\left(x_8\right)\right\}$$

Substituting the values

$$\int\limits_{a=0}^{b=2}\left(x^2+1\right)dx=0.25\begin{Bmatrix}1.0625+1.25+1.5625+2\\+2.5625+3.25+4.0625+5\end{Bmatrix}$$

$$\int\limits_{a=0}^{b=2}\left(x^2+1\right)dx=0.25\left\{20.75\right\}=5.1875$$

---

d). $n=2 \quad ,a=1 \quad b=5$

Sol: Given $f\left(x\right)=x^2+1$, which is continuous

And $a=1,b=5$ with $n=2$ therefore subintervals are $\left[x_0,x_1\right],\left[x_1,x_2\right]$ Right end points of each subinterval x_1,x_2 Width of each subinterval

$\Delta x=\frac{5-1}{2}=\frac{4}{2}=2$ Since $x_i=x_o+i.\Delta x$

At $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$ $\quad f\left(x_1\right)=\left(x_1\right)^2+1$
$x_1=1+1\left(2\right)$ $\quad f\left(3\right)=\left(3\right)^2+1$
$x_1=1+2$ $\quad f\left(3\right)=9+1$
$x_1=3$ $\quad f\left(3\right)=10$

At $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$ $\quad f\left(x_2\right)=\left(x_2\right)^2+1$
$x_2=1+2\left(2\right)$ $\quad f\left(5\right)=\left(5\right)^2+1$
$x_2=1+4$ $\quad f\left(5\right)=25+1$
$x_2=5=b$ $\quad f\left(5\right)=26$

Since b is the end point of the given interval
Now using the definition of definite integral

$$\int\limits_{x=a}^{x=b}f\left(x\right)dx=\Delta x\left\{f\left(x_1\right)+f\left(x_2\right)\right\}$$

Substituting the values

$$\int\limits_{a=1}^{b=5}\left(x^2+1\right)dx=2\left\{10+26\right\}$$

$$\int\limits_{a=1}^{b=5}\left(x^2+1\right)dx=2\left\{36\right\}=72$$

Q3. In each case, determine actual value of integral using

$$\int\limits_{x=a}^{x=b}f\left(x\right)dx=\lim_{n\to\infty}\frac{b-a}{n}\left[f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+\cdots+f\left(x_n\right)\right]$$

a). $\int\limits_{x=0}^{x=3} 3x\,dx$

Sol: Given $f(x)=3x$, which is continuous

And $a=x_o=0, b=x_n=3$, subintervals are $[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n]$ Right end points of each subinterval $x_1,x_2,x_3,\ldots,x_n$

Width of each subinterval $\Delta x=\dfrac{3-0}{n}=\dfrac{3}{n}$

Since $x_i=x_o+i.\Delta x$

At $i=1$ corresponding value of function

$$x_1=x_o+1.\Delta x \qquad f(x_1)=3(x_1)$$
$$x_1=0+1\left(\frac{3}{n}\right) \qquad f\left(\frac{3}{n}\right)=3\left(\frac{3}{n}\right)$$
$$x_1=0+\frac{3}{n}=\frac{3}{n} \qquad f\left(\frac{3}{n}\right)=\frac{9}{n}$$

At $i=2$ corresponding value of function

$$x_2=x_o+2.\Delta x \qquad f(x_2)=3(x_2)$$
$$x_2=0+2\left(\frac{3}{n}\right) \qquad f\left(\frac{6}{n}\right)=3\left(\frac{6}{n}\right)$$
$$x_2=0+\frac{6}{n}=\frac{6}{n} \qquad f\left(\frac{6}{n}\right)=\frac{18}{n}$$

At $i=3$ corresponding value of function

$$x_3=x_o+3.\Delta x \qquad f(x_3)=3(x_3)$$
$$x_3=0+3\left(\frac{3}{n}\right) \qquad f\left(\frac{9}{n}\right)=3\left(\frac{9}{n}\right)$$
$$x_3=0+\frac{9}{n}=\frac{9}{n} \qquad f\left(\frac{9}{n}\right)=\frac{27}{n}$$

And so on

At $i=n$ corresponding value of function

$$x_n=x_o+n.\Delta x \qquad f(x_n)=3(x_n)$$
$$x_n=0+n\left(\frac{3}{n}\right) \qquad f\left(\frac{3n}{n}\right)=3\left(\frac{3n}{n}\right)$$
$$x_n=0+\frac{3n}{n}=\frac{3n}{n} \qquad f\left(\frac{3n}{n}\right)=\frac{9n}{n}$$
$$x_n=3=b$$

Since b is the end point of the given interval

Now using the definition of definite integral

$$\int\limits_{x=a}^{x=b} f(x)\,dx=\lim_{n\to\infty}\Delta x\left\{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\right\}$$

Substituting the values

$$\int\limits_{a=0}^{b=3}(3x)\,dx=\lim_{n\to\infty}\frac{3}{n}\left\{\frac{9}{n}+\frac{18}{n}+\frac{27}{n}+\cdots+\frac{9n}{n}\right\}$$

$$\int\limits_{a=0}^{b=3}(3x)\,dx=\lim_{n\to\infty}\frac{3\times 9}{n^2}\left\{1+2+3+\cdots+n\right\}$$

$$\int\limits_{a=0}^{b=3}(3x)\,dx=\lim_{n\to\infty}\frac{27}{n^2}\left\{\frac{n(n+1)}{2}\right\}=\lim_{n\to\infty}\frac{27}{n^2}\left\{\frac{n^2+n}{2}\right\}$$

$$\int\limits_{a=0}^{b=3}(3x)\,dx=\frac{27}{2}\lim_{n\to\infty}\left\{\frac{n^2+n}{n^2}\right\}=\frac{27}{2}\lim_{n\to\infty}\left\{\frac{n^2}{n^2}+\frac{n}{h^2}\right\}$$

$$\int\limits_{a=0}^{b=3}(3x)\,dx=\frac{27}{2}\lim_{n\to\infty}\left\{1+\frac{1}{n}\right\}=\frac{27}{2}\left\{1+\frac{1}{\infty}\right\}$$

$$\int\limits_{a=0}^{b=3}(3x)\,dx=\frac{27}{2}\{1+0\}=\frac{27}{2}=13.5$$

---

b). $\int\limits_{x=0}^{x=3}(2x-4)\,dx$

Sol: Given $f(x)=2x-4$, which is continuous

And $a=x_o=0, b=x_n=3$, subintervals are $[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n]$ Right end points of each subinterval $x_1,x_2,x_3,\ldots,x_n$ Width of each subinterval $\Delta x=\dfrac{3-0}{n}=\dfrac{3}{n}$ Since $x_i=x_o+i.\Delta x$

At $i=1$ corresponding value of function

$$x_1=x_o+1.\Delta x \qquad f(x_1)=2(x_1)-4$$
$$x_1=0+1\left(\frac{3}{n}\right) \qquad f\left(\frac{3}{n}\right)=2\left(\frac{3}{n}\right)-4$$
$$x_1=0+\frac{3}{n}=\frac{3}{n} \qquad f\left(\frac{3}{n}\right)=\frac{6}{n}-4$$

At $i=2$ corresponding value of function

$$x_2=x_o+2.\Delta x \qquad f(x_2)=2(x_2)-4$$
$$x_2=0+2\left(\frac{3}{n}\right) \qquad f\left(\frac{6}{n}\right)=2\left(\frac{6}{n}\right)-4$$
$$x_2=0+\frac{6}{n}=\frac{6}{n} \qquad f\left(\frac{6}{n}\right)=\frac{12}{n}-4$$

At $i=3$ corresponding value of function

$$x_3=x_o+3.\Delta x \qquad f(x_3)=2(x_3)-4$$
$$x_3=0+3\left(\frac{3}{n}\right) \qquad f\left(\frac{9}{n}\right)=2\left(\frac{9}{n}\right)-4$$
$$x_3=0+\frac{9}{n}=\frac{9}{n} \qquad f\left(\frac{9}{n}\right)=\frac{18}{n}-4$$

And so on

At $i=n$ corresponding value of function

$$x_n=x_o+n.\Delta x \qquad f(x_n)=2(x_n)-4$$
$$x_n=0+n\left(\frac{3}{n}\right) \qquad f\left(\frac{3n}{n}\right)=2\left(\frac{3n}{n}\right)-4$$
$$x_n=0+\frac{3n}{n}=\frac{3n}{n} \qquad f\left(\frac{3n}{n}\right)=\frac{6n}{n}-4$$
$$x_n=3=b$$

Since b is the end point of the given interval

Now using the definition of definite integral

$$\int\limits_{x=a}^{x=b} f(x)\,dx=\lim_{n\to\infty}\Delta x\left\{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\right\}$$

Substituting the values

$$\int\limits_{a=0}^{b=3}(2x-4)\,dx=\lim_{n\to\infty}\frac{3}{n}\left\{\frac{6}{n}-4+\frac{12}{n}-4+\frac{18}{n}-4+\cdots+\frac{6n}{n}-4\right\}$$

$$I=\lim_{n\to\infty}\frac{3}{n}\left\{\frac{6}{n}+\frac{12}{n}+\frac{18}{n}+\cdots+\frac{6n}{n}-4-4-4-\cdots-4\right\}$$

$$I=\lim_{n\to\infty}\frac{3}{n}\left[\frac{6}{n}\{1+2+3+\cdots+n\}-4\{1+1+1+\cdots+1\}\right]$$

$$\int_{a=0}^{b=3}(2x-4)\,dx=\lim_{n\to\infty}\frac{3}{n}\left[\frac{6}{n}\left\{\frac{n(n+1)}{2}\right\}-4n\right]$$

$$\int_{a=0}^{b=3}(2x-4)\,dx=\lim_{n\to\infty}\frac{3}{n}\left[\frac{6}{2}\left\{\frac{n(n+1)}{n}\right\}-4n\right]$$

$$I=\lim_{n\to\infty}\frac{3}{n}\left[3\{n+1\}-4n\right]=\lim_{n\to\infty}\frac{3}{n}\left[3n+3-4n\right]$$

$$\int_{a=0}^{b=3}(2x-4)\,dx=\lim_{n\to\infty}\frac{3}{n}\left[3-n\right]=\lim_{n\to\infty}3\left[\frac{3}{n}-\frac{n}{n}\right]$$

$$\int_{a=0}^{b=3}(2x-4)\,dx=\lim_{n\to\infty}3\left[\frac{3}{n}-1\right]=3\left[\frac{3}{\infty}-1\right]$$

$$\int_{a=0}^{b=3}(2x-4)\,dx=3\left[0-1\right]=-3$$

---

c). $\displaystyle\int_{x=0}^{x=2}x^2\;dx$

Sol: Given $f(x)=x^2$, which is continuous

And $a=x_o=0, b=x_n=2$, subintervals are

$[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n]$ Right end points of each subinterval $x_1,x_2,x_3,\ldots,x_n$ Width of each subinterval $\Delta x=\frac{2-0}{n}=\frac{2}{n}$ Since $x_i=x_o+i.\Delta x$

At $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$ $\qquad f(x_1)=(x_1)^2$

$x_1=0+1\left(\frac{2}{n}\right)$ $\qquad f\left(\frac{2}{n}\right)=\left(\frac{2}{n}\right)^2$

$x_1=0+\frac{2}{n}=\frac{2}{n}$ $\qquad f\left(\frac{2}{n}\right)=\frac{4}{n^2}$

At $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$ $\qquad f(x_2)=(x_2)^2$

$x_2=0+2\left(\frac{2}{n}\right)$ $\qquad f\left(\frac{4}{n}\right)=\left(\frac{4}{n}\right)^2$

$x_2=0+\frac{4}{n}=\frac{4}{n}$ $\qquad f\left(\frac{4}{n}\right)=\frac{16}{n^2}$

At $i=3$ corresponding value of function

$x_3=x_o+3.\Delta x$ $\qquad f(x_3)=(x_3)^2$

$x_3=0+3\left(\frac{2}{n}\right)$ $\qquad f\left(\frac{6}{n}\right)=\left(\frac{6}{n}\right)^2$

$x_3=0+\frac{6}{n}=\frac{6}{n}$ $\qquad f\left(\frac{6}{n}\right)=\frac{36}{n^2}$

And so on

At $i=n$ corresponding value of function

$x_n=x_o+n.\Delta x$ $\qquad f(x_n)=(x_n)^2$

$x_n=0+n\left(\frac{2}{n}\right)$ $\qquad f\left(\frac{2n}{n}\right)=\left(\frac{2n}{n}\right)^2$

$x_n=0+\frac{2n}{n}=\frac{2n}{n}$ $\qquad f\left(\frac{2n}{n}\right)=\frac{4n^2}{n^2}$

$x_n=2=b$

Since b is the end point of the given interval

Now using the definition of definite integral

$$\int_{x=a}^{x=b}f(x)\,dx=\lim_{n\to\infty}\Delta x\left\{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)\right\}$$

Substituting the values

$$\int_{a=0}^{b=2}x^2dx=\lim_{n\to\infty}\frac{2}{n}\left\{\frac{4}{n^2}+\frac{16}{n^2}+\frac{36}{n^2}+\cdots+\frac{4n^2}{n^2}\right\}$$

$$\int_{a=0}^{b=2}x^2dx=\lim_{n\to\infty}\frac{2\times4}{n\times n^2}\left\{1+4+9+\cdots+n^2\right\}$$

$$\int_{a=0}^{b=2}x^2dx=\lim_{n\to\infty}\frac{8}{n^3}\left\{1^2+2^2+3^2+\cdots+n^2\right\}$$

$$\int_{a=0}^{b=2}x^2dx=\lim_{n\to\infty}\frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6}\right\}$$

$$\int_{a=0}^{b=2}x^2dx=\lim_{n\to\infty}\frac{8}{6}\left\{\frac{n\left(2n^2+n+2n+1\right)}{n^3}\right\}$$

$$\int_{a=0}^{b=2}x^2dx=\frac{4}{3}\lim_{n\to\infty}\left\{\frac{2n^2+3n+1}{n^2}\right\}=\frac{4}{3}\lim_{n\to\infty}\left\{\frac{2n^2}{n^2}+\frac{3n}{n^2}+\frac{1}{n^2}\right\}$$

$$\int_{a=0}^{b=2}x^2dx=\frac{4}{3}\lim_{n\to\infty}\left\{2+\frac{3}{n}+\frac{1}{n^2}\right\}$$

$$\int_{a=0}^{b=2}x^2dx=\frac{4}{3}\left\{2+\frac{3}{\infty}+\frac{1}{\infty^2}\right\}=\frac{4}{3}\left\{2+0+0\right\}=\frac{8}{3}$$

---

d). $\displaystyle\int_{x=2}^{x=3}\left(x^2-4\right)dx$

Sol: Given $f(x)=x^2-4$, which is continuous

And $a=x_o=2, b=x_n=3$, subintervals are

$[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n]$

Right end points of each subinterval

$x_1,x_2,x_3,\ldots,x_n$

Width of each subinterval $\Delta x=\frac{3-2}{n}=\frac{1}{n}$

Since $x_i=x_o+i.\Delta x$

At $i=1$ corresponding value of function

$x_1=x_o+1.\Delta x$ $\qquad f(x_1)=(x_1)^2-4$

$x_1=2+1\left(\frac{1}{n}\right)$ $\qquad f\left(2+\frac{1}{n}\right)=\left(2+\frac{1}{n}\right)^2$

$x_1=2+\frac{1}{n}$ $\qquad f\left(\frac{2}{n}\right)=4+\frac{4}{n}+\frac{1}{n^2}-4$

$f\left(\frac{2}{n}\right)=\frac{4}{n}+\frac{1}{n^2}$

At $i=2$ corresponding value of function

$x_2=x_o+2.\Delta x$ $\qquad f(x_2)=(x_2)^2-4$

$x_2=2+2\left(\frac{1}{n}\right)$ $\qquad f\left(2+\frac{2}{n}\right)=\left(2+\frac{2}{n}\right)^2-4$

$x_2=2+\frac{2}{n}$ $\qquad f\left(\frac{4}{n}\right)=4+\frac{8}{n}+\frac{4}{n^2}-4$

$f\left(\frac{4}{n}\right)=\frac{8}{n}+\frac{4}{n^2}$

At $i=3$ corresponding value of function

$f(x_3)=(x_3)^2-4$

$x_3=x_o+3.\Delta x$ $\qquad f\left(2+\frac{3}{n}\right)=\left(2+\frac{3}{n}\right)^2-4$

$x_3=2+3\left(\frac{1}{n}\right)$ $\qquad f\left(2+\frac{3}{n}\right)=4+\frac{12}{n}+\frac{9}{n^2}-4$

$x_3=2+\frac{3}{n}$ $\qquad f\left(2+\frac{3}{n}\right)=\frac{12}{n}+\frac{9}{n^2}$

And so on

At $i = n$ corresponding value of function

$$x_n = x_o + n.\Delta x \qquad f(x_n) = (x_n)^2 - 4$$

$$x_n = 2 + n\left(\frac{1}{n}\right) \qquad f\left(2+\frac{n}{n}\right) = \left(2+\frac{n}{n}\right)^2 - 4$$

$$x_n = 2 + \frac{n}{n} \qquad f\left(2+\frac{n}{n}\right) = 4 + \frac{4n}{n} + \frac{n^2}{n^2} - 4$$

$$x_n = 2 + 1 = 3 = b \qquad f\left(2+\frac{n}{n}\right) = \frac{4n}{n} + \frac{n^2}{n^2}$$

Since b is the end point of the given interval

Now using the definition of definite integral

$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{n\to\infty} \Delta x\left\{ f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right\}$$

Substituting the values

$$\int_{a=0}^{b=3} \left(x^2-4\right)dx = \lim_{n\to\infty} \frac{1}{n}\left\{ \frac{4}{n} + \frac{1}{n^2} + \frac{8}{n} + \frac{4}{n^2} + \frac{12}{n} + \frac{9}{n^2} + \cdots + \frac{4n}{n} + \frac{n^2}{n^2} \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \lim_{n\to\infty} \frac{1}{n}\left\{ \begin{array}{l} \frac{4}{n} + \frac{8}{n} + \frac{12}{n} + \cdots + \frac{4n}{n} \\ + \frac{1}{n^2} + \frac{4}{n^2} + \frac{9}{n^2} + \cdots + \frac{n^2}{n^2} \end{array} \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \lim_{n\to\infty} \frac{1}{n}\left\{ \begin{array}{l} \frac{4}{n}(1+2+3+\cdots+n) \\ + \frac{1}{n^2}\left(1+4+9+\cdots+n^2\right) \end{array} \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \lim_{n\to\infty} \frac{1}{n}\left\{ \begin{array}{l} \frac{4}{n}\frac{n(n+1)}{2} \\ + \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6} \end{array} \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \lim_{n\to\infty} \left\{ \frac{4}{2}\left(\frac{n+1}{n}\right) + \frac{1}{6}\frac{n+1}{n}\frac{2n+1}{n} \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \lim_{n\to\infty} \left\{ 2\left(\frac{n}{n}+\frac{1}{n}\right) + \frac{1}{6}\left(\frac{n}{n}+\frac{1}{n}\right)\left(\frac{2n}{n}+\frac{1}{n}\right) \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \lim_{n\to\infty} \left\{ 2\left(1+\frac{1}{n}\right) + \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \left\{ 2\left(1+\frac{1}{\infty}\right) + \frac{1}{6}\left(1+\frac{1}{\infty}\right)\left(2+\frac{1}{\infty}\right) \right\}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = \left\{ 2(1+0) + \frac{1}{6}(1+0)(2+0) \right\} = 2 + \frac{2}{6}$$

$$\int_{a=2}^{b=3} \left(x^2-4\right)dx = 2 + \frac{1}{3} = \frac{3}{3}.\frac{2}{} + \frac{1}{3} = \frac{6+1}{3} = \frac{7}{3}$$

Linearity rule

$$\int_{x=a}^{x=b} \left[rf(x) + sg(x) \right]dx = r\int_{x=a}^{x=b} f(x)\,dx + s\int_{x=a}^{x=b} g(x)\,dx$$

Equality rule

If $f(x) = g(y)$ then $\int_{x=a}^{x=b} f(x)\,dx = \int_{x=a}^{x=b} g(y)\,dy$

Subdivision rule

If $c \in [a,b]$ or $a \le c \le b$ then

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{x=a}^{x=c} f(x)\,dx + \int_{x=c}^{x=b} f(x)\,dx$$

Dominance rule

If $f(x) \le g(y)$ then $\int_{x=a}^{x=b} f(x)\,dx \le \int_{x=a}^{x=b} g(y)\,dy$

Point Rule

If $a = b$ then $\int_{x=a}^{x=a} f(x)\,dx = 0$

Opposite rule If $b \le a$

$$\int_{x=a}^{x=b} f(x)\,dx = -\int_{x=b}^{x=a} f(x)\,dx$$

Even and Odd rule

If $x \in [-a,a]$ and $-a \le 0 \le a$

Then

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(-x)\,d(-x) + \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = +\int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

When

$f(-x) = f(x)$ *for even*

$f(-x) = -f(x)$ *for odd*

Definite integral

$$\int_{x=a}^{x=b} f(x)\,dx = \left|F(x)\right|_{x=a}^{x=b} = F(b) - F(a)$$

Exercise 5.5

Q1. Evaluate the following definite integrals:

a). $\int_{3}^{4} 5\,dx$

Sol: Given $\int_{3}^{4} 5\,dx = 5\int_{3}^{4} dx$

$$\int_{3}^{4} 5\,dx = 5\left|x\right|_{3}^{4} = 5[4-3] = 5[1] = 5$$

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b). $\int_{12}^{20} dx$

Sol: Given $\int_{12}^{20} dx = \left|x\right|_{12}^{20}$

$$\int_{12}^{20} dx = 20 - 12 = 8$$

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c). $\int_{1}^{2} \left(2x^{-2} - 3\right)dx$

Sol: Given $\int_{1}^{2} \left(2x^{-2} - 3\right)dx$

$$\int_{1}^{2}\left(2x^{-2}-3\right)dx=2\int_{1}^{2}x^{-2}dx-3\int_{1}^{2}dx$$

$$\int_{1}^{2}\left(2x^{-2}-3\right)dx=2\left|\frac{x^{-2+1}}{-2+1}\right|_{1}^{2}-3|x|_{1}^{2}$$

$$\int_{1}^{2}\left(2x^{-2}-3\right)dx=2\left|\frac{x^{-1}}{-1}\right|_{1}^{2}-3[2-1]$$

$$\int_{1}^{2}\left(2x^{-2}-3\right)dx=-2\left|\frac{1}{x}\right|_{1}^{2}-3[1]=-2\left[\frac{1}{2}-\frac{1}{1}\right]-3$$

$$\int_{1}^{2}\left(2x^{-2}-3\right)dx=-2\left[\frac{1}{2}-\frac{1}{1}.\frac{2}{2}\right]-3=-2\left[\frac{1-2}{2}\right]-3$$

$$\int_{1}^{2}\left(2x^{-2}-3\right)dx=-[-1]-3=1-3=-2$$

---

d). $\int_{1}^{4}3\sqrt{x}\ dx$

Sol: Given $\int_{1}^{4}3\sqrt{x}\ dx$

$$\int_{1}^{4}3\sqrt{x}\ dx=3\int_{1}^{4}x^{\frac{1}{2}}\,dx$$

$$\int_{1}^{4}3\sqrt{x}\ dx=3\left|\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{1}^{4}=3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_{1}^{4}$$

$$\int_{1}^{4}3\sqrt{x}\ dx=2\left[x^{\frac{3}{2}}\right]_{1}^{4}=2\left[4^{\frac{3}{2}}-1^{\frac{3}{2}}\right]$$

$$\int_{1}^{4}3\sqrt{x}\ dx=2\left[2^{2\times\frac{3}{2}}-1\right]=2\left[2^{3}-1\right]$$

$$\int_{1}^{4}3\sqrt{x}\ dx=2[8-1]=2[7]=14$$

---

e). $\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx$

Sol: Given $\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx$

$$\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx=6\int_{2}^{3}\left(x^{2}-4\right)^{5}2x\ dx$$

$$\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx=6\left|\frac{\left(x^{2}-4\right)^{5+1}}{5+1}\right|_{2}^{3}$$

$$\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx=6\left|\frac{\left(x^{2}-4\right)^{6}}{6}\right|_{2}^{3}$$

$$\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx=\left|\left(x^{2}-4\right)^{6}\right|_{2}^{3}$$

$$\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx=\left[\left(3^{2}-4\right)^{6}-\left(2^{2}-4\right)^{6}\right]$$

$$\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx=\left[(9-4)^{6}-(4-4)^{6}\right]$$

$$\int_{2}^{3}12\left(x^{2}-4\right)^{5}x\ \ dx=\left[(5)^{6}-(0)^{6}\right]=15625$$

---

f). $\int_{-6}^{0}\sqrt{4-2x}\ dx$

Sol: Given $\int_{-6}^{0}\sqrt{4-2x}\ dx$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{2}\int_{-6}^{0}(4-2x)^{\frac{1}{2}}(-2dx)$$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{2}\left|\frac{(4-2x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-6}^{0}$$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{2}\left|\frac{(4-2x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-6}^{0}=\frac{-1}{3}\left|(4-2x)^{\frac{3}{2}}\right|_{-6}^{0}$$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{3}\left[(4-2(0))^{\frac{3}{2}}-(4-2(-6))^{\frac{3}{2}}\right]$$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{3}\left[(4)^{\frac{3}{2}}-(4+12)^{\frac{3}{2}}\right]$$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{3}\left[\left(2^{2}\right)^{\frac{3}{2}}-\left(4^{2}\right)^{\frac{3}{2}}\right]$$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{3}\left[2^{3}-4^{3}\right]$$

$$\int_{-6}^{0}\sqrt{4-2x}\ dx=\frac{-1}{3}[8-64]=\frac{-1}{3}[-56]=\frac{56}{3}$$

---

g). $\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx$

Sol: Given $\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\int_{-1}^{7}\frac{x+2-2}{\sqrt{x+2}}dx$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\int_{-1}^{7}\left\{\frac{x+2}{\sqrt{x+2}}-\frac{2}{\sqrt{x+2}}\right\}dx$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\int_{-1}^{7}\left\{\sqrt{x+2}-\frac{2}{\sqrt{x+2}}\right\}dx$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\int_{-1}^{7}(x+2)^{\frac{1}{2}}dx-2\int_{-1}^{7}(x+2)^{\frac{-1}{2}}dx$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\left|\frac{(x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{7}-2\left|\frac{(x+2)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right|_{-1}^{7}$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\left[\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{7}-2\left|\frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}}\right|_{-1}^{7}$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\frac{2}{3}\left[(x+2)^{\frac{3}{2}}\right]_{-1}^{7}-4\left|\sqrt{x+2}\right|_{-1}^{7}$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\frac{2}{3}\left[(7+2)^{\frac{3}{2}}-(-1+2)^{\frac{3}{2}}\right]-4\left[\sqrt{7+2}-\sqrt{-1+2}\right]$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\frac{2}{3}\left[(9)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right]-4\left[\sqrt{9}-\sqrt{1}\right]$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\frac{2}{3}\left[\left(3^{2}\right)^{\frac{3}{2}}-1\right]-4[3-1]=\frac{2}{3}\left[3^{3}-1\right]-8$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\frac{2}{3}[27-1]-8=\frac{2}{3}[26]-8=\frac{52}{3}-8.\frac{3}{3}$$

$$\int_{-1}^{7}\frac{x}{\sqrt{x+2}}dx=\frac{52-24}{3}=\frac{28}{3}=9.\bar{3}$$

---

h). $\int_{0}^{1}\left(e^{2x}-2x\right)^{2}\left(e^{2x}-1\right)dx$

Sol: Given $\int_{0}^{1}\left(e^{2x}-2x\right)^{2}\left(e^{2x}-1\right)dx$

Let $u=e^{2x}-2x$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}e^{2x}-2\tfrac{d}{dx}x$$

$$\tfrac{d}{dx}u=e^{2x}\tfrac{d}{dx}(2x)-2(1)$$

$$\tfrac{d}{dx}u=2e^{2x}\tfrac{d}{dx}x-2$$

$$\tfrac{d}{dx}u=2e^{2x}-2$$

$$\frac{du}{dx}=2\left(e^{x}-1\right)$$

$$\frac{du}{2}=\left(e^{x}-1\right)dx$$

Substituting values of u and du in given integral

$$\int_{0}^{1}\left(e^{2x}-2x\right)^{2}\left(e^{2x}-1\right)dx=\int_{0}^{1}u^{2}\frac{du}{2}$$

$$\int_{0}^{1}\left(e^{2x}-2x\right)^{2}\left(e^{2x}-1\right)dx=\frac{1}{2}\int_{0}^{1}u^{2}\,du$$

$$\int_{0}^{1}\left(e^{2x}-2x\right)^{2}\left(e^{2x}-1\right)dx=\frac{1}{2}\left|\frac{u^{3}}{3}\right|_{0}^{1}=\frac{1}{6}\left|u^{3}\right|_{0}^{1}$$

Putting the value of u

$$\int_{0}^{1}\left(e^{2x}-2x\right)^{2}\left(e^{2x}-1\right)dx=\frac{1}{6}\left|\left(e^{2x}-2x\right)^{3}\right|_{0}^{1}$$

$$I=\frac{1}{6}\left[\left(e^{2(1)}-2(1)\right)^{3}-\left(e^{2(0)}-2(0)\right)^{3}\right]$$

$$I=\frac{1}{6}\left[\left(e^{2}-2\right)^{3}-(1-0)^{3}\right]$$

$$I=\frac{1}{6}\left[\left(e^{2}-2\right)^{3}-1\right]$$

$$I=25.91809$$

Q2. Evaluate the following definite integrals:

a). $\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx$

Sol: Given $\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx$

Let $u=2x^{2}-3$ Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=2\tfrac{d}{dx}x^{2}-\tfrac{d}{dx}3$$

$$\tfrac{d}{dx}u=2(2x)\tfrac{d}{dx}x-0$$

$$\tfrac{du}{dx}=4x$$

$$\tfrac{du}{4}=x\,dx$$

Substituting values of u and du in given integral

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\int_{2}^{3}\sqrt{2x^{2}-3}\,x\,dx$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\int_{2}^{3}\sqrt{u}\,\frac{du}{4}=\frac{1}{4}\int_{2}^{3}u^{\frac{1}{2}}du$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\frac{1}{4}\left|\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{2}^{3}$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\frac{1}{4}\left|\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_{2}^{3}=\frac{1}{2\times3}\left|u^{\frac{3}{2}}\right|_{2}^{3}$$

Putting the value of u

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\frac{1}{6}\left|\left(2x^{2}-3\right)^{\frac{3}{2}}\right|_{2}^{3}$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\frac{1}{6}\left[\left(2(3)^{2}-3\right)^{\frac{3}{2}}-\left(2(2)^{2}-3\right)^{\frac{3}{2}}\right]$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\frac{1}{6}\left[\left(2(9)-3\right)^{\frac{3}{2}}-\left(2(4)-3\right)^{\frac{3}{2}}\right]$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\frac{1}{6}\left[(18-3)^{\frac{3}{2}}-(8-3)^{\frac{3}{2}}\right]$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=\frac{1}{6}\left[(15)^{\frac{3}{2}}-(5)^{\frac{3}{2}}\right]$$

$$\int_{2}^{3}x\sqrt{2x^{2}-3}\;dx=7.819068$$

---

b). $\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx$

Sol: Given $\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx$

Let $u=3x^{2}+2$

Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=3\tfrac{d}{dx}x^{2}+\tfrac{d}{dx}2$$

$$\tfrac{d}{dx}u=3(2x)\tfrac{d}{dx}x+0$$

$$\frac{du}{dx}=6x \qquad \Rightarrow\frac{du}{6}=x\,dx$$

Substituting values of u and du in given integral

$$\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx=\int_{0}^{1}\sqrt{3x^{2}+2}\,x\,dx$$

$$\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx=\int_{0}^{1}\sqrt{u}\,\frac{du}{6}=\frac{1}{6}\int_{0}^{1}u^{\frac{1}{2}}du$$

$$\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx=\frac{1}{6}\left|\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{0}^{1}$$

$$\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx=\frac{1}{6}\left|\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_{0}^{1}=\frac{1}{3\times3}\left|u^{\frac{3}{2}}\right|_{0}^{1}$$

Putting the value of u

$$\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx=\frac{1}{9}\left|\left(3x^{2}+2\right)^{\frac{3}{2}}\right|_{0}^{1}$$

$$\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx=\frac{1}{9}\left[\left(3(1)^{2}+2\right)^{\frac{3}{2}}-\left(3(0)^{2}+2\right)^{\frac{3}{2}}\right]$$

$$\int_{0}^{1}x\sqrt{3x^{2}+2}\;dx=\frac{1}{9}\left[(3+2)^{\frac{3}{2}}-(0+2)^{\frac{3}{2}}\right]$$

$$\int_0^1 x\sqrt{3x^2+2}\ dx=\frac{1}{9}\left[(5)^{\frac{3}{2}}-(2)^{\frac{3}{2}}\right]$$

$$\int_0^1 x\sqrt{3x^2+2}\ dx=0.92799$$

---

c). $\int_0^1 \frac{x-1}{x^2-2x+3}dx$

Sol: Given $\int_0^1 \frac{x-1}{x^2-2x+3}dx$

Let $u=x^2-2x+3$, Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}x^2-2\tfrac{d}{dx}x+\tfrac{d}{dx}3$$

$$\tfrac{d}{dx}u=(2x)\tfrac{d}{dx}x-2+0$$

$$\tfrac{d}{dx}u=2x-2$$

$$\frac{du}{dx}=2(x-1)$$

$$\frac{du}{2}=(x-1)dx$$

Substituting values of u and du in given integral

$$\int_0^1 \frac{x-1}{x^2-2x+3}dx=\int_0^1\frac{1}{u}\frac{du}{2}$$

$$\int_0^1 \frac{x-1}{x^2-2x+3}dx=\frac{1}{2}\int_0^1\frac{1}{u}du$$

$$\int_0^1 \frac{x-1}{x^2-2x+3}dx=\frac{1}{2}\left|\ln u\right|_0^1$$

Putting the value of u

$$\int_0^1 \frac{x-1}{x^2-2x+3}dx=\frac{1}{2}\left|\ln\left(x^2-2x+3\right)\right|_0^1$$

$$I=\frac{1}{2}\left[\ln\left(1^2-2(1)+3\right)-\ln\left(0^2-2(0)+3\right)\right]_0^1$$

$$I=\frac{1}{2}\left[\ln(1-2+3)-\ln(0-0+3)\right]$$

$$\int_0^1 \frac{x-1}{x^2-2x+3}dx=\frac{1}{2}[\ln 2-\ln 3]=\frac{1}{2}\ln\left[\frac{2}{3}\right]$$

$$\int_0^1 \frac{x-1}{x^2-2x+3}dx=-0.2027$$

---

d). $\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx$

Sol: Given $\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx$

Let $u=e^{-x}+e^x$, Differentiating w. r. t "x"

$$\tfrac{d}{dx}u=\tfrac{d}{dx}e^{-x}+\tfrac{d}{dx}e^x$$

$$\tfrac{d}{dx}u=e^{-x}\tfrac{d}{dx}(-x)+e^x\tfrac{d}{dx}x$$

$$\tfrac{d}{dx}u=-e^{-x}+e^x$$

$$\frac{du}{dx}=-\left(e^{-x}-e^x\right)$$

$$-du=\left(e^{-x}-e^x\right)dx$$

Substituting values of u and du in given integral

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=\int_{-1}^1\frac{-du}{u^2}$$

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=-\int_{-1}^1 u^{-2}du$$

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=-\left|\frac{u^{-2+1}}{-2+1}\right|_{-1}^1=-\left|\frac{u^{-1}}{-1}\right|_{-1}^1$$

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=\left|\frac{1}{u}\right|_{-1}^1$$

Putting the value of u

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=\left|\frac{1}{e^{-x}+e^x}\right|_{-1}^1$$

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=\left[\frac{1}{e^{-1}+e^1}-\frac{1}{e^{-(-1)}+e^{(-1)}}\right]$$

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=\left[\frac{1}{e^{-1}+e^1}-\frac{1}{e^1+e^{-1}}\right]$$

$$\int_{-1}^1 \frac{e^{-x}-e^x}{\left(e^{-x}+e^x\right)^2}dx=0$$

Q3. Evaluate the following definite integrals:

a). $\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx$

Sol: Given $\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx$

Let $u=\frac{x}{2}+\pi$, Differentiating w. r. t "x"

$$\frac{d}{dx}u=\frac{1}{2}\frac{d}{dx}x+\frac{d}{dx}\pi$$

$$\frac{du}{dx}=\frac{1}{2}+0$$

$$2du=dx$$

Substituting values of u and du in given integral

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=\int_{\frac{\pi}{2}}^{\pi}\cos(u)2du$$

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=2\int_{\frac{\pi}{2}}^{\pi}\cos(u)du$$

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=2\left|\sin u\right|_{\frac{\pi}{2}}^{\pi}$$

Putting the value of u

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=2\left|\sin\left(\frac{x}{2}+\pi\right)\right|_{\frac{\pi}{2}}^{\pi}$$

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=2\left[\sin\left(\frac{\pi}{2}+\pi\right)-\sin\left(\frac{\frac{\pi}{2}}{2}+\pi\right)\right]$$

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=2\left[\sin\left(\frac{3\pi}{2}\right)-\sin\left(\frac{5\pi}{4}\right)\right]$$

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=2\left[-1-\left(\frac{-\sqrt{2}}{2}\right)\right]$$

$$\int_{\frac{\pi}{2}}^{\pi}\cos\left(\frac{x}{2}+\pi\right)dx=2\left[\frac{-2+\sqrt{2}}{2}\right]=-2+\sqrt{2}$$

---

b). $\int_{0.75}^{2.5}x\cos x^2\ dx$

Sol: Given $\int_{0.75}^{2.5}x\cos x^2\ dx$

Let $u=x^2$ Differentiating w. r. t "x"

$\frac{d}{dx}u=\frac{d}{dx}x^2$

$\frac{d}{dx}u=2x\frac{d}{dx}x$

$\frac{du}{dx}=2x \qquad \Rightarrow \frac{du}{2}=x\,dx$

Substituting values of u and du in given integral

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\int_{0.75}^{2.5}\cos x^2\,x\,dx$$

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\int_{0.75}^{2.5}\cos u\ \frac{du}{2}$$

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\frac{1}{2}\int_{0.75}^{2.5}\cos u\,du$$

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\frac{1}{2}\left|\sin u\right|_{0.75}^{2.5}$$

Putting the value of u

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\frac{1}{2}\left|\sin x^2\right|_{0.75}^{2.5}$$

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\frac{1}{2}\left[\sin(2.5)^2-\sin(0.75)^2\right]$$

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\frac{1}{2}\left[\sin(6.25)-\sin(0.5625)\right]$$

$$\int_{0.75}^{2.5}x\cos x^2\ dx=\frac{1}{2}\left[-0.0332-0.5333\right]=-0.28325$$

---

c). $\int_{0}^{\frac{\pi}{4}}\sec^2\theta\ d\theta$

Sol: Given $\int_{0}^{\frac{\pi}{4}}\sec^2\theta\ d\theta$

Let $u=\tan\theta$ Differentiating w r t θ

$\frac{d}{d\theta}u=\frac{d}{d\theta}\tan\theta$

$\frac{du}{d\theta}=\sec^2\theta$

$du=\sec^2\theta\,d\theta$

Substituting values of u and du in given integral

$$\int_{0}^{\frac{\pi}{4}}\sec^2\theta\ d\theta=\int_{0}^{\frac{\pi}{4}}du$$

$$\int_{0}^{\frac{\pi}{4}}\sec^2\theta\ d\theta=\left|u\right|_{0}^{\frac{\pi}{4}}$$

Putting the value of u

$$\int_{0}^{\frac{\pi}{4}}\sec^2\theta\ d\theta=\left|\tan\theta\right|_{0}^{\frac{\pi}{4}}$$

$$\int_{0}^{\frac{\pi}{4}}\sec^2\theta\ d\theta=\left[\tan\tfrac{\pi}{4}-\tan 0\right]$$

$$\int_{0}^{\frac{\pi}{4}}\sec^2\theta\ d\theta=\left[1-0\right]=1$$

---

d). $\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx$

Sol: Given $\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx$

Let $u=2\pi x$ Differentiating w. r. t "x"

$\frac{d}{dx}u=2\pi\frac{d}{dx}x$

$\frac{du}{dx}=2\pi$

$\frac{du}{2\pi}=dx$

Substituting values of u and du in given integral

$$\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx=\int_{0}^{\frac{\pi}{4}}\tan u\ \frac{du}{2\pi}$$

$$\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx=\frac{-1}{2\pi}\int_{0}^{\frac{\pi}{4}}\frac{-\sin u}{\cos u}du$$

$$\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx=\frac{-1}{2\pi}\left|\ln\cos u\right|_{0}^{\frac{\pi}{4}}$$

$$\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx=\frac{-1}{2\pi}\left|\ln\cos u\right|_{0}^{\frac{\pi}{4}}$$

Putting the value of u

$$\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx=\frac{-1}{2\pi}\left|\ln\cos(2\pi x)\right|_{0}^{\frac{\pi}{4}}$$

$$I=\frac{-1}{2\pi}\left[\ln\cos\left\{2\pi\left(\frac{\pi}{4}\right)\right\}-\ln\cos\left\{2\pi(0)\right\}\right]$$

$$\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx=\frac{-1}{2\pi}\left[\ln\cos\left\{\frac{\pi^2}{2}\right\}-\ln\cos\left\{0\right\}\right]$$

$$\int_{0}^{\frac{\pi}{4}}\tan 2\pi x\ dx=\frac{-1}{2\pi}\left[\ln\cos\left\{\frac{\pi^2}{2}\right\}-\ln(1)\right]$$

e). $\int_0^1 \frac{1}{x^2+1}\,dx$

Sol: Given $\int_0^1 \frac{1}{x^2+1}\,dx$

Let $x=\tan\theta$ Differentiating w r t θ

$\frac{d}{d\theta}x=\frac{d}{d\theta}\tan\theta$

$\frac{d}{d\theta}x=\sec^2\theta\frac{d}{d\theta}\theta$

$\frac{dx}{d\theta}=\sec^2\theta \qquad \Rightarrow dx=\sec^2\theta\ d\theta$

Substituting values of x and dx in given integral

$$\int_0^1 \frac{1}{x^2+1}\,dx=\int_0^1\frac{\sec^2\theta d\theta}{\tan^2\theta+1} \qquad \therefore \tan\theta=x$$

$$\int_0^1 \frac{1}{x^2+1}\,dx=\int_0^1\frac{\sec^2\theta d\theta}{\sec^2\theta} \qquad \theta=\tan^{-1}(x)$$

$$\int_0^1 \frac{1}{x^2+1}\,dx=\int_0^1 d\theta=|\theta|_0^1$$

Putting the value of θ

$$\int_0^1 \frac{1}{x^2+1}\,dx=\left|\tan^{-1}(x)\right|_0^1$$

$$\int_0^1 \frac{1}{x^2+1}\,dx=\tan^{-1}(1)-\tan^{-1}(0)$$

$$\int_0^1 \frac{1}{x^2+1}\,dx=\frac{\pi}{4}-0=\frac{\pi}{4}$$

f). $\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx$

Sol: Given $\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx$

Let $x=\sec\theta$,Differentiating with respect to θ

$\frac{d}{d\theta}x=\frac{d}{d\theta}\sec\theta$

$\frac{d}{d\theta}x=\sec\theta\tan\theta\frac{d}{d\theta}\theta$

$\frac{dx}{d\theta}=\sec\theta\tan\theta \qquad \Rightarrow dx=\sec\theta\tan\theta\, d\theta$

Substituting values of x and dx in given integral

$$\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx=\int_{\sqrt2}^2\frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\sqrt{\sec^2\theta-1}}$$

$$\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx=\int_{\sqrt2}^2\frac{\tan\theta d\theta}{\sqrt{\tan^2\theta}} \qquad \therefore \sec\theta=x$$

$$\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx=\int_{\sqrt2}^2\frac{\tan\theta d\theta}{\tan\theta} \qquad \theta=\sec^{-1}(x)$$

$$\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx=\int_{\sqrt2}^2 d\theta=|\theta|_{\sqrt2}^2$$

Putting the value of θ

$$\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx=\left|\sec^{-1}(x)\right|_{\sqrt2}^2$$

$$\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx=\sec^{-1}(2)-\sec^{-1}\left(\sqrt2\right)$$

$$\int_{\sqrt2}^2 \frac{1}{x\sqrt{x^2-1}}\,dx=\frac{\pi}{3}-\frac{\pi}{4}=\frac{4\pi-3\pi}{12}=\frac{\pi}{12}$$

Q4. Evaluate the following definite integrals:

a). $\int_1^2 \frac{5t^2-3t+18}{t\left(9-t^2\right)}\,dt$

Sol: Given $\int_1^2 \frac{5t^2-3t+18}{t\left(9-t^2\right)}\,dt$

Take a fraction only $\frac{5t^2-3t+18}{t\left(9-t^2\right)}=\frac{5t^2-3t+18}{t(3-t)(3+t)}$

fraction is proper and has three linear factors

$$\frac{5t^2-3t+18}{t(3-t)(3+t)}=\frac{A}{t}+\frac{B}{3-t}+\frac{C}{3+t}..............(1)$$

Multiply each term by $t\left(9-t^2\right)$ we have

$$5t^2-3t+18=A\left(9-t^2\right)+Bt(3+t)+Ct(3-t)........(2)$$

Put $t=0$ in equation (2) we get

$$5(0)^2-3(0)+18=A\left(9-0^2\right)+B(0)(3+0)+C(0)(3-0)$$

$$5(0)-0+18=A(9-0)+0+0$$

$$0-0+18=9A$$

$$9A=18 \qquad \Rightarrow A=2$$

Put $t=3$ in equation (2) we get

$$5(3)^2-3(3)+18=A\left(9-3^2\right)+B(3)(3+3)+C(3)(3-3)$$

$$5(9)-9+18=A(9-9)+3B(6)+3C(0)$$

$$45+9=A(0)+18B+0$$

$$54=18B \qquad \Rightarrow B=\tfrac{54}{18}=3$$

Put $t=-3$ in equation (2) we get

$$5(-3)^2-3(-3)+18=A\left(9-(-3)^2\right)+B(-3)(3+(-3))$$
$$+C(-3)(3-(-3))$$

$$5(9)+9+18=A(9-9)-3B(3-3)-3C(3+3)$$

$$45+27=A(0)-3B(0)-3C(6)$$

$$72=-18C \qquad \Rightarrow C=-4$$

Putting values of A,B and C in equation (1) we get

$$\frac{5t^2-3t+18}{t\left(9-t^2\right)}=\frac{2}{t}+\frac{3}{3-t}-\frac{4}{3+t}$$

Substituting the value in the given integral

$$\int_1^2\frac{5t^2-3t+18}{t\left(9-t^2\right)}\,dt=\int_1^2\left(\frac{2}{t}+\frac{3}{3-t}-\frac{4}{3+t}\right)dt$$

$$\int_1^2\frac{5t^2-3t+18}{t\left(9-t^2\right)}\,dt=2\int_1^2\frac{1}{t}dt+3\int_1^2\frac{1}{3-t}\,dt-4\int_1^2\frac{1}{3+t}dt$$

$$\int_1^2\frac{5t^2-3t+18}{t\left(9-t^2\right)}\,dt=\int_1^2\left(\frac{2}{t}-3\frac{-1}{3-t}-4\frac{1}{3+t}\right)dt$$

$$\int_1^2\frac{5t^2-3t+18}{t\left(9-t^2\right)}\,dt=\left|2\ln t-3\ln(3-t)-4\ln(3+t)\right|_1^2$$

$$I=\ln\left(\frac{4}{625}\right)-\ln\left(\frac{1}{8\times256}\right)$$

$$I=2.57316$$

b). $\int_2^3\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)\left(x^2-1\right)}dx$

Sol: Given $\int_2^3\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)\left(x^2-1\right)}dx$

Take fraction which are given in the integral

$$\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)\left(x^2-1\right)}=\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{x^3-x-x^2+1}$$

This is improper so we have to divide it

$$\begin{array}{r|l}
 & 4x^2+x-1 \\ \hline
x^3-x^2-x+1 & 4x^5-3x^4-6x^3+4x^2+6x-1 \\
 & \underline{\pm4x^5\mp4x^4\mp4x^3\pm4x^2} \\
 & x^4-2x^3 \qquad +6x-1 \\
 & \underline{\pm x^4\mp\ x^3\mp x^2\pm x} \\
 & -x^3+x^2+5x-1 \\
 & \underline{\mp x^3\pm x^2\pm\ x\mp1} \\
 & 4x
\end{array}$$

Therefore the given improper fraction can be written as the sum of polynomials and proper fraction

$$\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)(x^2-1)}$$

$$=4x^2+x-1+\frac{4x}{(x-1)(x^2-1)}\text{............}(1)$$

Take $\frac{4x}{(x^2-1)(x-1)}$ which is proper, so we decompose into partial fractions according to factors of denominator

$$\frac{4x}{(x^2-1)(x-1)}=\frac{4x}{(x^2-1^2)(x-1)}$$

$$\frac{4x}{(x+1)(x-1)(x-1)}=\frac{4x}{(x+1)(x-1)^2}$$

$$\frac{4x}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\text{...........}(2)$$

Multiply each term by $(x+1)(x-1)^2$ we have

$$4x=A(x-1)^2+B(x+1)(x-1)+C(x+1)\text{.......}(3)$$

Put $x=1$ in equation (3) we get

$$4(1)=A(1-1)^2+B(1+1)(1-1)+C(1+1)$$

$$4=A(0)^2+B(2)(0)+C(2)$$

$$4=0+0+2C$$

$$2C=4$$

$$C=\frac{4}{2}=2$$

Put $x=-1$ in equation (3) we get

$$4(-1)=A(-1-1)^2+B(-1+1)(-1-1)+C(-1+1)$$

$$-4=A(-2)^2+B(0)(-2)+C(0)$$

$$-4=4A$$

$$A=\frac{-4}{4}=-1$$

Choose $x=0$ & putting value of A and C in eq (3)

$$4(0)=-1(0-1)^2+B(0+1)(0-1)+2(0+1)$$

$$0=-1(1)^2+B(1)(-1)+2(1)$$

$$0=-1-B+2$$

$$B=-1+2=1$$

Putting values of A,B and C in equation (2) we get

$$\frac{4x}{(x+1)(x-1)^2}=\frac{-1}{x+1}+\frac{1}{x-1}+\frac{2}{(x-1)^2}$$

Therefore equation (1) becomes

$$\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)(x^2-1)}$$

$$=4x^2+x-1-\frac{1}{x+1}+\frac{1}{x-1}+\frac{2}{(x-1)^2}$$

Integrating both sides according to their limits

$$\int_2^3\left(\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)(x^2-1)}\right)dx$$

$$=\int_2^3\left(4x^2+x-1-\frac{1}{x+1}+\frac{1}{x-1}+\frac{2}{(x-1)^2}\right)dx$$

$$I=\left|\frac{4x^3}{3}+\frac{x^2}{2}-x-\ln(x+1)+\ln(x-1)+2\frac{(x-1)^{-2+1}}{-2+1}\right|_2^3$$

$$I=\left|\frac{4x^3}{3}+\frac{x^2}{2}-x-\frac{2}{x-1}+\ln\left(\frac{x-1}{x+1}\right)\right|_2^3$$

$$I=\left[\frac{4\times3^3}{3}+\frac{3^2}{2}-3-\frac{2}{3-1}+\ln\left(\frac{3-1}{3+1}\right)\right]$$

$$-\left[\frac{4\times2^3}{3}+\frac{2^2}{2}-2-\frac{2}{2-1}+\ln\left(\frac{2-1}{2+1}\right)\right]$$

$$I=\left[36+\frac{9}{2}-3-1+\ln\left(\frac{1}{2}\right)\right]-\left[\frac{32}{3}+2-2-2+\ln\left(\frac{1}{3}\right)\right]$$

$$I=28.23879844$$

---

c). $\int_3^5\frac{x^2-2}{(x-2)^2}\,dx$

Sol: Given $\int_3^5\frac{x^2-2}{(x-2)^2}\,dx$

Take fraction which are given in the integral

$$\frac{x^2-2}{(x-2)^2}=\frac{x^2-2}{x^2-4x+4}$$

$$\frac{x^2-2}{(x-2)^2}=\frac{x^2-4x+4+4x-4-2}{x^2-4x+4}$$

$$\frac{x^2-2}{(x-2)^2}=\frac{x^2-4x+4}{x^2-4x+4}+\frac{4x-6}{x^2-4x+4}$$

$$\frac{x^2-2}{(x-2)^2}=1+\frac{4x-6}{(x-2)^2}\text{..................}(1)$$

Take $\frac{4x-6}{(x-2)^2}$ which is proper, so we decompose into partial fractions according to factors of denominator

$$\frac{4x-6}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}\text{...........}(2)$$

Multiply each term by $(x-2)^2$ we have

$$4x-6=A(x-2)+B\text{....................}(3)$$

Put $x=2$ in equation (3) we get

$$4(2)-6=A(2-2)+B$$

$$8-6=A(0)+B$$

$$B=2$$

Choose $x=0$ & putting value of B in eq (3)

$$4(0)-6=A(0-2)+2$$

$$0-6=-2A+2$$

$$-2-6=-2A$$

$$-2A=-8$$

$$A=\frac{-8}{-2}=4$$

Putting the values of A and B in equation (2) we get

$$\frac{4x-6}{(x-2)^2}=\frac{4}{x-2}+\frac{2}{(x-2)^2}$$

Therefore equation (1) becomes

$$\frac{x^2-2}{(x-2)^2}=1+\frac{4x-6}{(x-2)^2}$$

$$\frac{x^2-2}{(x-2)^2}=1+\frac{4}{x-2}+\frac{2}{(x-2)^2}$$

Integrating both sides according to their limits

$$\frac{x^2-2}{(x-2)^2}=1+\frac{4}{x-2}+\frac{2}{(x-2)^2}$$

$$\int_3^5\frac{x^2-2}{(x-2)^2}\ dx=\int_3^5 dx+4\int_3^5\frac{1}{x-2}\ dx+2\int_3^5(x-2)^{-2}\ dx$$

$$\int_3^5\frac{x^2-2}{(x-2)^2}\ dx=|x|_3^5+4|\ln(x-2)|_3^5+2\left|\frac{(x-2)^{-2+1}}{-2+1}\right|_3^5$$

$$\int_3^5\frac{x^2-2}{(x-2)^2}\ dx=|x|_3^5+4|\ln(x-2)|_3^5+2\left|\frac{(x-2)^{-1}}{-1}\right|_3^5$$

$$I=[5-3]+4[\ln(5-2)-\ln(3-2)]-2\left[\frac{1}{5-2}-\frac{1}{3-2}\right]$$

$$I=2+4[\ln3-\ln1]-2\left[\frac{1}{3}-\frac{1}{1}\right]$$

$$I=2+4[\ln3-0]-\frac{2}{3}+2$$

$$I=2+4(1.0986)-\tfrac{2}{3}+2=7.7278$$

---

d). $\displaystyle\int_1^2\frac{4}{t^3+4t}\ dt$

Sol: Given $\displaystyle\int_1^2\frac{4}{t^3+4t}\ dt$

Take a fraction only $\dfrac{4}{t^3+4t}=\dfrac{4}{t\left(t^2+4\right)}$ which is proper fraction having two factors linear & quadratic

$$\frac{4}{t\left(t^2+4\right)}=\frac{A}{t}+\frac{Bt+C}{t^2+4}\text{.....................}(1)$$

Multiply each term by $t(t+4)$ we have

$$4=A\left(t^2+4\right)+(Bt+C)t\text{.....................}(2)$$

$$4=At^2+4A+Bt^2+Ct$$

$$4=At^2+Bt^2+Ct+4A$$

Comparing the coefficients

constant	coefficients of t	coefficients of t^2
$4A=4$	$C=0$	$A+B=0$
$\Rightarrow A=1$		$B=-4$

Putting values of A,B and C in equation (1) we get

$$\frac{4}{t\left(t^2+4\right)}=\frac{1}{t}+\frac{-t+0}{t^2+4}$$

Substituting the value in the given integral

$$\int_1^2\frac{4}{t^3+4t}\ dt=\int_1^2\frac{1}{t}\ dt-\frac{1}{2}\int_1^2\frac{2t}{t^2+4}dt$$

$$\int_1^2\frac{4}{t^3+4t}\ dt=|\ln(t)|_1^2-\frac{1}{2}\left|\ln\left(t^2+4\right)\right|_1^2$$

$$\int_1^2\frac{4}{t^3+4t}\ dt=[\ln2-\ln1]-\frac{1}{2}\left[\ln\left(2^2+4\right)-\ln\left(1^2+4\right)\right]$$

$$\int_1^2\frac{4}{t^3+4t}\ dt=\ln2-0-\frac{1}{2}[\ln8-\ln5]=0.458145365$$

---

e). $\displaystyle\int_1^3\ln(2x+1)\ dx$

Sol: Given $\displaystyle\int_1^3\ln(2x+1)\ dx$

$$I=\ln(2x+1)\int_1^3 dx-\int_1^3\tfrac{d}{dx}\ln(2x+1)\left(\int_1^3 dx\right)dx$$

$$\int_1^3\ln(2x+1)\ dx=|x\ln(2x+1)|_1^3-2\int_1^3\frac{x}{2x+1}dx$$

$$I=\left[3\ln(2(3)+1)-1\ln(2(1)+1)\right]-\int_1^3\frac{2x}{2x+1}dx$$

$$I=[3\ln7-\ln3]-\int_1^3\frac{2x+1-1}{2x+1}dx$$

$$I=[3\ln7-\ln3]-\left\{\int_1^3\frac{2x+1}{2x+1}dx-\int_1^3\frac{1}{2x+1}dx\right\}$$

$$I=[3\ln7-\ln3]-\int_1^3 dx+\frac{1}{2}\int_1^3\frac{2}{2x+1}dx$$

$$I=[3\ln7-\ln3]-|x|_1^3+\frac{1}{2}|\ln(2x+1)|_1^3$$

$$I=[3\ln7-\ln3]-[3-1]$$
$$+\frac{1}{2}\left[\ln\{2(3)+1\}-\ln\{2(1)+1\}\right]$$

$$I=[3\ln7-\ln3]-[2]+\frac{1}{2}[\ln7-\ln3]$$

$$I=[3(1.6094)-1.0986]-2+0.5[1.6094-1.0986]$$

I = 3.162767089

---

f). $\displaystyle\int_1^4\frac{\ln x}{x^3}dx$

Sol: Given $\displaystyle\int_1^4\frac{\ln x}{x^3}dx$

$$\int_1^4\frac{\ln x}{x^3}dx=\ln x\int_1^4 x^{-3}dx-\int_1^4\tfrac{d}{dx}\ln x\left(\int_1^4 x^{-3}dx\right)dx$$

$$\int_1^4\frac{\ln x}{x^3}dx=\left|\ln x\frac{x^{-3+1}}{-3+1}\right|_1^4-\int_1^4\frac{1}{x}\left(\frac{x^{-3+1}}{-3+1}\right)dx$$

$$\int_1^4\frac{\ln x}{x^3}dx=\frac{-1}{2}\left|x^{-2}\ln x\right|_1^4+\frac{1}{2}\int_1^4\frac{1}{x}x^{-2}dx$$

$$\int_1^4\frac{\ln x}{x^3}dx=\frac{-1}{2}\left|\frac{\ln x}{x^2}\right|_1^4+\frac{1}{2}\int_1^4 x^{-3}dx$$

$$\int_1^4\frac{\ln x}{x^3}dx=\frac{-1}{2}\left|\frac{\ln x}{x^2}\right|_1^4+\frac{1}{2}\left|\frac{x^{-3+1}}{-3+1}\right|_1^4$$

$$\int_1^4\frac{\ln x}{x^3}dx=\frac{-1}{2}\left[\frac{\ln4}{4^2}-\frac{\ln1}{1^2}\right]-\frac{1}{4}\left|x^{-2}\right|_1^4$$

$$\int_1^4\frac{\ln x}{x^3}dx=\frac{-1}{2}\left[\frac{\ln4}{4^2}-\frac{0}{1^2}\right]-\frac{1}{4}\left[\frac{1}{4^2}-\frac{1}{1^2}\right]$$

$$\int_1^4 \frac{\ln x}{x^3}dx = \frac{-1}{2}\left[\frac{\ln 4}{16}-0\right]-\frac{1}{4}\left[\frac{1}{16}-\frac{1}{1}.\frac{16}{16}\right]$$

$$\int_1^4 \frac{\ln x}{x^3}dx = \frac{-1}{2}\left[\frac{\ln 4}{16}\right]-\frac{1}{4}\left[\frac{1-16}{16}\right]$$

$$\int_1^4 \frac{\ln x}{x^3}dx = \frac{-\ln 4}{32}-\frac{1}{4}\left[\frac{-15}{16}\right]=\frac{-\ln 4}{32}+\frac{15}{64}$$

$$\int_1^4 \frac{\ln x}{x^3}dx = 0.1910$$

---

g). $\int_1^2 x(x-1)^6\,dx$

Sol: Given $\int_1^2 x(x-1)^6\,dx$

$$I = x\int_1^2 (x-1)^6\,dx-\int_1^2 \tfrac{d}{dx}x\left(\int_1^2 (x-1)^6\,dx\right)dx$$

$$\int_1^2 x(x-1)^6\,dx = x\frac{(x-1)^{6+1}}{6+1}-\int_1^2 1\left(\frac{(x-1)^{6+1}}{6+1}\right)dx$$

$$\int_1^2 x(x-1)^6\,dx = \frac{1}{7}\left|x(x-1)^7\right|_1^2-\frac{1}{7}\int_1^2 (x-1)^7\,dx$$

$$I = \frac{1}{7}\left[2(2-1)^7-1(1-1)^7\right]-\frac{1}{7}\left|\frac{(x-1)^{7+1}}{7+1}\right|_1^2$$

$$I = \tfrac{1}{7}\left[2(1)^7-1(0)^7\right]-\tfrac{1}{49}\left[(2-1)^8-(1-1)^8\right]$$

$$\int_1^2 x(x-1)^6\,dx = \frac{1}{7}[2-0]-\frac{1}{56}\left[(1)^8-(0)^8\right]$$

$$\int_1^2 x(x-1)^6\,dx = \frac{2}{7}-\frac{1}{56}[1-0]=\frac{8}{8}.\frac{2}{7}-\frac{1}{56}$$

$$\int_1^2 x(x-1)^6\,dx = \frac{16}{56}-\frac{1}{56}=\frac{15}{56}=0.2679$$

---

h). $\int_0^1 (x-3)e^x dx$

Sol: Given $\int_0^1 (x-3)e^x dx$

$$\int_0^1 (x-3)e^x dx = (x-3)\int_0^1 e^x dx-\int_0^1 \tfrac{d}{dx}(x-3)\left(\int_0^1 e^x dx\right)dx$$

$$\int_0^1 (x-3)e^x dx = \left|(x-3)e^x\right|_0^1-\int_0^1 (1-0)e^x dx$$

$$\int_0^1 (x-3)e^x dx = \left[(1-3)e^1-(0-3)e^0\right]-\int_0^1 e^x dx$$

$$\int_0^1 (x-3)e^x dx = -2e^1+3e^o-\left|e^x\right|_0^1$$

$$\int_0^1 (x-3)e^x dx = -2e^1+3e^o-\left[e^1-e^0\right]$$

$$\int_0^1 (x-3)e^x dx = -2e+3-\left[e-1\right]$$

$$\int_0^1 (x-3)e^x dx = -2e+3-e+1$$

$$\int_0^1 (x-3)e^x dx = -3e+4 = -4.1548$$

Q5. Use the definite integral to find out the area between the curve $f(x)$ and x-axis over the indicated interval $[a,b]$

a). $f(x)=4-x^2,\quad [0,3]$

Sol: Given $f(x)=4-x^2$, With interval $[0,3]$

First find out the x-intercepts of the curve by taking $f(x)=0$

i.e., $4-x^2=0 \qquad x^2=4$

Taking square root on both sides

$\sqrt{x^2}=\pm\sqrt{4} \qquad \Rightarrow x=\pm 2$

Therefore subintervals are $[0,2],[2,3]$

To identify the function is positive or negative in the sub-interval, take any point in the sub-intervals

Take $1\in[0,2]$ then the function at $x=1$

$f(1)=4-(1)^2=4-1=3$ is positive

And $3\in[2,3]$ then the function at $x=3$

$f(3)=4-(3)^2=4-9=-5$ is negative

Total area of the region in the $[0,3]$ is the sum of the area of the sub-regions in subintervals $[0,2],[2,3]$

$$\int_0^3 (4-x^2)dx = \int_0^2 (4-x^2)dx-\int_2^3 (4-x^2)dx$$

$$\int_0^3 (4-x^2)dx = \left|4x-\frac{x^3}{3}\right|_0^2-\left|4x-\frac{x^3}{3}\right|_2^3$$

$$\int_0^3 (4-x^2)dx = \left[\left\{4(2)-\frac{(2)^3}{3}\right\}-\left\{4(0)-\frac{(0)^3}{3}\right\}\right]$$
$$-\left[\left\{4(3)-\frac{(3)^3}{3}\right\}-\left\{4(2)-\frac{(2)^3}{3}\right\}\right]$$

$$I=\left\{8-\frac{8}{3}\right\}-\{0-0\}-\left\{12-\frac{27}{3}\right\}+\left\{8-\frac{8}{3}\right\}$$

$$I=2\left\{\frac{3}{3}.\frac{8}{1}-\frac{8}{3}\right\}-\{12-9\}$$

$$I=2\left\{\frac{24-8}{3}\right\}-3=2\left(\frac{16}{3}\right)-\frac{3}{1}.\frac{3}{3}$$

$$I=\frac{32-9}{3}=\frac{23}{3}$$

---

b). $f(x)=x^2-5x+6 \qquad [0,3]$

Sol: Given $f(x)=x^2-5x+6$, With interval $[0,3]$

First find out x-intercepts of the curve by taking

$f(x)=0$ i.e., $x^2-5x+6=0$

$x^2-3x-2x+6=0$

$x(x-3)-2(x-3)=0$

$(x-2)(x-3)=0$

Either $x-2=0$ or $x-3=0$

$x=2 \qquad x=3$

Therefore subintervals are $[0,2],[2,3]$

To identify the function is positive or negative in the sub-interval, take any point in the sub-intervals

Take $1\in[0,2]$ then the function at $x=1$

$f(1)=(1)^2-5(1)+6=1-5+6=2$ is positive

And $3\in[2,3]$ then the function at $x=3$

$f(2.5)=(2.5)^2-5(2.5)+6=6.25-12.5+6=-0.25$

is negative. Total area of the region in the $[0,3]$ is sum of area of sub-regions in subintervals $[0,2],[2,3]$

$$\int_0^3\left(x^2-5x+6\right)dx=\int_0^2\left(x^2-5x+6\right)dx-\int_2^3\left(x^2-5x+6\right)dx$$

$$I=\left|\frac{x^3}{3}-\frac{5x^2}{2}+6x\right|_0^2-\left|\frac{x^3}{3}-\frac{5x^2}{2}+6x\right|_2^3$$

$$I=\left[\left\{\frac{2^3}{3}-\frac{5(2)^2}{2}+6(2)\right\}-\left\{\frac{0^3}{3}-\frac{5(0)^2}{2}+6(0)\right\}\right]$$

$$-\left[\left\{\frac{3^3}{3}-\frac{5(3)^2}{2}+6(3)\right\}-\left\{\frac{2^3}{3}-\frac{5(2)^2}{2}+6(2)\right\}\right]$$

$$I=\left\{\frac{8}{3}-\frac{5(4)}{2}+12\right\}-\{0-0+0\}$$

$$-\left\{\frac{27}{3}-\frac{5(9)}{2}+18\right\}+\left\{\frac{8}{3}-\frac{5(4)}{2}+12\right\}$$

$$I=2\left\{\frac{8}{3}-10+12\right\}-\left\{9-\frac{45}{2}+18\right\}$$

$$I=2\left\{\frac{8}{3}+\frac{2}{1}\right\}-\left\{\frac{27}{1}-\frac{45}{2}\right\}$$

$$I=2\left\{\frac{8}{3}+\frac{2}{1}.\frac{3}{3}\right\}-\left\{\frac{2}{2}.\frac{27}{1}-\frac{45}{2}\right\}$$

$$I=2\left\{\frac{8+6}{3}\right\}-\left\{\frac{54-45}{2}\right\}=2\left(\frac{14}{3}\right)-\frac{9}{2}$$

$$I=\frac{2}{2}.\frac{28}{3}-\frac{9}{2}.\frac{3}{3}=\frac{56-27}{6}=\frac{29}{6}$$

---

c). $f(x)=x^2-6x+8$ $[0,4]$

Sol: Given $f(x)=x^2-6x+8,$ With interval $[0,4]$

First find out the x-intercepts of the curve by taking $f(x)=0$

i.e., $x^2-6x+8=0$

$x^2-4x-2x+8=0$

$x(x-4)-2(x-4)=0$

$(x-2)(x-4)=0$

Either $x-2=0$ or $x-4=0$

$x=2$ $x=4$

Therefore subintervals are $[0,2],[2,4]$

To identify the function is positive or negative in the sub-interval, take any point in the sub-intervals

Take $1\in[0,2]$ then the function at $x=1$

$f(1)=(1)^2-6(1)+8=1-6+8=3$ is + ve

And $3\in[2,4]$ then the function at $x=3$

$f(3)=(3)^2-6(3)+8=9-18+8=-1$ is - ve

Total area of region in the $[0,4]$ is the sum of \ area of the sub-regions in the subintervals $[0,2],[2,4]$

$$\int_0^4\left(x^2-6x+8\right)dx=\int_0^2\left(x^2-6x+8\right)dx-\int_2^4\left(x^2-6x+8\right)dx$$

$$I=\left|\frac{x^3}{3}-\frac{6x^2}{2}+8x\right|_0^2-\left|\frac{x^3}{3}-\frac{6x^2}{2}+8x\right|_2^4$$

$$I=\left[\left\{\frac{2^3}{3}-3(2)^2+8(2)\right\}-\left\{\frac{0^3}{3}-3(0)^2+8(0)\right\}\right]$$

$$-\left[\left\{\frac{4^3}{3}-3(4)^2+8(4)\right\}-\left\{\frac{2^3}{3}-3(2)^2+8(2)\right\}\right]$$

$$I=\left\{\frac{8}{3}-3(4)+16\right\}-\{0-0+0\}$$

$$-\left\{\frac{64}{3}-3(16)+32\right\}+\left\{\frac{8}{3}-3(4)+16\right\}$$

$$I=2\left\{\frac{8}{3}-12+16\right\}-\left\{\frac{64}{3}-48+32\right\}$$

$$I=2\left\{\frac{8}{3}+\frac{4}{1}\right\}-\left\{\frac{64}{3}-16\right\}$$

$$I=2\left\{\frac{8}{3}+\frac{4}{1}.\frac{3}{3}\right\}-\left\{\frac{64}{3}-\frac{16}{1}.\frac{3}{3}\right\}$$

$$I=2\left\{\frac{8+12}{3}\right\}-\left\{\frac{64-48}{3}\right\}=2\left(\frac{20}{3}\right)-\left(\frac{16}{3}\right)$$

$$I=\frac{40}{3}-\frac{16}{3}=\frac{40-16}{6}=\frac{24}{3}=8$$

---

d). $f(x)=5x-x^2$ $[1,3]$

Sol: Given $f(x)=5x-x^2$ With interval $[1,3]$

First find out the x-intercepts of the curve by taking $f(x)=0$

i.e., $5x-x^2=0$

$x(5-x)=0$

Either or

$x=0$ $5-x=0$

$x=5$

Therefore subintervals are not possible in given interval

The total area of the region in the $[1,3]$

$$\int_1^3\left(5x-x^2\right)dx=\left|\frac{5x^2}{2}-\frac{x^3}{3}\right|_1^3$$

$$I=\left[\left\{\frac{5(3)^2}{2}-\frac{3^3}{3}\right\}-\left\{\frac{5(1)^2}{2}-\frac{1^3}{3}\right\}\right]$$

$$I=\left\{\frac{5(9)}{2}-\frac{27}{3}\right\}-\left\{\frac{5}{2}-\frac{1}{3}\right\}$$

$$I=\frac{45}{2}-\frac{27}{3}-\frac{5}{2}+\frac{1}{3}$$

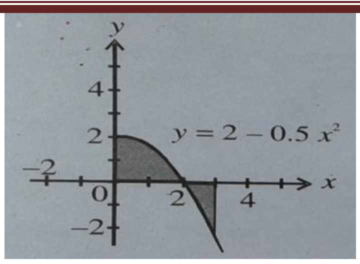
$$I=\frac{45}{2}-\frac{5}{2}+\frac{1}{3}-\frac{27}{3}$$

$$I=\frac{45-5}{2}+\frac{1-27}{3}$$

$$I=\frac{40}{2}-\frac{26}{3}=\frac{3}{3}.\frac{20}{1}-\frac{26}{3}=\frac{60-26}{3}=\frac{34}{3}$$

Q6. Set up definite integral in problems a to d that represent the indicated shaded areas;

a).



Sol: Given $f\left(x\right) = 2 - 0.5x^2$ with interval $\left[0,3\right]$

From the figure positive subinterval is $\left[0,2\right]$

And the negative subinterval is $\left[2,3\right]$

The total area of the region in the $\left[0,3\right]$ is the sum of area of the sub-regions in the subintervals $\left[0,2\right],\left[2,3\right]$

$$\int_0^3\left(2-0.5x^2\right)dx = \int_0^2\left(2-\frac{x^2}{2}\right)dx - \int_2^3\left(2-\frac{x^2}{2}\right)dx$$

$$\int_0^3\left(2-0.5x^2\right)dx = \left|2x-\frac{x^3}{2\times 3}\right|_0^2 - \left|2x-\frac{x^3}{2\times 3}\right|_2^3$$

$$\int_0^3\left(2-0.5x^2\right)dx = \left|2x-\frac{x^3}{6}\right|_0^2 - \left|2x-\frac{x^3}{6}\right|_2^3$$

$$I = \left[\left\{2(2)-\frac{(2)^3}{6}\right\}-\left\{2(0)-\frac{(0)^3}{6}\right\}\right]$$
$$-\left[\left\{2(3)-\frac{(3)^3}{6}\right\}-\left\{2(2)-\frac{(2)^3}{6}\right\}\right]$$

$$I = \left\{4-\frac{8}{6}\right\}-\left\{0-\frac{0}{6}\right\}-\left\{6-\frac{27}{6}\right\}+\left\{4-\frac{8}{6}\right\}$$

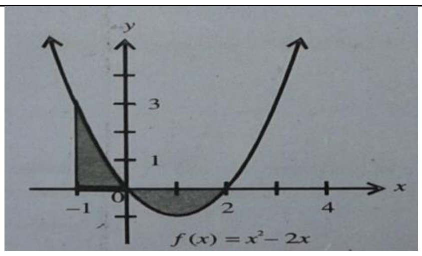
$$I = 2\left\{4-\frac{8}{6}\right\}-\left\{6-\frac{27}{6}\right\}$$

$$I = 2\left\{\frac{3}{3}.\frac{4}{1}-\frac{4}{3}\right\}-\left\{\frac{2}{2}.\frac{6}{1}-\frac{9}{2}\right\}$$

$$I = 2\left\{\frac{12-4}{3}\right\}-\left\{\frac{12-9}{2}\right\} = 2\left\{\frac{8}{3}\right\}-\left\{\frac{3}{2}\right\}$$

$$I = \frac{2}{2}.\frac{16}{3}-\frac{3}{2}.\frac{3}{3} = \frac{32-9}{6} = \frac{23}{6}$$

b).



Sol: Given $f\left(x\right) = x^2 - 2x$ with interval $\left[-1,2\right]$

From the figure positive subinterval is $\left[-1,0\right]$

And the negative subinterval is $\left[0,2\right]$

The total area of the region in the $\left[-1,2\right]$ is the sum of area of the sub-regions in the subintervals $\left[-1,0\right],\left[0,2\right]$

$$\int_{-1}^2\left(x^2-2x\right)dx = \int_{-1}^0\left(x^2-2x\right)dx - \int_0^2\left(x^2-2x\right)dx$$

$$\int_{-1}^2\left(x^2-2x\right)dx = \left|\frac{x^3}{3}-x^2\right|_{-1}^0 - \left|\frac{x^3}{3}-x^2\right|_0^2$$

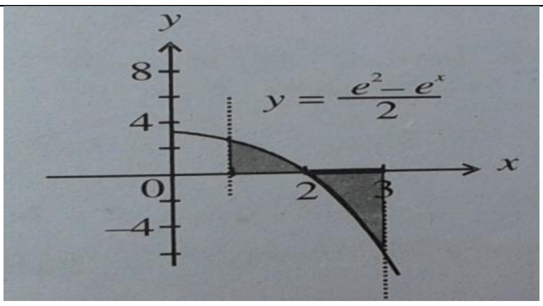
$$I = \left[\left\{\frac{(0)^3}{3}-(0)^2\right\}-\left\{\frac{(-1)^3}{3}-(-1)^2\right\}\right]$$
$$-\left[\left\{\frac{(2)^3}{3}-(2)^2\right\}-\left\{\frac{(0)^3}{3}-(0)^2\right\}\right]$$

$$I = \{0\}-\left\{\frac{-1}{3}-1\right\}-\left\{\frac{8}{3}-4\right\}+\{0\}$$

$$I = \frac{1}{3}+1-\frac{8}{3}+4 = \frac{1}{3}-\frac{8}{3}+1+4$$

$$I = \frac{-7}{3}+5 = \frac{-7+15}{3} = \frac{8}{3}$$

c).



Sol: Given $f\left(x\right) = \frac{e^2-e^x}{2}$ with interval $\left[1,3\right]$

From the figure positive subinterval is $\left[1,2\right]$

And the negative subinterval is $\left[2,3\right]$

The total area of the region in the $\left[1,3\right]$ is the sum of area of the sub-regions in the subintervals $\left[1,2\right],\left[2,3\right]$

$$\int_1^3\frac{e^2-e^x}{2}dx = \frac{1}{2}\int_1^3\left(e^2-e^x\right)dx$$

$$\int_1^3\frac{e^2-e^x}{2}dx = \frac{1}{2}\left[\int_1^2\left(e^2-e^x\right)dx - \int_2^3\left(e^2-e^x\right)dx\right]$$

$$\int_1^3\frac{e^2-e^x}{2}dx = \frac{1}{2}\left[\left|xe^2-e^x\right|_1^2 - \left|xe^2-e^x\right|_2^3\right]$$

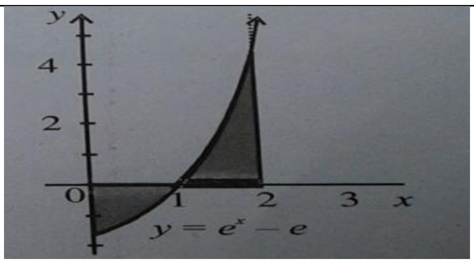
$$\int_1^3\frac{e^2-e^x}{2}dx = \frac{1}{2}\left[\begin{array}{l}\left\{\left(2e^2-e^2\right)-\left(1e^2-e^1\right)\right\}\\ \quad -\left\{\left(3e^2-e^3\right)-\left(2e^2-e^2\right)\right\}\end{array}\right]$$

$$\int_1^3\frac{e^2-e^x}{2}dx = \frac{1}{2}\left[\left\{e^2-e^2+e^1\right\}-\left\{3e^2-e^3-e^2\right\}\right]$$

$$\int_1^3\frac{e^2-e^x}{2}dx = \frac{1}{2}\left[\left\{e^1\right\}-\left\{2e^2-e^3\right\}\right]$$

$$\int_1^3\frac{e^2-e^x}{2}dx = \frac{1}{2}\left[e^1-2e^2+e^3\right] = 4.0129$$

d).



Sol: Given $f\left(x\right) = e^x - e$ with interval $\left[0,2\right]$

From the figure negative subinterval is $\left[0,1\right]$

And the positive subinterval is $\left[1,2\right]$

The total area of the region in the $\left[0,2\right]$ is the sum of area of the sub-regions in subintervals $\left[0,1\right],\left[1,2\right]$

$$\int_0^2 \left(e^x - e\right)dx = -\int_0^1 \left(e^x - e\right)dx + \int_1^2 \left(e^x - e\right)dx$$

$$\int_0^2 \left(e^x - e\right)dx = -\left|e^x - xe\right|_0^1 + \left|e^x - xe\right|_1^2$$

$$I = -\left[\left\{e^1 - 1e\right\} - \left\{e^0 - 0.e\right\}\right] + \left[\left\{e^2 - 2e\right\} - \left\{e^1 - 1e\right\}\right]$$

$$I = -\left[\left\{e - e\right\} - \left\{1 - 0\right\}\right] + \left[\left\{e^2 - 2e\right\} - \left\{e - e\right\}\right]$$

$$I = -\left[\left\{0\right\} - \left\{1\right\}\right] + \left[\left\{e^2 - 2e\right\} - \left\{0\right\}\right]$$

$$I = -\left[-1\right] + \left[e^2 - 2e\right] = 1 + e^2 - 2e = 2.9525$$

Q7. An oil tanker is leaking oil at a rate given in barrels per hour by $\frac{dL}{dt} = \frac{80\ln(t+1)}{(t+1)}$ where t is the time in hours after the tanker hits a hidden rock (when t=0)

Sol: Given $\frac{dL}{dt} = \frac{80\ln(t+1)}{(t+1)}$

$$dL = \frac{80\ln(t+1)}{(t+1)}dt$$

Integrating both sides

$$\int dL = \int \frac{80\ln(t+1)}{(t+1)}dt$$

Let $u = \ln(t+1)$ Differentiating w r t "t"

$$\tfrac{d}{dt}u = \tfrac{d}{dt}\ln(t+1)$$

$$\tfrac{d}{dt}u = \frac{1}{t+1}\tfrac{d}{dt}(t+1)$$

$$\frac{du}{dt} = \frac{1}{t+1}$$

$$du = \frac{dt}{t+1}$$

Putting the value of u and du in given integral

$$\int dL = \int \frac{80\ln(t+1)}{(t+1)}dt = 80\int \ln(t+1)\frac{dt}{(t+1)}$$

$$L = 80\int u\,du = 80\frac{u^2}{2} = 40u^2$$

Putting the value of u

$$L = 40\left[\ln(t+1)\right]^2$$

---

a). Find total number of barrels that ship will leak on first day

Sol: start time $a = t_o = 0$ hour and end time $b = t_n = 24$ hours

Therefore given integral

$$\int_0^{24} dL(t) = \int_0^{24} \frac{80\ln(t+1)}{(t+1)}dt$$

$$\left|L(t)\right|_0^{24} = 40\left|\left\{\ln(t+1)\right\}^2\right|_0^{24}$$

$$L(24) - L(0) = 40\left[\left\{\ln(24+1)\right\}^2 - \left\{\ln(0+1)\right\}^2\right]$$

$$L(24) - L(0) = 40\left[\left\{\ln(25)\right\}^2 - \left\{\ln(1)\right\}^2\right]$$

$$L(24) - L(0) = 40\left[\left\{3.2189\right\}^2 - \left\{0\right\}^2\right]$$

$$L(24) - L(0) = 40\left[10.3612\right] = 414.4465$$

---

b). Find the total number of barrels that the ship will leak on the second day

Solution: start time $a = t_o = 24$ hours and end time $b = t_n = 48$ hours

Therefore the given integral

$$\int_{24}^{48} dL(t) = \int_{24}^{48} \frac{80\ln(t+1)}{(t+1)}dt$$

$$\left|L(t)\right|_{24}^{48} = 40\left|\left\{\ln(t+1)\right\}^2\right|_{24}^{48}$$

$$L(48) - L(24) = 40\left[\left\{\ln(48+1)\right\}^2 - \left\{\ln(24+1)\right\}^2\right]$$

$$L(48) - L(24) = 40\left[\left\{\ln(49)\right\}^2 - \left\{\ln(25)\right\}^2\right]$$

$$L(48) - L(24) = 40\left[\left\{3.8918\right\}^2 - \left\{3.2189\right\}^2\right]$$

$$L(48) - L(24) = 40\left[15.1463 - 10.3612\right] = 191.404$$

---

c). what happening over the long run to the amount of oil leaked per day?

Solution: Oil leaked at first day = 414.4465 barrels

Oil leaked at second day = 191.404 barrels

Oil leaking from the tank is decreasing to 0 (zero)
