Chapter 4

For domain

We can take all values for the domain **not including**

i). non zero **denominator** i.e., $\frac{1}{x\neq 0}$

ii). positive **radicand** i.e., $\sqrt{x\geq 0}$

iii). log(natural or common) $\ln x$ with $x>0$

Exercise 4.1

Q1. Find domain for the following vector functions:

a). $F(t)=2t\,i-3t\,j+t^{-1}k$

Solution: We have $F(t)=2t\,i-3t\,j+t^{-1}k$

Denominator $t\neq 0$

Domain of the given vector function $F(t)$ is set of real number expect $t\neq 0$

b). $F(t)=(1-t)i+\sqrt{t}\,j-(t-2)^{-1}k$

Solution: We have $F(t)=(1-t)i+\sqrt{t}\,j-\frac{1}{t-2}k$

Denominator $t-2\neq 0$ and the radicand $t\geq 0$

Domain of the given vector function $F(t)$ is set of real number except $t\neq 2$ & $t\geq 0$

c). $F(t)=\sin t\,i+\cos t\,j+\tan t\,k$

Solution: We have $F(t)=\sin t\,i+\cos t\,j+\frac{\sin t}{\cos t}k$

Denominator $\cos t\neq 0$

$t\neq \cos^{-1}(0)$

$t\neq \frac{\pi}{2}$

$t\neq \frac{\pi}{2}(2n+1)$ For $n\in\Re$

Domain of the given vector function $F(t)$ is set of real number except $t\neq\frac{\pi}{2}(2n+1),\ n\in\Re$

d). $F(t)=\cos t\,i-\cot t\,j+\operatorname{cosec} t\,k$

Solution: We have $F(t)=\cos t\,i-\frac{\cos t}{\sin t}j+\frac{1}{\sin t}k$

Denominator $\sin t\neq 0$

$t\neq\sin^{-1}(0)$

$t\neq\pi$

$t\neq n\pi$ For $n\in\Re$

Domain of the given vector function $F(t)$ is set of real number except $t\neq n\pi,\ n\in\Re$

e). $F(t)+G(t)$, where

$F(t)=3t\,i+t^{-1}k,\qquad G(t)=5t\,i+\sqrt{10-t}\,j$

Solution: $F(t)=3t\,i+t^{-1}k, G(t)=5t\,i+\sqrt{10-t}\,j$

$F(t)+G(t)=\left(3t\,i+t^{-1}k\right)+\left(5t\,i+\sqrt{10-t}\,j\right)$

$F(t)+G(t)=3t\,i+t^{-1}k+5t\,i+\sqrt{10-t}\,j$

$F(t)+G(t)=3t\,i+5t\,i+\sqrt{10-t}\,j+t^{-1}k$

$F(t)+G(t)=8t\,i+\sqrt{10-t}\,j+\frac{1}{t}k$

Denominator	radicand
$t\neq 0$	$10-t\geq 0$
	$10\geq t$

Domain of the given vector function $F(t)+G(t)$, is set of real number except $t\neq 0$ & $t\leq 10$

f). $F(t)-G(t)$, where

$F(t)=\ln t\,i+3t\,j-t^2k,\qquad G(t)=i+5t\,j-t^2k$

Solution: $F(t)=\ln t\,i+3t\,j-t^2k, G(t)=i+5t\,j-t^2k$

$F(t)-G(t)=\left(\ln t\,i+3t\,j-t^2k\right)-\left(i+5t\,j-t^2k\right)$

$F(t)-G(t)=\ln t\,i+3t\,j-t^2k-i-5t\,j+t^2k$

$F(t)-G(t)=\ln t\,i-i+3t\,j-5t\,j-t^2k+t^2k$

$F(t)-G(t)=(\ln t-1)i-2t\,j$

For Natural log $t>0$

Domain of the given vector function $F(t)-G(t)$ is set of real number $t>0$

g). $F(t)\times G(t)$, where

$F(t)=t^2i-t\,j+2t\,k,\ G(t)=(t+2)^{-1}i+(t+4)j-\sqrt{-t}\,k$

Sol: Given $F(t)\times G(t)=\begin{vmatrix} i & j & k\\ t^2 & -t & 2t\\ (t+2)^{-1} & t+4 & -\sqrt{-t}\end{vmatrix}$

$F(t)\times G(t)=i\left(t\sqrt{-t}-2t(t+4)\right)-j\left(-t^2\sqrt{-t}-2t(t+2)^{-1}\right)$
$+k\left(t^2(t+4)+t(t+2)^{-1}\right)$

$F(t)\times G(t)=i\left(t\sqrt{-t}-2t(t+4)\right)-j\left(-t^2\sqrt{-t}-\frac{2t}{t+2}\right)$
$+k\left(t^2(t+4)+\frac{t}{t+2}\right)$

Denominator	Radicand
$t+2\neq 0$	$-t\geq 0$
$t\neq -2$	$t\leq 0$

Domain of the given vector function $F(t)\times G(t)$ is set of real number except $t\neq -2$ & $t\leq 0$

Q2. Sketch the following vector function:

a). $F(t)=2t\,i+t^2j$

Solution: We have $F(t)=2t\,i+t^2j$

At $t=-2$ $F(-2)=2(-2)i+(-2)^2j$

$F(-2)=-4i+4j$

At $t=-1$ $F(-1)=2(-1)i+(-1)^2j$

$F(-1)=-2i+j$

At $t=0$ $\quad F(0)=2(0)i+(0)^2 j$

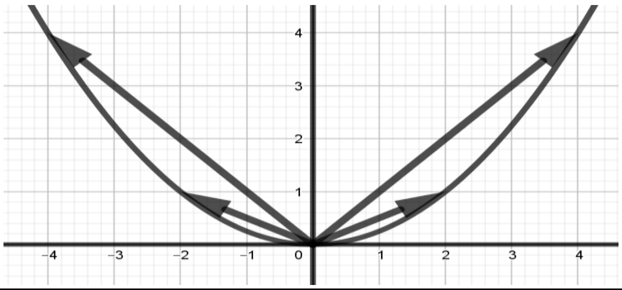
$F(0)=0i+0j$

At $t=1$ $\quad F(1)=2(1)i+(1)^2 j$

$F(1)=2i+j$

At $t=2$ $\quad F(2)=2(2)i+(2)^2 j$

$F(2)=4i+4j$



b). $\quad G(t)=\sin t\, i-\cos t\, j$

Solution: We have $G(t)=\sin t\, i-\cos t\, j$

At $t=$ 0 radian $\quad G(0)=\sin(0)i-\cos(0)j$

$G(0)=0.i-1j$

At $t=\frac{\pi}{2}$ $\quad G\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)i-\cos\left(\frac{\pi}{2}\right)j$

$G\left(\frac{\pi}{2}\right)=1i-0j$

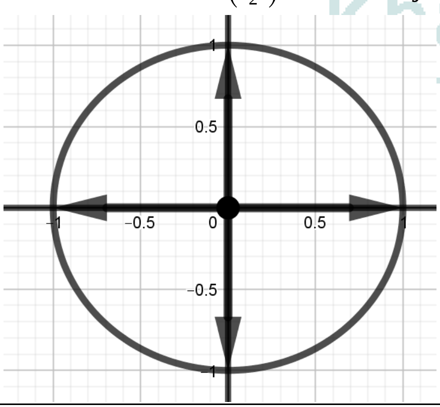
At $t=\pi$ $\quad G(\pi)=\sin(\pi)i-\cos(\pi)j$

$G(\pi)=0i-(-1)j$

$G(\pi)=0.i+1j$

At $t=\frac{3\pi}{2}$ $\quad G\left(\frac{3\pi}{2}\right)=\sin\left(\frac{3\pi}{2}\right)i-\cos\left(\frac{3\pi}{2}\right)j$

$G\left(\frac{3\pi}{2}\right)=-1.i-0.j$



Q3. Perform the operation of the following expressions with $F(t)=2t\,i-5\,j+t^2\,k$,

$G(t)=(1-t)i+\frac{1}{t}k$, $\quad H(t)=\sin t\, i+e^t\, j$

a). $\quad 2F(t)-3G(t)$

Sol: $2F(t)-3G(t)=2\left[2t\,i-5\,j+t^2\,k\right]-3\left[(1-t)i+\frac{1}{t}k\right]$

$$2F(t)-3G(t)=4t\,i-10\,j+2t^2\,k-(3-3t)i-\frac{3}{t}k$$

$$2F(t)-3G(t)=4t\,i-(3-3t)i-10\,j+2t^2\,k-\frac{3}{t}k$$

$$2F(t)-3G(t)=(4t-3+3t)i-10\,j+\left(2t^2-\frac{3}{t}\right)k$$

$$2F(t)-3G(t)=(7t-3)i-10\,j+\left(2t^2-\frac{3}{t}\right)k$$

b). $\quad F(t).G(t)$

Sol: $F(t).G(t)=\left[2t\,i-5\,j+t^2\,k\right].\left[(1-t)i+0\,j+\frac{1}{t}k\right]$

$$F(t).G(t)=2t(1-t)-5(0)+t^2\left(\frac{1}{t}\right)$$

$$F(t).G(t)=2t-2t^2+t$$

$$F(t).G(t)=3t-2t^2$$

c). $\quad G(t).H(t)$

Sol: $G(t).H(t)=\left[(1-t)i+\frac{1}{t}k\right].\left[\sin t\, i+e^t\, j+0\,k\right]$

$$G(t).H(t)=(1-t)\sin t-0.e^t+\frac{1}{t}.0$$

$$G(t).H(t)=(1-t)\sin t$$

d). $\quad F(t)\times H(t)$

Sol: $F(t)\times H(t)=\begin{vmatrix} i & j & k \\ 2t & -5 & t^2 \\ \sin t & e^t & 0 \end{vmatrix}$

$$F(t)\times H(t)=i\left(0-t^2e^t\right)-j\left(0-t^2\sin t\right)+k\left(2te^t+5\sin t\right)$$

$$F(t)\times H(t)=-t^2e^t\,i+t^2\sin t\,j+\left(2te^t+5\sin t\right)k$$

e). $\quad 2e^tF(t)+tG(t)+10H(t)$

Solution: We have $2e^tF(t)+tG(t)+10H(t)$

$$2e^tF(t)+tG(t)+10H(t)=2e^t\left[2t\,i-5\,j+t^2\,k\right]+t\left[(1-t)i+\frac{1}{t}k\right]+10\left[\sin t\,i+e^t\,j\right]$$

$$2e^tF(t)+tG(t)+10H(t)=4te^t\,i-10e^t\,j+2e^tt^2\,k+\left(t-t^2\right)i+k+10\sin t\,i+10e^t\,j$$

$$2e^tF(t)+tG(t)+10H(t)=4te^t\,i+\left(t-t^2\right)i+10\sin t\,i-10e^t\,j+10e^t\,j+2e^tt^2\,k+k$$

$$2e^tF(t)+tG(t)+10H(t)=\left(4te^t+t-t^2+10\sin t\right)i+\left(2e^tt^2+1\right)k$$

Q4. Evaluate limits of the following expressions:

a). $\quad \lim_{t\to1}\left[3t\,i+e^{2t}\,j+\sin\pi t\,k\right]$

Solution: We have $\lim_{t\to1}\left[3t\,i+e^{2t}\,j+\sin\pi t\,k\right]$

$$=3\lim_{t\to1}t\,i+\lim_{t\to1}e^{2t}\,j+\lim_{t\to1}\sin\pi t\,k$$

$$=3(1)i+e^{2(1)}\,j+\sin(1\pi)k$$

$$=3i+e^2\,j+\sin\pi\,k$$

$$=3i+e^2\,j \qquad \because \sin\pi=0$$

b). $\quad \lim_{t\to0}\left[\frac{\sin t\,i-t\,k}{t^2+t-1}\right]$

Solution: We have $\lim_{t\to0}\left[\frac{\sin t\,i-t\,k}{t^2+t-1}\right]$

$$=\frac{\lim_{t\to0}\sin t\,i-\lim_{t\to0}t\,k}{\lim_{t\to0}t^2+\lim_{t\to0}t-\lim_{t\to0}1}$$

$$=\frac{\sin(0)i-(0)k}{(0)^2+(0)-1}=\frac{0-0}{0+0-1}=\frac{0}{-1}=0$$

c). $\lim_{t\to1}\left[\frac{t^3-1}{t-1}i+\frac{t^2-3t+2}{t^2+t-2}j+\left(t^2+1\right)e^{t-1}k\right]$

Solution: Let $A=\frac{t^3-1}{t-1}$, $B=\frac{t^2-3t+2}{t^2+t-2}$ and

$C=\left(t^2+1\right)e^{t-1}$ then the given function will be

$$\lim_{t\to1}\left[\frac{t^3-1}{t-1}i+\frac{t^2-3t+2}{t^2+t-2}j+\left(t^2+1\right)e^{t-1}k\right]=\lim_{t\to1}\left[Ai+Bj+Ck\right]$$

Now apply the limit individually

$$\lim_{t\to1}A=\lim_{t\to1}\frac{t^3-1}{t-1}\qquad \frac{0}{0}\text{ Form}$$

$$\lim_{t\to1}A=\lim_{t\to1}\frac{t^3-1^3}{t-1}$$

$$\lim_{t\to1}A=\lim_{t\to1}\frac{(t-1)\left(t^2+t.1+1^2\right)}{t-1}$$

$$\lim_{t\to1}A=\lim_{t\to1}\left(t^2+t+1\right)$$

$$\lim_{t\to1}A=\lim_{t\to1}t^2+\lim_{t\to1}t+\lim_{t\to1}1$$

$$\lim_{t\to1}A=(1)^2+1+1=3$$

Now $\lim_{t\to1}B=\lim_{t\to1}\frac{t^2-3t+2}{t^2+t-2}\qquad \frac{0}{0}\text{ Form}$

$$\lim_{t\to1}B=\lim_{t\to1}\frac{t^2-2t-t+2}{t^2+2t-t-2}$$

$$\lim_{t\to1}B=\lim_{t\to1}\frac{t(t-2)-1(t-2)}{t(t+2)-1(t+2)}$$

$$\lim_{t\to1}B=\lim_{t\to1}\frac{(t-1)(t-2)}{(t-1)(t+2)}$$

$$\lim_{t\to1}B=\lim_{t\to1}\frac{t-2}{t+2}$$

$$\lim_{t\to1}B=\frac{\lim_{t\to1}t-\lim_{t\to1}2}{\lim_{t\to1}t+\lim_{t\to1}2}$$

$$\lim_{t\to1}B=\frac{1-2}{1+2}=\frac{-1}{3}$$

Now $\lim_{t\to1}C=\lim_{t\to1}\left(t^2+1\right)\lim_{t\to1}e^{t-1}$

$$\lim_{t\to1}C=\left(1^2+1\right)e^{1-1}$$

$$\lim_{t\to1}C=(1+1)e^0$$

$$\lim_{t\to1}C=(2)(1)=2$$

Substituting the values

$$\lim_{t\to1}\left[Ai+Bj+Ck\right]=\lim_{t\to1}Ai+\lim_{t\to1}Bj+\lim_{t\to1}Ck$$

$$\lim_{t\to1}\left[Ai+Bj+Ck\right]=3i+\left(\frac{-1}{3}\right)j+2k$$

$$\lim_{t\to1}\left[Ai+Bj+Ck\right]=3i-\frac{1}{3}j+2k$$

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d). $\lim_{t\to0}\left[\frac{te^t}{1-e^t}i+\frac{e^{t-1}}{\cos t}j\right]$

Solution: we have $\lim_{t\to0}\left[\frac{te^t}{1-e^t}i+\frac{e^{t-1}}{\cos t}j\right]$

Let $A=\frac{te^t}{1-e^t}, B=\frac{e^{t-1}}{\cos t}$

then the given function will be

$$\lim_{t\to0}\left[\frac{te^t}{1-e^t}i+\frac{e^{t-1}}{\cos t}j\right]=\lim_{t\to0}\left[Ai+Bj\right]$$

Now apply the limit individually

$$\lim_{t\to0}A=\lim_{t\to0}\frac{te^t}{1-e^t}\qquad \frac{0}{0}\text{ Form}$$

Apply L Hopital's Rule

$$\lim_{t\to0}A=\lim_{t\to0}\frac{\frac{d}{dt}\left(te^t\right)}{\frac{d}{dt}\left(1-e^t\right)}$$

$$\lim_{t\to0}A=\lim_{t\to0}\frac{t\frac{d}{dt}e^t+e^t\frac{d}{dt}t}{\frac{d}{dt}1-\frac{d}{dt}e^t}$$

$$\lim_{t\to0}A=\lim_{t\to0}\frac{te^t+e^t}{0-e^t}\qquad \therefore \frac{d}{dt}e^t=e^t\frac{d}{dt}t=e^t$$

$$\lim_{t\to0}A=\lim_{t\to0}\frac{e^t(t+1)}{-e^t}$$

$$\lim_{t\to0}A=\lim_{t\to0}-(t+1)$$

$$\lim_{t\to0}A=-\left(\lim_{t\to0}t+\lim_{t\to0}1\right)$$

$$\lim_{t\to0}A=-(0+1)=-1$$

Now $\lim_{t\to0}B=\lim_{t\to0}\frac{e^{t-1}}{\cos t}$

$$\lim_{t\to0}B=\frac{\lim_{t\to0}e^{t-1}}{\lim_{t\to0}\cos t}$$

$$\lim_{t\to0}B=\frac{e^{0-1}}{\cos(0)}=\frac{e^{-1}}{1}=\frac{1}{e}$$

Substituting the values

$$\lim_{t\to0}\left[\frac{te^t}{1-e^t}i+\frac{e^{t-1}}{\cos t}j\right]=\lim_{t\to0}\left[Ai+Bj\right]$$

$$\lim_{t\to0}\left[\frac{te^t}{1-e^t}i+\frac{e^{t-1}}{\cos t}j\right]=\lim_{t\to0}Ai+\lim_{t\to0}Bj$$

$$\lim_{t\to0}\left[\frac{te^t}{1-e^t}i+\frac{e^{t-1}}{\cos t}j\right]=-1i+\frac{1}{e}j=i+\frac{1}{e}j$$

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e). $\lim_{t\to0}\left[\frac{\sin t}{t}i+\frac{1-\cos t}{t}j+e^{1-t}k\right]$

Sol: $\lim_{t\to0}\left[\frac{\sin t}{t}i+\frac{1-\cos t}{t}j+e^{1-t}k\right]$

Let $A=\frac{\sin t}{t}$, $B=\frac{1-\cos t}{t}$, $C=e^{1-t}$

Then the given function is

$$\lim_{t\to0}\left[\frac{\sin t}{t}i+\frac{1-\cos t}{t}j+e^{1-t}k\right]=\lim_{t\to0}\left[Ai+Bj+Ck\right]$$

Now apply the limit individually

$$\lim_{t\to0}A=\lim_{t\to0}\frac{\sin t}{t}\qquad \frac{0}{0}\text{ Form}$$

Apply L Hopital's Rule

$$\lim_{t\to0}A=\lim_{t\to0}\frac{\frac{d}{dt}\sin t}{\frac{d}{dt}t}$$

$$\lim_{t\to0}A=\lim_{t\to0}\frac{\cos t}{1}=\cos(0)=1$$

Now $\lim_{t\to0} B = \lim_{t\to0}\dfrac{1-\cos t}{t}$ $\quad \dfrac{0}{0}$ Form

Apply L Hopital's Rule

$$\lim_{t\to0} B = \lim_{t\to0}\frac{\frac{d}{dt}(1-\cos t)}{\frac{d}{dt}t}$$

$$\lim_{t\to0} B = \lim_{t\to0}\frac{\frac{d}{dt}1-\frac{d}{dt}\cos t}{1}$$

$$\lim_{t\to0} B = \lim_{t\to0}\frac{0-(-\sin t)\frac{d}{dt}t}{1}$$

$$\lim_{t\to0} B = \lim_{t\to0}(\sin t) = \sin(0) = 0$$

Now $\lim_{t\to0} C = \lim_{t\to0} e^{1-t} = e^{1-0}$

$\lim_{t\to0} C = e$

Substituting the values

$$\lim_{t\to0}\left[\frac{\sin t}{t}i+\frac{1-\cos t}{t}j+e^{1-t}k\right] = \lim_{t\to0}[Ai+Bj+Ck]$$

$$\lim_{t\to0}\left[\frac{\sin t}{t}i+\frac{1-\cos t}{t}j+e^{1-t}k\right] = \lim_{t\to0}Ai+\lim_{t\to0}Bj+\lim_{t\to0}Ck$$

$$\lim_{t\to0}\left[\frac{\sin t}{t}i+\frac{1-\cos t}{t}j+e^{1-t}k\right] = (1)i+(0)j+ek$$

$$\lim_{t\to0}\left[\frac{\sin t}{t}i+\frac{1-\cos t}{t}j+e^{1-t}k\right] = i+ek$$

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f). $\lim_{t\to0}\left[\dfrac{\sin 3t}{\sin 2t}i+\dfrac{\ln(\sin t)}{\ln(\tan t)}j+t^t k\right]$

Sol: $\lim_{t\to0}\left[\dfrac{\sin 3t}{\sin 2t}i+\dfrac{\ln(\sin t)}{\ln(\tan t)}j+t^t k\right]$

Let $A=\dfrac{\sin 3t}{\sin 2t}$, $\quad B=\dfrac{\ln(\sin t)}{\ln(\tan t)}$, $\quad C=t^t$

Then the given function is

$$\lim_{t\to0}\left[\frac{\sin 3t}{\sin 2t}i+\frac{\ln(\sin t)}{\ln(\tan t)}j+t^t k\right] = \lim_{t\to0}[Ai+Bj+Ck]$$

Now apply the limit individually

$\lim_{t\to0} A = \lim_{t\to0}\dfrac{\sin 3t}{\sin 2t}$ $\quad \dfrac{0}{0}$ Form

Apply L Hopital's Rule

$$\lim_{t\to0} A = \lim_{t\to0}\frac{\frac{d}{dt}\sin 3t}{\frac{d}{dt}\sin 2t}$$

$$\lim_{t\to0} A = \lim_{t\to0}\frac{\cos 3t\frac{d}{dt}(3t)}{\cos 2t\frac{d}{dt}(2t)} = \lim_{t\to0}\frac{3\cos 3t}{2\cos 2t}$$

$$\lim_{t\to0} A = \frac{3\lim_{t\to0}\cos 3t}{2\lim_{t\to0}\cos 2t} = \frac{3\cos(0)}{2\cos(0)} = \frac{3}{2}$$

Now $\lim_{t\to0} B = \lim_{t\to0}\dfrac{\ln(\sin t)}{\ln(\tan t)}$ $\quad \dfrac{0}{0}$ Form

Apply L Hopital's Rule

$$\lim_{t\to0} B = \lim_{t\to0}\frac{\frac{d}{dt}\ln(\sin t)}{\frac{d}{dt}\ln(\tan t)} = \lim_{t\to0}\frac{\frac{1}{\sin t}\frac{d}{dt}\sin t}{\frac{1}{\tan t}\frac{d}{dt}\tan t}$$

$$\lim_{t\to0} B = \lim_{t\to0}\frac{\frac{1}{\sin t}\cos t}{\frac{\cos t}{\sin t}\sec^2 t} = \lim_{t\to0}\frac{\cos t}{\cos t.\sec^2 t}$$

$$\lim_{t\to0} B = \lim_{t\to0}\frac{1}{\sec^2 t} = \lim_{t\to0}\cos^2 t$$

$$\lim_{t\to0} B = \{\cos(0)\}^2 = \{1\}^2 = 1$$

Since $C=t^t$ Taking ln (natural log) on both ssides

$\ln C = \ln t^t$

$\ln C = t\ln t$

$\ln C = \dfrac{\ln t}{t^{-1}}$

Apply the limit on both sides

$\lim_{t\to0}\{\ln C\} = \lim_{t\to0}\left\{\dfrac{\ln t}{t^{-1}}\right\}$ $\quad \dfrac{\infty}{\infty}$ Form

Apply L Hopital's Rule

$$\lim_{t\to0}\{\ln C\} = \lim_{t\to0}\frac{\frac{d}{dt}\ln t}{\frac{d}{dt}t^{-1}}$$

$$\lim_{t\to0}\{\ln C\} = \lim_{t\to0}\frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim_{t\to0}\frac{1}{t}\div\frac{-1}{t^2}$$

$$\lim_{t\to0}\{\ln C\} = \lim_{t\to0}\frac{1}{t}\times\frac{t^2}{-1} = \lim_{t\to0}(-t) = -0$$

$$\lim_{t\to0}\{\ln C\} = 0$$

$$\lim_{t\to0} C = e^0 = 1$$

Substituting the values

$$\lim_{t\to0}\left[\frac{\sin 3t}{\sin 2t}i+\frac{\ln(\sin t)}{\ln(\tan t)}j+t^t k\right] = \lim_{t\to0}[Ai+Bj+Ck]$$

$$\lim_{t\to0}\left[\frac{\sin 3t}{\sin 2t}i+\frac{\ln(\sin t)}{\ln(\tan t)}j+t^t k\right] = \lim_{t\to0}Ai+\lim_{t\to0}Bj+\lim_{t\to0}Ck$$

$$\lim_{t\to0}\left[\frac{\sin 3t}{\sin 2t}i+\frac{\ln(\sin t)}{\ln(\tan t)}j+t^t k\right] = \left(\frac{3}{2}\right)i+(1)j+(1)k$$

$$\lim_{t\to0}\left[\frac{\sin 3t}{\sin 2t}i+\frac{\ln(\sin t)}{\ln(\tan t)}j+t^t k\right] = \frac{3}{2}i+j+k$$

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g). $\lim_{t\to1}\left[2t\,i-3\,j+e^t k\right]$

Solution: We have $\lim_{t\to1}\left[2t\,i-3\,j+e^t k\right]$

$$\lim_{t\to1}\left[2t\,i-3\,j+e^t k\right] = \lim_{t\to1}2t\,i-\lim_{t\to1}3\,j+\lim_{t\to1}e^t k$$

$$\lim_{t\to1}\left[2t\,i-3\,j+e^t k\right] = 2(1)i-3\,j+e^1 k$$

$$\lim_{t\to1}\left[2t\,i-3\,j+e^t k\right] = 2i-3\,j+ek$$

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h). $\lim_{t\to2}\left[\left(2i-t\,j+e^t k\right)\times\left(t^2 i+4\sin t\,j\right)\right]$

Sol: $\lim_{t\to2}\left[\left(2i-t\,j+e^t k\right)\times\left(t^2 i+4\sin t\,j\right)\right]$

$$=\lim_{t\to2}\begin{vmatrix} i & j & k \\ 2 & -t & e^t \\ t^2 & 4\sin t & 0 \end{vmatrix}$$

$$=\lim_{t\to2}\left\{i\left(0-4e^t\sin t\right)-j\left(0-e^t t^2\right)+k\left(8\sin t+t^3\right)\right\}$$

$$=\lim_{t\to2}\left\{-4e^t\sin t\,i+e^t t^2\,j+\left(8\sin t+t^3\right)k\right\}$$

$$=-4e^2\sin(2)i+e^2(2)^2\,j+\left(8\sin(2)+(2)^3\right)k$$

$$=-4e^2\sin(2)i+4e^2\,j+\left(8\sin(2)+8\right)k$$

$$=-4e^2\sin(2)i+4e^2\,j+8\left(\sin(2)+1\right)k$$

Q5. Test the continuity of the following expressions for all value of t

a). $F(t)=t\,i+3\,j-(1-t)k$

Solution: We have $F(t)=t\,i+3\,j-(1-t)k$

Domain of the given vector function $F(t)$ is set of real number, therefore the given function is continuous in the whole set of real numbers

---

b). $G(t)=t\,i-t^{-1}k$

Solution: We have $G(t)=t\,i-\frac{1}{t}k$

Denominator $t\neq 0$ then Domain of the given vector function $F(t)$ is set of real number except $t=0$, therefore the given function is continuous in the whole set of real numbers except $t=0$

---

c). $G(t)=\dfrac{i+2\,j}{t^2+t}$

Solution: We have $G(t)=\dfrac{i+2\,j}{t^2+t}$

Denominator $t^2+t\neq 0$

$t(t+1)\neq 0$

Either $t\neq 0$ or $t+1\neq 0$, $t\neq -1$

Domain of the given vector function $F(t)$ is set of real number except $t=0$ & $t=-1$, therefore the given function is continuous in the whole set of real numbers except $t=0$ & $t=-1$

---

d). $F(t)=e^t\sin t\,i+e^t\cos t\,k$

Solution: We have $F(t)=e^t\sin t\,i+e^t\cos t\,k$

Domain of the given vector function $F(t)$ is set of real number, therefore the given function is continuous in the whole set of real numbers

---

e). $F(t)=e^t\left(t\,i+t^{-1}\,j+3k\right)$

Solution: We have $F(t)=e^t\left(t\,i+\frac{1}{t}\,j+3k\right)$

Denominator $t\neq 0$ then Domain of the given vector function $F(t)$ is set of real number except $t=0$, therefore the given function is continuous in the whole set of real numbers except $t=0$

---

f). $G(t)=\dfrac{t\,i+\sqrt{t}\,j}{\sqrt{t^2+t}}$

Solution: We have $G(t)=\dfrac{t\,i+\sqrt{t}\,j}{\sqrt{t^2+t}}$

Denominator is radicand $\sqrt{t^2+t}$ Take

$t^2+t\neq 0$

$t(t+1)\neq 0$

Either $t\neq 0$ or $t+1\neq 0$, $t\neq -1$

Domain of the given vector function $F(t)$ is set of real number except $t=0$ & $t=-1$, therefore the given function is continuous in the whole set of real numbers except $t=0$ & $t=-1$

Example 4.4.2 If $F(t)=i+e^t\,j+t^2k$, and $G(t)=3t^2i+e^{-t}\,j-2tk$ then find out $\frac{d}{dt}(F.G)$

Solution: we have $F(t)=i+e^t\,j+t^2k$, and $G(t)=3t^2i+e^{-t}\,j-2tk$

First we find $F.G$

$$F(t).G(t)=\left(i+e^t\,j+t^2k\right)\left(3t^2i+e^{-t}\,j-2tk\right)$$

$$F(t).G(t)=3t^2+e^{t-t}+t^2(-2t)$$

$$F(t).G(t)=3t^2+1-2t^3$$

Now differentiating with respect to t

$$\tfrac{d}{dt}\left[F(t).G(t)\right]=\tfrac{d}{dt}\left(3t^2+1-2t^3\right)$$

$$\tfrac{d}{dt}\left[F(t).G(t)\right]=3\tfrac{d}{dt}t^2+\tfrac{d}{dt}1-2\tfrac{d}{dt}t^3$$

$$\tfrac{d}{dt}\left[F(t).G(t)\right]=3(2t)+0-2\left(3t^2\right)$$

$$\tfrac{d}{dt}\left[F(t).G(t)\right]=6t-6t^2$$

Exercise 4.2

Q1. Find the vector derivative $F'(t)$ of the following vector functions:

a). $F(t)=t\,i+t^2\,j+\left(t+t^3\right)k$

Solution: We have $F(t)=t\,i+t^2\,j+\left(t+t^3\right)k$

Differentiating both sides with respect to t

$$\tfrac{d}{dt}F(t)=\tfrac{d}{dt}\left\{t\,i+t^2\,j+\left(t+t^3\right)k\right\}$$

$$F'(t)=\tfrac{d}{dt}t\,i+\tfrac{d}{dt}t^2\,j+\tfrac{d}{dt}\left(t+t^3\right)k$$

$$F'(t)=(1)i+2t\tfrac{d}{dt}t\,j+\left(\tfrac{d}{dt}t+\tfrac{d}{dt}t^3\right)k$$

$$F'(t)=i+2t\,j+\left(1+3t^2\tfrac{d}{dt}t\right)k$$

$$F'(t)=i+2t\,j+\left(1+3t^2\right)k$$

---

b). $F(s)=\left(s\,i+s^2\,j+s^2\,k\right)+\left(2s^2\,i-s\,j+3k\right)$

Solution: Since

$$F(s)=\left(s\,i+s^2\,j+s^2\,k\right)+\left(2s^2\,i-s\,j+3k\right)$$

$$F(s)=s\,i+s^2\,j+s^2\,k+2s^2\,i-s\,j+3k$$

$$F(s)=s\,i+2s^2\,i+s^2\,j-s\,j+s^2\,k+3k$$

$$F(s)=\left(s+2s^2\right)i+\left(s^2-s\right)j+\left(s^2+3\right)k$$

Differentiating both sides with respect to s

$$\tfrac{d}{ds}F(s)=\tfrac{d}{ds}\left\{\left(s+2s^2\right)i+\left(s^2-s\right)j+\left(s^2+3\right)k\right\}$$

$$F'(s)=\tfrac{d}{ds}\left(s+2s^2\right)i+\tfrac{d}{ds}\left(s^2-s\right)j+\tfrac{d}{ds}\left(s^2+3\right)k$$

$$F'(s)=\left(\tfrac{d}{ds}s+2\tfrac{d}{ds}s^2\right)i+\left(\tfrac{d}{ds}s^2-\tfrac{d}{ds}s\right)j+\left(\tfrac{d}{ds}s^2+\tfrac{d}{ds}3\right)k$$

$$F'(s)=\left(1+2(2s)\tfrac{d}{ds}s\right)i+\left(2s\tfrac{d}{ds}s-1\right)j+\left(2s\tfrac{d}{ds}s+0\right)k$$

$$F'(s)=(1+4s)i+(2s-1)j+2s\,k$$

---

c). $F(\theta)=\cos\theta\left[i+\tan\theta\,j+3k\right]$

Solution: We have $F(\theta)=\cos\theta\left[i+\tan\theta\,j+3k\right]$

$$F(\theta)=\cos\theta\left[i+\frac{\sin\theta}{\cos\theta}\,j+3k\right]$$

$$F(\theta)=\cos\theta\,i+\sin\theta\,j+3\cos\theta\,k$$

Differentiating both sides with respect to θ

$$\tfrac{d}{d\theta}F(\theta)=\tfrac{d}{d\theta}\{\cos\theta\,i+\sin\theta\,j+3\cos\theta\,k\}$$

$$F'(\theta)=\tfrac{d}{d\theta}\cos\theta\,i+\tfrac{d}{d\theta}\sin\theta\,j+3\tfrac{d}{d\theta}\cos\theta\,k$$

$$F'(\theta)=-\sin\theta\tfrac{d}{d\theta}\theta\,i+\cos\theta\tfrac{d}{d\theta}\theta\,j+3(-\sin\theta)\tfrac{d}{d\theta}\theta\,k$$

$$F'(\theta)=-\sin\theta\,i+\cos\theta\,j-3\sin\theta\,k$$

Q2. Find $F'(t)$ and $F''(t)$ of following vectors functions:

a). $F(t)=t^2\,i+t^{-1}\,j+e^{2t}\,k$

Solution: We have $F(t)=t^2\,i+t^{-1}\,j+e^{2t}\,k$

Differentiating both sides with respect to t

$$\tfrac{d}{dt}F(t)=\tfrac{d}{dt}\{t^2\,i+t^{-1}\,j+e^{2t}\,k\}$$

$$F'(t)=\tfrac{d}{dt}t^2\,i+\tfrac{d}{dt}t^{-1}\,j+\tfrac{d}{dt}e^{2t}\,k$$

$$F'(t)=2t\tfrac{d}{dt}t\,i+\left(-t^{-1-1}\right)\tfrac{d}{dt}t\,j+e^{2t}\tfrac{d}{dt}(2t)k$$

$$F'(t)=2t\,i-t^{-2}\,j+2e^{2t}\tfrac{d}{dt}(t)k$$

$$F'(t)=2t\,i-t^{-2}\,j+2e^{2t}\,k$$

Again differentiating both sides with respect to x

$$\tfrac{d}{dt}F'(t)=\tfrac{d}{dt}\{2t\,i-t^{-2}\,j+2e^{2t}\,k\}$$

$$F''(t)=2\tfrac{d}{dt}t\,i-\tfrac{d}{dt}t^{-2}\,j+2\tfrac{d}{dt}e^{2t}\,k$$

$$F''(t)=2i-\left(-2t^{-2-1}\right)\tfrac{d}{dt}t\,j+2e^{2t}\tfrac{d}{dt}(2t)k$$

$$F''(t)=2i+2t^{-3}\,j+4e^{2t}\tfrac{d}{dt}(t)k$$

$$F''(t)=2i+2t^{-3}\,j+4e^{2t}\,k$$

---

b). $F(s)=\left(1-2s^2\right)i+s(\cos s)\,j-s\,k$

Solution: Since $F(s)=\left(1-2s^2\right)i+s(\cos s)\,j-s\,k$

$$\tfrac{d}{ds}F(s)=\tfrac{d}{ds}\left\{\left(1-2s^2\right)i+s(\cos s)\,j-s\,k\right\}$$

$$F'(s)=\tfrac{d}{ds}\left(1-2s^2\right)i+\tfrac{d}{ds}\{s(\cos s)\}\,j-\tfrac{d}{ds}s\,k$$

$$F'(s)=\left(\tfrac{d}{ds}1-2\tfrac{d}{ds}s^2\right)i+\left\{s\tfrac{d}{ds}(\cos s)+\cos s\tfrac{d}{ds}s\right\}j-(1)k$$

$$F'(s)=\left(0-2(2s)\tfrac{d}{ds}s\right)i+\left\{s(-\sin s)\tfrac{d}{ds}s+\cos s\right\}j-k$$

$$F'(s)=-4s\,i+\{-s\sin s+\cos s\}\,j-k$$

Again differentiating both sides with respect to s

$$\tfrac{d}{ds}F'(s)=\tfrac{d}{ds}\left[-4s\,i+\{-s\sin s+\cos s\}\,j-k\right]$$

$$F''(s)=-4\tfrac{d}{ds}s\,i+\tfrac{d}{ds}\{-s\sin s+\cos s\}\,j-\tfrac{d}{ds}k$$

$$F''(s)=-4i+\left\{-\tfrac{d}{ds}(s\sin s)+\tfrac{d}{ds}\cos s\right\}j-0$$

$$F''(s)=-4i+\left\{-\left(s\tfrac{d}{ds}\sin s+\sin s\tfrac{d}{ds}s\right)-\sin s\right\}j$$

$$F''(s)=-4i+\{-(s\cos s+\sin s)-\sin s\}\,j$$

$$F''(s)=-4i-(s\cos s+\sin s+\sin s)\,j$$

$$F''(s)=-4i-(s\cos s+2\sin s)\,j$$

---

c). $F(s)=\sin s\,i+\cos s\,j+s^2\,k$

Solution: We have $F(s)=\sin s\,i+\cos s\,j+s^2\,k$

Differentiating both sides with respect to s

$$\tfrac{d}{ds}F(s)=\tfrac{d}{ds}\left\{\sin s\,i+\cos s\,j+s^2\,k\right\}$$

$$F'(s)=\tfrac{d}{ds}\sin s\,i+\tfrac{d}{ds}\cos s\,j+\tfrac{d}{ds}s^2\,k$$

$$F'(s)=\cos s\tfrac{d}{ds}s\,i+(-\sin s)\tfrac{d}{ds}s\,j+2s\tfrac{d}{ds}s\,k$$

$$F'(s)=\cos s\,i-\sin s\,j+2s\,k$$

Again differentiating both sides with respect to s

$$\tfrac{d}{ds}F'(s)=\tfrac{d}{ds}\{\cos s\,i-\sin s\,j+2s\,k\}$$

$$F''(s)=\tfrac{d}{ds}\cos s\,i-\tfrac{d}{ds}\sin s\,j+2\tfrac{d}{ds}s\,k$$

$$F''(s)=-\sin s\tfrac{d}{ds}s\,i-\cos s\tfrac{d}{ds}s\,j+2k$$

$$F''(s)=-\sin s\,i-\cos s\,j+2k$$

---

d). $F(\theta)=\sin^2\theta\,i+\cos 2\theta\,j+\theta^2\,k$

Sol: We have $F(\theta)=\sin^2\theta\,i+\cos 2\theta\,j+\theta^2\,k$

Differentiating both sides with respect to θ

$$\tfrac{d}{d\theta}F(\theta)=\tfrac{d}{d\theta}\left\{\sin^2\theta\,i+\cos 2\theta\,j+\theta^2\,k\right\}$$

$$F'(\theta)=\tfrac{d}{d\theta}\sin^2\theta\,i+\tfrac{d}{d\theta}\cos 2\theta\,j+\tfrac{d}{d\theta}\theta^2\,k$$

$$F'(\theta)=2\sin\theta\tfrac{d}{d\theta}\sin\theta\,i+(-\sin 2\theta)\tfrac{d}{d\theta}(2\theta)\,j+2\theta\tfrac{d}{d\theta}\theta\,k$$

$$F'(\theta)=2\sin\theta\cos\theta\tfrac{d}{d\theta}\theta\,i-2\sin 2\theta\tfrac{d}{d\theta}\theta\,j+2\theta\,k$$

$$F'(\theta)=\sin 2\theta\,i-2\sin 2\theta\,j+2\theta\,k$$

$$\therefore 2\sin\theta\cos\theta=\sin 2\theta$$

Again differentiating both sides with respect to x

$$\tfrac{d}{d\theta}F'(\theta)=\tfrac{d}{d\theta}\{\sin 2\theta\,i-2\sin 2\theta\,j+2\theta\,k\}$$

$$F''(\theta)=\tfrac{d}{d\theta}\sin 2\theta\,i-2\tfrac{d}{d\theta}\sin 2\theta\,j+2\tfrac{d}{d\theta}\theta\,k$$

$$F''(\theta)=\cos 2\theta\tfrac{d}{d\theta}(2\theta)\,i-2\cos 2\theta\tfrac{d}{d\theta}(2\theta)\,j+2k$$

$$F''(\theta)=2\cos 2\theta\tfrac{d}{d\theta}\theta\,i-4\cos 2\theta\tfrac{d}{d\theta}\theta\,j+2k$$

$$F''(\theta)=2\cos 2\theta\,i-4\cos 2\theta\,j+2k$$

Q3. Differentiate the following scalar functions:

a). $f(x)=\left[x\,i+(x+1)\,j\right].\left[2x\,i-3x^2\,j\right]$

Sol: Since $f(x)=\left[x\,i+(x+1)\,j\right].\left[2x\,i-3x^2\,j\right]$

$$\tfrac{d}{dx}f(x)=\tfrac{d}{dx}\left\{\left[x\,i+(x+1)\,j\right].\left[2x\,i-3x^2\,j\right]\right\}$$

$$f'(x)=\left[x\,i+(x+1)\,j\right]\tfrac{d}{dx}\left[2x\,i-3x^2\,j\right]+\left[2x\,i-3x^2\,j\right]\tfrac{d}{dx}\left[x\,i+(x+1)\,j\right]$$

$$f'(x)=\left[x\,i+(x+1)\,j\right]\left[2\tfrac{d}{dx}x\,i-3\tfrac{d}{dx}x^2\,j\right]+\left[2x\,i-3x^2\,j\right]\left[\tfrac{d}{dx}x\,i+\tfrac{d}{dx}(x+1)\,j\right]$$

$$f'(x)=\left[x\,i+(x+1)\,j\right]\left[2i-3(2x)\tfrac{d}{dx}x\,j\right]+\left[2x\,i-3x^2\,j\right]\left[i+\left(\tfrac{d}{dx}x+\tfrac{d}{dx}1\right)j\right]$$

$$f'(x)=\left[x\,i+(x+1)\,j\right]\left[2i-6x\,j\right]+\left[2x\,i-3x^2\,j\right]\left[i+(1+0)\,j\right]$$

$$f'(x)=\left[x\,i+(x+1)\,j\right]\left[2i-6x\,j\right]+\left[2x\,i-3x^2\,j\right]\left[i+j\right]$$

$$f'(x)=\{2x-6x(x+1)\}+\left\{2x-3x^2\right\}$$

$$f'(x)=2x-6x^2-6x+2x-3x^2$$

$$f'(x)=2x-6x+2x-6x^2-3x^2$$

$$f'(x)=-2x-9x^2$$

---

b). $f(x)=\left[\cos x\,i+x\,j-x\,k\right].\left[\sec x\,i-x^2\,j+2x\,k\right]$

Sol: Since $f(x)=\left[\cos x\,i+x\,j-x\,k\right].\left[\sec x\,i-x^2\,j+2x\,k\right]$

Differentiating both sides with respect to x

$\frac{d}{dx} f(x) = \frac{d}{dx}\{[\cos x\,i + x\,j - x\,k].[\sec x\,i - x^2 j + 2x\,k]\}$

$f'(x) = [\cos x\,i + x\,j - x\,k]\frac{d}{dx}[\sec x\,i - x^2 j + 2x\,k]$
$\quad + [\sec x\,i - x^2 j + 2x\,k]\frac{d}{dx}[\cos x\,i + x\,j - x\,k]$

$f'(x) = [\cos x\,i + x\,j - x\,k][\frac{d}{dx}\sec x\,i - \frac{d}{dx}x^2 j + 2\frac{d}{dx}x\,k]$
$\quad + [\sec x\,i - x^2 j + 2x\,k][\frac{d}{dx}\cos x\,i + \frac{d}{dx}x\,j - \frac{d}{dx}x\,k]$

$f'(x) = [\cos x\,i + x\,j - x\,k][\sec x\tan x\frac{d}{dx}x\,i - 2x\frac{d}{dx}x\,j + 2k]$
$\quad + [\sec x\,i - x^2 j + 2x\,k][-\sin x\frac{d}{dx}x\,i + j - k]$

$f'(x) = [\cos x\,i + x\,j - x\,k][\sec x\tan x\,i - 2x\,j + 2k]$
$\quad + [\sec x\,i - x^2 j + 2x\,k][-\sin x\,i + j - k]$

$f'(x) = \cos x\sec x\tan x - 2x^2 - 2x - \sin x\sec x - x^2 - 2x$

$f'(x) = \cos x.\frac{1}{\cos x}\tan x - \sin x\frac{1}{\cos x} - 2x^2 - x^2 - 2x - 2x$

$f'(x) = \tan x - \tan x - 3x^2 - 4x$

$f'(x) = -3x^2 - 4x$

c). $g(x) = |\sin x\,i - 2x\,j + \cos x\,k|$

Solution: We have $g(x) = |\sin x\,i - 2x\,j + \cos x\,k|$

$g(x) = \sqrt{(\sin x)^2 + (-2x)^2 + (\cos x)^2}$

$g(x) = \sqrt{\sin^2 x + 4x^2 + \cos^2 x}$

$g(x) = \sqrt{\sin^2 x + \cos^2 x + 4x^2}$

$g(x) = \sqrt{1 + 4x^2}$

Differentiating both sides with respect to x

$\frac{d}{dx} g(x) = \frac{d}{dx}[1 + 4x^2]^{\frac{1}{2}}$

$g'(x) = \frac{1}{4}[1 + 4x^2]^{\frac{1}{2}-1}\frac{d}{dx}[1 + 4x^2]$

$g'(x) = \frac{1}{4}[1 + 4x^2]^{\frac{-1}{2}}[8x]$

$g'(x) = \frac{2x}{\sqrt{1 + 4x^2}}$

Q4. Find the particle's velocity, acceleration, speed and direction of motion for the indicated value of t, when the position vector of a particle's in space at time t is $R(t)$

a). $R(t) = t\,i + t^2 j + 2t\,k$ at $t = 1$

Solution: Since the rate of change of displacement with respect to time is called velocity i.e., $V(t) = \frac{d}{dt}R(t)$

Therefore differentiate $R(t)$ with respect to time t

$V(t) = \frac{d}{dt}R(t) = \frac{d}{dt}t\,i + \frac{d}{dt}t^2 j + 2\frac{d}{dt}t\,k$

$V(t) = i + 2t\frac{d}{dt}t\,j + 2k$

$V(t) = i + 2t\,j + 2k$

The velocity at time $t = 1$

$V(1) = i + 2(1)\,j + 2k$

$V(1) = i + 2\,j + 2k$

And the speed

$|V(t)| = \sqrt{1^2 + 2^2 + 2^2}$

$|V(t)| = \sqrt{1 + 4 + 4}$

$|V(t)| = \sqrt{9} = 3$

Speed at time $t = 1$

$|V(1)| = 3$

The direction of motion at time t = 1

$\frac{V(1)}{|V(1)|} = \frac{i + 2\,j + 2k}{3} = \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k$

Since the rate of change of velocity with respect to time is called acceleration i.e., $A(t) = \frac{d}{dt}V(t)$

Therefore differentiate $V(t)$ with respect to time t

$A(t) = \frac{d}{dt}V(t) = \frac{d}{dt}i + 2\frac{d}{dt}t\,j + \frac{d}{dt}2k$

$A(t) = \frac{d}{dt}V(t) = 0 + 2\,j + 0$

$A(t) = \frac{d}{dt}V(t) = 2\,j$

Acceleration at time $t = 1$

$A(1) = 2\,j$

b). $R(t) = (1 - 2t)i - t^2\,j + e^t\,k$ at $t = 0$

Solution: Since the rate of change of displacement with respect to time is called velocity i.e., $V(t) = \frac{d}{dt}R(t)$

Therefore differentiate $R(t)$ with respect to time t

$V(t) = \frac{d}{dt}R(t) = \frac{d}{dt}(1 - 2t)i - \frac{d}{dt}t^2\,j + \frac{d}{dt}e^t\,k$

$V(t) = \frac{d}{dt}R(t) = (\frac{d}{dt}1 - 2\frac{d}{dt}t)i - 2t\frac{d}{dt}t\,j + e^t\frac{d}{dt}t\,k$

$V(t) = \frac{d}{dt}R(t) = (0 - 2)i - 2t\,j + e^t\,k$

$V(t) = \frac{d}{dt}R(t) = -2i - 2t\,j + e^t\,k$

The velocity at time $t = 0$

$V(0) = -2i - 2(0)\,j + e^0\,k$

$V(0) = -2i - 0\,j + 1k$

$V(0) = -2i + k$

And the speed

$|V(t)| = \sqrt{(-2)^2 + (-2t)^2 + (e^t)^2}$

$|V(t)| = \sqrt{4 + 4t^2 + e^{2t}}$

Speed at time $t = 0$

$|V(0)| = \sqrt{4 + 4(0)^2 + e^{2(0)}}$

$|V(0)| = \sqrt{4 + 4(0) + e^0}$

$|V(0)| = \sqrt{4 + 0 + 1} = \sqrt{5}$

The direction of motion at time $t = 0$

$\frac{V(0)}{|V(0)|} = \frac{-2i + k}{\sqrt{5}} = \frac{-2}{\sqrt{5}}i + \frac{1}{\sqrt{5}}k$

Since the rate of change of velocity with respect to time is called acceleration i.e., $V(t) = \frac{d}{dt}R(t)$

Therefore differentiate $V(t)$ with respect to time t

$A(t)=\frac{d}{dt}V(t)=-\frac{d}{dt}2i-2\frac{d}{dt}t\,j+\frac{d}{dt}e^t\,k$

$A(t)=\frac{d}{dt}V(t)=-(0)i-2\,j+e^t\,\frac{d}{dt}t\,k$

$A(t)=\frac{d}{dt}V(t)=-2\,j+e^t\,k$

Acceleration at time $t=0$

$A(0)=-2\,j+e^0\,k$

$A(0)=-2\,j+1k$

$A(0)=-2\,j+k$

c). $R(t)=\cos t\,i+\sin t\,j+3t\,k$ at $t=\frac{\pi}{4}$

Sol: Since the rate of change of displacement with respect to time is called velocity i.e. $V(t)=\frac{d}{dt}R(t)$

Therefore differentiate $R(t)$ with respect to time t

$V(t)=\frac{d}{dt}R(t)=\frac{d}{dt}\cos t\,i+\frac{d}{dt}\sin t\,j+3\frac{d}{dt}t\,k$

$V(t)=\frac{d}{dt}R(t)=-\sin t\frac{d}{dt}t\,i+\cos t\frac{d}{dt}t\,j+3k$

$V(t)=\frac{d}{dt}R(t)=-\sin t\,i+\cos t\,j+3k$

The velocity at time $t=\frac{\pi}{4}$

$V\left(\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)i+\cos\left(\frac{\pi}{4}\right)j+3k$

$V\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}i+\frac{\sqrt{2}}{2}j+3k$

And the speed

$|V(t)|=\sqrt{\sin^2 t+\cos^2 t+9}$

$|V(t)|=\sqrt{1+9}=\sqrt{10}$

Speed at time $t=\frac{\pi}{4}$

$\left|V\left(\frac{\pi}{4}\right)\right|=\sqrt{10}$

The direction of motion at time $t=\frac{\pi}{4}$

$\frac{V\left(\frac{\pi}{4}\right)}{\left|V\left(\frac{\pi}{4}\right)\right|}=\frac{-\frac{\sqrt{2}}{2}i+\frac{\sqrt{2}}{2}j+3k}{\sqrt{10}}$

$\frac{V\left(\frac{\pi}{4}\right)}{\left|V\left(\frac{\pi}{4}\right)\right|}=-\frac{\sqrt{2}}{2\sqrt{10}}i+\frac{\sqrt{2}}{2\sqrt{10}}j+\frac{3}{\sqrt{10}}k$

$\frac{V\left(\frac{\pi}{4}\right)}{\left|V\left(\frac{\pi}{4}\right)\right|}=-\frac{1}{2\sqrt{5}}i+\frac{1}{2\sqrt{5}}j+\frac{3}{\sqrt{10}}k$

Since the rate of change of velocity with respect to time is called acceleration i.e., $V(t)=\frac{d}{dt}R(t)$

Therefore differentiate $V(t)$ with respect to time t

$A(t)=\frac{d}{dt}V(t)=-\frac{d}{dt}\sin t\,i+\frac{d}{dt}\cos t\,j+\frac{d}{dt}3k$

$A(t)=\frac{d}{dt}V(t)=-\cos t\frac{d}{dt}t\,i-\sin t\frac{d}{dt}t\,j+0$

$A(t)=\frac{d}{dt}V(t)=-\cos t\,i-\sin t\,j$

Acceleration at time $t=\frac{\pi}{4}$

$A\left(\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)i-\sin\left(\frac{\pi}{4}\right)j$

$A\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}i-\frac{\sqrt{2}}{2}j$

Q5. If $v=2i-j+5k$ and $w=i+2j-3k$ are the two vector functions, then evaluate the following derivatives problems;

a). $\frac{d}{dt}(v+tw)$

Solution: We have to find $\frac{d}{dt}(v+tw)$

$v+tw=(2i-j+5k)+t(i+2j-3k)$

$v+tw=2i-j+5k+ti+2t\,j-3t\,k$

$v+tw=2i+ti-j+2t\,j+5k-3t\,k$

$v+tw=(2+t)i-(1-2t)j+(5-3t)k$

Differentiating both sides with respect to t

$\frac{d}{dt}(v+tw)=\frac{d}{dt}(2+t)i-\frac{d}{dt}(1-2t)j+\frac{d}{dt}(5-3t)k$

$\frac{d}{dt}(v+tw)=\left(\frac{d}{dt}2+\frac{d}{dt}t\right)i-\left(\frac{d}{dt}1-2\frac{d}{dt}t\right)j+\left(\frac{d}{dt}5-3\frac{d}{dt}t\right)k$

$\frac{d}{dt}(v+tw)=(0+1)i-(0-2)j+(0-3)k$

$\frac{d}{dt}(v+tw)=i+2j-3k$

b). $\frac{d^2}{dt^2}(v.\,t^4w)$

Solution: We have to find $\frac{d^2}{dt^2}(v.\,t^4w)$

$v.t^4w=(2i-j+5k).t^4(i+2j-3k)$

$v.t^4w=(2i-j+5k).(t^4i+2t^4j-3t^4k)$

Differentiating both sides with respect to t

$\frac{d}{dt}\{v.t^4w\}=\frac{d}{dt}\{(2i-j+5k).(t^4i+2t^4j-3t^4k)\}$

$\frac{d}{dt}\{v.t^4w\}=(2i-j+5k).\frac{d}{dt}(t^4i+2t^4j-3t^4k)$
$\qquad+(t^4i+2t^4j-3t^4k)\frac{d}{dt}(2i-j+5k)$

$\frac{d}{dt}\{v.t^4w\}=(2i-j+5k).\left(\frac{d}{dt}t^4i+2\frac{d}{dt}t^4j-3\frac{d}{dt}t^4k\right)$
$\qquad+(t^4i+2t^4j-3t^4k)\left(\frac{d}{dt}2i-\frac{d}{dt}j+\frac{d}{dt}5k\right)$

$\frac{d}{dt}\{v.t^4w\}=(2i-j+5k).\left(4t^3\frac{d}{dt}t\,i+8t^3\frac{d}{dt}t\,j-3(4t^3)\frac{d}{dt}t\,k\right)$
$\qquad+(t^4i+2t^4j-3t^4k)(0i-0j+0k)$

$\frac{d}{dt}\{v.t^4w\}=(2i-j+5k).(4t^3i+8t^3j-12t^3k)$
$\qquad+(t^4i+2t^4j-3t^4k)(0)$

$\frac{d}{dt}\{v.t^4w\}=(2i-j+5k).(4t^3i+8t^3j-12t^3k)+0$

$\frac{d}{dt}\{v.t^4w\}=(2i-j+5k).(4t^3i+8t^3j-12t^3k)$

Differentiating again with respect to t

$\frac{d}{dt}\frac{d}{dt}\{v.t^4w\}=\frac{d}{dt}\{(2i-j+5k).(4t^3i+8t^3j-12t^3k)\}$

$\frac{d^2}{dt^2}\{v.t^4w\}=(2i-j+5k)\frac{d}{dt}(4t^3i+8t^3j-12t^3k)$
$\qquad+(4t^3i+8t^3j-12t^3k)\frac{d}{dt}(2i-j+5k)$

$\frac{d^2}{dt^2}\{v.t^4w\}=(2i-j+5k)\left(4\frac{d}{dt}t^3i+8\frac{d}{dt}t^3j-12\frac{d}{dt}t^3k\right)$
$\qquad+(4t^3i+8t^3j-12t^3k)(0)$

$\frac{d^2}{dt^2}\{v.t^4w\}=(2i-j+5k)\left(4(3t^2)i+8(3t^2)j-12(3t^2)k\right)$

$\frac{d^2}{dt^2}\{v.t^4w\}=(2i-j+5k)(12t^2i+24t^2j-36t^2k)$

Now we will multiply the RHS

$\frac{d^2}{dt^2}\{v.t^4w\}=(2i-j+5k)(12t^2i+24t^2j-36t^2k)$

$\frac{d^2}{dt^2}\{v.t^4w\}=(2)(12t^2)+(-1)(24t^2)+(5)(-36t^2)$

$\frac{d^2}{dt^2}\{v.t^4w\}=24t^2-24t^2-180t^2$

$\frac{d^2}{dt^2}\{v.t^4w\}=-180t^2$

c). $\frac{d^2}{dt^2}\left(t|v|+t^2|w|\right)$

Solution: We have to find $\frac{d^2}{dt^2}\left(t|v|+t^2|w|\right)$

$t|v|+t^2|w|=t|2i-j+5k|+t^2|i+2j-3k|$

$t|v|+t^2|w|=t\sqrt{(2)^2+(-1)^2+(5)^2}+t^2\sqrt{(1)^2+(2)^2+(-3)^2}$

$t|v|+t^2|w|=t\sqrt{4+1+25}+t^2\sqrt{1+4+9}$

$t|v|+t^2|w|=t\sqrt{30}+t^2\sqrt{14}$

$t|v|+t^2|w|=\sqrt{30}t+\sqrt{14}t^2$

Differentiating both sides with respect to t

$\frac{d}{dt}\left\{t|v|+t^2|w|\right\}=\sqrt{30}\frac{d}{dt}t+\sqrt{14}\frac{d}{dt}t^2$

$\frac{d}{dt}\left\{t|v|+t^2|w|\right\}=\sqrt{30}+\sqrt{14}(2t)\frac{d}{dt}t$

$\frac{d}{dt}\left\{t|v|+t^2|w|\right\}=\sqrt{30}+2\sqrt{14}\ t$

Again differentiating with respect to t

$\frac{d}{dt}\frac{d}{dt}\left\{t|v|+t^2|w|\right\}=\frac{d}{dt}\sqrt{30}+2\sqrt{14}\ \frac{d}{dt}t$

$\frac{d^2}{dt^2}\left\{t|v|+t^2|w|\right\}=0+2\sqrt{14}$

$\frac{d^2}{dt^2}\left\{t|v|+t^2|w|\right\}=2\sqrt{14}$

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d). $\frac{d}{dt}\left(tv\times t^2w\right)$

Solution: We have to find $\frac{d}{dt}\left(tv\times t^2w\right)$

$tv\times t^2w=t(2i-j+5k)\times t^2(i+2j-3k)$

$tv\times t^2w=(2ti-tj+5tk)\times(t^2i+2t^2j-3t^2k)$

Differentiating both sides with respect to x

$\frac{d}{dt}\left(tv\times t^2w\right)=\frac{d}{dt}\left\{(2ti-tj+5tk)\times(t^2i+2t^2j-3t^2k)\right\}$

$\frac{d}{dt}\left(tv\times t^2w\right)=(2ti-tj+5tk)\times\frac{d}{dt}(t^2i+2t^2j-3t^2k)$
$\quad+\frac{d}{dt}(2ti-tj+5tk)\times(t^2i+2t^2j-3t^2k)$

$\frac{d}{dt}\left(tv\times t^2w\right)=(2ti-tj+5tk)\times\left(\frac{d}{dt}t^2i+2\frac{d}{dt}t^2j-3\frac{d}{dt}t^2k\right)$
$\quad+\left(2\frac{d}{dt}ti-\frac{d}{dt}tj+5\frac{d}{dt}tk\right)\times(t^2i+2t^2j-3t^2k)$

$\frac{d}{dt}\left(tv\times t^2w\right)=(2ti-tj+5tk)\times(2ti+2(2t)j-3(2t)k)$
$\quad+(2i-j+5k)\times(t^2i+2t^2j-3t^2k)$

$\frac{d}{dt}\left(tv\times t^2w\right)=(2ti-tj+5tk)\times(2ti+4tj-6tk)$
$\quad+(2i-j+5k)\times(t^2i+2t^2j-3t^2k)$

$$\frac{d}{dt}\left(tv\times t^2w\right)=\begin{vmatrix}i&j&k\\2t&-t&5t\\2t&4t&-6t\end{vmatrix}+\begin{vmatrix}i&j&k\\2&-1&5\\t^2&2t^2&-3t^2\end{vmatrix}$$

$\frac{d}{dt}\left(tv\times t^2w\right)=i(6t^2-20t^2)-j(-12t^2-10t^2)+k(8t^2+2t^2)$
$\quad+i(3t^2-10t^2)-j(-6t^2-5t^2)+k(4t^2+t^2)$

$\frac{d}{dt}\left(tv\times t^2w\right)=-14t^2i+22t^2j+10t^2k$
$\quad-7t^2i+11t^2j+5t^2k$

$\frac{d}{dt}\left(tv\times t^2w\right)=-21t^2i+33t^2j+15t^2k$

Q6. Verify following indicated equations for vector functions, when $F(t)=(3+t^2)i-(\cos 3t)j+t^{-1}k$ and $G(t)=\sin(2-t)i-e^{2t}k$

a). $(3F-2G)'(t)=3F'(t)-2G'(t)$

Sol: We verify $(3F-2G)'(t)=3F'(t)-2G'(t)$

For LHS

$3F(t)-2G(t)=3\left\{(3+t^2)i-(\cos 3t)j+t^{-1}k\right\}$
$\quad-2\left\{\sin(2-t)i-e^{2t}k\right\}$

$3F(t)-2G(t)=3(3+t^2)i-3(\cos 3t)j+3t^{-1}k$
$\quad-2\sin(2-t)i+2e^{2t}k$

$3F(t)-2G(t)=(9+3t^2-2\sin(2-t))i-3(\cos 3t)j$
$\quad+(3t^{-1}+2e^{2t})k$

Differentiating with respect to t

$\frac{d}{dt}\left\{3F(t)-2G(t)\right\}=\frac{d}{dt}(9+3t^2-2\sin(2-t))i$
$\quad-3\frac{d}{dt}(\cos 3t)j+\frac{d}{dt}(3t^{-1}+2e^{2t})k$

$\frac{d}{dt}\left\{3F-2G\right\}(t)=\left(\frac{d}{dt}9+3\frac{d}{dt}t^2-2\frac{d}{dt}\sin(2-t)\right)i$
$\quad-3\left(\frac{d}{dt}\cos 3t\right)j+\left(3\frac{d}{dt}t^{-1}+2\frac{d}{dt}e^{2t}\right)k$

$\frac{d}{dt}(3F-2G)(t)=\left(0+6t-2\cos(2-t)\frac{d}{dt}(2-t)\right)i$
$\quad-3\left(-\sin 3t\frac{d}{dt}3t\right)j+\left(-3t^{-1-1}\frac{d}{dt}t+2e^{2t}\frac{d}{dt}(2t)\right)k$

$\frac{d}{dt}(3F-2G)(t)=(6t+2\cos(2-t))i$
$\quad+9\sin 3t\ j+(-3t^{-2}+4e^{2t})k$................(1)

For RHS

$3\frac{d}{dt}F(t)-2\frac{d}{dt}G(t)=3\frac{d}{dt}\left\{(3+t^2)i-(\cos 3t)j+t^{-1}k\right\}$
$\quad-2\frac{d}{dt}\left\{\sin(2-t)i-e^{2t}k\right\}$

$3\frac{d}{dt}F(t)-2\frac{d}{dt}G(t)=3\left\{\frac{d}{dt}(3+t^2)i-\frac{d}{dt}(\cos 3t)j+\frac{d}{dt}t^{-1}k\right\}$
$\quad-2\left\{\frac{d}{dt}\sin(2-t)i-\frac{d}{dt}e^{2t}k\right\}$

$3\frac{d}{dt}F(t)-2\frac{d}{dt}G(t)=3\left\{\left(\frac{d}{dt}3+\frac{d}{dt}t^2\right)i+\sin 3t\frac{d}{dt}(3t)j-t^{-2}\frac{d}{dt}t\,k\right\}$
$\quad-2\left\{\cos(2-t)\frac{d}{dt}(2-t)i-e^{2t}\frac{d}{dt}(2t)k\right\}$

$3\frac{d}{dt}F(t)-2\frac{d}{dt}G(t)=3\left\{(0+2t)i+3\sin 3t\ j-t^{-2}k\right\}$
$\quad-2\left\{-\cos(2-t)i-2e^{2t}k\right\}$

$3\frac{d}{dt}F(t)-2\frac{d}{dt}G(t)=6ti+9\sin 3t\ j-3t^{-2}k$
$\quad+2\cos(2-t)i+4e^{2t}k$

$3\frac{d}{dt}F(t)-2\frac{d}{dt}G(t)=6ti+2\cos(2-t)i+9\sin 3t\ j$
$\quad-3t^{-2}k+4e^{2t}k$

$3\frac{d}{dt}F(t)-2\frac{d}{dt}G(t)=\left\{6t+2\cos(2-t)\right\}i$
$\quad+9\sin 3t\ j+\left\{-3t^{-2}+4e^{2t}\right\}k$................(2)

Form equations (1) and (2) we get

$(3F-2G)'(t)=3F'(t)-2G'(t)$

---

b). $(F.G)'(t)=(F'.G)(t)+(F.G')(t)$

Solution: For LHS

$F(t).G(t)=\left[(3+t^2)i-(\cos 3t)j+t^{-1}k\right].\left[\sin(2-t)i-e^{2t}k\right]$

$F(t).G(t)=(3+t^2)\sin(2-t)-0.(\cos 3t)-e^{2t}t^{-1}$

$F(t).G(t)=(3+t^2)\sin(2-t)-e^{2t}t^{-1}$

Now differentiating with respect to t

$$\tfrac{d}{dt}\{F(t).G(t)\}=\tfrac{d}{dt}\{(3+t^2)\sin(2-t)-e^{2t}t^{-1}\}$$

$$\tfrac{d}{dt}\{F(t).G(t)\}=(3+t^2)\tfrac{d}{dt}\sin(2-t)$$
$$+\sin(2-t)\tfrac{d}{dt}(3+t^2)-\{e^{2t}\tfrac{d}{dt}t^{-1}+t^{-1}\tfrac{d}{dt}e^{2t}\}$$

$$\tfrac{d}{dt}\{F(t).G(t)\}=(3+t^2)\cos(2-t)\tfrac{d}{dt}\sin(2-t)$$
$$+\sin(2-t)\left(\tfrac{d}{dt}3+\tfrac{d}{dt}t^2\right)-e^{2t}\tfrac{d}{dt}t^{-1}-t^{-1}e^{2t}\tfrac{d}{dt}(2t)$$

$$\tfrac{d}{dt}\{F(t).G(t)\}=-(3+t^2)\cos(2-t)$$
$$+\sin(2-t)(0+2t\tfrac{d}{dt}t)-(-t^{-1-1})e^{2t}\tfrac{d}{dt}t-2t^{-1}e^{2t}$$

$$\tfrac{d}{dt}\{F(t).G(t)\}=-(3+t^2)\cos(2-t)$$
$$+2t\sin(2-t)+t^{-2}e^{2t}\tfrac{d}{dt}t-2t^{-1}e^{2t}\ldots\ldots\ldots\ldots.(1)$$

For RHS

$$\tfrac{d}{dt}F(t).G(t)+F(t).\tfrac{d}{dt}G(t)=\tfrac{d}{dt}F.G+F.\tfrac{d}{dt}G$$

$$F'.G+F.G'=\left[\tfrac{d}{dt}(3+t^2)i-\tfrac{d}{dt}(\cos 3t)j+\tfrac{d}{dt}t^{-1}k\right].\left[\sin(2-t)i-e^{2t}k\right]$$
$$\left[(3+t^2)i-(\cos 3t)j+t^{-1}k\right].\left[\tfrac{d}{dt}\sin(2-t)i-\tfrac{d}{dt}e^{2t}k\right]$$

$$F'.G+F.G'=\left[(0+2t)i-(-\sin 3t)\tfrac{d}{dt}(3t)j-t^{-2}k\right].\left[\sin(2-t)i-e^{2t}k\right]$$
$$\left[(3+t^2)i-(\cos 3t)j+t^{-1}k\right].\left[\cos(2-t)\tfrac{d}{dt}(2-t)i-e^{2t}\tfrac{d}{dt}(2t)k\right]$$

$$F'.G+F.G'=\left[2t\,i+3\sin 3\mathrm{t}\,j-t^{-2}k\right].\left[\sin(2-t)i-e^{2t}k\right]$$
$$\left[(3+t^2)i-(\cos 3t)j+t^{-1}k\right].\left[-\cos(2-t)i-2e^{2t}k\right]$$

$$F'.G+F.G'=2t\sin(2-t)+0.3\sin 3\mathrm{t}+e^{2t}t^{-2}$$
$$-(3+t^2)\cos(2-t)-0.(\cos 3t)-2e^{2t}t^{-1}$$

$$F'.G+F.G'=2t\sin(2-t)+e^{2t}t^{-2}$$
$$-(3+t^2)\cos(2-t)-2e^{2t}t^{-1}\ldots\ldots\ldots(2)$$

Form equations (1) and (2) we get

$$(F.G)'(t)=(F'.G)(t)+(\mathrm{F}.G')(t)$$

Q7. If $F(t)$ and $G(t)$ are differentiable vector functions of t, then prove that

a). $(F.G)'(t)=(F'.G)(t)+(\mathrm{F}.G')(t)$

Solution: Let $F(t)=f_1i+f_2j+f_3k$

and $G(t)=g_1i+g_2j+g_3k$

Where f_i,g_i are the functions of t , For LHS

$$F(t).G(t)=(f_1i+f_2j+f_3k)(g_1i+g_2j+g_3k)$$

$$F.G=f_1g_1+f_2g_2+f_3g_3$$

Differentiating with respect to t

$$\tfrac{d}{dt}(F.G)=\tfrac{d}{dt}\{f_1g_1+f_2g_2+f_3g_3\}$$

$$(F.G)'=\tfrac{d}{dt}(f_1g_1)+\tfrac{d}{dt}(f_2g_2)+\tfrac{d}{dt}(f_3g_3)$$

$$(F.G)'=f_1\tfrac{d}{dt}g_1+g_1\tfrac{d}{dt}f_1+f_2\tfrac{d}{dt}g_2$$
$$+g_2\tfrac{d}{dt}f_2+f_3\tfrac{d}{dt}g_3+g_3\tfrac{d}{dt}f_3$$

$$(F.G)'=f_1g_1'+g_1f_1'+f_2g_2'$$
$$+g_2f_2'+f_3g_3'+g_3f_3'\ldots\ldots\ldots\ldots.(1)$$

For the RHS

$$(F'.G)(t)+(\mathrm{F}.G')(t)=F'.G+\mathrm{F}.G'$$

Substituting the values

$$F'.G+\mathrm{F}.G'=\tfrac{d}{dt}(f_1i+f_2j+f_3k)(g_1i+g_2j+g_3k)$$
$$+(f_1i+f_2j+f_3k)\tfrac{d}{dt}(g_1i+g_2j+g_3k)$$

$$F'.G+\mathrm{F}.G'=(f_1'i+f_2'j+f_3'k)(g_1i+g_2j+g_3k)$$
$$+(f_1i+f_2j+f_3k)(g_1'i+g_2'j+g_3'k)$$

$$F'.G+\mathrm{F}.G'=f_1'g_1+f_2'g_2+f_3'g_3$$
$$+f_1g_1'+f_2g_2'+f_3g_3'$$

Rearranging

$$F'.G+\mathrm{F}.G'=f_1g_1'+f_1'g_1+f_2'g_2$$
$$+f_2g_2'+f_3'g_3+f_3g_3'\ldots\ldots\ldots\ldots.(2)$$

From equations (1) and (2) we get

$$(F.G)'(t)=(F'.G)(t)+(\mathrm{F}.G')(t)$$

b). $(F\times G)'(t)=(F'\times G)(t)+(\mathrm{F}\times G')(t)$

Sol: Let $F(t)=f_1i+f_2j+f_3k$ &

$G(t)=g_1i+g_2j+g_3k$

Where f_i,g_i are the functions of t, For LHS

$$F(t)\times G(t)=(f_1i+f_2j+f_3k)\times(g_1i+g_2j+g_3k)$$

$$F(t)\times G(t)=\begin{vmatrix} i & j & k\\ f_1 & f_2 & f_3\\ g_1 & g_2 & g_3\end{vmatrix}$$

$$F(t)\times G(t)=i(f_2g_3-f_3g_2)-j(f_1g_3-f_3g_1)$$
$$+k(f_1g_2-f_2g_1)$$

Differentiating with respect to t

$$\tfrac{d}{dt}\{F(t)\times G(t)\}=i\tfrac{d}{dt}(f_2g_3-f_3g_2)-j\tfrac{d}{dt}(f_1g_3-f_3g_1)$$
$$+k\tfrac{d}{dt}(f_1g_2-f_2g_1)$$

$$\tfrac{d}{dt}\{F\times G\}=i\{\tfrac{d}{dt}(f_2g_3)-\tfrac{d}{dt}(f_3g_2)\}$$
$$-j\{\tfrac{d}{dt}(f_1g_3)-\tfrac{d}{dt}(f_3g_1)\}+k\{\tfrac{d}{dt}(f_1g_2)-\tfrac{d}{dt}(f_2g_1)\}$$

$$\tfrac{d}{dt}\{F\times G\}=i\{(f_2\tfrac{d}{dt}g_3+g_3\tfrac{d}{dt}f_2)-(f_3\tfrac{d}{dt}g_2+g_2\tfrac{d}{dt}f_3)\}$$
$$-j\{(f_1\tfrac{d}{dt}g_3+g_3\tfrac{d}{dt}f_1)-(f_3\tfrac{d}{dt}g_1+g_1\tfrac{d}{dt}f_3)\}$$
$$+k\{(f_1\tfrac{d}{dt}g_2+g_2\tfrac{d}{dt}f_1)-(f_2\tfrac{d}{dt}g_1+g_1\tfrac{d}{dt}f_2)\}$$

$$\tfrac{d}{dt}\{F\times G\}=i\{(f_2g_3'+g_3f_2')-(f_3g_2'+g_2f_3')\}$$
$$-j\{(f_1g_3'+g_3f_1')-(f_3g_1'+g_1f_3')\}$$
$$+k\{(f_1g_2'+g_2f_1')-(f_2g_1'+g_1f_2')\}$$

$$\tfrac{d}{dt}\{F\times G\}=i\{f_2g_3'+g_3f_2'-f_3g_2'-g_2f_3'\}$$
$$-j\{f_1g_3'+g_3f_1'-f_3g_1'-g_1f_3'\}$$
$$+k\{f_1g_2'+g_2f_1'-f_2g_1'-g_1f_2'\}\ldots\ldots\ldots\ldots\ldots.(1)$$

For the RHS

$$F'\times G+\mathrm{F}\times G'=\tfrac{d}{dt}F\times G+\mathrm{F}\times\tfrac{d}{dt}G$$

Substituting the values

$$F'\times G+\mathrm{F}\times G'=\tfrac{d}{dt}(f_1i+f_2j+f_3k)\times(g_1i+g_2j+g_3k)$$
$$+(f_1i+f_2j+f_3k)\times\tfrac{d}{dt}(g_1i+g_2j+g_3k)$$

$$F'\times G+\mathrm{F}\times G'=\left(\tfrac{d}{dt}f_1i+\tfrac{d}{dt}f_2j+\tfrac{d}{dt}f_3k\right)\times(g_1i+g_2j+g_3k)$$
$$+(f_1i+f_2j+f_3k)\times\left(\tfrac{d}{dt}g_1i+\tfrac{d}{dt}g_2j+\tfrac{d}{dt}g_3k\right)$$

$$F'\times G+\mathrm{F}\times G'=(f_1'i+f_2'j+f_3'k)\times(g_1i+g_2j+g_3k)$$
$$+(f_1i+f_2j+f_3k)\times(g_1'i+g_2'j+g_3'k)$$

.

$$F'\times G+F\times G'=\begin{vmatrix} i & j & k \\ f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \end{vmatrix}+\begin{vmatrix} i & j & k \\ f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \end{vmatrix}$$

$$F'\times G+F\times G'=i\left(f_2'g_3-f_3'g_2\right)-j\left(f_1'g_3-f_3'g_1\right)+k\left(f_1'g_2-f_2'g_1\right)+i\left(f_2g_3'-f_3g_2'\right)-j\left(f_1g_3'-f_3g_1'\right)+k\left(f_1g_2'-f_2g_1'\right)$$

$$F'\times G+F\times G'=i\left(f_2'g_3-f_3'g_2\right)+i\left(f_2g_3'-f_3g_2'\right)-j\left(f_1'g_3-f_3'g_1\right)-j\left(f_1g_3'-f_3g_1'\right)+k\left(f_1'g_2-f_2'g_1\right)+k\left(f_1g_2'-f_2g_1'\right)$$

$$F'\times G+F\times G'=i\left(f_2'g_3-f_3'g_2+f_2g_3'-f_3g_2'\right)-j\left(f_1'g_3-f_3'g_1+f_1g_3'-f_3g_1'\right)+k\left(f_1'g_2-f_2'g_1+f_1g_2'-f_2g_1'\right)$$

$$F'\times G+F\times G'=i\left(f_2'g_3+f_2g_3'-f_3'g_2-f_3g_2'\right)-j\left(f_1'g_3+f_1g_3'-f_3'g_1-f_3g_1'\right)+k\left(f_1'g_2+f_1g_2'-f_2'g_1-f_2g_1'\right)\text{..........}(2)$$

From equations (1) and (2) we get

$$\left(F\times G\right)'(t)=\left(F'\times G\right)(t)+\left(F\times G'\right)(t)$$

Q8. If $F(t)$ is a differentiable vector functions of t, such that $F(t)\neq 0$ then show that

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{\left[F(t)F'(t)\right]F(t)}{|F(t)|^3}$$

Solution: Let $F(t)=f_1i+f_2j+f_3k$

Where f_1, f_2, f_3 are the functions of t

$$|F(t)|=|f_1i+f_2j+f_3k|$$

$$|F(t)|=\sqrt{f_1^2+f_2^2+f_3^2}$$

Differentiating with respect to t

$$\tfrac{d}{dt}|F(t)|=\tfrac{d}{dt}\sqrt{f_1^2+f_2^2+f_3^2}$$

$$\tfrac{d}{dt}|F(t)|=\tfrac{d}{dt}\left(f_1^2+f_2^2+f_3^2\right)^{\frac{1}{2}}$$

$$\tfrac{d}{dt}|F(t)|=\tfrac{1}{2}\left(f_1^2+f_2^2+f_3^2\right)^{\frac{1}{2}-1}\tfrac{d}{dt}\left(f_1^2+f_2^2+f_3^2\right)$$

$$\tfrac{d}{dt}|F(t)|=\tfrac{1}{2}\left(f_1^2+f_2^2+f_3^2\right)^{\frac{-1}{2}}\tfrac{d}{dt}\left(f_1^2+f_2^2+f_3^2\right)$$

$$\tfrac{d}{dt}|F(t)|=\frac{1}{2}\frac{\tfrac{d}{dt}\left(f_1^2+f_2^2+f_3^2\right)}{\left(f_1^2+f_2^2+f_3^2\right)^{\frac{1}{2}}}$$

$$\tfrac{d}{dt}|F(t)|=\frac{1}{2}\frac{\tfrac{d}{dt}\left\{F(t).F(t)\right\}}{\sqrt{f_1^2+f_2^2+f_3^2}}$$

$$\tfrac{d}{dt}|F(t)|=\frac{1}{2}\frac{\tfrac{d}{dt}\left\{F(t)\right\}^2}{|F(t)|}$$

$$\tfrac{d}{dt}|F(t)|=\frac{1}{2}\frac{2F(t)\tfrac{d}{dt}F(t)}{|F(t)|}$$

$$\tfrac{d}{dt}|F(t)|=\frac{F(t)\tfrac{d}{dt}F(t)}{|F(t)|}$$

$$\tfrac{d}{dt}|F(t)|=\frac{F(t)F'(t)}{|F(t)|}\text{.........}(1)$$

We have to show that

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{\left[F(t)F'(t)\right]F(t)}{|F(t)|^3}$$

Take LHS

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{d}{dt}\left[F(t)\left\{|F(t)|\right\}^{-1}\right]$$

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\left\{|F(t)|\right\}^{-1}\frac{d}{dt}F(t)+F(t)\frac{d}{dt}\left\{|F(t)|\right\}^{-1}$$

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\left\{|F(t)|\right\}^{-1}F'(t)-F(t)\left\{|F(t)|\right\}^{-1-1}\frac{d}{dt}|F(t)|$$

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{F(t)}{|F(t)|^2}\frac{d}{dt}|F(t)|$$

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{F(t)}{|F(t)|^2}\frac{F(t)F'(t)}{|F(t)|}$$

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{F(t)\left[F(t)F'(t)\right]}{|F(t)|^3}=RHS$$

Hence proved
