Chapter No 3

Higher Order Derivatives successive derivatives of some functions are gathered to obtain the general form of nth derivatives

First derivative of $y = f(x)$ is denoted by $y' = f'(x)$

Second derivative of $f(x)$ is denoted by $y'' = f''(x)$

Third derivative of $f(x)$ is denoted by $y''' = f'''(x)$

derivative $n \geq 4$ holds notation $f^{(n)}(x), n = 4,5,6,...$

Exercise 3.1

Q1.Find the indicated higher derivatives of the following functions:

a). $f(x) = 3x^3 + 4x + 5,$ $f''(x)$

Sol: Given $f(x) = 3x^3 + 4x + 5$

Differentiating with respect to x

$\frac{d}{dx} f(x) = 3\frac{d}{dx} x^3 + 4\frac{d}{dx} x + \frac{d}{dx} 5$

$f'(x) = 3(3x^2)\frac{d}{dx} x + 4(1) + 0$

$f'(x) = 9x^2 + 4$

Again differentiating with respect to x

$\frac{d}{dx} f'(x) = 9\frac{d}{dx} x^2 + \frac{d}{dx} 4$

$f''(x) = 9(2x)\frac{d}{dx} x + 0$

$f''(x) = 18x$

b). $f(x) = x + \frac{1}{x}$ $f'''(x)$

Sol: Given $f(x) = x + \frac{1}{x} = x + x^{-1}$

Differentiating with respect to x

$\frac{d}{dx} f(x) = \frac{d}{dx} x + \frac{d}{dx} x^{-1}$

$f'(x) = 1 + (-x^{-1-1})$

$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$

Again differentiating with respect to x

$\frac{d}{dx} f'(x) = \frac{d}{dx} 1 - \frac{d}{dx} x^{-2}$

$f''(x) = 0 - (-2x^{-2-1})\frac{d}{dx} x$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

Again differentiating with respect to x

$\frac{d}{dx} f''(x) = 2\frac{d}{dx} x^{-3}$

$f'''(x) = 2(-3x^{-3-1})\frac{d}{dx} x$

$f'''(x) = -6x^{-4} = \frac{-6}{x^4}$

c). $f(x) = 1 + \frac{2}{x} - \frac{3}{x^2}$ $f''(x)$

Sol: Given $f(x) = 1 + \frac{2}{x} - \frac{3}{x^2} = 1 + 2x^{-1} - x^{-2}$

Differentiating with respect to x

$\frac{d}{dx} f(x) = \frac{d}{dx} 1 + 2\frac{d}{dx} x^{-1} - 3\frac{d}{dx} x^{-2}$

$f'(x) = 0 + 2(-x^{-1-1})\frac{d}{dx} x - 3(-2x^{-2-1})\frac{d}{dx} x$

$f'(x) = -2x^{-2} + 6x^{-3} = \frac{-2}{x^2} + \frac{6}{x^3}$

Again differentiating with respect to x

$\frac{d}{dx} f'(x) = -2\frac{d}{dx} x^{-2} + 6\frac{d}{dx} x^{-3}$

$f''(x) = -2(-2x^{-2-1})\frac{d}{dx} x + 6(-3x^{-3-1})\frac{d}{dx} x$

$f''(x) = 4x^{-3} - 18x^{-4} = \frac{4}{x^3} - \frac{18}{x^4}$

d). $s(t) = \sqrt{5t+7}$ $s''(t)$

Sol: Given $s(t) = \sqrt{5t+7}$

Differentiating with respect to x

$\frac{d}{dt} s(t) = \frac{d}{dt}(5t+7)^{\frac{1}{2}}$

$s'(t) = \frac{1}{2}(5t+7)^{\frac{1}{2}-1}\frac{d}{dt}(5t+7)$

$s'(t) = \frac{1}{2}(5t+7)^{\frac{-1}{2}}\left(5\frac{d}{dt} t + \frac{d}{dt} 7\right)$

$s'(t) = \frac{1}{2}(5t+7)^{\frac{-1}{2}}(5(1)+0) = \frac{5}{2}(5t+7)^{\frac{-1}{2}}$

$s'(t) = \frac{5}{2(5t+7)^{\frac{1}{2}}} = \frac{5}{2\sqrt{5t+7}}$

Again differentiating with respect to x

$\frac{d}{dt} s'(t) = \frac{5}{2}\frac{d}{dt}(5t+7)^{\frac{-1}{2}}$

$s''(t) = \frac{5}{2}\frac{-1}{2}(5t+7)^{\frac{-1}{2}-1}\frac{d}{dt}(5t+7)$

$s''(t) = \frac{-5}{4}(5t+7)^{\frac{-3}{2}}\left(5\frac{d}{dt} t + \frac{d}{dt} 7\right)$

$s''(t) = \frac{-5}{4}(5t+7)^{\frac{-3}{2}}(5(1)+0)$

$s''(t) = \frac{-25}{4(5t+7)^{\frac{3}{2}}}$

e). $y = \frac{x+1}{x-1},$ y''

Sol: Given $y = \frac{x+1}{x-1},$

Differentiating with respect to x

$\frac{d}{dx} y = \frac{d}{dx}\left(\frac{x+1}{x-1}\right)$

$y' = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$

$y' = \frac{(x-1)\left(\frac{d}{dx} x + \frac{d}{dx} 1\right) - (x+1)\left(\frac{d}{dx} x - \frac{d}{dx} 1\right)}{(x-1)^2}$

$y' = \frac{(x-1)(1+0) - (x+1)(1-0)}{(x-1)^2}$

$y' = \frac{x-1-x-1}{(x-1)^2}$

$y' = \frac{-2}{(x-1)^2} = -2(x-1)^{-2}$

Again differentiating with respect to x

$\frac{d}{dx} y' = -2\frac{d}{dx}(x-1)^{-2}$

$y'' = -2(-2)(x-1)^{-2-1}\frac{d}{dx}(x-1)$

$y'' = 4(x-1)^{-3}\left(\frac{d}{dx} x - \frac{d}{dx} 1\right)$

$y'' = 4(x-1)^{-3}(1-0)$

$y'' = 4(x-1)^{-3} = \dfrac{4}{(x-1)^3}$

f). $y = (x+3)(x^2+7x+2)^2$ y''

Sol: Given $y = (x+3)(x^2+7x+2)^2$

Differentiating with respect to x

$\frac{d}{dx}y = \frac{d}{dx}\left\{(x+3)(x^2+7x+2)^2\right\}$

$y' = (x+3)\frac{d}{dx}(x^2+7x+2)^2 + (x^2+7x+2)^2\frac{d}{dx}(x+3)$

$y' = 2(x+3)(x^2+7x+2)^1\frac{d}{dx}(x^2+7x+2) + (x^2+7x+2)^2(1+0)$

$y' = 2(x+3)(x^2+7x+2)(2x+7) + (x^2+7x+2)^2$

$y' = 2(x+3)(x^2+7x+2)(2x+7) + (x^2+7x+2)^2$

$y' = 2(x+3)(2x+7)(x^2+7x+2) + (x^2+7x+2)^2$

$y' = 2(2x^2+7x+6x+21)(x^2+7x+2) + (x^2+7x+2)^2$

$y' = 2(2x^2+13x+21)(x^2+7x+2) + (x^2+7x+2)^2$

$y' = (x^2+7x+2)\left\{2(2x^2+13x+21) + (x^2+7x+2)\right\}$

$y' = (x^2+7x+2)\left\{4x^2+26x+42+x^2+7x+2\right\}$

$y' = (x^2+7x+2)\left\{5x^2+33x+44\right\}$

Again differentiating with respect to x

$y'' = (x^2+7x+2)\frac{d}{dx}(5x^2+33x+44) + (5x^2+33x+44)\frac{d}{dx}(x^2+7x+2)$

$y'' = (x^2+7x+2)(10x+33) + (5x^2+33x+44)(2x+7)$

Q2.Find the indicated higher derivatives of the following trigonometric functions:

a). $y = \tan x$ y'''

Sol: Given $y = \tan x$

Differentiating with respect to x

$\frac{d}{dx}y = \frac{d}{dx}\tan x$

$y' = \sec^2 x\frac{d}{dx}x$

$y' = \sec^2 x$

$y' = 1+\tan^2 x$

$y' = 1+y^2$ $\quad y = \tan x$

Again differentiating with respect to x

$\frac{d}{dx}y' = \frac{d}{dx}1 + \frac{d}{dx}y^2$

$y'' = 0 + 2y\frac{d}{dx}y$

$y'' = 2y\ y'$

$y'' = 2y(1+y^2)$ $\quad \therefore y' = 1+y^2$

$y'' = 2y+2y^3$

Again differentiating with respect to x

$\frac{d}{dx}y'' = 2\frac{d}{dx}y + 2\frac{d}{dx}y^3$

$y''' = 2\frac{d}{dx}y + 2(3y^2)\frac{d}{dx}y$

$y''' = \left[2+6y^2\right]\frac{d}{dx}y$

$y''' = \left[2+6y^2\right]y'$

$y''' = \left[2+6\tan^2 x\right]\sec^2 x$ $\quad y' = \sec^2 x$

$y''' = \left[2+6\tan^2 x+6-6\right]\sec^2 x$

$y''' = \left[6\tan^2 x+6-4\right]\sec^2 x$

$y''' = \left[6(\tan^2 x+1)-4\right]\sec^2 x$

$y''' = \left[6\sec^2 x-4\right]\sec^2 x$

$y''' = 6\sec^4 x - 4\sec^2 x$

b). $y = \ln\sin x$ y'''

Sol: Given $y = \ln\sin x$

Differentiating with respect to x

$\frac{d}{dx}y = \frac{d}{dx}\ln\sin x$

$y' = \dfrac{1}{\sin x}\frac{d}{dx}\sin x$

$y' = \dfrac{\cos x}{\sin x} = \cot x$

Again differentiating with respect to x

$\frac{d}{dx}y' = \frac{d}{dx}\cot x$

$y'' = -\operatorname{cosec}^2 x\frac{d}{dx}x$

$y'' = -\operatorname{cosec}^2 x$

Again differentiating with respect to x

$\frac{d}{dx}y'' = -\frac{d}{dx}\operatorname{cosec}^2 x$

$y''' = -2\operatorname{cosec} x\frac{d}{dx}\operatorname{cosec} x$

$y''' = -2\operatorname{cosec} x(-\operatorname{cosec} x\cot x)\frac{d}{dx}x$

$y''' = +2\operatorname{cosec}^2 x\cot x$

c). $y = \sqrt{\sec 2x}$ y'''

Sol: Given $y = \sqrt{\sec 2x} = \{\sec 2x\}^{\frac{1}{2}}$

Differentiating with respect to x

$\frac{d}{dx}y = \frac{d}{dx}\{\sec 2x\}^{\frac{1}{2}}$

$y' = \dfrac{1}{2}\{\sec 2x\}^{\frac{1}{2}-1}\frac{d}{dx}\{\sec 2x\}$

$y' = \dfrac{1}{2}\{\sec 2x\}^{\frac{-1}{2}}\sec 2x\tan 2x\frac{d}{dx}(2x)$

$y' = \dfrac{2\sec 2x\tan 2x}{2\{\sec 2x\}^{\frac{1}{2}}}$

$y' = \{\sec 2x\}^{1-\frac{1}{2}}\tan 2x$ $\quad \because \{\sec 2x\}^{\frac{1}{2}} = \sqrt{\sec 2x}$

$y' = \sqrt{\sec 2x}\tan 2x$ ………………..(1)

Again differentiating with respect to x

$\frac{d}{dx}y' = \frac{d}{dx}\left\{\sqrt{\sec 2x}.\tan 2x\right\}$

$y'' = \sqrt{\sec 2x}\frac{d}{dx}\tan 2x + \tan 2x\frac{d}{dx}\sqrt{\sec 2x}$

$y'' = \sqrt{\sec 2x}\sec^2 2x\frac{d}{dx}(2x) + \tan 2x\left\{\sqrt{\sec 2x}.\tan 2x\right\}$

$y'' = \sqrt{\sec 2x}\left\{2\sec^2 2x + \tan^2 2x\right\}$

$y'' = \sqrt{\sec 2x}\left\{2\sec^2 2x + \sec^2 2x - 1\right\}$

$y'' = \sqrt{\sec 2x}\left\{3\sec^2 2x - 1\right\}$

Again differentiating with respect to x

$\frac{d}{dx}y'' = \sqrt{\sec 2x}\frac{d}{dx}\left\{3\sec^2 2x-1\right\} + \left\{3\sec^2 2x-1\right\}\frac{d}{dx}\sqrt{\sec 2x}$

$y''' = \sqrt{\sec 2x}\left\{6\sec 2x\frac{d}{dx}\sec 2x\right\} + \left\{3\sec^2 2x-1\right\}\sqrt{\sec 2x}\tan 2x$

$y''' = \sqrt{\sec 2x}\left\{6\times 2\sec^2 2x\tan 2x\right\} + \left\{3\sec^2 2x-1\right\}\sqrt{\sec 2x}\tan 2x$

$y''' = \sqrt{\sec 2x}\tan 2x\left\{12\sec^2 2x + 3\sec^2 2x - 1\right\}$

$y''' = \sqrt{\sec 2x}\left\{15\sec^2 2x - 1\right\}\tan 2x$

$y''' = \left[15\{\sec 2x\}^{\frac{5}{2}} - \{\sec 2x\}^{\frac{1}{2}}\right]\tan 2x$

d). $y = \dfrac{1}{x}$ y^{iv}

Sol: Given $y = x^{-1}$ Differentiating w r t x

$\frac{d}{dx}y = \frac{d}{dx}x^{-1}$

$y' = -x^{-2}$

Again differentiating with respect to x

$\frac{d}{dx}y' = -\frac{d}{dx}x^{-2}$

$y'' = -(-2)x^{-3}$

$y'' = 2x^{-3}$

Again differentiating with respect to x

$\frac{d}{dx}y'' = 2\frac{d}{dx}x^{-3}$

$y''' = 2(-3)x^{-4}$

$y''' = -6x^{-4}$

Again differentiating with respect to x

$y^{iv} = -6(-4)x^{-5}$

$y^{iv} = 24x^{-5}$

$y^{iv} = \dfrac{24}{x^5}$

e). $y = \sin(\sin x)$ y''

Sol: Given $y = \sin(\sin x)$

Differentiating with respect to x

$\frac{d}{dx}y = \frac{d}{dx}\sin(\sin x)$

$y' = \cos(\sin x)\frac{d}{dx}\sin x$

$y' = \cos(\sin x)\cos x$

Again differentiating with respect to x

$\frac{d}{dx}y' = \frac{d}{dx}\{\cos(\sin x)\cos x\}$

$y'' = \cos(\sin x)\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\cos(\sin x)$

$y'' = -\cos(\sin x)\sin x - \cos x\sin(\sin x)\frac{d}{dx}\sin x$

$y'' = -\cos(\sin x)\sin x - \cos x\sin(\sin x)\cos x$

$y'' = -\cos(\sin x)\sin x - \cos^2 x\sin(\sin x)$

Q3. Use implicit rule to find out the second derivative of the following functions:

a). $y = x + \tan^{-1} y$ y''

Sol: Given $y = x + \tan^{-1} y$ Differentiating w r t x

$\frac{d}{dx}y = \frac{d}{dx}x + \frac{d}{dx}\tan^{-1} y$

$y' = 1 + \dfrac{1}{1+y^2}\frac{d}{dx}y$

$y' + \dfrac{1}{1+y^2}y' = 1$

$\left\{\dfrac{1+y^2-1}{1+y^2}\right\}y' = 1$

$y' = \dfrac{1+y^2}{y^2} = \dfrac{1}{y^2} + 1$

$y' = y^{-2} + 1$

Again differentiating with respect to x

$\dfrac{d}{dx}y' = \dfrac{d}{dx}y^{-2} + \dfrac{d}{dx}1$

$y'' = -2y^{-3}\frac{d}{dx}y$

$y'' = -2y^{-3}.y'$

$y'' = -2y^{-3}\left(\dfrac{1+y^2}{y^2}\right)$

$y'' = -2\left(\dfrac{1+y^2}{y^5}\right)$

b). $x^2 + y^2 = r^2$ y''

Sol: Given $x^2 + y^2 = r^2$ Differentiating w r t x

$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}r^2$

$2x\frac{d}{dx}x + 2y\frac{d}{dx}y = 0$

$2x + 2y\ y' = 0$

$2y\ y' = -2x$

$y' = -\frac{2x}{2y} = -\frac{x}{y}$

Again differentiating with respect to x

$\frac{d}{dx}y' = -\frac{d}{dx}\left(\frac{x}{y}\right)$

$y'' = -\dfrac{y\frac{d}{dx}x - x\frac{d}{dx}y}{y^2}$

$y'' = -\dfrac{1}{y^2}[y - x\ y']$

$y'' = -\frac{1}{y^2}\left[y - x\left(-\frac{x}{y}\right)\right]$ $\therefore y' = -\frac{x}{y}$

$y'' = -\frac{1}{y^2}\left[y + \frac{x^2}{y}\right]$

$y'' = -\dfrac{1}{y^2}\left[\dfrac{y^2+x^2}{y}\right]$

$y'' = -\dfrac{r^2}{y^3}$ $\therefore y^2 + x^2 = r^2$

c). $y^2 - 2xy = 0$ y''

Sol: Given $y^2 - 2xy = 0$

Differentiating with respect to x

$\frac{d}{dx}y^2 - 2\frac{d}{dx}(xy) = \frac{d}{dx}0$

$2y\frac{d}{dx}y - 2\left(x\frac{d}{dx}y + y\frac{d}{dx}x\right) = 0$

$2y\ y' - 2(x\ y' + y) = 0$

$2y\ y' - 2x\ y' - 2y = 0$

$2y\ y' - 2x\ y' = 2y$

$2y'(y-x) = 2y$

$y' = \dfrac{2y}{2(y-x)}$

$y' = \dfrac{y}{(y-x)}$

Again differentiating with respect to x

$\dfrac{d}{dx}y' = \dfrac{d}{dx}\left\{\dfrac{y}{(y-x)}\right\}$

$y'' = \dfrac{(y-x)\frac{d}{dx}y - y\frac{d}{dx}(y-x)}{(y-x)^2}$

$$y''=\frac{1}{(y-x)^2}\left[(y-x)\tfrac{d}{dx}y-y\left(\tfrac{d}{dx}y-\tfrac{d}{dx}x\right)\right]$$

$$y''=\frac{1}{(y-x)^2}\left[(y-x)\tfrac{d}{dx}y-y\left(\tfrac{d}{dx}y-1\right)\right]$$

$$y''=\frac{1}{(y-x)^2}\left[y\tfrac{d}{dx}y-x\tfrac{d}{dx}y-y\tfrac{d}{dx}y+y\right]$$

$$y''=\frac{1}{(y-x)^2}\left[-x\tfrac{d}{dx}y+y\right]$$

$$y''=\frac{1}{(y-x)^2}\left[-\frac{xy}{y-x}+y\right] \qquad y'=\frac{y}{y-x}$$

$$y''=\frac{1}{(y-x)^2}\left[\frac{-xy+y(y-x)}{y-x}\right]$$

$$y''=\frac{1}{(y-x)^3}\left[-xy+y^2-xy\right]$$

$$y''=\frac{1}{(y-x)^3}\left[y^2-2xy\right]$$

$$y''=\frac{1}{(y-x)^3}[0] \qquad \therefore y^2-2xy=0$$

$$y''=0$$

2nd method

We have $y^2-2xy=0$

Or $y^2=2xy$

Or $y=2x$

Differentiating with respect to x

$y'=2$

Differentiating with respect to x

$y''=0$

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d). $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ $\qquad y''$

Sol: Given $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$

Differentiating with respect to x

$$\frac{d}{dx}\frac{x^2}{a^2}+\frac{d}{dx}\frac{y^2}{b^2}-\frac{d}{dx}1=\frac{d}{dx}0$$

$$\frac{1}{a^2}\frac{d}{dx}x^2+\frac{1}{b^2}\frac{d}{dx}y^2-0=0$$

$$\frac{2x}{a^2}\frac{d}{dx}x+\frac{2y}{b^2}\frac{d}{dx}y=0$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$$

$$\frac{2y}{b^2}y'=-\frac{2x}{a^2}$$

$$y'=-\frac{b^2}{a^2}\cdot\frac{2x}{2y}$$

$$y'=-\frac{b^2}{a^2}\cdot\frac{x}{y}$$

Again differentiating with respect to x

$$\frac{d}{dx}y'=-\frac{b^2}{a^2}\cdot\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$y''=-\frac{b^2}{a^2}\cdot\left(\frac{y\frac{d}{dx}x-x\frac{d}{dx}y}{y^2}\right)$$

$$y''=-\frac{b^2}{a^2y^2}\cdot(y-x\,y')$$

$$y''=-\frac{b^2}{a^2y^2}\cdot\left(y-x\left(-\frac{b^2}{a^2}\frac{x}{y}\right)\right) \qquad \therefore y'=-\frac{b^2}{a^2}\frac{x}{y}$$

$$y''=-\frac{b^2}{a^2y^2}\cdot\left(y+\frac{b^2x^2}{a^2y}\right)$$

$$y''=-\frac{b^2}{a^2y^2}\cdot\left(\frac{a^2y^2+b^2x^2}{a^2y}\right)$$

$$y''=-\frac{b^2\left(a^2y^2+b^2x^2\right)}{a^4y^3}$$

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e). $\sec x\cos y=C$ $\qquad y''$

Sol: Given $\sec x\cos y=C$

$\dfrac{\cos y}{\cos x}=C$ Differentiating with respect to x

$$\frac{d}{dx}\frac{\cos y}{\cos x}=\frac{d}{dx}C$$

$$\frac{\cos x\frac{d}{dx}\cos y-\cos y\frac{d}{dx}\cos x}{\cos^2 x}=0$$

$$-\sin y\cos x\tfrac{d}{dx}y+\sin x\cos y\tfrac{d}{dx}x=0.\cos^2 x$$

$$-\sin y\cos x\tfrac{d}{dx}y=-\sin x\cos y$$

$$\tfrac{d}{dx}y=\frac{-\sin x\cos y}{-\cos x\sin y}$$

$$y'=\frac{\tan x}{\tan y}$$

Again differentiating with respect to x

$$\tfrac{d}{dx}y'=\tfrac{d}{dx}\left(\frac{\tan x}{\tan y}\right)$$

$$y''=\frac{1}{\tan^2 y}\left\{\tan y\tfrac{d}{dx}\tan x-\tan x\tfrac{d}{dx}\tan y\right\}$$

$$y''=\frac{1}{\tan^2 y}\left\{\tan y\sec^2 x\tfrac{d}{dx}x-\tan x\sec^2 y\tfrac{d}{dx}y\right\}$$

$$y''=\frac{1}{\tan^2 y}\left\{\tan y\sec^2 x-\tan x\sec^2 y\frac{\tan x}{\tan y}\right\}$$

$$y''=\frac{1}{\tan^2 y}\left\{\frac{\tan^2 y\sec^2 x-\sec^2 y\tan^2 x}{\tan y}\right\}$$

$$y''=\frac{\tan^2 y\sec^2 x-\sec^2 y\tan^2 x}{\tan^3 y}$$

$$y''=\frac{\tan^2 y\left(1+\tan^2 x\right)-\left(1+\tan^2 y\right)\tan^2 x}{\tan^3 y}$$

$$y''=\frac{\tan^2 y+\tan^2 y\tan^2 x-\tan^2 x-\tan^2 x\tan^2 y}{\tan^3 y}$$

$$y''=\frac{\tan^2 y-\tan^2 x}{\tan^3 y}$$

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f). $e^x+x=e^y+y$ $\qquad y''$

Sol: Given $e^x+x=e^y+y$

Differentiating with respect to x

$$\tfrac{d}{dx}e^x+\tfrac{d}{dx}x=\tfrac{d}{dx}e^y+\tfrac{d}{dx}y$$

$$e^x+1=e^y\tfrac{d}{dx}y+\tfrac{d}{dx}y$$

$$e^x+1=\left(e^y+1\right)\tfrac{d}{dx}y$$

$$y'=\frac{e^x+1}{e^y+1}$$

Again differentiating with respect to x

$$\frac{d}{dx}y'=\frac{d}{dx}\left(\frac{e^x+1}{e^y+1}\right)$$

$$y''=\frac{\left(e^y+1\right)\tfrac{d}{dx}\left(e^x+1\right)-\left(e^x+1\right)\tfrac{d}{dx}\left(e^y+1\right)}{\left(e^y+1\right)^2}$$

$$y''=\frac{\left(e^y+1\right)\left(\tfrac{d}{dx}e^x+\tfrac{d}{dx}1\right)-\left(e^x+1\right)\left(\tfrac{d}{dx}e^y+\tfrac{d}{dx}1\right)}{\left(e^y+1\right)^2}$$

$$y''=\frac{\left(e^y+1\right)\left(e^x+0\right)-\left(e^x+1\right)\left(e^y\tfrac{d}{dx}y+0\right)}{\left(e^y+1\right)^2}$$

$$y''=\frac{1}{\left(e^y+1\right)^2}\left[e^x\left(e^y+1\right)-\left(e^x+1\right)e^y\tfrac{d}{dx}y\right]$$

$$y''=\frac{1}{\left(e^y+1\right)^2}\left[e^x\left(e^y+1\right)-\left(e^x+1\right)e^y\frac{e^x+1}{e^y+1}\right]$$

$$y''=\frac{1}{\left(e^y+1\right)^2}\left[\frac{e^x\left(e^y+1\right)^2-e^y\left(e^x+1\right)^2}{\left(e^y+1\right)}\right]$$

$$y''=\frac{1}{\left(e^y+1\right)^3}\left[e^x\left(e^{2y}+1+2e^y\right)-e^y\left(e^{2x}+1+2e^x\right)\right]$$

$$y''=\frac{1}{\left(e^y+1\right)^3}\left[e^{2y+x}+e^x+2e^{x+y}-e^{2x+y}-e^y-2e^{x+y}\right]$$

$$y''=\frac{1}{\left(e^y+1\right)^3}\left[e^x-e^{2x+y}-e^y+e^{2y+x}+2e^{x+y}-2e^{x+y}\right]$$

$$y''=\frac{1}{\left(e^y+1\right)^3}\left[e^x\left(1-e^{x+y}\right)-e^y\left(1-e^{x+y}\right)\right]$$

$$y''=\frac{\left(1-e^{x+y}\right)\left(e^x-e^y\right)}{\left(e^y+1\right)^3}$$

Q4. Use parametric differentiation to find out $\frac{d^2y}{dx^2}$ for the following parametric functions $x(t)$ and $y(t)$

a). $x=4t^2+1,\quad y=6t^3+1$

Sol: Given $x=4t^2+1,\quad y=6t^3+1$

Differentiating with respect to t

$$\tfrac{d}{dt}x=4\tfrac{d}{dt}t^2+\tfrac{d}{dt}1,\qquad \tfrac{d}{dt}y=6\tfrac{d}{dt}t^3+\tfrac{d}{dt}1$$

$$\tfrac{d}{dt}x=4\left(2t\right)\tfrac{d}{dt}t+0,\qquad \tfrac{d}{dt}y=6\left(3t^{3-1}\right)\tfrac{d}{dt}t+0$$

$$\tfrac{d}{dt}x=8t,\qquad \tfrac{d}{dt}y=18t^2$$

Reciprocal of $\frac{dx}{dt}$ is $\frac{dt}{dx}$ i.e., $\frac{dt}{dx}=\frac{1}{8t}$

Substituting the value of $\frac{dt}{dx}$ and $\frac{dy}{dt}$ in the following

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{18t^2}{8t}=\frac{9}{4}t$$

Differentiating $\frac{dy}{dx}$ with respect to x

$$\frac{d}{dx}\frac{dy}{dx}=\frac{9}{4}\frac{d}{dx}t$$

$$\frac{d^2y}{dx^2}=\frac{9}{4}\frac{d}{dt}t.\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{9}{4}\frac{dt}{dx}$$

Substituting the value of $\frac{dt}{dx}$

$$\frac{d^2y}{dx^2}=\frac{9}{4}.\frac{1}{8t}$$

$$\frac{d^2y}{dx^2}=\frac{9}{32t}$$

---

b). $x=3at^2+2,\qquad y=6t^4+9$

Sol: Given $x=3at^2+2,\qquad y=6t^4+9$

Differentiating with respect to t

$$\tfrac{d}{dt}x=3a\tfrac{d}{dt}t^2+\tfrac{d}{dt}2,\qquad \tfrac{d}{dt}y=6\tfrac{d}{dt}t^4+\tfrac{d}{dt}9$$

$$\tfrac{d}{dt}x=3a\left(2t^{2-1}\right)\tfrac{d}{dt}t+0,\quad \tfrac{d}{dt}y=6\left(4t^{4-1}\right)\tfrac{d}{dt}t+0$$

$$\tfrac{d}{dt}x=6at,\qquad \tfrac{d}{dt}y=24t^3$$

Reciprocal of $\frac{dx}{dt}$ is $\frac{dt}{dx}$ i.e., $\frac{dt}{dx}=\frac{1}{6at}$

Substituting the value of $\frac{dt}{dx}$ and $\frac{dy}{dt}$ in the following

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{24t^3}{6at}$$

$$\frac{dy}{dx}=\frac{4}{a}t^2$$

Differentiating $\frac{dy}{dx}$ with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{4}{a}\frac{d}{dx}t^2$$

$$\frac{d^2y}{dx^2}=\frac{4}{a}\frac{d}{dt}t^2\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{4}{a}\left(2t\right)\frac{dt}{dx}$$

Substituting the value of $\frac{dt}{dx}$

$$\frac{d^2y}{dx^2}=\frac{8t}{a}\frac{1}{6at}$$

$$\frac{d^2y}{dx^2}=\frac{4}{3a^2}$$

---

c). $x=a\left(t-\sin t\right),\qquad y=a\left(1-\cos t\right)$

Sol: Given $x=a\left(t-\sin t\right), y=a\left(1-\cos t\right)$

Differentiating with respect to t

$$\tfrac{d}{dt}x=a\tfrac{d}{dt}\left(t-\sin t\right),\qquad \tfrac{d}{dt}y=a\tfrac{d}{dt}\left(1-\cos t\right)$$

$$\tfrac{d}{dt}x=a\left(\tfrac{d}{dt}t-\tfrac{d}{dt}\sin t\right),\quad \tfrac{d}{dt}y=a\left(\tfrac{d}{dt}1-\tfrac{d}{dt}\cos t\right)$$

$$\tfrac{d}{dt}x=a\left(1-\cos t\right),\qquad \tfrac{d}{dt}y=a\sin t$$

Reciprocal of $\frac{dx}{dt}$ is $\frac{dt}{dx}$ i.e., $\frac{dt}{dx}=\frac{1}{a\left(1-\cos t\right)}$

Substituting the value of $\frac{dt}{dx}$ and $\frac{dy}{dt}$ in the following

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$
$$\frac{dy}{dx}=\frac{a\sin t}{1}\frac{1}{a(1-\cos t)}$$
$$\frac{dy}{dx}=\frac{\sin t}{1-\cos t}$$

Differentiating $\frac{dy}{dx}$ with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{\sin t}{1-\cos t}\right)$$
$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{(1-\cos t)\frac{d}{dt}\sin t-\sin t\frac{d}{dt}(1-\cos t)}{(1-\cos t)^2}\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{(1-\cos t)\cos t-\sin t(0+\sin t)}{(1-\cos t)^2}\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{\cos t-\cos^2 t-\sin^2 t}{(1-\cos t)^2}\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{-(\cos^2 t+\sin^2 t)+\cos t}{(1-\cos t)^2}\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{-1+\cos t}{(1-\cos t)^2}\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{-(1-\cos t)}{(1-\cos t)^2}\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{-1}{(1-\cos t)}\frac{dt}{dx}$$

Substituting the value of $\frac{dt}{dx}$

$$\frac{d^2y}{dx^2}=\frac{-1}{(1-\cos t)}.\frac{1}{a(1-\cos t)}$$
$$\frac{d^2y}{dx^2}=\frac{-1}{a(1-\cos t)^2}$$

---

d). $x=a\cos 2t,\qquad y=b\sin 2t$

Sol: Given $x=a\cos 2t,\qquad y=b\sin 2t$

Differentiating with respect to t

$\frac{d}{dt}x=a\frac{d}{dt}\cos 2t,\qquad \frac{d}{dt}y=b\frac{d}{dt}\sin 2t$

$\frac{d}{dt}x=-a\sin 2t\frac{d}{dt}(2t),\qquad \frac{d}{dt}y=b\cos 2t\frac{d}{dt}(2t)$

$\frac{d}{dt}x=-2a\sin 2t,\qquad \frac{d}{dt}y=2b\cos 2t$

Reciprocal of $\frac{dx}{dt}$ is $\frac{dt}{dx}$ i.e., $\frac{dt}{dx}=\frac{1}{-2a\sin 2t}$

Substituting the value of $\frac{dt}{dx}$ and $\frac{dy}{dt}$ in the following

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$
$$\frac{dy}{dx}=\frac{2b\cos 2t}{-2a\sin 2t}=\frac{-b}{a}\cot 2t$$

Differentiating $\frac{dy}{dx}$ with respect to x

$$\frac{d^2y}{dx^2}=\frac{-b}{a}\frac{d}{dx}(\cot 2t)$$
$$\frac{d^2y}{dx^2}=\frac{-b}{a}\frac{d}{dt}(\cot 2t)\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{-b}{a}\left(-\operatorname{cosec}^2 2t\right)\frac{d}{dt}(2t)\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{2b}{a\sin^2 2t}\frac{dt}{dx}$$

Substituting the value of $\frac{dt}{dx}$

$$\frac{d^2y}{dx^2}=\frac{2b}{a\sin^2 2t}\frac{-1}{2a\sin 2t}$$
$$\frac{d^2y}{dx^2}=\frac{-b}{a^2\sin^3 2t}$$

---

e). $x=\frac{3at}{1+t^3},\qquad y=\frac{3at^2}{1+t^3}$

Sol: Given $x=\frac{3at}{1+t^3},\qquad y=\frac{3at^2}{1+t^3}$

Differentiating x with respect to t

$$\frac{d}{dt}x=\frac{d}{dt}\frac{3at}{1+t^3}$$
$$\frac{dx}{dt}=\frac{(1+t^3)\frac{d}{dt}(3at)-3at\frac{d}{dt}(1+t^3)}{(1+t^3)^2}$$
$$\frac{dx}{dt}=\frac{(1+t^3)(3a)-3at(0+3t^2)}{(1+t^3)^2}$$
$$\frac{dx}{dt}=\frac{3a+3at^3-9at^3}{(1+t^3)^2}$$
$$\frac{dx}{dt}=\frac{3a-6at^3}{(1+t^3)^2}$$

Reciprocal of $\frac{dx}{dt}$ is $\frac{dt}{dx}$ i.e., $\frac{dt}{dx}=\frac{(1+t^3)^2}{3a-6at^3}$

Differentiating y with respect to t

$$\frac{d}{dt}y=\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)$$
$$\frac{dy}{dt}=\frac{(1+t^3)\frac{d}{dt}(3at^2)-3at^2\frac{d}{dt}(1+t^3)}{(1+t^3)^2}$$
$$\frac{dy}{dt}=\frac{(1+t^3)(6at)-3at^2(0+3t^2)}{(1+t^3)^2}$$
$$\frac{dy}{dt}=\frac{6at+6at^4-9at^4}{(1+t^3)^2}$$
$$\frac{dy}{dt}=\frac{6at-3at^4}{(1+t^3)^2}=\frac{3at(2-t^3)}{(1+t^3)^2}$$

Substituting the value of $\frac{dt}{dx}$ and $\frac{dy}{dt}$ in the following

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$
$$\frac{dy}{dx}=\frac{3at(2-t^3)}{(1+t^3)^2}.\frac{(1+t^3)^2}{3a(1-2t^3)}$$
$$\frac{dy}{dx}=\frac{t(2-t^3)}{(1-2t^3)}=\frac{2t-t^4}{(1-2t^3)}$$

Differentiating $\frac{dy}{dx}$ with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{2t-t^4}{1-2t^3}\right)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{2t-t^4}{1-2t^3}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2t^3\right)\frac{d}{dt}\left(2t-t^4\right)-\left(2t-t^4\right)\frac{d}{dt}\left(1-2t^3\right)}{\left(1-2t^3\right)^2}\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2t^3\right)\left(2-4t^3\right)-\left(2t-t^4\right)\left(0-6t^2\right)}{\left(1-2t^3\right)^2}\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{2-4t^3-4t^3+8t^6+12t^3-6t^6}{\left(1-2t^3\right)^2}\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{2+4t^3+2t^6}{\left(1-2t^3\right)^2}\frac{dt}{dx}$$

Substituting the value of $\frac{dt}{dx}$

$$\frac{d^2y}{dx^2}=\frac{2\left(1+2t^3+t^6\right)}{\left(1-2t^3\right)^2}\cdot\frac{\left(1+t^3\right)^2}{3a\left(1-2t^3\right)}$$

$$\frac{d^2y}{dx^2}=\frac{2\left(1+2.1.t^3+\left(t^3\right)^2\right)}{3a\left(1-2t^3\right)^3}\cdot\frac{\left(1+t^3\right)^2}{1}$$

$$\frac{d^2y}{dx^2}=\frac{2\left(1+t^3\right)^4}{3a\left(1-2t^3\right)^3}$$

---

f). $x=a\frac{1-t^2}{1+t^2},\quad y=b\frac{2t}{1+t^2}$

Sol: Given $x=a\frac{1-t^2}{1+t^2},\quad y=b\frac{2t}{1+t^2}$

Differentiating x with respect to t

$$\frac{dx}{dt}=a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$\frac{dx}{dt}=a\frac{\left(1+t^2\right)\frac{d}{dt}\left(1-t^2\right)-\left(1-t^2\right)\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$\frac{dx}{dt}=a\frac{\left(1+t^2\right)\left(0-2t\right)-\left(1-t^2\right)\left(0+2t\right)}{\left(1+t^2\right)^2}$$

$$\frac{dx}{dt}=a\frac{-2t\left(1+t^2\right)-2t\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

$$\frac{dx}{dt}=a\frac{-2t-2t^3-2t+2t^3}{\left(1+t^2\right)^2}$$

$$\frac{dx}{dt}=a\frac{-4t}{\left(1+t^2\right)^2}=\frac{-4at}{\left(1+t^2\right)^2}$$

Reciprocal of $\frac{dx}{dt}$ is $\frac{dt}{dx}$ i.e., $\frac{dt}{dx}=-\frac{\left(1+t^2\right)^2}{4at}$

Differentiating y with respect to t

$$\frac{dy}{dt}=b\frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$\frac{dy}{dt}=b\frac{\left(1+t^2\right)\frac{d}{dt}\left(2t\right)-2t\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dt}=b\frac{\left(1+t^2\right)\left(2\right)-2t\left(0+2t\right)}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dt}=b\frac{2+2t^2-4t^2}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dt}=b\frac{2-2t^2}{\left(1+t^2\right)^2}$$

Substituting the value of $\frac{dt}{dx}$ and $\frac{dy}{dt}$ in the following

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx}=b\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}\cdot\left[-\frac{\left(1+t^2\right)^2}{4at}\right]$$

$$\frac{dy}{dx}=-\frac{b\left(1-t^2\right)}{2at}$$

Differentiating $\frac{dy}{dx}$ with respect to x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{-b}{2a}\frac{d}{dx}\left(\frac{1-t^2}{t}\right)$$

$$\frac{d^2y}{dx^2}=\frac{-b}{2a}\frac{d}{dt}\left(\frac{1-t^2}{t}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{-b}{2a}\left(\frac{t\frac{d}{dt}\left(1-t^2\right)-\left(1-t^2\right)\frac{d}{dt}t}{t^2}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{-b}{2a}\left(\frac{t\left(0-2t\right)-\left(1-t^2\right)}{t^2}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{-b}{2a}\left(\frac{-2t^2-1+t^2}{t^2}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{-b}{2a}\left(\frac{-1-t^2}{t^2}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{b}{2a}\left(\frac{1+t^2}{t^2}\right)\frac{dt}{dx}$$

Substituting the value of $\frac{dt}{dx}$

$$\frac{d^2y}{dx^2}=\frac{b}{2a}\left(\frac{1+t^2}{t^2}\right)\left[-\frac{\left(1+t^2\right)^2}{4at}\right]$$

$$\frac{d^2y}{dx^2}=-\frac{b\left(1+t^2\right)^3}{8a^2t^3}$$

Equation of tangent passing through point $P\left(x_1,y_1\right)$ having slope m

$y-y_1=m\left(x-x_1\right)$ where $m=\left.\frac{dy}{dx}\right|_{x=x_1}=\left.y'\right|_{x=x_1}$

Equation of Normal passing through point $P\left(x_1,y_1\right)$ having slope m

$y-y_1=\frac{-1}{m}\left(x-x_1\right)$ where $m=\left.\frac{dy}{dx}\right|_{x=x_1}=\left.y'\right|_{x=x_1}$

Angles between two curves

Let m_1 & m_2 are slopes of two curves, angle b/w curves

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

Taylor series

$$f(x)=f\left(x_o\right)+\left(x-x_o\right)f'\left(x_0\right)+\frac{\left(x-x_o\right)^2}{2!}f''\left(x_0\right)+\frac{\left(x-x_o\right)^3}{3!}f'''\left(x_0\right)+\cdots$$

Maclaurin series

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

Example 3.2.2 Use Taylor series to approximate value of a function $f(x)=e^x$ at a point $x_o=2$

Sol: Given $f(x)=e^x \qquad \Rightarrow f(2)=e^2$

First derivative

$\frac{d}{dx}f(x)=\frac{d}{dx}e^x$

$f'(x)=e^x \qquad \Rightarrow f'(2)=e^2$

Second derivative

$\frac{d}{dx}f'(x)=\frac{d}{dx}e^x$

$f''(x)=e^x \qquad \Rightarrow f''(2)=e^2$

Third derivative

$\frac{d}{dx}f''(x)=\frac{d}{dx}e^x$

$f'''(x)=e^x \qquad \Rightarrow f'''(2)=e^2$

First four term of Taylor's series at $x_o=1$

$$f(x)=f(4)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\cdots$$

Substituting values in the above relation

$$f(x)=e^2+(x-2)e^2+\frac{(x-2)^2}{2!}e^2+\frac{(x-2)^3}{3!}e^2+\cdots$$

$$f(x)=e^2\left\{1+x-2+\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}+\cdots\right\}$$

$$f(x)=e^2\left\{x-1+\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}+\cdots\right\}$$

Application of Derivatives

Derivatives are use to determine

i). Tangent and Normal lines

ii). Angles between two curves

iii). Maximum and Minimum values of functions with in intervals where the function is increasing or decreasing

Exercise 3.2

Q1. In each case, find an equation of tangent line to the curve at the indicated value of x.

a). $y=\sqrt{x+1} \qquad x=3$

Sol: Given $y=\sqrt{x+1}$

At $x_1=3$ the corresponding value of the given function

$y_1=\sqrt{3+1}=\sqrt{4}=2$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}(x+1)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}(x+1)^{\frac{1}{2}-1}\frac{d}{dx}(x+1)$$

$$\frac{dy}{dx}=\frac{1}{2}(x+1)^{\frac{-1}{2}}\left(\frac{d}{dx}x+\frac{d}{dx}1\right)$$

$$\frac{dy}{dx}=\frac{1}{2(x+1)^{\frac{1}{2}}}(1+0)$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x+1}}$$

Slope at the point $x_1=3$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=3}=\frac{1}{2\sqrt{3+1}}=\frac{1}{2\sqrt{4}}=\frac{1}{2(2)}$$

$m=\frac{1}{4}$

Equation of **tangent** passing through point $P(x_1,y_1)=(3,2)$

$y-y_1=m(x-x_1)$

$y-2=\frac{1}{4}(x-3)$

$4(y-2)=(x-3)$

$4y-8=x-3$

$x-4y-3+8=0$

$x-4y+5=0$

b). $y=\sin(2x+\pi) \qquad x=0$

Sol: Given $y=\sin(2x+\pi)$

At $x_1=0$ the corresponding value of the given function

$y_1=\sin(2(0)+\pi)=\sin(0+\pi)=\sin(\pi)=0$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}\sin(2x+\pi)$$

$$\frac{dy}{dx}=\cos(2x+\pi)\frac{d}{dx}(2x+\pi)$$

$$\frac{dy}{dx}=\cos(2x+\pi)(2+0)$$

$$\frac{dy}{dx}=2\cos(2x+\pi)$$

Slope at the point $x_1=0$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=0}=2\cos(2(0)+\pi)=2\cos(\pi)$$

$m=-2$

Equation of **tangent** passing through point $P(x_1,y_1)=(0,0)$

$y-y_1=m(x-x_1)$

$y-0=-2(x-0)$

$y=-2x$

$2x+y=0$

c). $y=x^2e^{-x} \qquad x=1$

Sol: Given $y=x^2e^{-x}$

At $x_1=1$ the corresponding value of the given function

$$y_1=(1)^2e^{-1}=\frac{1}{e}$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}\left(x^2e^{-x}\right)$$

$$\frac{dy}{dx}=x^2\frac{d}{dx}e^{-x}+e^{-x}\frac{d}{dx}x^2$$

$$\frac{dy}{dx}=-x^2e^{-x}+2xe^{-x}$$

Slope at the point $x_1=1$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=1}=-(1)^2e^{-1}+2(1)e^{-1}=-e^{-1}+2e^{-1}=e^{-1}$$

$$m=\frac{1}{e}$$

Equation of **tangent** passing through $P(x_1,y_1)=\left(1,\frac{1}{e}\right)$

$y-y_1=m(x-x_1)$

$$y-\frac{1}{e}=\frac{1}{e}(x-1)$$

Multiply by e

$$ey-1=x-1$$

$$x-ey-1+1=0$$

$$x-ey=0$$

---

d). $y=\dfrac{2x+1}{x+2}$ $x=2$

Sol: Given $y=\dfrac{2x+1}{x+2}$

At $x_1=2$ the corresponding value of the given function

$$y_1=\frac{2(2)+1}{(2)+2}=\frac{4+1}{2+2}=\frac{5}{4}$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}\left(\frac{2x+1}{x+2}\right)$$

$$\frac{dy}{dx}=\frac{(x+2)\frac{d}{dx}(2x+1)-(2x+1)\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$\frac{dy}{dx}=\frac{(x+2)(2+0)-(2x+1)(1+0)}{(x+2)^2}$$

$$\frac{dy}{dx}=\frac{2x+4-2x-1}{(x+2)^2}=\frac{3}{(x+2)^2}$$

Slope at the point $x_1=2$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=2}=\frac{3}{(2+2)^2}=\frac{3}{4^2}\Rightarrow m=\frac{3}{16}$$

Equation of **tangent** passing through point $P(x_1,y_1)=\left(2,\frac{5}{4}\right)$

$$y-y_1=m(x-x_1)$$

$$y-\frac{5}{4}=\frac{3}{16}(x-2)$$

Multiply be 16

$$16y-5(4)=3(x-2)$$

$$16y-20=3x-6$$

$$3x-16y-6+20=0$$

$$3x-16y+14=0$$

---

e). $y=\dfrac{x}{x^2+1}$ $x=1$

Sol: Given $y=\dfrac{x}{x^2+1}$

At $x_1=1$ the corresponding value of the given function

$$y_1=\frac{1}{(1)^2+1}=\frac{1}{1+1}=\frac{1}{2}$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$\frac{dy}{dx}=\frac{(x^2+1)\frac{d}{dx}x-x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\frac{dy}{dx}=\frac{(x^2+1)-x(2x+0)}{(x^2+1)^2}$$

$$\frac{dy}{dx}=\frac{x^2+1-2x^2}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}$$

Slope at the point $x_1=1$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=1}=\frac{1-(1)^2}{((1)^2+1)^2}=\frac{1-1}{(1+1)^2}=\frac{0}{2^2}\qquad\Rightarrow m=0$$

Equation of **tangent** passing through point $P(x_1,y_1)=\left(1,\frac{1}{2}\right)$

$$y-y_1=m(x-x_1)$$

$$y-\tfrac{1}{2}=0(x-1)$$

$$y-\tfrac{1}{2}=0$$

$$y=\tfrac{1}{2}$$

---

f). $y=3\sin x-\cos x$ $x=\pi$

Sol: Given $y=3\sin x-\cos x$

At $x_1=\pi$ the corresponding value of the given function

$$y_1=3\sin\pi-\cos\pi=3(0)-(-1)=1$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}(3\sin x-\cos x)$$

$$\frac{dy}{dx}=3\frac{d}{dx}\sin x-\frac{d}{dx}\cos x$$

$$\frac{dy}{dx}=3\cos x+\sin x$$

Slope at the point $x_1=\pi$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=\pi}=3\cos\pi+\sin\pi=3(-1)+(0)$$

$$m=-3$$

Equation of **tangent** passing through point $P(x_1,y_1)=(\pi,1)$

$$y-y_1=m(x-x_1)$$

$$y-1=-3(x-\pi)$$

$$y-1=-3x+3\pi$$

$$3x+y-1-3\pi=0$$

---

g). $y=2\ln x$ $x=e$

Sol: Given $y=2\ln x$

At $x_1=e$ the corresponding value of the given function

$$y_1=2\ln e=2(1)=2$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=2\frac{d}{dx}\ln x$$

$$\frac{dy}{dx}=\frac{2}{x}$$

Slope at the point $x_1=e$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=e}=\frac{2}{e}$$

Equation of **tangent** passing through point $P(x_1,y_1)=(e,2)$

$$y-y_1=m(x-x_1)$$

$$y-2=\tfrac{2}{e}(x-e)$$

Multiply by e

$ey-2e=2(x-e)$

$ey-2e=2x-2e$

$2x-ey-2e+2e=0$

$2x-ey=0$

h). $y=3e^{x}+e^{-x}$ $x=0$

Sol: Given $y=3e^{x}+e^{-x}$

At $x_1=0$ the corresponding value of the given function

$y_1=3e^{0}+e^{-0}=3(1)+(1)=3+1=4$

Now differentiating given function with respect to x

$\frac{d}{dx}y=3\frac{d}{dx}e^{x}+\frac{d}{dx}e^{-x}$

$\frac{dy}{dx}=3e^{x}-e^{-x}$

Slope at the point $x_1=0$ we have

$m=\left.\frac{dy}{dx}\right|_{x_1=0}=3e^{0}-e^{-0}=3(1)-(1)=3-1$

$m=2$

Equation of **tangent** passing through point $P(x_1,y_1)=(0,4)$

$y-y_1=m(x-x_1)$

$y-4=2(x-0)$

$y-4=2x$

$2x-y+4=0$

Q2. In each case, find an equation of normal line to the curve at the indicated value of x.

a). $y=xe^{x}$ $x=1$

Sol: Given $y=xe^{x}$

At $x_1=1$ the corresponding value of the given function

$y_1=(1)e^{1}=e$

Now differentiating given function with respect to x

$\frac{d}{dx}y=\frac{d}{dx}\left(xe^{x}\right)$

$\frac{dy}{dx}=x\frac{d}{dx}e^{x}+e^{x}\frac{d}{dx}x$

$\frac{dy}{dx}=xe^{x}+e^{x}$

Slope at the point $x_1=1$ we have

$m=\left.\frac{dy}{dx}\right|_{x_1=1}=(1)e^{1}+e^{1}=e+e$

$m=2e$

Equation of **Normal** passing through $P(x_1,y_1)=(1,e)$

$y-y_1=\frac{-1}{m}(x-x_1)$

$y-e=\frac{-1}{2e}(x-1)$ Multiply by 2e

$2ey-2e^{2}=-(x-1)$

$2ey-2e^{2}=-x+1$

$x+2ey-2e^{2}-1=0$

b). $y=2\sin 3x$ $x=\pi$

Sol: Given $y=2\sin 3x$

At $x_1=\pi$ the corresponding value of the given function

$y_1=2\sin 3\pi=2(0)=0$

Now differentiating given function with respect to x

$\frac{d}{dx}y=2\frac{d}{dx}\sin 3x=2\cos 3x\frac{d}{dx}(3x)=2\times 3\cos 3x$

$\frac{dy}{dx}=6\cos 3x$

Slope at the point $x_1=$ we have

$m=\left.\frac{dy}{dx}\right|_{x_1=\pi}=6\cos 3\pi=6(-1)$

$m=-6$

Equation of **Normal** passing through point $P(x_1,y_1)=(\pi,0)$

$y-y_1=\frac{-1}{m}(x-x_1)$

$y-0=\frac{-1}{-6}(x-\pi)$

$6y=(x-\pi)$

$6y=x-\pi$

$x-6y-\pi=0$

c). $y=2\ln x$ $x=1$

Sol: Given $y=2\ln x$

At $x_1=1$ the corresponding value of the given function

$y_1=2\ln(1)=2(0)=0$

Now differentiating given function with respect to x

$\frac{d}{dx}y=2\frac{d}{dx}\ln x$

$\frac{dy}{dx}=\frac{2}{x}$

Slope at the point $x_1=1$ we have

$m=\left.\frac{dy}{dx}\right|_{x_1=1}=\frac{2}{1}\Rightarrow m=2$

Equation of **Normal** passing through $P(x_1,y_1)=(1,0)$

$y-y_1=\frac{-1}{m}(x-x_1)$

$y-0=\frac{-1}{2}(x-1)$

$2y=-x+1$

$x+2y-1=0$

d). $y=(2x+1)^{6}$ $x=0$

Sol: Given $y=(2x+1)^{6}$

At $x_1=0$ the corresponding value of the given function

$y_1=(2(0)+1)^{6}=(0+1)^{6}=1$

Now differentiating given function with respect to x

$\frac{d}{dx}y=\frac{d}{dx}(2x+1)^{6}=6(2x+1)^{6-1}\frac{d}{dx}(2x+1)$

$\frac{dy}{dx}=6(2x+1)^{5}(2(1)+0)=12(2x+1)^{5}$

Slope at the point $x_1=0$ we have

$m=\left.\frac{dy}{dx}\right|_{x_1=0}=12(2(0)+1)^{5}=12(0+1)^{5}$

$m=12$

Equation of **Normal** passing through $P(x_1,y_1)=(0,1)$

$y-y_1=\frac{-1}{m}(x-x_1)$

$y-1=\frac{-1}{12}(x-0)$

$12(y-1)=-x$

$x+12y-12=0$

e). $y=\dfrac{e^x+1}{x}$ $x=1$

Sol: Given $y=\dfrac{e^x+1}{x}$

At $x_1=1$ the corresponding value of the given function

$$y_1=\frac{e^1+1}{1}=e+1$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}\left(\frac{e^x+1}{x}\right)=\frac{x\frac{d}{dx}\left(e^x+1\right)-\left(e^x+1\right)\frac{d}{dx}(x)}{x^2}$$

$$\frac{dy}{dx}=\frac{x\left(e^x+0\right)-\left(e^x+1\right)}{x^2}=\frac{xe^x-e^x-1}{x^2}$$

Slope at the point $x_1=1$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=1}=\frac{(1)e^1-e^1-1}{(1)^2}=e-e-1$$

$$m=-1$$

Equation of **Normal** passing through point $P(x_1,y_1)=(1,e+1)$

$$y-y_1=\tfrac{-1}{m}(x-x_1)$$

$$y-(e+1)=\frac{-1}{-1}(x-1)$$

$$y-e-1=x-1$$

$$x-y+e+1-1=0$$

$$x-y+e=0$$

f). $y=\cos(x-\pi)$ $x=\dfrac{\pi}{2}$

Sol: Given $y=\cos(x-\pi)$

At $x_1=\dfrac{\pi}{2}$ the corresponding value of the given function

$$y_1=\cos\left(\frac{\pi}{2}-\pi\right)=\cos\left(-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}\cos(x-\pi)=-\sin(x-\pi)\frac{d}{dx}(x-\pi)$$

$$\frac{dy}{dx}=-\sin(x-\pi)(1-0)=-\sin(x-\pi)$$

Slope at the point $x_1=\dfrac{\pi}{2}$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=\frac{\pi}{2}}=-\sin\left(\frac{\pi}{2}-\pi\right)=-\sin\left(-\frac{\pi}{2}\right)$$

$$m=-(-1)=1$$

Equation of **Normal** passing through point $P(x_1,y_1)=\left(\dfrac{\pi}{2},0\right)$

$$y-y_1=\tfrac{-1}{m}(x-x_1)$$

$$y-0=\frac{-1}{1}\left(x-\frac{\pi}{2}\right)$$

$$y=-x+\frac{\pi}{2}$$

$$x+y-\frac{\pi}{2}=0$$

g). $y=x^3\ln x$ $x=1$

Sol: Given $y=x^3\ln x$

At $x_1=1$ the corresponding value of the given function

$$y_1=(1)^3\ln(1)=(1)(0)=0$$

Now differentiating given function with respect to x

$$\frac{d}{dx}y=\frac{d}{dx}\left(x^3\ln x\right)$$

$$\frac{dy}{dx}=x^3\frac{d}{dx}\ln x+\ln x\frac{d}{dx}x^3$$

$$\frac{dy}{dx}=\frac{x^3}{x}\frac{d}{dx}x+3x^2\ln x\frac{d}{dx}x$$

$$\frac{dy}{dx}=x^2+3x^2\ln x$$

Slope at the point $x_1=1$ we have

Equation of **Normal** passing through $P(x_1,y_1)=(1,0)$

$y-y_1=\tfrac{-1}{m}(x-x_1)$ putting

$$y-0=-1(x-1)$$

$$y=-x+1$$

$$x+y-1=0$$

h). $y=\sqrt{x^2+1}$ $x=2$

Sol: Given $y=\sqrt{x^2+1}$

At $x_1=2$ the corresponding value of the given function

$$y_1=\sqrt{(2)^2+1}=\sqrt{4+1}=\sqrt{5}$$

Now differentiating given function with respect to x

$$\tfrac{d}{dx}y=\tfrac{d}{dx}\left(x^2+1\right)^{\frac{1}{2}}=\tfrac{1}{2}\left(x^2+1\right)^{\frac{1}{2}-1}\tfrac{d}{dx}\left(x^2+1\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(x^2+1\right)^{\frac{-1}{2}}(2x+0)=\frac{x}{\sqrt{x^2+1}}$$

Slope at the point $x_1=$ we have

$$m=\left.\frac{dy}{dx}\right|_{x_1=2}=\frac{2}{\sqrt{(2)^2+1}}=\frac{2}{\sqrt{4+1}}$$

$$m=\frac{2}{\sqrt{5}}\qquad\Rightarrow\frac{1}{m}=\frac{\sqrt{5}}{2}$$

Equation of **Normal** passing through point

$$P(x_1,y_1)=\left(2,\sqrt{5}\right)$$

$$y-y_1=\tfrac{-1}{m}(x-x_1)$$

$$y-\sqrt{5}=\tfrac{-\sqrt{5}}{2}(x-2)$$

$$2y-2\sqrt{5}=-\sqrt{5}x+2\sqrt{5}$$

$$\sqrt{5}x+2y-2\sqrt{5}-2\sqrt{5}=0$$

$$\sqrt{5}x+2y-4\sqrt{5}=0$$

Q3a). Find an equation of tangent line to the curve $x^2+y^2=13$ at $(-2,3)$

Sol: Given $x^2+y^2=13$

Differentiating given function with respect to x

$$\tfrac{d}{dx}x^2+\tfrac{d}{dx}y^2=\tfrac{d}{dx}13$$

$$2x\tfrac{d}{dx}x+2y\tfrac{d}{dx}y=0$$

$$2x+2y\tfrac{d}{dx}y=0$$

$2y\frac{d}{dx}y=-2x$

$\frac{d}{dx}y=\dfrac{-2x}{2y}=\dfrac{-x}{y}$

Slope at the point $x_1=-2, y_1=3$ we have

$m=\left.\dfrac{dy}{dx}\right|_{(-2,3)}=\dfrac{-(-2)}{3}$

$m=\frac{2}{3}$

Equation of **tangent** passing through $P(x_1,y_1)=(-2,3)$

$y-y_1=m(x-x_1)$

$y-3=\dfrac{2}{3}(x-(-2))$

Multiply by 3

$3(y-3)=2(x+2)$

$3y-9=2x+4$

$2x-3y+4+9=0$

$2x-3y+13=0$

---

b). Find an equation of tangent line to the curve $\sin(x-y)=xy$ at $(0,\pi)$

Sol: Given $\sin(x-y)=xy$

Differentiating given function with respect to x

$\frac{d}{dx}\sin(x-y)=\frac{d}{dx}(xy)$

$\cos(x-y)\frac{d}{dx}(x-y)=x\frac{d}{dx}y+y\frac{d}{dx}x$

$\cos(x-y)\left(\frac{d}{dx}x-\frac{d}{dx}y\right)=x\frac{d}{dx}y+y$

$\cos(x-y)\left(1-\frac{d}{dx}y\right)=x\frac{d}{dx}y+y$

$\cos(x-y)-\cos(x-y)\frac{d}{dx}y=x\frac{d}{dx}y+y$

$\cos(x-y)-y=x\frac{d}{dx}y+\cos(x-y)\frac{d}{dx}y$

$\cos(x-y)-y=\{x+\cos(x-y)\}\frac{d}{dx}y$

$\dfrac{dy}{dx}=\dfrac{\cos(x-y)-y}{\cos(x-y)+x}$

Slope at the point $x_1=0, y_1=\pi$ we have

$m=\left.\dfrac{dy}{dx}\right|_{(0,\pi)}=\dfrac{\cos(0-\pi)-\pi}{\cos(0-\pi)+0}=\dfrac{\cos(-\pi)-\pi}{\cos(-\pi)}=\dfrac{-1-\pi}{-1}$

$m=1+\pi$

Equation of **tangent** passing through $P(x_1,y_1)=(0,\pi)$

$y-y_1=m(x-x_1)$ Putting the values

$y-\pi=(1+\pi)(x-0)$

$y-\pi=(1+\pi)x$

$(1+\pi)x-y+\pi=0$

---

c). Find an equation of normal line to the curve $x^2+2xy=y^3$ at $(1,-1)$

Sol: Given $x^2+2xy=y^3$

Now differentiating given function with respect to x

$\frac{d}{dx}x^2+2\frac{d}{dx}(xy)=\frac{d}{dx}y^3$

$2x\frac{d}{dx}x+2\left(x\frac{d}{dx}y+y\frac{d}{dx}x\right)=3y^2\frac{d}{dx}y$

$2x+2\left(x\frac{d}{dx}y+y\right)=3y^2\frac{d}{dx}y$

$2x+2x\frac{d}{dx}y+2y=3y^2\frac{d}{dx}y$

$2x\frac{d}{dx}y-3y^2\frac{d}{dx}y=-2x-2y$

$\left(2x-3y^2\right)\frac{d}{dx}y=-2x-2y$

$\frac{d}{dx}y=\dfrac{-2x-2y}{2x-3y^2}$

Slope at the point $x_1=1, y_1=-1$ we have

$m=\left.\dfrac{dy}{dx}\right|_{(1,-1)}=\dfrac{-2(1)-2(-1)}{2(1)-3(-1)^2}=\dfrac{-2+2}{2-3}$

$m=0$

Equation of **Normal** passing through $P(x_1,y_1)=(1,-1)$

$y-(-1)=\frac{-1}{0}(x-1)$

$0(y+1)=-x+1$

$x=1$

Q4a). Show that the first four terms in the Taylor series expansion of $f(x)=\tan x$ about $x_o=\frac{\pi}{4}$ are $1+2\left(x-\dfrac{\pi}{4}\right)+2\left(x-\dfrac{\pi}{4}\right)^2+\dfrac{8}{3}\left(x-\dfrac{\pi}{4}\right)^3$

Sol $f(x)=\tan x$ $\qquad f\left(\dfrac{\pi}{4}\right)=\tan\left(\dfrac{\pi}{4}\right)=1$

First derivative

$\frac{d}{dx}f(x)=\frac{d}{dx}\tan x$

$f'(x)=\sec^2 x$ $\qquad f'\left(\dfrac{\pi}{4}\right)=\sec^2\left(\dfrac{\pi}{4}\right)=2$

Second derivative

$\frac{d}{dx}f'(x)=\frac{d}{dx}\sec^2 x$

$f''(x)=2\sec x\frac{d}{dx}\sec x$

$f''(x)=2\sec x(\sec x\tan x)\frac{d}{dx}x$

$f''(x)=2\sec^2 x\tan x$

$\Rightarrow f''\left(\dfrac{\pi}{4}\right)=2\sec^2\left(\dfrac{\pi}{4}\right)\tan\left(\dfrac{\pi}{4}\right)=4$

Third derivative

$\frac{d}{dx}f''(x)=2\frac{d}{dx}\left(\sec^2 x\tan x\right)$

$f'''(x)=2\sec^2 x\frac{d}{dx}\tan x+2\tan x\frac{d}{dx}\sec^2 x$

$f'''(x)=2\sec^2 x\left(\sec^2 x\right)+2\tan x(2\sec x)\frac{d}{dx}\sec x$

$f'''(x)=2\sec^4 x+2\tan x(2\sec x)\sec x\tan x$

$f'''(x)=2\sec^4 x+4\tan^2 x\sec^2 x$

$\Rightarrow f'''\left(\frac{\pi}{4}\right)=2\sec^4\left(\frac{\pi}{4}\right)+4\tan^2\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right)$

$\Rightarrow f'''\left(\frac{\pi}{4}\right)=2(4)+4(1)(2)=8+8=16$

First four term of Taylor's series at $x_o=\frac{\pi}{4}$

$f(x)=f\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right)+\dfrac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\dfrac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)+\cdots$

Substituting values in the above relation

$f(x)=1+2\left(x-\dfrac{\pi}{4}\right)+\dfrac{4}{2!}\left(x-\dfrac{\pi}{4}\right)^2+\dfrac{16}{3!}\left(x-\dfrac{\pi}{4}\right)^3+\cdots$

$\tan x=1+2\left(x-\dfrac{\pi}{4}\right)+\dfrac{4}{2\times1}\left(x-\dfrac{\pi}{4}\right)^2+\dfrac{16}{3\times2\times1}\left(x-\dfrac{\pi}{4}\right)^3+\cdots$

$\tan x=1+2\left(x-\dfrac{\pi}{4}\right)+2\left(x-\dfrac{\pi}{4}\right)^2+\dfrac{8}{3}\left(x-\dfrac{\pi}{4}\right)^3+\cdots$

---

b). Show that the first four terms in the Taylor series expansion of $f(x)=\sqrt{x}$ about $x=4$ are

$$2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$$

Sol: $f(x)=\sqrt{x}$ $\Rightarrow f(4)=\sqrt{4}=2$

First derivative $\frac{d}{dx}f(x)=\frac{d}{dx}x^{\frac{1}{2}}$

$$f'(x)=\tfrac{1}{2}x^{\frac{1}{2}-1}\tfrac{d}{dx}x$$

$$f'(x)=\tfrac{1}{2}x^{\frac{-1}{2}}=\frac{1}{2\sqrt{x}} \qquad \Rightarrow f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{2\times 2}=\frac{1}{4}$$

Second derivative $\frac{d}{dx}f'(x)=\frac{1}{2}\frac{d}{dx}x^{\frac{-1}{2}}$

$$f''(x)=\tfrac{1}{2}\left(\tfrac{-1}{2}\right)x^{\frac{-1}{2}-1}\tfrac{d}{dx}x$$

$$f''(x)=\tfrac{-1}{4}x^{\frac{-3}{2}} \qquad \Rightarrow f''(4)=\tfrac{-1}{4}(4)^{\frac{-3}{2}}=\tfrac{-1}{4}\left(2^2\right)^{\frac{-3}{2}}$$

$$\Rightarrow f''(4)=\frac{-1}{4}\left(2^{-3}\right)=\frac{-1}{4\times 8}=\frac{-1}{32}$$

Third derivative $\frac{d}{dx}f''(x)=\frac{-1}{4}\frac{d}{dx}x^{\frac{-3}{2}}$

$$f'''(x)=\tfrac{-1}{4}\left(\tfrac{-3}{2}\right)x^{\frac{-3}{2}-1}\tfrac{d}{dx}x$$

$$f'''(x)=\tfrac{3}{8}x^{\frac{-5}{2}} \qquad \Rightarrow f'''(4)=\tfrac{3}{8}(4)^{\frac{-5}{2}}=\tfrac{3}{8}\left(2^2\right)^{\frac{-5}{2}}$$

$$\Rightarrow f'''(4)=\tfrac{3}{8}\left(2^{-5}\right)=\tfrac{3}{8}\left(\tfrac{1}{32}\right)=\tfrac{3}{256}$$

First four term of Taylor's series at $x_o=4$

$$f(x)=f(4)+(x-4)f'(4)+\frac{(x-4)^2}{2!}f''(4)+\frac{(x-4)^3}{3!}f'''(4)+\cdots$$

Substituting values in the above relation

$$f(x)=2+\frac{1}{4}(x-4)+\frac{-1}{32}\frac{(x-4)^2}{2\times 1}+\frac{3}{256}\frac{(x-4)^3}{3\times 2\times 1}+\cdots$$

$$f(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{3}{512}(x-4)^3+\cdots$$

---

c). Show that first four terms in the Taylor series expansion of $f(x)=x+e^x$ about $x=1$ are

$$(1+e)x+e\left[\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}+\cdots\right]$$

Sol: $f(x)=x+e^x$ $\Rightarrow f(1)=(1)+e^1=1+e$

First derivative $\frac{d}{dx}f(x)=\frac{d}{dx}x+\frac{d}{dx}e^x$

$$f'(x)=1+e^x \qquad \Rightarrow f'(1)=1+e^1=1+e$$

Second derivative $\frac{d}{dx}f'(x)=\frac{d}{dx}1+\frac{d}{dx}e^x$

$$f''(x)=0+e^x \qquad \Rightarrow f''(1)=e^1=e$$

Third derivative $\frac{d}{dx}f''(x)=\frac{d}{dx}e^x$

$$f'''(x)=e^x \qquad \Rightarrow f'''(1)=e^1=e$$

First four term of Taylor's series at $x_o=1$

$$f(x)=f(4)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\cdots$$

Substituting values in the above relation

$$f(x)=1(1+e)+(x-1)(1+e)+\frac{(x-1)^2}{2!}e+\frac{(x-1)^3}{3!}e+\cdots$$

$$f(x)=(1+x-1)(1+e)+\frac{(x-1)^2}{2!}e+\frac{(x-1)^3}{3!}e+\cdots$$

$$f(x)=x(1+e)+e\left[\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\cdots\right]$$

Q5. Find Maclaurin series expansion for following functions.

a). $f(x)=\dfrac{1}{1+x}$

Sol: Given $f(x)=\dfrac{1}{1+x}=(1+x)^{-1}$

$$f(x)=(1+x)^{-1} \qquad \Rightarrow f(0)=(1+0)^{-1}=1$$

First derivative $\frac{d}{dx}f(x)=\frac{d}{dx}(1+x)^{-1}$

$$f'(x)=-(1+x)^{-2}\tfrac{d}{dx}(1+x)$$

$$f'(x)=-(1+x)^{-2} \qquad \Rightarrow f'(0)=-(1+0)^{-2}=-1$$

Second derivative $\frac{d}{dx}f'(x)=-\frac{d}{dx}(1+x)^{-2}$

$$f''(x)=2(1+x)^{-3} \qquad \Rightarrow f''(0)=2(1+0)^{-3}=2$$

Third derivative $\frac{d}{dx}f''(x)=2\frac{d}{dx}(1+x)^{-3}$

$$f'''(x)=-6(1+x)^{-4}\tfrac{d}{dx}(1+x)$$

$$\Rightarrow f'''(0)=-6(1+0)^{-4}=-6$$

First four term of Maclaurin's series

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

Substituting values in the above relation

$$f(x)=1+x(1)+\frac{x^2}{2}(2)+\frac{x^3}{6}(-6)+\cdots$$

$$f(x)=1-x+x^2-x^3+\cdots$$

---

b). $f(x)=\sin^2 x$

Sol: $f(x)=\sin^2 x$ $\Rightarrow f(0)=\{\sin(0)\}^2=0$

Frist derivative $\frac{d}{dx}f(x)=\frac{d}{dx}\sin^2 x$

$$f'(x)=2\sin x\tfrac{d}{dx}\sin x$$

$$f'(x)=2\sin x\cos x$$

$$f'(x)=\sin 2x \qquad \Rightarrow f'(0)=\sin(2.0)=0$$

Second derivative $\frac{d}{dx}f'(x)=\frac{d}{dx}\sin 2x$

$$f''(x)=\cos 2x\tfrac{d}{dx}2x$$

$$f''(x)=2\cos 2x \qquad \Rightarrow f''(0)=2\cos(2.0)=2$$

Third derivative $\frac{d}{dx}f''(x)=2\frac{d}{dx}\cos 2x$

$$f'''(x)=-2\sin 2x\tfrac{d}{dx}(2x)$$

$$f'''(x)=-4\sin 2x \qquad \Rightarrow f'''(0)=-4\sin(2.0)=0$$

fourth derivative $\frac{d}{dx}f''(x)=-4\frac{d}{dx}\sin 2x$

$$f^{iv}(x)=-4\cos 2x\tfrac{d}{dx}(2x)$$

$$f^{iv}(x)=-8\cos 2x \qquad \Rightarrow f^{iv}(0)=-8\cos(2.0)=-8$$

Fifth derivative $\frac{d}{dx}f^{iv}(x)=-8\frac{d}{dx}\cos 2x$

$$f^{v}(x)=8\sin 2x\tfrac{d}{dx}(2x)$$

$$f^{v}(x)=16\sin 2x \qquad \Rightarrow f^{v}(0)=16\sin(2.0)=0$$

Sixth derivative $\frac{d}{dx}f^{v}(x)=16\frac{d}{dx}\sin 2x$

$$f^{vi}(x)=16\cos 2x\tfrac{d}{dx}(2x)$$

$$f^{vi}(x)=32\cos 2x \qquad \Rightarrow f^{vi}(0)=32\cos(2.0)=32$$

First six term of Maclaurin's series

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\cdots$$

Substituting values in the above relation

$$f(x)=0+x(0)+\frac{x^2}{2}(2)+\frac{x^3}{6}(0)+\frac{x^4}{24}(-8)+\frac{x^5}{120}(0)+\frac{x^6}{720}(32)+\cdots$$

$$f(x)=x^2-\frac{x^4}{3}+\frac{2x^6}{45}+\cdots$$

c). $f(x)=\cosh x$

Sol: $f(x)=\cosh x \quad \Rightarrow f(0)=\cosh(0)=1$

Frist derivative $\frac{d}{dx}f(x)=\frac{d}{dx}\cosh x$

$f'(x)=\sinh x \quad \Rightarrow f'(0)=\sinh(0)=0$

Second derivative $\frac{d}{dx}f'(x)=\frac{d}{dx}\sinh x$

$f''(x)=\cosh x \quad \Rightarrow f''(0)=\cosh(0)=1$

Third derivative $\frac{d}{dx}f''(x)=\frac{d}{dx}\cosh x$

$f'''(x)=\sinh x \quad \Rightarrow f'''(0)=\sinh(0)=0$

fourth derivative $\frac{d}{dx}f''(x)=\frac{d}{dx}\sinh x$

$f^{iv}(x)=\cosh x \quad \Rightarrow f^{iv}(0)=\cosh(0)=1$

Fifth derivative $\frac{d}{dx}f^{iv}(x)=\frac{d}{dx}\cosh x$

$f^{v}(x)=\sinh x \quad \Rightarrow f^{v}(0)=\sinh(0)=0$

Sixth derivative $\frac{d}{dx}f^{v}(x)=\frac{d}{dx}\sinh x$

$f^{vi}(x)=\cosh x \quad \Rightarrow f^{vi}(0)=\cosh(0)=1$

First six term of Maclaurin's series

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\cdots$$

Substituting values in the above relation

$$f(x)=1+x(0)+\frac{x^2}{2}(1)+\frac{x^3}{6}(0)+\frac{x^4}{24}(1)+\frac{x^5}{120}(0)+\frac{x^6}{720}(1)+\cdots$$

$$f(x)=1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\cdots$$

d). $f(x)=\ln(1-4x)$

Sol: Given

$f(x)=\ln(1-4x) \quad \Rightarrow f(0)=\ln(1-0)=\ln(1)=0$

Frist derivative $\frac{d}{dx}f(x)=\frac{d}{dx}\ln(1-4x)$

$f'(x)=\frac{-4}{1-4x} \quad \Rightarrow f'(0)=\frac{-4}{1-0}=-4$

Second derivative $\frac{d}{dx}f'(x)=-4\frac{d}{dx}(1-4x)^{-1}$

$f''(x)=\frac{4(-4)}{(1-4x)^2} \quad \Rightarrow f''(0)=-16$

Third derivative $\frac{d}{dx}f''(x)=-16\frac{d}{dx}(1-4x)^{-2}$

$f'''(x)=\frac{32(-4)}{(1-4x)^3} \quad \Rightarrow f'''(0)=-128$

fourth derivative $\frac{d}{dx}f''(x)=-128\frac{d}{dx}(1-4x)^{-3}$

$f^{iv}(x)=\frac{384(-4)}{(1-4x)^4} \quad \Rightarrow f^{iv}(0)=-1536$

First four terms of Maclaurin's series

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\cdots$$

Substituting values in the above relation

$$f(x)=0+x(-4)+\frac{x^2}{2}(-16)+\frac{x^3}{6}(-128)+\frac{x^4}{24}(-1536)+\cdots$$

$$f(x)=-4x-8x^2-\frac{64}{3}x^3-64x^4+\cdots$$

Q6a). Use Maclaurin series for e^x to show that the sum of infinite series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$ is e

Solution: $f(x)=e^x \quad \Rightarrow f(0)=e^0=1$

First derivative $\frac{d}{dx}f(x)=\frac{d}{dx}e^x$

$f'(x)=e^x \quad \Rightarrow f'(0)=e^0=1$

Second derivative $\frac{d}{dx}f'(x)=\frac{d}{dx}e^x$

$f''(x)=e^x \quad \Rightarrow f''(0)=e^0=1$

Third derivative $\frac{d}{dx}f''(x)=\frac{d}{dx}e^x$

$f'''(x)=e^x \quad \Rightarrow f'''(0)=e^0=1$

First four term of Maclaurin's series at $x_o=1$

$$e^x=f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

Substituting values in the above relation

$$e^x=f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

At $x=1$ we get

$$f(1)=e^1=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

b). Use part (a) to find out the value of e that must be accurate to 4 decimal places.

Sol: $f(x)=e^x \quad \Rightarrow f(0)=e^0=1$

First derivative $\frac{d}{dx}f(x)=\frac{d}{dx}e^x$

$f'(x)=e^x \quad \Rightarrow f'(0)=e^0=1$

Second derivative $\frac{d}{dx}f'(x)=\frac{d}{dx}e^x$

$f''(x)=e^x \quad \Rightarrow f''(0)=e^0=1$

Third derivative $\frac{d}{dx}f''(x)=\frac{d}{dx}e^x$

$f'''(x)=e^x \quad \Rightarrow f'''(0)=e^0=1$

First four term of Maclaurin's series at $x_o=1$

$$e^x=f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

Substituting values in the above relation

$$e^x=f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

At $x=1$ we get

$$f(1)=e^1=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\cdots$$

$$e=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040}+\cdots$$

$$e=2+0.5+0.1666+0.0416+0.0083+0.0013+0.0001+\cdots$$

$$e=2.7179$$

Q7. Find the angle of intersection between the following curves:

a). $x^2-y^2=a^2, \quad x^2+y^2=a^2\sqrt{2}$

Solution: Adding the system

$$x^2-y^2=a^2$$
$$\pm x^2\pm y^2=\pm a^2\sqrt{2}$$
$$-2y^2=a^2-a^2\sqrt{2}$$
$$2y^2=a^2\sqrt{2}-a^2$$
$$y^2=a^2\frac{\sqrt{2}-1}{2}$$

Putting the value of y^2 in equation (1) we have

$$x^2-a^2\frac{\sqrt{2}-1}{2}=a^2$$

$$x^2=a^2+a^2\frac{\sqrt{2}-1}{2}$$

$$x^2=\frac{2a^2+a^2\sqrt{2}-a^2}{2}$$

$$x^2=\frac{a^2\sqrt{2}+a^2}{2}=a^2\frac{\sqrt{2}+1}{2}$$

So we have

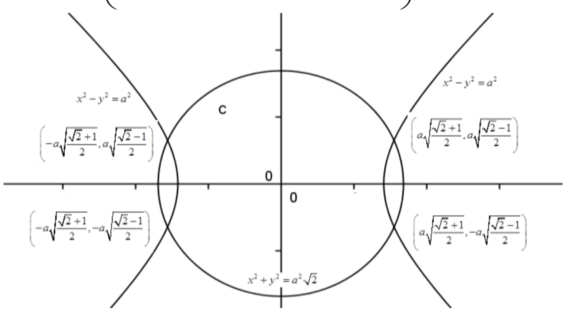
$$x^2=a^2\frac{\sqrt{2}+1}{2} \qquad y^2=a^2\frac{\sqrt{2}-1}{2}$$

Taking square root on both sides we get

$$x=\pm a\sqrt{\frac{\sqrt{2}+1}{2}} \qquad y=\pm a\sqrt{\frac{\sqrt{2}-1}{2}}$$

Therefore the points of intersections of the given curves are

$$(x,y)=\left(\pm a\sqrt{\frac{\sqrt{2}+1}{2}},\pm a\sqrt{\frac{\sqrt{2}-1}{2}}\right)=(\pm1.21a,\pm0.46a)$$



Differentiating the given system of equations

$$\frac{d}{dx}x^2-\frac{d}{dx}y^2=\frac{d}{dx}a^2, \qquad \frac{d}{dx}x^2+\frac{d}{dx}y^2=\frac{d}{dx}a^2\sqrt{2}$$

$$2x\frac{d}{dx}x-2y\frac{d}{dx}y=0, \qquad 2x\frac{d}{dx}x+2y\frac{d}{dx}y=0$$

$$2x-2y\frac{d}{dx}y=0, \qquad 2x+2y\frac{d}{dx}y=0$$

$$-2y\frac{d}{dx}y=-2x, \qquad 2y\frac{d}{dx}y=-2x$$

$$\frac{d}{dx}y=\frac{-2x}{-2y}, \qquad \frac{d}{dx}y=\frac{-2x}{2y}$$

$$\frac{d}{dx}y=\frac{-x}{-y}, \qquad \frac{d}{dx}y=\frac{-x}{y}$$

Slope at the point $(1.21a,0.46a)$

$$\left.\frac{dy}{dx}\right|_{(1.21a,0.46a)}=\frac{-1.21a}{-0.46a} \qquad \left.\frac{dy}{dx}\right|_{(1.21a,0.46a)}=\frac{-1.21a}{0.46a}$$

$$m_1=2.63 \qquad m_2=-2.63$$

An angle between the two curve is

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$ Substituting the values

$$\tan\theta=\frac{2.63-(-2.63)}{1-(2.63)(-2.63)}=\frac{2.63+2.63}{1-6.92}=\frac{5.26}{-5.92}$$

$$\theta=\tan^{-1}\left(\frac{5.26}{5.92}\right)=0.72643\,radian=43^o35'\approx45^o$$

b). $y^2=ax \qquad x^3+y^3=3axy$

Sol: Given

$$y^2=ax\ldots\ldots\ldots\ldots\ldots\ldots(1) \qquad \Rightarrow y=\sqrt{ax}\ldots\ldots\ldots(3)$$

$$x^3+y^3=3axy\ldots\ldots\ldots(2)$$

Putting the value of x in equation (2) we have

$$x^3+\left(\sqrt{ax}\right)^3=3ax\sqrt{ax}$$

$$x^3+a^{\frac{3}{2}}x^{\frac{3}{2}}=3a^{1+\frac{1}{2}}x^{1+\frac{1}{2}}$$

$$x^3=3a^{\frac{3}{2}}x^{\frac{3}{2}}-a^{\frac{3}{2}}x^{\frac{3}{2}}$$

$$x^3=2a^{\frac{3}{2}}x^{\frac{3}{2}}$$

Taking cube root on both sides

$$x=2^{\frac{1}{3}}a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$x-2^{\frac{1}{3}}a^{\frac{1}{2}}x^{\frac{1}{2}}=0$$

$$x^{\frac{1}{2}}\left(x^{\frac{1}{2}}-2^{\frac{1}{3}}a^{\frac{1}{2}}\right)=0$$

either

$$x^{\frac{1}{2}}=0$$

$$x=0$$

or

$$x^{\frac{1}{2}}-2^{\frac{1}{3}}a^{\frac{1}{2}}=0$$

$$x^{\frac{1}{2}}=2^{\frac{1}{3}}a^{\frac{1}{2}}$$

$$x=2^{\frac{2}{3}}a$$

The corresponding values of x we get from equation (3)

$$y=\sqrt{a(0)} \qquad y=\sqrt{a\left(2^{\frac{2}{3}}a\right)}$$

$$y=0 \qquad y=2^{\frac{1}{3}}a$$

Therefore points of intersection are $(0,0)$ & $\left(2^{\frac{2}{3}}a,2^{\frac{1}{3}}a\right)$

Now differentiating the given system of equations

$$\frac{d}{dx}y^2=a\frac{d}{dx}x \qquad \frac{d}{dx}x^3+\frac{d}{dx}y^3=3a\frac{d}{dx}(xy)$$

$$2y\frac{d}{dx}y=a \qquad 3x^2\frac{d}{dx}x+3y^2\frac{d}{dx}y=3a\left(x\frac{d}{dx}y+y\frac{d}{dx}x\right)$$

$$\frac{dy}{dx}=\frac{a}{2y} \qquad 3x^2+3y^2\frac{dy}{dx}=3a\left(x\frac{d}{dx}y+y\right)$$

$$3x^2+3y^2\frac{dy}{dx}=3ax\frac{dy}{dx}+3ay$$

$$3y^2\frac{dy}{dx}-3ax\frac{dy}{dx}=3ay-3x^2$$

$$3\left(y^2-ax\right)\frac{dy}{dx}=3\left(ay-x^2\right)$$

$$\frac{dy}{dx}=\frac{3\left(ay-x^2\right)}{3\left(y^2-ax\right)}=\frac{\left(ay-x^2\right)}{\left(y^2-ax\right)}$$

Slope at the point $\left(2^{\frac{2}{3}}a,2^{\frac{1}{3}}a\right)$

$$\left.\frac{dy}{dx}\right|_{\left(2^{\frac{2}{3}}a,2^{\frac{1}{3}}a\right)}=\frac{a}{2.2^{\frac{1}{3}}a} \qquad \left.\frac{dy}{dx}\right|_{\left(2^{\frac{2}{3}}a,2^{\frac{1}{3}}a\right)}=\frac{a\left(2^{\frac{1}{3}}a\right)-\left(2^{\frac{2}{3}}a\right)^2}{\left(2^{\frac{1}{3}}a\right)^2-a\left(2^{\frac{2}{3}}a\right)}$$

$$m_1=2^{\frac{-4}{3}} \qquad m_2=\frac{2^{\frac{1}{3}}a^2-2^{\frac{4}{3}}a^2}{2^{\frac{2}{3}}a^2-2^{\frac{2}{3}}a^2}=\frac{2^{\frac{1}{3}}a^2-2^{\frac{4}{3}}a^2}{0}$$

$$m_2=\frac{1}{0}$$

An angle between the two curve is

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$ Substituting the values

$$\tan\theta=\frac{2^{\frac{-1}{3}}-\frac{1}{0}}{1-\left(2^{\frac{-1}{3}}\right)\left(\frac{1}{0}\right)}=\frac{\frac{0-2^{\frac{-1}{3}}}{0}}{\frac{0-2^{\frac{-1}{3}}}{0}}=\frac{2^{\frac{-1}{3}}}{2^{\frac{-1}{3}}}=1$$

$$\theta=\tan^{-1}(1)=45^o$$

Increasing function

$x\in[a,b]$ or $a\le x\le b\Rightarrow f(a)<f(b)$

Decreasing function

$x\in[a,b]$ or $a\le x\le b\Rightarrow f(a)>f(b)$

Extreme points of the function:

An extreme points is a vertex of the function

Or Extreme points of function where $f'(x)=0$

Critical value and critical points

$f(x)$ is defined at a number c

Either $f'(c)=0$ or $f'(c)$ does not exists

Then **c** is called **critical value**

And the points on a graph $(c,f(c))$ is **critical points**

Exercise 3.3

Q1a). What is the first derivative test?
Answer: First derivative test is use for relative extrema
Steps for relative extrema
i). Find all critical values of $f(x)$
i.e., find all numbers which are defined $f'(c)=0$
or which are undefined $f'(c)$
ii). The $P(c, f(c))$ is relative maximum if
$f'(x)>0$ (rising) for all x in (a,c) and $f'(x)<0$ (falling) for all x in an open interval (c,b)
iii). The $P(c, f(c))$ is relative minimum if
$f'(x)<0$ (falling) for all x in (a,c) and $f'(x)>0$ (rising) for all x in an open interval (c,b)
iv). The $P(c, f(c))$ is not an extremum if $f'(x)$ has the same sign in (a,c) and (c,b)

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b). what is the relationship between the graph of a function and the graph of its derivative?
Answer:*. if $f(x)$ is differentiable on (a,b) i.e.,
i) $f(x)$ is strictly increasing on (a,b) if $f'(x)>0$
ii) $f(x)$ is strictly decreasing on (a,b) if $f'(x)<0$
* $f(x)$ in increasing on (a,b) if tangent lines to its graph at $(x_1, f(x_1))$ makes positive slope $f(x_1)>0$
* $f(x)$ in decreasing on (a,b) if tangent lines to its graph at $(x_1, f(x_1))$ makes negative slope $f(x_1)<0$
* $f(x)$ is neither increasing nor decreasing on (a,b) if the tangent lines to its graph at point $(x_1, f(x_1))$ makes zero slope $f(x_1)=0$

---

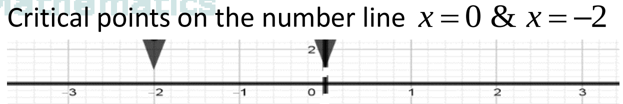
c). What is the second derivative test?
Answer: Let $f(x)$ be a function such that
$f'(c)=0$ and $f''(x)$ exists with $c\in(a,b)$
i. If $f''(c)>0$, then there is a **relative minimum** at $x=c$ and the graph of $f(x)$ is **concave up** in the neighborhood of point $(c, f(c))$
ii. If $f''(c)<0$, then there is a **relative maximum** at $x=c$ and graph of $f(x)$ is **concave down** in the neighborhood of point $(c, f(c))$
iii. If $f''(c)=0$, then second derivative test fails and gives no information.

---

d). what is the relationship between concavity, points of inflection and second derivative test?
Answer: The graph of the function is concave upward on (a,b) where $f''(x)>0$ and it is concave downward where $f''(x)<0$
And the point of inflection is in between the concave upward and concave downward
Through the second derivative test we find the relative maximum and relative minimum as well

---

Q2. Find the critical value of the given functions and show where the function is increasing and where it is decreasing. Plot each critical point and label it as a relative maximum, a relative minimum, or neither
a). $f(x)=x^3+3x^2+1$
Sol: Given $f(x)=x^3+3x^2+1$ ....(1)
Differentiate with respect to x
$\frac{d}{dx}f(x)=\frac{d}{dx}x^3+3\frac{d}{dx}x^2+\frac{d}{dx}1$
$f'(x)=(3x^2)\frac{d}{dx}x+3(2x)\frac{d}{dx}x+0$
$f'(x)=3x^2+6x$ ....(2)
For the critical values take $f'(x)=0$ we get
$3x^2+6x=0$
$3x(x+2)=0$
Either or
$3x=0$
$x=\frac{0}{3}$
$x=0$
$x+2=0$
$x=-2$
The critical values are $x=0$ & $x=-2$
To find critical point at $x=0$ we get
$f(0)=(0)^3+3(0)^2+1$
$f(0)=0+0+1=1$
To find critical point at $x=-2$ we get
$f(-2)=(-2)^3+3(-2)^2+1$
$f(-2)=-8+3(4)+1$
$f(-2)=-8+12+1=5$
Thus the critical points are $(0,1)$ & $(-2,5)$
Critical points on the number line $x=0$ & $x=-2$



There are three interval $(-\infty,-2]\cup[-2,0]\cup[0,\infty)$
we have to check given function is increasing or decreasing
First we check interval $(-\infty,-2]$ take any points in interval, let $x_1=-3$ & $x_2=-2$ i.e., $x_1<x_2$
So $f(-3)=(-3)^3+3(-3)^2+1=1$
And $f(-2)=(-2)^3+3(-2)^2+1=5$
We have $x_1<x_2 \Rightarrow f(x_1)<f(x_2)$
The given function is increasing on $(-\infty,-2]$
Or simply we use first derivative test $-3\in(-\infty,-2]$
$f'(-3)=3(-3)^2+6(-3)=27-18=9>0$
So function is increasing on $(-\infty,-2]$
Now we check interval $[-2,0]$ take any points in interval, let $x_1=-2$ & $x_2=0$ i.e., $x_1<x_2$
So $f(-2)=(-2)^3+3(-2)^2+1=5$
And $f(0)=(0)^3+3(0)^2+1=1$

We have $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

The given function is decreasing at $[-2,0]$

Or simply we use first derivative test $-1 \in [-2,0]$

$f'(-1) = 3(-1)^2 + 6(-1) = 3-6 = -3 < 0$

So function is decreasing on $[-2,0]$

Now we check on the interval $[0,\infty)$ take any points in between in interval, let $x_1 = 0$ & $x_2 = 2$ i.e., $x_1 < x_2$

So $f(0) = (0)^3 + 3(0)^2 + 1 = 1$

And $f(2) = (2)^3 + 3(2)^2 + 1 = 21$

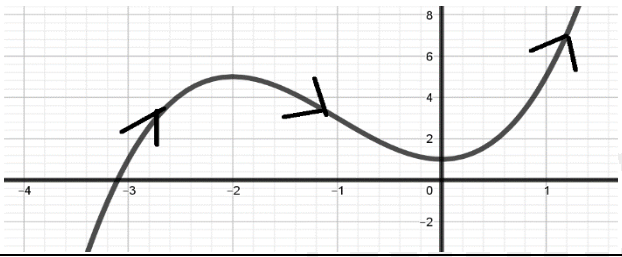
We have $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

The given function is increasing at $[0,\infty)$

Or simply we use first derivative test $3 \in [0,\infty)$

$f'(3) = 3(3)^2 + 6(3) = 27 - 18 = 9 > 0$

So function is increasing on $[0,\infty)$ And the graph



b). $f(x) = x^3 + 35x^2 - 125x - 9.375$

Sol: We have $f(x) = x^3 + 35x^2 - 125x - 9.375$

Differentiate with respect to x

$\frac{d}{dx} f(x) = \frac{d}{dx} x^3 + 35\frac{d}{dx} x^2 - 125\frac{d}{dx} x - \frac{d}{dx} 9.375$

$f'(x) = (3x^2)\frac{d}{dx} x + 35(2x)\frac{d}{dx} x - 125 - 0$

$f'(x) = 3x^2 + 70x - 125$

For the critical values take $f'(x) = 0$ we get

$3x^2 + 70x - 125 = 0$

$3x^2 + 75x - 5x - 125 = 0$

$3x(x+25) - 5(x+25) = 0$

$(3x-5)(x+25) = 0$

Either or

$3x - 5 = 0$ $x + 25 = 0$

$x = \frac{5}{3}$ $x = -25$

The critical values are $x = \frac{5}{3}$ & $x = -25$

To find critical point at $x = \frac{5}{3}$ we get

$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 + 35\left(\frac{5}{3}\right)^2 - 125\left(\frac{5}{3}\right) - 9.375$

$f\left(\frac{5}{3}\right) = \frac{125}{27} + 35\left(\frac{25}{9}\right) - \frac{625}{3} - 9.375$

$f\left(\frac{5}{3}\right) = -115.856$

To find critical point at $x = -25$ we get

$f(-25) = (-25)^3 + 35(-25)^2 - 125(-25) - 9.375$

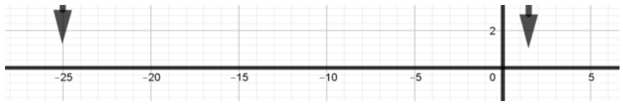
$f(-25) = -15625 + 35(625) + 3125 - 9.375$

$f(-25) = 9\,365.625$

Thus critical points are

$\left(\frac{5}{3}, -115.856\right)$ & $(-25, 9365.625)$

critical points on the number line $x = \frac{5}{3}$ & $x = -25$



There are three intervals $(-\infty,-25]$, $\left[-25,\frac{5}{3}\right]$ & $\left[\frac{5}{3},\infty\right)$ where we have to check the given function is increasing or decreasing

By using First derivative test we take any point

Let $-30 \in (-\infty,-25]$

$f'(-30) = 3(-30)^2 + 70(-30) - 125$

$f'(-30) = 2700 - 2100 - 125 = 475 > 0$

So function is increasing at $(-\infty,-25]$

By using First derivative test we take any point

Let $-20 \in \left[-25,\frac{5}{3}\right]$

$f'(-20) = 3(-20)^2 + 70(-20) - 125$

$f'(-20) = 1200 - 1400 - 125 = -325 < 0$

So function is decreasing at $\left[-25,\frac{5}{3}\right]$

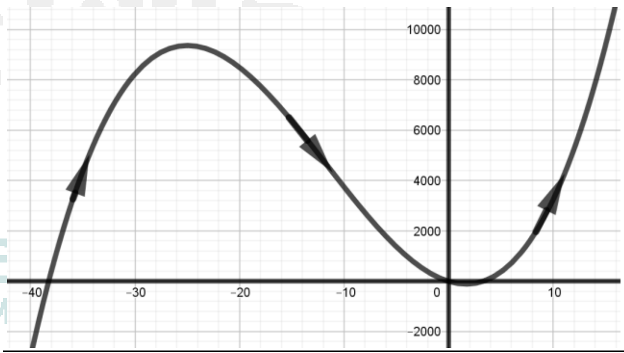
By using First derivative test we take any point

Let $2 \in \left[\frac{5}{3},\infty\right)$

$f'(2) = 3(2)^2 + 70(2) - 125$

$f'(2) = 12 + 140 - 125 = 27 > 0$

So function is increasing at $\left[\frac{5}{3},\infty\right)$ And the graph



Q3. Find critical values of the following functions

a). $f(x) = 2x^3 - 3x^2 - 72x + 15$

Sol: Given $f(x) = 2x^3 - 3x^2 - 72x + 15$

Differentiate with respect to x

$\frac{d}{dx} f(x) = 2\frac{d}{dx} x^3 - 3\frac{d}{dx} x^2 - 72\frac{d}{dx} x + \frac{d}{dx} 15$

$f'(x) = 2(3x^2)\frac{d}{dx} x - 3(2x)\frac{d}{dx} x - 72 - 0$

$f'(x) = 6x^2 - 6x - 72$

For the critical values take $f'(x) = 0$ we get

$6x^2 - 6x - 72 = 0$

$6(x^2 - x - 12) = 0$

$x^2 - x - 12 = 0$

$x^2 - 4x + 3x - 12 = 0$

$x(x-4) + 3(x-4) = 0$

$(x+3)(x-4) = 0$

Either or

$x + 3 = 0$ $x - 4 = 0$

$x = -3$ $x = 4$

The critical values are -3 and 4

b). $f(x)=\frac{1}{3}x^3-x^2-15x+6$

Sol: Given $f(x)=\frac{1}{3}x^3-x^2-15x+6$

Differentiate with respect to x

$\frac{d}{dx}f(x)=\frac{1}{3}\frac{d}{dx}x^3-\frac{d}{dx}x^2-15\frac{d}{dx}x+\frac{d}{dx}6$

$f'(x)=\frac{1}{3}(3x^2)\frac{d}{dx}x-(2x)\frac{d}{dx}x-15+0$

$f'(x)=x^2-2x-15$

For the critical values take $f'(x)=0$ we get

$x^2-2x-15=0$

$x^2-5x+3x-15=0$

$x(x-5)+3(x-5)=0$

$(x+3)(x-5)=0$

Either or

$x+3=0$ $x-5=0$

$x=-3$ $x=5$

The critical values are -3 and 5

c). $f(x)=6x^{\frac{2}{3}}-4x$

Sol: Given $f(x)=6x^{\frac{2}{3}}-4x$

Differentiate with respect to x

$\frac{d}{dx}f(x)=6\frac{d}{dx}x^{\frac{2}{3}}-4\frac{d}{dx}x$

$f'(x)=6\left(\frac{2}{3}x^{\frac{2}{3}-1}\right)\frac{d}{dx}x-4$

$f'(x)=4x^{\frac{-1}{3}}-4$

For the critical values take $f'(x)=0$ we get

$4x^{\frac{-1}{3}}-4=0$

$4x^{\frac{-1}{3}}=4$

$x^{\frac{-1}{3}}=1$

$\frac{1}{x^{\frac{1}{3}}}=1$ $\Rightarrow x\neq 0$

$1=x^{\frac{1}{3}}$

Taking cube on both sides

$\left(x^{\frac{1}{3}}\right)^3=(1)^3$

$x=1$

The critical values are 0 & 1

d). $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$

Sol: Given $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$

Differentiate with respect to x

$\frac{d}{dx}f(x)=3\frac{d}{dx}x^{\frac{4}{3}}-12\frac{d}{dx}x^{\frac{1}{3}}$

$f'(x)=3\left(\frac{4}{3}\right)x^{\frac{4}{3}-1}\frac{d}{dx}x-12\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}\frac{d}{dx}x$

$f'(x)=4x^{\frac{1}{3}}-4x^{\frac{-2}{3}}$

For the critical values take $f'(x)=0$ we get

$4x^{\frac{1}{3}}-4x^{\frac{-2}{3}}=0$

$4\left(x^{\frac{1}{3}}-x^{\frac{-2}{3}}\right)=0$

$4x^{\frac{-2}{3}}(x-1)=0$

Either or

$4x^{\frac{-2}{3}}=0$ $x-1=0$

$x=0$ $x=1$

The critical values are 0 and 1

Q4. Determine whether given function has a relative maximum, a relative minimum or neither at the given critical values for following problems

a). $f(x)=(x^3-3x+1)^7=0$ at $x=1;\ x=-1$

Sol: Given $f(x)=(x^3-3x+1)^7=0$

Differentiate with respect to x

$\frac{d}{dx}f(x)=\frac{d}{dx}(x^3-3x+1)^7$

$f'(x)=7(x^3-3x+1)^{7-1}\frac{d}{dx}(x^3-3x+1)$

$f'(x)=7(x^3-3x+1)^6\left(\frac{d}{dx}x^3-3\frac{d}{dx}x+\frac{d}{dx}1\right)$

$f'(x)=7(x^3-3x+1)^6\left(3x^2\frac{d}{dx}x-3(1)+0\right)$

$f'(x)=7(x^3-3x+1)^6(3x^2-3)$

$f'(x)=21(x^3-3x+1)^6(x^2-1)$

Again differentiating with respect to x

$\frac{d}{dx}f'(x)=21\frac{d}{dx}\left\{(x^3-3x+1)^6(x^2-1)\right\}$

$f''(x)=21\left\{(x^3-3x+1)^6\frac{d}{dx}(x^2-1)+(x^2-1)\frac{d}{dx}(x^3-3x+1)^6\right\}$

$f''(x)=21(x^3-3x+1)^6\left(\frac{d}{dx}x^2-\frac{d}{dx}1\right)$
$+21\times 6(x^2-1)(x^3-3x+1)^5\frac{d}{dx}(x^3-3x+1)$

$f''(x)=21(x^3-3x+1)^6\left(2x\frac{d}{dx}x-0\right)$
$+126(x^2-1)(x^3-3x+1)^5\left(\frac{d}{dx}x^3-3\frac{d}{dx}x+\frac{d}{dx}1\right)$

$f''(x)=21\times 2x(x^3-3x+1)^6+126(x^2-1)(x^3-3x+1)^5(3x^2-3)$

$f''(x)=42x(x^3-3x+1)^6+126\times 3(x^2-1)^2(x^3-3x+1)^5$

$f''(x)=42x(x^3-3x+1)^6+378(x^2-1)^2(x^3-3x+1)^5$

$f''(x)=42(x^3-3x+1)^5\left\{x(x^3-3x+1)+9(x^2-1)^2\right\}$

The $f''(x)$ at the critical point/given point $x=1$

$f''(1)=42\left((1)^3-3(1)+1\right)^5\left\{(1)\left((1)^3-3(1)+1\right)+9\left((1)^2-1\right)^2\right\}$

$f''(1)=42(1-3+1)^5\left\{(1-3+1)+9(1-1)^2\right\}$

$f''(1)=42(-1)^5\left\{(-1)+9(0)^2\right\}$

$f''(1)=42(-1)\{-1+0\}$

$f''(1)=42>0$

Through the second derivative test $f''(1)>0$ then there is a relative minimum at $x=1$

The $f''(x)$ at the critical point/given point $x=-1$

$f''(-1)=42\left((-1)^3-3(-1)+1\right)^5\left\{(-1)\left((-1)^3-3(-1)+1\right)+9\left((-1)^2-1\right)^2\right\}$

$f''(-1)=42(-1-3+1)^5\left\{-(-1-3+1)+9(1-1)^2\right\}$

$f''(-1)=42(-3)^5\left\{-(-3)+9(0)^2\right\}$

$f''(-1)=42(-243)\{3+0\}$

$f''(1)=-30618<0$

Through the second derivative test $f''(-1)<0$ then there is a relative maximum at $x=-1$

b). $f(x)=(x^4-4x+2)^5$ at $x=1$

Sol: Given $f(x)=(x^4-4x+2)^5$

$$\tfrac{d}{dx}f(x)=\tfrac{d}{dx}(x^4-4x+2)^5$$
$$f'(x)=5(x^4-4x+2)^4\tfrac{d}{dx}(x^4-4x+2)$$
$$f'(x)=5(x^4-4x+2)^4\left(\tfrac{d}{dx}x^4-4\tfrac{d}{dx}x+\tfrac{d}{dx}2\right)$$
$$f'(x)=5(x^4-4x+2)^4\left(4x^3\tfrac{d}{dx}x-4+0\right)$$
$$f'(x)=5(x^4-4x+2)^4(4x^3-4)$$

Again differentiating with respect to x

$$\tfrac{d}{dx}f'(x)=5\tfrac{d}{dx}\left\{(x^4-4x+2)^4(4x^3-4)\right\}$$
$$f''(x)=5\left\{(x^4-4x+2)^4\tfrac{d}{dx}(4x^3-4)+(4x^3-4)\tfrac{d}{dx}(x^4-4x+2)^4\right\}$$
$$f''(x)=5(x^4-4x+2)^4\left(4\tfrac{d}{dx}x^3-\tfrac{d}{dx}4\right)$$
$$+5\times4(4x^3-4)(x^4-4x+2)^3\tfrac{d}{dx}(x^4-4x+2)$$
$$f''(x)=5(x^4-4x+2)^4\left(4(3x^2)\tfrac{d}{dx}x-0\right)$$
$$+20(4x^3-4)(x^4-4x+2)^3\left(\tfrac{d}{dx}x^4-4\tfrac{d}{dx}x+\tfrac{d}{dx}2\right)$$
$$f''(x)=5\times12x^2(x^4-4x+2)^4+20(x^4-4x+2)^3(4x^3-4)^2$$
$$f''(x)=20(x^4-4x+2)^3\left\{3x^2(x^4-4x+2)+(4x^3-4)^2\right\}$$

The $f''(x)$ at the critical point/given point $x=1$

$$f''(1)=20\left((1)^4-4(1)+2\right)^3\left\{3(1)^2\left((1)^4-4(1)+2\right)+\left(4(1)^3-4\right)^2\right\}$$
$$f''(1)=20(1-4+2)^3\left\{3(1-4+2)+(4-4)^2\right\}$$
$$f''(1)=20(-1)^3\left\{3(-1)+(0)^2\right\}$$
$$f''(1)=20(-1)\{-3+0\}$$
$$f''(1)=60>0$$

Through the second derivative test $f''(1)>0$ then there is a relative minimum at $x=1$

c). $f(x)=(x^2-4)^4(x^2-1)^3$ at $x=1;\ x=2$

Sol: Given $f(x)=(x^2-4)^4(x^2-1)^3$

$$\tfrac{d}{dx}f(x)=\tfrac{d}{dx}\left\{(x^2-4)^4(x^2-1)^3\right\}$$
$$f'(x)=(x^2-4)^4\tfrac{d}{dx}(x^2-1)^3+(x^2-1)^3\tfrac{d}{dx}(x^2-4)^4$$
$$f'(x)=3(x^2-4)^4(x^2-1)^2\tfrac{d}{dx}(x^2-1)$$
$$+4(x^2-1)^3(x^2-4)^3\tfrac{d}{dx}(x^2-4)$$
$$f'(x)=3(x^2-4)^4(x^2-1)^2\left(\tfrac{d}{dx}x^2-\tfrac{d}{dx}1\right)$$
$$+4(x^2-1)^3(x^2-4)^3\left(\tfrac{d}{dx}x^2-\tfrac{d}{dx}4\right)$$
$$f'(x)=3\times2x(x^2-4)^4(x^2-1)^2+4\times2x(x^2-1)^3(x^2-4)^3$$
$$f'(x)=6x(x^2-4)^4(x^2-1)^2+8x(x^2-1)^3(x^2-4)^3$$
$$f'(x)=2x(x^2-1)^2(x^2-4)^3\left\{3(x^2-4)+4(x^2-1)\right\}$$
$$f'(x)=2x(x^2-1)^2(x^2-4)^3\left\{3x^2-12+4x^2-4\right\}$$
$$f'(x)=2x(x^2-1)^2(x^2-4)^3(7x^2-16)$$

Again differentiating with respect to x

$$\tfrac{d}{dx}f'(x)=\tfrac{d}{dx}\left[\left\{2x(x^2-1)^2\right\}\left\{(x^2-4)^3(7x^2-16)\right\}\right]$$
$$f''(x)=\left\{2x(x^2-1)^2\right\}\tfrac{d}{dx}\left\{(x^2-4)^3(7x^2-16)\right\}$$
$$+\left\{(x^2-4)^3(7x^2-16)\right\}\tfrac{d}{dx}\left\{2x(x^2-1)^2\right\}$$
$$f''(x)=\left\{2x(x^2-1)^2\right\}\left\{(x^2-4)^3\tfrac{d}{dx}(7x^2-16)+(7x^2-16)\tfrac{d}{dx}(x^2-4)^3\right\}$$
$$+\left\{(x^2-4)^3(7x^2-16)\right\}\left\{2x\tfrac{d}{dx}(x^2-1)^2+(x^2-1)^2\tfrac{d}{dx}(2x)\right\}$$
$$f''(x)=\left\{2x(x^2-1)^2\right\}\left\{(x^2-4)^3(14x)+3(2x)(7x^2-16)(x^2-4)^2\right\}$$
$$+\left\{(x^2-4)^3(7x^2-16)\right\}\left\{4x(2x)(x^2-1)+2(x^2-1)^2\right\}$$

The $f''(x)$ at the critical point/given point $x=1$

$$f''(1)=\left\{2(1)(1^2-1)^2\right\}\left\{(1^2-4)^3(14)+3(2)(7(1)^2-16)(1^2-4)^2\right\}$$
$$+\left\{(1^2-4)^3(7(1)^2-16)\right\}\left\{4(2)(1^2-1)+2(1^2-1)^2\right\}$$
$$f''(1)=\left\{2(1)(0)^2\right\}\left\{(-3)^3(14)+3(2)(-9)(-3)^2\right\}$$
$$+\left\{(-3)^3(-9)\right\}\left\{4(2)(0)+2(0)^2\right\}$$
$$f''(1)=\{0\}\{(-27)(14)+6(-9)(9)\}+\{(-27)(-9)\}\{0+0\}$$
$$f''(1)=0+0$$
$$f''(1)=0$$

So second derivative test $f''(1)=0$ fails at $x=1$

Now to check $f''(x)$ at the critical point $x=2$

$$f''(2)=\left\{2(2)(2^2-1)^2\right\}\left\{(2^2-4)^3(14\times2)+3(2\times2)(7(2)^2-16)(2^2-4)^2\right\}$$
$$+\left\{(2^2-4)^3(7(2)^2-16)\right\}\left\{4\times2(2\times2)(2^2-1)+2(2^2-1)^2\right\}$$
$$f''(2)=\left\{4(3)^2\right\}\left\{(0)^3(28)+12(12)(0)^2\right\}+\left\{(0)^3(12)\right\}\left\{8(4)(3)+2(3)^2\right\}$$
$$f''(2)=\{36\}\{0+0\}+\{0\}\{96+18\}$$
$$f''(2)=0+0=0$$

So second derivative test $f''(2)=0$ fails at $x=2$

d). $f(x)=\sqrt[3]{x^3-48}$ at $x=4$

Sol: Given $f(x)=\sqrt[3]{x^3-48}$ Differentiate w.r.t. x

$$\tfrac{d}{dx}f(x)=\tfrac{d}{dx}(x^3-48)^{\frac{1}{3}}$$
$$f'(x)=\tfrac{1}{3}(x^3-48)^{\frac{1}{3}-1}\tfrac{d}{dx}(x^3-48)$$
$$f'(x)=\tfrac{1}{3}(x^3-48)^{\frac{-2}{3}}\left(\tfrac{d}{dx}x^3-\tfrac{d}{dx}48\right)$$
$$f'(x)=\tfrac{1}{3}(x^3-48)^{\frac{-2}{3}}\left(3x^2\tfrac{d}{dx}x-0\right)$$
$$f'(x)=\tfrac{1}{3}(3x^2)(x^3-48)^{\frac{-2}{3}}$$
$$f'(x)=x^2(x^3-48)^{\frac{-2}{3}}$$

Again differentiating with respect to x

$$\tfrac{d}{dx}f'(x)=\tfrac{d}{dx}\left\{x^2(x^3-48)^{\frac{-2}{3}}\right\}$$
$$f''(x)=x^2\tfrac{d}{dx}(x^3-48)^{\frac{-2}{3}}+(x^3-48)^{\frac{-2}{3}}\tfrac{d}{dx}x^2$$
$$f''(x)=\tfrac{-2}{3}x^2(x^3-48)^{\frac{-2}{3}-1}\tfrac{d}{dx}(x^3-48)+2x(x^3-48)^{\frac{-2}{3}}$$
$$f''(x)=\tfrac{-2}{3}x^2(x^3-48)^{\frac{-5}{3}}\left(\tfrac{d}{dx}x^3-\tfrac{d}{dx}48\right)+2x(x^3-48)^{\frac{-2}{3}}$$
$$f''(x)=\tfrac{-2}{3}x^2(3x^2)(x^3-48)^{\frac{-5}{3}}+2x(x^3-48)^{\frac{-2}{3}}$$
$$f''(x)=-2x^4(x^3-48)^{\frac{-5}{3}}+2x(x^3-48)^{\frac{-2}{3}}$$

The $f''(x)$ at the critical point/given point $x=4$

$f''(4)=-2(4)^4(4^3-48)^{\frac{-5}{3}}+2(4)(4^3-48)^{\frac{-2}{3}}$

$f''(4)=-2(2^2)^4(16)^{\frac{-5}{3}}+2^{1+2}(16)^{\frac{-2}{3}}$

$f''(4)=-2(2^8)(2^4)^{\frac{-5}{3}}+2^3(2^4)^{\frac{-2}{3}}$

$f''(4)=-2^{1+8-\frac{20}{3}}+2^{3-\frac{8}{3}}$

$f''(4)=-2^{9-\frac{20}{3}}+2^{3-\frac{8}{3}}$

$f''(4)=-2^{\frac{27-20}{3}}+2^{\frac{9-8}{3}}$

$f''(4)=-2^{\frac{7}{3}}+2^{\frac{1}{3}}=-3.78<0$

Through the second derivative test $f''(4)<0$

then there is a relative maximum at $x=4$

Q5. Find all relative extrema of following functions

a). $f(x)=x^3-3x^2+1$

Sol: Given $f(x)=x^3-3x^2+1$

Differentiate with respect to x

$\frac{d}{dx}f(x)=\frac{d}{dx}x^3-3\frac{d}{dx}x^2+\frac{d}{dx}1$

$f'(x)=3x^2\frac{d}{dx}x-3(2x)\frac{d}{dx}x+0$

$f'(x)=3x^2-6x$

For the critical values take $f'(x)=0$ we get

$3x^2-6x=0$

$3x(x-2)=0$

Either $3x=0$, $x=0$ or $x-2=0$, $x=2$

The critical values are 0 and 2

Differentiating $f'(x)$ with respect to x

$\frac{d}{dx}f'(x)=3\frac{d}{dx}x^2-6\frac{d}{dx}x$

$f''(x)=3(2x)\frac{d}{dx}x-6$

$f''(x)=6x-6$

The $f''(x)$ at the critical point/given point $x=2$

$f''(2)=6(2)-6$

$f''(2)=12-6=6>0$

Through the second derivative test $f''(2)>0$

then there is a relative minimum at $x=2$

Now to find minimum value at $x=2$ in the given function

$f(2)=(2)^3-3(2)^2+1$

$f(2)=8-3(4)+1=8-12+1=-3$

Minimum value is -3 at the critical value at $x=2$

The $f''(x)$ at the critical point/given point $x=0$

$f''(0)=6(0)-6$

$f''(0)=0-6=-6<0$

Through the second derivative test $f''(0)<0$

then there is a relative maximum at $x=0$

Now to find maximum value at $x=0$ in the given function

$f(0)=(0)^3-3(0)^2+1$

$f(0)=0-0+1=1$

Maximum value is 1 at the critical value at $x=0$

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b). $f(x)=x^3+6x^2+9x+2$

Sol: Given $f(x)=x^3+6x^2+9x+2$

Differentiate with respect to x

$\frac{d}{dx}f(x)=\frac{d}{dx}x^3+6\frac{d}{dx}x^2+9\frac{d}{dx}x+\frac{d}{dx}2$

$f'(x)=3x^2\frac{d}{dx}x+6(2x)\frac{d}{dx}x+9+0$

$f'(x)=3x^2+12x+9$

For the critical values take $f'(x)=0$ we get

$3x^2+12x+9=0$

$3(x^2+4x+3)=0$

$x^2+4x+3=0$

$x^2+3x+x+3=0$

$x(x+3)+1(x+3)=0$

$(x+1)(x+3)=0$

Either $x+1=0$, $x=-1$ or $x+3=0$, $x=-3$

The critical values are -1 and -3

Differentiating $f'(x)$ with respect to x

$\frac{d}{dx}f'(x)=3\frac{d}{dx}x^2+12\frac{d}{dx}x+\frac{d}{dx}9$

$f''(x)=3(2x)\frac{d}{dx}x+12+0$

$f''(x)=6x+12$

The $f''(x)$ at the critical point/given point $x=-1$

$f''(-1)=6(-1)+12$

$f''(-1)=-6+12=6>0$

Through the second derivative test $f''(-1)>0$

then there is a relative minimum at $x=-1$

Now to find minimum value at $x=-1$ in the given function

$f(-1)=(-1)^3+6(-1)^2+9(-1)+2$

$f(-1)=(-1)+6(1)-9+2$

$f(-1)=-1+6-7$

$f(-1)=-2$

$\therefore$ minimum value of function is -2 at critical $x=-1$

The $f''(x)$ at the critical point/given point $x=-3$

$f''(-3)=6(-3)+12$

$f''(-3)=-18+12=-6<0$

Through the second derivative test $f''(-3)<0$

then there is a relative maximum at $x=-3$

Now to find maximum value at $x=-3$ in the given function

$f(-3)=(-3)^3+6(-3)^2+9(-3)+2$

$f(-3)=(-27)+6(9)-27+2$

$f(-1)=-27+57-25$

$f(-1)=27-25=2$

Therefore the maximum value of the function is 2 at the critical value $x=-3$

Q6a). Suppose a $f(x)$ is differential function with derivative $f'(x)=(x-1)^2(x-2)(x-4)(x+5)^4$

Find all critical values of $f(x)$ and determine whether each corresponds to a relative maximum, a relative minimum or neither

Solution: $f'(x)=(x-1)^2(x-2)(x-4)(x+5)^4$

For the critical values take $f'(x)=0$ we get

$(x-1)^2(x-2)(x-4)(x+5)^4=0$

Either $(x-1)^2=0$, $x-1=0$, $x=1$ or $x-2=0$, $x=2$

Or $x-4=0$, $x=4$ or $(x+5)^4=0$, $x+5=0$, $x=-5$

The critical values are 1,2,4 and -5

Differentiating $f'(x)$ with respect to x

$$\tfrac{d}{dx}f'(x)=\tfrac{d}{dx}\left[\left\{(x-1)^2(x-2)\right\}\left\{(x-4)(x+5)^4\right\}\right]$$

$$f''(x)=\left\{(x-1)^2(x-2)\right\}\tfrac{d}{dx}\left\{(x-4)(x+5)^4\right\}+\left\{(x-4)(x+5)^4\right\}\tfrac{d}{dx}\left\{(x-1)^2(x-2)\right\}$$

$$f''(x)=\left\{(x-1)^2(x-2)\right\}\left\{(x-4)\tfrac{d}{dx}(x+5)^4+(x+5)^4\tfrac{d}{dx}(x-4)\right\}+\left\{(x-4)(x+5)^4\right\}\left\{(x-1)^2\tfrac{d}{dx}(x-2)+(x-2)\tfrac{d}{dx}(x-1)^2\right\}$$

$$f''(x)=\left\{(x-1)^2(x-2)\right\}\left\{4(x-4)(x+5)^3+(x+5)^4\right\}+\left\{(x-4)(x+5)^4\right\}\left\{(x-1)^2+2(x-2)(x-1)\right\}$$

The $f''(x)$ at the critical point/given point $x=1$

$$f''(1)=\left\{(1-1)^2(1-2)\right\}\left\{4(1-4)(1+5)^3+(1+5)^4\right\}+\left\{(1-4)(1+5)^4\right\}\left\{(1-1)^2+2(1-2)(1-1)\right\}$$

$$f''(1)=\left\{(0)^2(-1)\right\}\left\{4(-3)(6)^3+(6)^4\right\}+\left\{(-3)(6)^4\right\}\left\{(0)^2+2(-1)(0)\right\}$$

$$f''(1)=\{0\}\left\{4(-3)(6)^3+(6)^4\right\}+\left\{(-3)(6)^4\right\}\{0+0\}$$

$$f''(1)=0$$

So, second derivative test $f''(1)=0$ fails at $x=1$

The $f''(x)$ at the critical point/given point $x=2$

$$f''(2)=\left\{(2-1)^2(2-2)\right\}\left\{4(2-4)(2+5)^3+(2+5)^4\right\}+\left\{(2-4)(2+5)^4\right\}\left\{(2-1)^2+2(2-2)(2-1)\right\}$$

$$f''(2)=\left\{(1)^2(0)\right\}\left\{4(-2)(7)^3+(7)^4\right\}+\left\{(-2)(7)^4\right\}\left\{(1)^2+2(0)(-1)\right\}$$

$$f''(2)=\{0\}\left\{4(-2)(7)^3+(7)^4\right\}+\left\{(-2)(2401)\right\}\{1+0\}$$

$$f''(2)=0-4802=-4802<0$$

Through the second derivative test $f''(2)<0$ then there is a relative maximum at $x=2$

The $f''(x)$ at the critical point/given point $x=4$

$$f''(4)=\left\{(4-1)^2(4-2)\right\}\left\{4(4-4)(4+5)^3+(4+5)^4\right\}+\left\{(4-4)(4+5)^4\right\}\left\{(4-1)^2+2(4-2)(4-1)\right\}$$

$$f''(4)=\left\{(3)^2(2)\right\}\left\{4(0)(9)^3+(9)^4\right\}+\left\{(0)(9)^4\right\}\left\{(3)^2+2(2)(3)\right\}$$

$$f''(4)=\{36\}\{0+6561\}+\{0\}\{9+12\}=236196$$

Through the second derivative test $f''(4)>0$ then there is a relative minimum at $x=4$

The $f''(x)$ at the critical point/given point $x=-5$

$$f''(-5)=\left\{(-5-1)^2(-5-2)\right\}\left\{4(-5-4)(-5+5)^3+(-5+5)^4\right\}+\left\{(-5-4)(-5+5)^4\right\}\left\{(-5-1)^2+2(-5-2)(-5-1)\right\}$$

$$f''(-5)=\left\{(-6)^2(-7)\right\}\left\{4(-9)(0)^3+(0)^4\right\}+\left\{(-9)(0)^4\right\}\left\{(-6)^2+2(-7)(-6)\right\}$$

$$f''(-5)=\{(36)(-7)\}\{0+0\}+\{0\}\{36+84\}=0$$

So second derivative test $f''(-5)=0$ fails at $x=-5$

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b). Suppose $f(x)$ is differential function with derivative $f'(x)=\dfrac{(2x-1)(x+3)}{(x-1)^2}$

Find all critical values of $f(x)$ and determine whether each corresponds to a relative maximum, a relative minimum or neither

Sol: Given $f'(x)=\dfrac{(2x-1)(x+3)}{(x-1)^2}$

For the critical values take $f'(x)=0$ we get

$$\frac{(2x-1)(x+3)}{(x-1)^2}=0$$

$$(2x-1)(x+3)=0$$

Either $2x-1=0$, $2x=1$, $x=\frac{1}{2}$ or $x+3=0$, $x=-3$

For critical values for $f'(x)$ is undefined, we get

$(x-1)^2=0$

$x-1=0 \quad \Rightarrow x=1$

The critical values are -3 and ½ and the critical value that undefined at $x=1$

Differentiating $f'(x)$ with respect to x

$$\tfrac{d}{dx}f'(x)=\frac{d}{dx}\left[\frac{(2x-1)(x+3)}{(x-1)^2}\right]$$

$$f''(x)=\frac{(x-1)^2\tfrac{d}{dx}\{(2x-1)(x+3)\}-\{(2x-1)(x+3)\}\tfrac{d}{dx}(x-1)^2}{(x-1)^4}$$

$$f''(x)=\frac{1}{(x-1)^4}\left[\begin{matrix}(x-1)^2\left\{(2x-1)\tfrac{d}{dx}(x+3)+(x+3)\tfrac{d}{dx}(2x-1)\right\}\\-2(x-1)\{(2x-1)(x+3)\}\tfrac{d}{dx}(x-1)\end{matrix}\right]$$

$$f''(x)=\frac{1}{(x-1)^4}\left[\begin{matrix}(x-1)^2\left\{(2x-1)(1+0)+(x+3)(2-0)\right\}\\-2(x-1)\{(2x-1)(x+3)\}(1-0)\end{matrix}\right]$$

$$f''(x)=\frac{1}{(x-1)^4}\left[\begin{matrix}(x-1)^2\left\{(2x-1)+2(x+3)\right\}\\-2(x-1)\{(2x-1)(x+3)\}\end{matrix}\right]$$

The $f''(x)$ at the critical point at $x=-3$

$$f''(-3)=\frac{1}{(-3-1)^4}\left[\begin{matrix}(-3-1)^2\left\{(-6-1)+2(-3+3)\right\}\\-2(-3-1)\{(-6-1)(-3+3)\}\end{matrix}\right]$$

$$f''(-3)=\frac{1}{(-4)^4}\left[(-4)^2\{(-7)+2(0)\}-2(-4)\{(-7)(0)\}\right]$$

$$f''(-3)=\frac{1}{(-4)^4}\left[(-4)^2\{(-7)+0\}-2(-4)\{0\}\right]$$

$f''(-3)=\frac{1}{256}\left[16(-7)-0\right]=\frac{-7}{16}<0$

Through the second derivative test $f''(-3)<0$

then there is a relative maximum at $x=-3$

The $f''(x)$ at the critical point at $x=\frac{1}{2}$

$$f''\left(\frac{1}{2}\right)=\frac{1}{\left(\frac{1}{2}-1\right)^4}\begin{bmatrix}\left(\frac{1}{2}-1\right)^2\left\{(1-1)+2\left(\frac{1}{2}+3\right)\right\}\\-2\left(\frac{1}{2}-1\right)\left\{(1-1)\left(\frac{1}{2}+3\right)\right\}\end{bmatrix}$$

$$f''\left(\frac{1}{2}\right)=\frac{1}{\left(\frac{-1}{2}\right)^4}\left[\left(\frac{-1}{2}\right)^2\left\{0+2\left(\frac{7}{2}\right)\right\}-2\left(\frac{-1}{2}\right)\left\{0\left(\frac{7}{2}\right)\right\}\right]$$

$$f''\left(\frac{1}{2}\right)=\frac{16}{1}\left[\left(\frac{1}{4}\right)\{7\}-0\right]=28>0$$

Through the second derivative test $f''\left(\frac{1}{2}\right)>0$

then there is a relative minimum at $x=\frac{1}{2}$

Q7. A company has found through experience that increasing its advertising also increases its sales up to a point. The company believes that the mathematical model connecting profit in hundreds of dollars $P(x)$ and expenditures on advertising in thousands of dollars x is: $P(x)=80+108x-x^3 \qquad 0\le x\le 10$

a). Find the expenditure on advertising that leads to maximum profit.

Sol: Given $P(x)=80+108x-x^3$

Differentiate with respect to x

$\frac{d}{dx}P(x)=\frac{d}{dx}80+108\frac{d}{dx}x-\frac{d}{dx}x^3$

$P'(x)=0+108-3x^2\frac{d}{dx}x$

$P'(x)=108-3x^2$

For the critical values take $P'(x)$ we get

$108-3x^2=0$

$108=3x^2$

$3x^2=108$

$x^2=\frac{108}{3}=36$

Taking square root on both sides

$\sqrt{x^2}=\pm\sqrt{36}$

$x=\pm 6$

The critical values are 6 and -6

Differentiating $P'(x)$ with respect to x

$\frac{d}{dx}P'(x)=\frac{d}{dx}108-3\frac{d}{dx}x^2$

$P''(x)=0-3(2x)\frac{d}{dx}x$

$P''(x)=-6x$

The $P''(x)$ at the critical point/given point $x=6$

$P''(-1)=-6(6)$

$P''(-1)=-36<0$

Through second derivative test $P''(6)<0$ then there is a relative maximum expenditure at $x=6$ thousand dollars

b). Find the maximum profit.

Sol: Givento find maximum profit at expenditure $x=6$ thousand dollars in the given function

$P(6)=80+108(6)-(6)^3$

$P(6)=80+648-216=512$

Therefore the maximum value of the function is 512 thousand dollars at critical value $x=6$ thousand dollars

Q8. Total profit $P(x)$ (in thousands of dollars) from the sale of x hundred thousand of automobile tyres is approximated by $P(x)=-x^3+9x^2+120x-400$ $3\le x\le 15$ Find numbers of hundred thousand of tyres that must be sold to maximize profit. Find maximum profit

Sol: Given $P(x)=-x^3+9x^2+120x-400$

Differentiate with respect to x

$\frac{d}{dx}P(x)=-\frac{d}{dx}x^3+9\frac{d}{dx}x^2+120\frac{d}{dx}x-\frac{d}{dx}400$

$P'(x)=-3x^2\frac{d}{dx}x+9(2x)\frac{d}{dx}x+120-0$

$P'(x)=-3x^2+18x+120$

For the critical values take $P'(x)=0$ we get

$-3x^2+18x+120=0$

$-3\left(x^2-6x-40\right)=0$

$x^2-6x-40=0$

$x^2-10x+4x-40=0$

$x(x-10)+4(x-10)=0$

$(x-10)(x+4)$

Either or

$x-10=0 \qquad x+4=0$

$x=10 \qquad x=-4$

The critical values are 10 and -4

Differentiating $P'(x)$ with respect to x

$\frac{d}{dx}P'(x)=-3\frac{d}{dx}x^2+18\frac{d}{dx}x+\frac{d}{dx}120$

$P''(x)=-3(2x)\frac{d}{dx}x+18+0$

$P''(x)=-6x+18$

The $P''(x)$ at the critical point/given point $x=10$

$P''(10)=-6(10)+18$

$P''(10)=-60+18=-42<0$

Through the second derivative test $P''(10)<0$ then there is a relative maximum at $x=10$ thousand dollars

To find maximum profit at expenditure $x=10$ thousand dollars in the given function

$P(10)=-(10)^3+9(10)^2+120(10)-400$

$P(10)=-1000+9(100)+1200-400$

$P(10)=-1000+900+800$

$P(10)=-100+800=700$

Therefore maximum value of the function is 700 hundred thousand dollars at the critical value $x=10$ hundred thousand dollars

Q9. Find the percent of concentration of drug in the bloodstream x hours after the drug is administered is given by $K(x)=\frac{4x}{3x^2+27}$

Sol: Given $K(x)=\frac{4x}{3x^2+27}$

Differentiate with respect to x

$\frac{d}{dx}K(x)=\frac{d}{dx}\left\{\frac{4x}{3x^2+27}\right\}$

$$K'(x)=\frac{(3x^2+27)\frac{d}{dx}(4x)-4x\frac{d}{dx}(3x^2+27)}{(3x^2+27)^2}$$

$$K'(x)=\frac{4(3x^2+27)-4x(6x+0)}{(3x^2+27)^2}$$

$$K'(x)=\frac{12x^2+108-24x^2}{(3x^2+27)^2}=\frac{108-12x^2}{(3x^2+27)^2}$$

For the critical values take $K'(x)=0$ we get

$$\frac{108-12x^2}{(3x^2+27)^2}=0$$

$$108-12x^2=0$$
$$108=12x^2$$
$$12x^2=108$$
$$x^2=\tfrac{108}{12}=9$$

Taking square root on both sides

$$\sqrt{x^2}=\pm\sqrt{9}$$
$$x=\pm 3$$

The critical values are 3 and -3

For critical values for $K'(x)$ is undefined, we get

$$3x^2+27=0$$
$$3x^2=-27$$
$$x^2=-9$$

Taking square root on both sides

$$\sqrt{x^2}=\pm\sqrt{-9}$$
$$x=\pm 3i$$

Critical values which are undefined are imaginary so leave them.

Differentiating $P'(x)$ with respect to x

$$\tfrac{d}{dx}K'(x)=\tfrac{d}{dx}\left\{\frac{108-12x^2}{(3x^2+27)^2}\right\}$$

$$K''(x)=\frac{(3x^2+27)^2\frac{d}{dx}(108-12x^2)-(108-12x^2)\frac{d}{dx}(3x^2+27)^2}{(3x^2+27)^4}$$

$$K''(x)=\frac{(3x^2+27)^2(0-24x)-2(108-12x^2)(3x^2+27)\frac{d}{dx}(3x^2+27)}{(3x^2+27)^4}$$

$$K''(x)=\frac{-24x(3x^2+27)^2-12x(108-12x^2)(3x^2+27)}{(3x^2+27)^4}$$

$$K''(x)=\frac{(3x^2+27)\{-24x(3x^2+27)-12x(108-12x^2)\}}{(3x^2+27)^4}$$

$$K''(x)=\frac{-24x(3x^2+27)-12x(108-12x^2)}{(3x^2+27)^3}$$

The $K''(x)$ at the critical point $x=3$

$$K''(3)=\frac{-24(3)(3(3)^2+27)-12(3)(108-12(3)^2)}{(3(3)^2+27)^3}$$

$$K''(3)=\frac{-72(27+27)-36(108-108)}{(27+27)^3}$$

$$K''(3)=\frac{-72(54)-36(0)}{(54)^3}=\frac{-3888}{157\,464}<0$$

Through the second derivative test $K''(3)<0$ then there is a relative maximum at $x=3$

The $K''(x)$ at the critical point $x=-3$

$$K''(-3)=\frac{-24(-3)(3(-3)^2+27)-12(-3)(108-12(-3)^2)}{(3(-3)^2+27)^3}$$

$$K''(-3)=\frac{72(27+27)+36(108-108)}{(27+27)^3}$$

$$K''(-3)=\frac{72(54)+36(0)}{(54)^3}=\frac{3888}{157\,464}>0$$

Through the second derivative test $K''(-3)>0$ then there is a relative minimum at $x=-3$

We have the critical values -3 and 3 this means that domain we can split into three intervals

i.e., $\Re=(-\infty,-3]\cup[-3,3]\cup[3,\infty)$

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a). On what time intervals is the concentration of the drug increasing?

Sol: using first derivative test take any point $2\in[-3,3]$

$$K'(2)=\frac{108-12(2)^2}{\left[3(2)^2+27\right]^2}=\frac{108-48}{[12+27]^2}=\frac{68}{39^2}>0$$

So function is increasing on $[-3,3]$

---

b). On what intervals is it decreasing?

Sol: Given intervals $(-\infty,-3]\,\&\,[3,\infty)$

using first derivative test take $-5\in(-\infty,-3]$

$$K'(-5)=\frac{108-12(-5)^2}{\left[3(-5)^2+27\right]^2}=\frac{108-300}{[75+27]^2}=\frac{-192}{102^2}<0$$

So function is decreasing on the interval $[3,\infty)$

using first derivative test take $5\in[3,\infty)$

$$K'(5)=\frac{108-12(5)^2}{\left[3(5)^2+27\right]^2}=\frac{108-300}{[75+27]^2}=\frac{-192}{102^2}<0$$

So function is decreasing on the interval $[3,\infty)$

therefore the given function is decreasing on the intervals $(-\infty,-3]\,\&\,[3,\infty)$

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c). Find the maximum concentration.

Sol: Since maximum concentration at time $x=3$

$$K(3)=\frac{4(3)}{3(3)^2+27}=\frac{12}{27+27}$$

$$K(3)=\frac{12}{54}=0.22\%$$

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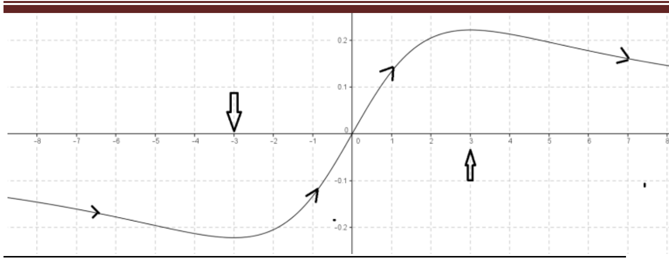
d). Find the minimum concentration.

Sol: Since minimum concentration at time $x=3$

$$K(-3)=\frac{4(-3)}{3(-3)^2+27}=\frac{-12}{27+27}$$

$$K(-3)=\frac{-12}{54}=-0.22\%$$



Q10. A diesel generator burns fuel at the rate of $G(x)=\frac{1}{48}\left(\frac{300}{x}+2x\right)$ gallons per hour when producing x thousand kilowatt hours of electricity. Suppose that fuel costs \$2.25 a gallon and find the value of x that leads to minimum total cost if the generator is operated for 32 hours. Find the minimum cost.

Solution: we have $G(x)=\frac{1}{48}\left(\frac{300}{x}+2x\right)$ is rate of fuel burns of the generator and cost of fuel \$2.25 per gallon for 32 hours

Therefore the cost of electricity produced is

$$C(x)=(32)(2.25)G(x)=\frac{128}{9}G(x)$$

$$C(x)=72\left[\frac{1}{48}\left(\frac{300}{x}+2x\right)\right]$$

$$C(x)=\frac{3}{2}\left(\frac{300}{x}+2x\right)$$

Differentiating with respect to x

$$\tfrac{d}{dx}C(x)=\tfrac{3}{2}\tfrac{d}{dx}\left(300x^{-1}+2x\right)$$

$$C'(x)=\frac{3}{2}\left(300\tfrac{d}{dx}x^{-1}+2\tfrac{d}{dx}x\right)$$

$$C'(x)=\frac{3}{2}\left(-300x^{-1-1}+2\right)$$

$$C'(x)=\frac{3}{2}\left(-300x^{-2}+2\right)$$

$$C'(x)=\frac{3}{2}\left(\frac{-300}{x^2}+2\right)$$

For the critical value put $G'(x)=0$

$$\frac{3}{2}\left(\frac{-300}{x^2}+2\right)=0$$

$$\frac{-300}{x^2}+2=0$$

$$\frac{-300}{x^2}=-2$$

$$-300=-2x^2$$

$$x^2=150$$

Taking square root on both sides

$$\sqrt{x^2}=\pm\sqrt{150}$$

$$x=\pm\sqrt{150}$$

Differentiating $C'(x)$ with respect to x

$$\tfrac{d}{dx}C'(x)=\frac{3}{2}\left(-300\tfrac{d}{dx}x^{-2}+\tfrac{d}{dx}2\right)$$

$$C''(x)=\frac{3}{2}\left(-300(-2)x^{-2-1}+0\right)$$

$$C''(x)=\frac{3}{2}\left(\frac{600}{x^3}+0\right)$$

$$C''(x)=\frac{900}{x^3}$$

The $C''(x)$ at the critical point $x=\sqrt{150}$

$$C''\left(\sqrt{150}\right)=\frac{900}{\left(\sqrt{150}\right)^3}=\frac{900}{\sqrt{150^3}}$$

$$C''\left(\sqrt{150}\right)=\frac{900}{\sqrt{3\,375\,000}}=0.4899>0$$

Through the second derivative test $C''\left(\sqrt{150}\right)>0$ then there is a relative minimum at $x=\sqrt{150}$

And the minimum cost is at $x=\sqrt{150}$

$$C\left(\sqrt{150}\right)=\frac{3}{2}\left(\frac{300}{\sqrt{150}}+2\sqrt{150}\right)$$

$$C\left(\sqrt{150}\right)=\$73.48$$