# Chapter 7

## Mathematical Induction & Binomial theorem

$S_1 = a_1 \qquad = \sum_{i=1}^{1} a_i$

$S_2 = a_1 + a_2 \qquad = \sum_{i=1}^{2} a_i$

$S_n = a_1 + a_2 + a_3 + \cdots a_n \qquad = \sum_{i=1}^{n} a_i$

$S_{n+1} = a_1 + a_2 + a_3 + \cdots a_n + a_{n+1} \qquad = \sum_{i=1}^{n+1} a_i$

$S_{n+1} = S_n + a_{n+1} \qquad = \sum_{i=1}^{n+1} a_i$

## Exercise 7.1

Establish formulas given below by mathematical induction

Q1. $1+5+9+\cdots+(4n-3) = n(2n-1)$

Rauf work: Here $a_n = 4n-3, S_n = n(2n-1)$

Replacing n by k + 1

Then $a_{k+1} = 4(k+1)-3$ & $S_{n+1} = (n+1)\left[2(n+1)-1\right]$

Solution; $1+5+9+\cdots+(4n-3) = n(2n-1)$

Or $\sum_{k=1}^{n}(4k-3) = n(2n-1)$

By mathematical induction Step 1; $n=1$ *i.e.* $a_1 = S_1$

$4(1)-3 = (1)(2(1)-1)$

$4-3 = 2-1$

$1=1$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1 + a_2 + a_3 + \cdots + a_k = S_k \qquad \Rightarrow \sum_{i=1}^{k} a_i = S_k$

*i.e.*, $1+5+9+\cdots+(4k-3) = k(2k-1)$

Step 3; For $n=k+1$ adding both sides by $a_{k+1} = 4k+1$

$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$

*i.e.*, $\sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$

$1+5+9+\cdots+(4k-3)+(4k+1) = k(2k-1)+(4k+1)$

Taking only RHS

$$\begin{aligned}
&= 2k^2 - k + 4k + 1\\
&= 2k^2 + 3k + 1\\
&= 2k^2 + 2k + k + 1\\
&= 2k(k+1) + 1(k+1)\\
&= (k+1)(2k+1)\\
&= (k+1)(2k+1+1-1)\\
&= (k+1)\{2(k+1)-1\}\\
&= S_{k+1}
\end{aligned}$$

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

Q2. $3+6+9+\cdots+3n = \frac{3n(n+1)}{2}$

Rauf work: Here $a_n = 3n, S_n = \frac{3n(n+1)}{2}$

Replacing n by k + 1

Then $a_{k+1} = 3(k+1), S_{k+1} = \frac{3(k+1)\{(k+1)+1\}}{2}$

Solution; $3+6+9+\cdots+3n = \frac{3n(n+1)}{2}$

Or $\sum_{k=1}^{n} 3k = \frac{3n(n+1)}{2}$

By mathematical induction Step 1; $n=1$ *i.e.* $a_1 = S_1$

$3(1) = \frac{3.1(1+1)}{2}$

$3 = \frac{3.2}{2}$

$3=3$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1 + a_2 + a_3 + \cdots + a_k = S_k \qquad \Rightarrow \sum_{i=1}^{k} a_i = S_k$

*i.e.*, $3+6+9+\cdots+3k = \frac{3k(k+1)}{2}$

Step 3; for $n=k+1$ adding both sides by $a_{k+1} = 3(k+1)$

$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$

*i.e.*, $\sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$

$$\begin{aligned}
3+6+9+\cdots+3k+3(k+1) &= \frac{3k(k+1)}{2} + 3(k+1)\\
&= (k+1)\left\{\frac{3k}{2}+3\right\}\\
&= \frac{(k+1)}{2}\{3k+6\}\\
&= \frac{3(k+1)(k+2)}{2}\\
&= \frac{3(k+1)\{(k+1)+1\}}{2}\\
&= S_{k+1}
\end{aligned}$$

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

Q3. $5+10+15+\cdots+5n = \frac{5n(n+1)}{2}$

Rauf work: Here $a_n = 5n, S_n = \frac{5n(n+1)}{2}$

Replacing n by k + 1

Then $a_{k+1} = 5(k+1), S_{k+1} = \frac{5(k+1)\{(k+1)+1\}}{2}$

Solution; $5+10+15+\cdots+5n = \frac{5n(n+1)}{2}$

Or $\sum_{k=1}^{n} 5k = \frac{5n(n+1)}{2}$

By mathematical induction Step 1; $n=1$ i.e. $a_1 = S_1$

$5(1) = \frac{5.1(1+1)}{2}$

$5 = \frac{5.2}{2}$

$5 = 5$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1 + a_2 + a_3 + \cdots + a_k = S_k \quad \Rightarrow \sum_{i=1}^{k} a_i = S_k$

i.e., $5 + 10 + 15 + \cdots + 5k = \frac{5k(k+1)}{2}$

Step 3; for $n = k+1$ adding b.S by $a_{k+1} = 5(k+1)$

$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$

i.e., $\sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$

$$5 + 10 + 15 + \cdots + 5k + 5(k+1) = \frac{5k(k+1)}{2} + 5(k+1)$$
$$= (k+1)\left\{\frac{5k}{2} + 5\right\}$$
$$= \frac{(k+1)}{2}\{5k+10\}$$
$$= \frac{5(k+1)(k+2)}{2}$$
$$= \frac{5(k+1)\{(k+1)+1\}}{2}$$
$= S_{k+1}$ $\therefore$ true for $n = k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q4 $a + (a+d) + (a+2d) + \cdots + \{a + (n-1)d\} = \frac{n}{2}\{2a + (n-1)d\}$

R work: $a_n = \{a + (n-1)d\}, S_n = \frac{n}{2}\{2a + (n-1)d\}$

Replacing n by k + 1

$a_{k+1} = \{a + (k+1-1)d\}, S_{k+1} = \frac{k+1}{2}\{2a + (k+1-1)d\}$

Sol; $a + (a+d) + (a+2d) + \cdots + \{a + (n-1)d\} = \frac{n}{2}\{2a + (n-1)d\}$

Or $\sum_{k=1}^{n} \{a + (k-1)d\} = \frac{n}{2}\{2a + (n-1)d\}$

By mathematical induction Step 1; $n=1$ i.e. $a_1 = S_1$

$\{a + (1-1)d\} = \frac{1}{2}\{2a + (1-1)d\}$

$a + 0 = \frac{1}{2}\{2a + 0.d\}$

$a = a$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1 + a_2 + a_3 + \cdots + a_k = S_k \quad \Rightarrow \sum_{i=1}^{k} a_i = S_k$

$a + (a+d) + (a+2d) + \cdots + \{a + (k-1)d\} = \frac{n}{2}\{2a + (k-1)d\}$

Step 3; for $n = k+1$ adding both sides by $a_{k+1} = a + kd$

$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$

i.e., $\sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$

$$a + (a+d) + \cdots + \{a + (k-1)d\} + a + kd$$
$$= \frac{k}{2}\{2a + (k-1)d\} + a + kd$$
$$= \frac{k\{2a + (k-1)d\} + 2a + 2kd}{2}$$
$$= \tfrac{1}{2}\{2ka + k^2d - kd + 2a + 2kd\}$$
$$= \tfrac{1}{2}\{2ka + 2a + k^2d + kd\}$$
$$= \tfrac{1}{2}\{2a(k+1) + kd(k+1)\}$$
$$= \frac{(k+1)}{2}\{2a + kd\}$$
$$= \frac{(k+1)}{2}\{2a + (k+1-1)d\}$$
$= S_{k+1}$ $\therefore$ true for $n = k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q5. $a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r-1}$

Rauf work: Here $a_n = ar^n, S_{n+1} = \frac{a(r^{n+1} - 1)}{r-1}$

Replacing n by k + 1

$a_{k+1} = ar^{k+1}, S_{k+2} = \frac{a(r^{k+2} - 1)}{r-1}$

Solution; $a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r-1}$

Or $\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1} - 1)}{r-1}$

By mathematical induction Step 1; $n=1$ $i.e. a_0 + a_1 = S_1$

$a + ar = \frac{a(r^2 - 1)}{r-1} = \frac{a(r+1)(r-1)}{r-1}$

$a + ar = a(r+1)$

$a + ar = a + ar$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_0 + a_1 + a_2 + a_3 + \cdots + a_k = S_{k+1} \Rightarrow \sum_{i=0}^{k} a_i = S_{k+1}$

i.e., $1 + 5 + 9 + \cdots + (4k-3) = k(2k-1)$

Step 3; for $n = k+1$ adding both sides by $a_{k+1} = ar^{k+1}$

$a_0 + a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_{k+1} + a_{k+1}$

i.e., $\sum_{i=0}^{k+1} a_i = S_{k+1} + a_{k+1}$

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1}$$
$$= \frac{a(r^{k+1} - 1)}{r-1} + ar^{k+1}$$
$$= \frac{a(r^{k+1} - 1) + ar^{k+1}(r-1)}{r-1}$$

$$= \frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}$$

$$= \frac{a\left(r^{k+2} - 1\right)}{r-1}$$

$= S_{k+2}$ $\therefore$ true for $n = k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q6. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n\left(4n^2 - 1\right)}{3}$

Rauf work: Here $a_n = (2n-1)^2$, $S_n = \dfrac{n\left(4n^2-1\right)}{3}$

Replacing n by k + 1

$$a_{k+1} = \left\{2(k+1) - 1\right\}^2, S_{k+1} = \frac{(k+1)\left\{4(k+1)^2 - 1\right\}}{3}$$

Solution; $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n\left(4n^2-1\right)}{3}$

Or $\displaystyle\sum_{k=1}^{n} (2k-1)^2 = \frac{n\left(4n^2-1\right)}{3}$

By mathematical induction Step 1; $n = 1$ $i.e.\ a_1 = S_1$

$$(2.1-1)^2 = \frac{1.\left(4.1^2 - 1\right)}{3}$$

$$(2-1)^2 = \frac{(4-1)}{3}$$

$$1^2 = \frac{3}{3}$$

$1 = 1$ true for $n = 1$

Step 2; suppose it is true for $n = k$

$$a_1 + a_2 + a_3 + \cdots + a_k = S_k \quad \Rightarrow \sum_{i=1}^{k} a_i = S_k$$

$$i.e., \quad 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k\left(4k^2-1\right)}{3}$$

Step 3; for $n = k+1$ adding b s by $a_{k+1} = (2k+1)^2$

$$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$$

$$i.e., \quad \sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$$

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2$$

$$= \frac{k\left(4k^2-1\right)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{(2k+1)}{3}\left\{k(2k-1) + 3(2k+1)\right\}$$

$$= \frac{(2k+1)}{3}\left\{2k^2 - k + 6k + 3\right\}$$

$$= \frac{(2k+1)}{3}\left\{2k^2 + 5k + 3\right\}$$

$$= \frac{(2k+1)}{3}\left\{2k^2 + 3k + 2k + 3\right\}$$

$$= \frac{(2k+1)}{3}\left\{k(2k+3) + 1(2k+3)\right\}$$

$$= \frac{(k+1)}{3}(2k+3)(2k+1)$$

$$= \frac{(k+1)}{3}\left\{4k^2 + 2k + 6k + 3\right\}$$

$$= \frac{(k+1)}{3}\left\{4k^2 + 8k + 4 - 1\right\}$$

$$= \frac{(k+1)}{3}\left\{4\left(k^2 + 2k + 1\right) - 1\right\}$$

$$= \frac{(k+1)}{3}\left\{4(k+1)^2 - 1\right\}$$

$= S_{k+1}$ $\therefore$ true for $n = k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q7 $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \dfrac{2}{3}n(n+1)(2n+1)$

Rauf work: Here $a_n = (2n)^2$, $S_n = \dfrac{2}{3}n(n+1)(2n+1)$

Replacing n by k + 1

$$a_{k+1} = \left\{2(k+1)\right\}^2, S_{k+1} = \frac{2}{3}(k+1)\left\{(k+1)+1\right\}\left\{2(k+1)+1\right\}$$

Sol; $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \dfrac{2}{3}n(n+1)(2n+1)$

Or $\displaystyle\sum_{k=1}^{n} (2k) = \frac{2}{3}n(n+1)(2n+1)$

By mathematical induction Step 1; $n = 1$ $i.e.\ a_1 = S_1$

$$(2.1)^2 = \frac{2}{3}1(1+1)(2.1+1)$$

$$(2)^2 = \frac{2}{3}(2)(3)$$

$4 = 4$ true for $n = 1$

Step 2; suppose it is true for $n = k$

$$a_1 + a_2 + a_3 + \cdots + a_k = S_k \quad \Rightarrow \sum_{i=1}^{k} a_i = S_k$$

i.e. $2^2 + 4^2 + 6^2 + \cdots + (2k)^2 = \dfrac{2}{3}k(k+1)(2k+1)$

Step 3; To show for $n = k+1$ adding both sides by

$$a_{k+1} = \left\{2(k+1)\right\}^2 = (2k+2)^2$$

$$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$$

$$i.e., \quad \sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$$

$$2^2 + 4^2 + 6^2 + \cdots + (2k)^2 + (2k+2)^2$$

$$= \frac{2}{3}k(k+1)(2k+1) + (2k+2)^2$$

$$= \frac{2}{3}k(k+1)(2k+1) + 4(k+1)^2$$

$$= \frac{2(k+1)}{3}\left\{k(2k+1) + 6(k+1)\right\}$$

$$= \frac{2(k+1)}{3}\left\{2k^2 + k + 6k + 6\right\}$$

$$=\frac{2(k+1)}{3}\{2k^2+7k+6\}$$
$$=\frac{2(k+1)}{3}\{2k^2+4k+3k+6\}$$
$$=\frac{2(k+1)}{3}\{2k(k+2)+3(k+2)\}$$
$$=\frac{2}{3}(k+1)(k+2)(2k+3)$$
$$=\frac{2}{3}(k+1)(k+2)\{2(k+1)+1\}$$
$= S_{k+1}$ $\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q8. $1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$

Here $a_n=n^3, S_n=\left(\frac{n(n+1)}{2}\right)^2$ Replacing n by k + 1

$a_{k+1}=(k+1)^3, S_{k+1}=\left[\frac{(k+1)\{(k+1)+1\}}{2}\right]^2$

Solution; $1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$

Or $\sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2$

By mathematical induction, Step 1; $n=1$ *i.e.* $a_1=S_1$

$$1^3=\left(\frac{1(1+1)}{2}\right)^2=\left(\frac{1\times 2}{2}\right)^2$$

$1=1$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1+a_2+a_3+\cdots+a_k=S_k \quad \Rightarrow \sum_{i=1}^{k}a_i=S_k$

i.e., $1^3+2^3+3^3+\cdots+k^3=\left(\frac{k(k+1)}{2}\right)^2$

Step 3; for $n=k+1$ adding both sides by $a_{k+1}=(k+1)^3$

$a_1+a_2+a_3+\cdots+a_k+a_{k+1}=S_k+a_{k+1}$

*i.e.,* $\sum_{i=1}^{k+1}a_i=S_k+a_{k+1}$

$1^3+2^3+3^3+\cdots+k^3+(k+1)^3$
$$=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3$$
$$=(k+1)^2\left\{\frac{k^2}{4}+k+1\right\}$$
$$=\frac{(k+1)^2}{4}\{k^2+4k+4\}$$
$$=\frac{(k+1)^2(k+2)^2}{4}$$
$$=\left(\frac{(k+1)(k+2)}{2}\right)^2$$
$=S_{k+1}$ $\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q9. $1(1!)+2(2!)+3(3!)+\cdots+n(n!)=(n+1)!-1$

Rauf work: Here $a_n=n(n!), S_n=(n+1)!-1$

Replacing n by k + 1

$a_{k+1}=(k+1)[(k+1)!], S_{k+1}=\{(k+1)+1\}!-1$

Sol; $1(1!)+2(2!)+3(3!)+\cdots+n(n!)=(n+1)!-1$

Or $\sum_{k=1}^{n}k(k!)=(n+1)!-1$

By mathematical induction Step 1; $n=1$ *i.e.* $a_1=S_1$

$1(1!)=(1+1)!-1$

$1\times 1=2!-1$

$1=2-1$

$1=1$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1+a_2+a_3+\cdots+a_k=S_k \quad \Rightarrow \sum_{i=1}^{k}a_i=S_k$

*i.e.,* $1(1!)+2(2!)+3(3!)+\cdots+k(k!)=(k+1)!-1$

Step 3 To show for $n=k+1$ adding BS by $a_{k+1}=(k+1)(k+1)!$

$a_1+a_2+a_3+\cdots+a_k+a_{k+1}=S_k+a_{k+1}$

*i.e.,* $\sum_{i=1}^{k+1}a_i=S_k+a_{k+1}$

$1(1!)+2(2!)+3(3!)+\cdots+k(k!)+(k+1)(k+1)!$
$$=(k+1)!-1+(k+1)(k+1)!$$
$$=(k+1)!+(k+1)(k+1)!-1$$
$$=(k+1)!\{1+k+1\}-1=(k+2)(k+1)!-1$$
$$=(k+2)!-1$$
$=S_{k+1}$ $\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q10. $1+2^1+2^2+\cdots+2^{n-1}=2^n-1$

Here $a_n=2^{n-1}, S_n=2^n-1$ Replacing n by k + 1

$a_{k+1}=2^{(k+1)-1}, S_{k+1}=2^{k+1}-1$

Solution; $1+2^1+2^2+\cdots+2^{n-1}=2^n-1$

Or $\sum_{k=1}^{n}2^{k-1}=2^n-1$

By mathematical induction Step 1; $n=1$

$2^{1-1}=2^1-1$ $\quad a_1=S_1$

$2^0=2-1$

$1=1$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1+a_2+a_3+\cdots+a_k=S_k \quad \Rightarrow \sum_{i=1}^{k}a_i=S_k$

$1+2^1+2^2+\cdots+2^{k-1}=2^k-1$

Step 3; for $n=k+1$ adding both sides by $a_{k+1}=2^k$

$a_1+a_2+a_3+\cdots+a_k+a_{k+1}=S_k+a_{k+1}$

i.e., $\sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$

$1+2^1+2^2+\cdots+2^{k-1}+2^k = 2^k - 1 + 2^k$

$\sum_{i=1}^{k+1} a_i = 2^k + 2^k - 1$

$\sum_{i=1}^{k+1} a_i = 2.2^k - 1$

$\sum_{i=1}^{k+1} a_i = 2^{k+1} - 1$

$\sum_{i=1}^{k+1} a_i = S_{k+1} \quad \therefore$ true for $n = k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q11. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$

Rauf work: Here $a_n = \frac{1}{n(n+1)}$, $\quad S_n = \frac{n}{n+1}$

Replacing n by k + 1

$a_{k+1} = \frac{1}{(k+1)\{(k+1)+1\}}, S_{k+1} = \frac{k+1}{(k+1)+1}$

Solution; $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$

Or $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

By mathematical induction Step 1; $n=1$ i.e. $a_1 = S_1$

$\frac{1}{1(1+1)} = \frac{1}{1+1}$

$\frac{1}{2} = \frac{1}{2} \quad \therefore$ true for $n=1$

Step 2; suppose it is true for $n = k$

$a_1 + a_2 + a_3 + \cdots + a_k = S_k \quad \Rightarrow \sum_{i=1}^{k} a_i = S_k$

i.e., $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}$

Step 3; To show for $n = k+1$ adding both sides by

$a_{k+1} = \frac{1}{(k+1)\{(k+1)+1\}} = \frac{1}{(k+1)(k+2)}$

$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$

i.e., $\sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{k+1}\left\{k + \frac{1}{k+2}\right\}$$

$$= \frac{1}{k+1}\left\{\frac{k(k+2)+1}{k+2}\right\}$$

$$= \frac{1}{k+1}\left\{\frac{k^2+2k+1}{k+2}\right\}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{(k+1)}{(k+2)}$$

$$= \frac{(k+1)}{(k+1)+1}$$

$= S_{k+1} \quad \therefore$ true for $n = k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q12 $1.2+2.3+3.4+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$

Rauf work: Here $a_n = n(n+1)$, $S_n = \frac{n(n+1)(n+2)}{3}$

Replacing n by k + 1

$a_k = (k+1)(k+1+1), S_{k+1} = \frac{(k+1)(k+1+1)(k+1+2)}{3}$

Sol: $1.2+2.3+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$

Or $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

By mathematical induction Step 1; $n=1$ i.e. $a_1 = S_1$

$1.(1+1) = \frac{1(1+1)(1+2)}{3}$

$2 = 2 \quad \therefore$ true for $n=1$

Step 2; suppose it is true for $n = k$

$a_1 + a_2 + a_3 + \cdots + a_k = S_k \quad \Rightarrow \sum_{i=1}^{k} a_i = S_k$ i.e.

$1.2+2.3+3.4+\cdots+k(k+1) = \frac{k(k+1)(k+2)}{3}$

Step 3; To show for $n = k+1$ adding bs by $a_{k+1} = (k+1)(k+2)$

$a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} = S_k + a_{k+1}$

i.e., $\sum_{i=1}^{k+1} a_i = S_k + a_{k+1}$

$1.2+2.3+3.4+\cdots+k(k+1)+(k+1)(k+2)$

$\sum_{i=1}^{k+1} a_i = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$\sum_{i=1}^{k+1} a_i = (k+1)(k+2)\left(\frac{k}{3}+1\right)$

$\sum_{i=1}^{k+1} a_i = (k+1)(k+2)\left(\frac{k+3}{3}\right)$

$\sum_{i=1}^{k+1} a_i = \frac{(k+1)(k+2)(k+3)}{3}$

$\sum_{i=1}^{k+1} a_i = S_{k+1} \quad \therefore$ true for $n = k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q13. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^n} = \frac{1}{2}\left[1-\frac{1}{3^n}\right]$

Rauf work; Here $a_n = \frac{1}{3^n}$, $\quad S_n = \frac{1}{2}\left[1-\frac{1}{3^n}\right]$

Replacing n by k + 1

$a_{k+1}=\frac{1}{3^{k+1}}$, $\quad S_{k+1}=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$

Solution: $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^n}=\frac{1}{2}\left[1-\frac{1}{3^n}\right]$

Or $\quad \sum_{k=1}^{n}\frac{1}{3^k}=\frac{1}{2}\left[1-\frac{1}{3^n}\right]$

By mathematical induction Step 1; $n=1$

$\frac{1}{3^1}=\frac{1}{2}\left[1-\frac{1}{3^1}\right]$ $\quad a_1=S_1$

$\frac{1}{3}=\frac{1}{2}\left[\frac{3-1}{3}\right]$

$\frac{1}{3}=\frac{1}{2}\left[\frac{2}{3}\right]$

$\frac{1}{3}=\frac{1}{3}$ $\quad\therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1+a_2+a_3+\cdots+a_k=S_k \quad \Rightarrow \sum_{i=1}^{k}a_i=S_k$

*i.e.,* $\quad \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^k}=\frac{1}{2}\left[1-\frac{1}{3^k}\right]$

Step 3; To show for $n=k+1$ adding bs by $a_{k+1}=\frac{1}{3^{k+1}}$

$a_1+a_2+a_3+\cdots+a_k+a_{k+1}=S_k+a_{k+1}$

*i.e.,* $\quad \sum_{i=1}^{k+1}a_i=S_k+a_{k+1}$

$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^k}+\frac{1}{3^{k+1}}$

$\sum_{i=1}^{k+1}a_i=\frac{1}{2}\left[1-\frac{1}{3^k}\right]+\frac{1}{3^{k+1}}$

$\sum_{i=1}^{k+1}a_i=\frac{1}{2}-\frac{1}{2.3^k}+\frac{1}{3^{k+1}}$

$\sum_{i=1}^{k+1}a_i=\frac{1}{2}-\frac{3-2}{2.3^{k+1}}$

$\sum_{i=1}^{k+1}a_i=\frac{1}{2}-\frac{1}{2.3^{k+1}}$

$\sum_{i=1}^{k+1}a_i=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$

$\sum_{i=1}^{k+1}a_i=S_{k+1}$ $\quad\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q14. $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{n+2}{3}=\binom{n+3}{4}$

Rauf work; Here $a_n=\binom{n+2}{3}$, $\quad S_n=\binom{n+3}{4}$

Replacing n by k + 1

$a_{k+1}=\binom{k+1+2}{3}$, $\quad S_{k+1}=\binom{k+1+3}{4}$

Solution: $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{n+2}{3}=\binom{n+3}{4}$

Or $\quad \sum_{k=1}^{n}\binom{k+2}{3}=\binom{n+3}{4}$

By mathematical induction Step 1; $n=1$

$\binom{1+2}{3}=\binom{1+3}{4}$ $\quad a_1=S_1$

$\binom{3}{3}=\binom{4}{4}$

$1=1$ $\quad\therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1+a_2+a_3+\cdots+a_k=S_k \quad \Rightarrow \sum_{i=1}^{k}a_i=S_k$

*i.e.,* $\quad \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}=\binom{k+3}{4}$

Step 3; for $n=k+1$ adding b.s by $a_{k+1}=\binom{k+3}{3}$

$a_1+a_2+a_3+\cdots+a_k+a_{k+1}=S_k+a_{k+1}$

*i.e.,* $\quad \sum_{i=1}^{k+1}a_i=S_k+a_{k+1}$

$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\cdots+\binom{k+2}{3}+\binom{k+3}{3}$

$\sum_{i=1}^{k+1}a_i=\binom{k+3}{4}+\binom{k+3}{3}$

$\sum_{i=1}^{k+1}a_i=\frac{(k+3)!}{(k+3-4)!4!}+\frac{(k+3)!}{(k+3-3)!3!}$

$\sum_{i=1}^{k+1}a_i=\frac{(k+3)!}{(k-1)!4!}+\frac{(k+3)!}{(k)!3!}$

$\sum_{i=1}^{k+1}a_i=\frac{(k+3)!}{(k-1)!4\times3!}+\frac{(k+3)!}{k(k-1)!3!}$

$\sum_{i=1}^{k+1}a_i=\frac{(k+3)!}{(k-1)!3!}\left(\frac{1}{4}+\frac{1}{k}\right)$

$\sum_{i=1}^{k+1}a_i=\frac{(k+3)!}{(k-1)!3!}\left(\frac{k+4}{4k}\right)$

$\sum_{i=1}^{k+1}a_i=\frac{(k+4)(k+3)!}{k(k-1)!4\times3!}$

$\sum_{i=1}^{k+1}a_i=\frac{(k+4)!}{k!4!}$

$\sum_{i=1}^{k+1}a_i=\binom{k+4}{4}$

$\sum_{i=1}^{k+1}a_i=S_{k+1}$ $\quad\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q15. $\binom{5}{5}+\binom{6}{5}+\binom{7}{5}+\cdots+\binom{n+4}{5}=\binom{n+5}{6}$

Rauf work; Here $a_n=\binom{n+4}{5}$, $\quad S_n=\binom{n+5}{6}$

Replacing n by k + 1

$$a_{k+1}=\binom{k+1+4}{5},\qquad S_{k+1}=\binom{k+1+5}{6}$$

Solution: $\binom{5}{5}+\binom{6}{5}+\binom{7}{5}+\cdots+\binom{n+4}{5}=\binom{n+5}{6}$

Or $\sum_{k=1}^{n}\binom{k+4}{5}=\binom{n+5}{6}$

By mathematical induction Step 1; $n=1$

$\binom{1+4}{5}=\binom{1+5}{6}$ $a_1=S_1$

$\binom{5}{5}=\binom{6}{6}$

$1=1$ $\therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1+a_2+a_3+\cdots+a_k=S_k \quad\Rightarrow \sum_{i=1}^{k}a_i=S_k$

*i.e.,* $\binom{5}{5}+\binom{6}{5}+\binom{7}{5}+\cdots+\binom{k+4}{5}=\binom{k+5}{6}$

Step 3; for $n=k+1$ adding b s by $a_{k+1}=\binom{k+5}{5}$

$a_1+a_2+a_3+\cdots+a_k+a_{k+1}=S_k+a_{k+1}$

*i.e.,* $\sum_{i=1}^{k+1}a_i=S_k+a_{k+1}$

$$\binom{5}{5}+\binom{6}{5}+\binom{7}{5}+\cdots+\binom{k+4}{5}+\binom{k+5}{5}$$

$$=\binom{k+5}{6}+\binom{k+5}{5}$$

$$=\frac{(k+5)!}{(k+5-6)!6!}+\frac{(k+5)!}{(k+5-5)!5!}$$

$$=\frac{(k+5)!}{(k-1)!6!}+\frac{(k+5)!}{(k)!5!}$$

$$=\frac{(k+5)!}{(k-1)!6\times5!}+\frac{(k+5)!}{k(k-1)!5!}$$

$$=\frac{(k+5)!}{(k-1)!5!}\left(\frac{1}{6}+\frac{1}{k}\right)$$

$$=\frac{(k+5)!}{(k-1)!5!}\left(\frac{k+6}{6k}\right)$$

$$=\frac{(k+6)(k+5)!}{k(k-1)!6\times5!}$$

$$=\frac{(k+6)!}{k!6!}=\binom{k+6}{6}$$

$=S_{k+1}$ $\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q16 $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n}{2}=\binom{n+1}{3}$ For $n\ge 2$

Rauf work: Here $a_n=\binom{n}{2}$, $\quad S_n=\binom{n+1}{3}$

Replacing n by k + 1

$$a_{k+1}=\binom{k+1}{2},\qquad S_{k+1}=\binom{k+1+1}{3}$$

Solution; $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n}{2}=\binom{n+1}{3}$

Or $\sum_{k=1}^{n}\binom{k}{2}=\binom{n+1}{3}$

By mathematical induction Step 1; $n=2$

$\binom{2}{2}=\binom{2+1}{3}$ $a_2=S_2$

$\binom{2}{2}=\binom{3}{3}$

$1=1$ $\therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$a_1+a_2+a_3+\cdots+a_k=S_k \quad\Rightarrow \sum_{i=1}^{k}a_i=S_k$

*i.e.,* $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{k}{2}=\binom{k+1}{3}$

Step 3; for $n=k+1$ adding bs by $a_{k+1}=\binom{k+1}{2}$

$a_1+a_2+a_3+\cdots+a_k+a_{k+1}=S_k+a_{k+1}$

*i.e.,* $\sum_{i=1}^{k+1}a_i=S_k+a_{k+1}$

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{k}{2}+\binom{k+1}{2}$$

$$\sum_{i=1}^{k+1}a_i=\binom{k+1}{3}+\binom{k+1}{2}$$

$$\sum_{i=1}^{k+1}a_i=\frac{(k+1)!}{(k+1-3)!3!}+\frac{(k+1)!}{(k+1-2)!2!}$$

$$\sum_{i=1}^{k+1}a_i=\frac{(k+1)!}{(k-2)!3!}+\frac{(k+1)!}{(k-1)!2!}$$

$$\sum_{i=1}^{k+1}a_i=\frac{(k+1)!}{(k-1)!3\times2!}+\frac{(k+5)!}{(k-1)(k-2)!2!}$$

$$\sum_{i=1}^{k+1}a_i=\frac{(k+1)!}{(k-2)!2!}\left(\frac{1}{3}+\frac{1}{k-1}\right)$$

$$\sum_{i=1}^{k+1}a_i=\frac{(k+1)!}{(k-2)!2!}\left(\frac{k-1+3}{3(k-1)}\right)$$

$$\sum_{i=1}^{k+1}a_i=\frac{(k+2)(k+1)!}{(k-1)(k-2)!3\times2!}$$

$$\sum_{i=1}^{k+1}a_i=\frac{(k+2)!}{(k-1)!3!}=\binom{k+2}{3}$$

$\sum_{i=1}^{k+1}a_i=S_{k+1}$ $\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q17. Show by mathematical induction that

i). $\dfrac{5^{2n}-1}{24}$ is an integer

Solution; Let $P(n)=\dfrac{5^{2n}-1}{24}$

By mathematical induction Step 1; $n=1$

$P(1)=\dfrac{5^2-1}{24}=\dfrac{25-1}{24}=\dfrac{24}{24}$

$P(1)=1(\text{an integer}) \therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$P(k)=\dfrac{5^{2k}-1}{24}$ (an integer)....(1)

Step 3; To show for $n=k+1$

i.e., To show

$P(k+1)=\dfrac{5^{2(k+1)}-1}{24}$ Is an integer

$=\dfrac{5^{2k+2}-1}{24}=\dfrac{5^{2k}.5^2-1}{24}$

Subtracting and adding 25

$P(k+1)=\dfrac{25.5^{2k}-25+25-1}{24}$

$=\dfrac{25.5^{2k}-25}{24}+\dfrac{25-1}{24}$

$=25.\dfrac{5^{2k}-1}{24}+\dfrac{24}{24}$

$P(k+1)=25.\dfrac{5^{2k}-1}{24}+1.....(2)$

In RHS of equation (2) $\dfrac{5^{2k}-1}{24}$ is an integer from (1)

So RHS of equation (2) is an integer, therefore $P(k+1)$ is an integer $\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q17ii). $\dfrac{10^{n+1}-9n-10}{81}$ is an integer

Solution; Let $P(n)=\dfrac{10^{n+1}-9n-10}{81}$

By mathematical induction Step 1; $n=1$

$P(1)=\dfrac{10^{1+1}-9.1-10}{81}=\dfrac{100-19}{81}=\dfrac{81}{81}$

$P(1)=1(\text{an integer}) \quad \therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$P(k)=\dfrac{10^{k+1}-9k-10}{81}$ (an integer)....(1)

Step 3; To show for $n=k+1$

i.e., To show

$P(k+1)=\dfrac{10^{k+1+1}-9(k+1)-10}{81}$ Is an integer

$P(k+1)=\dfrac{10.10^{k+1}-9k-9-10}{81}$

Subtracting and adding $90k$ and 100

$P(k+1)=\dfrac{10.10^{k+1}-90k-100+90k-9k-19+100}{81}$

$P(k+1)=\dfrac{10(10^{k+1}-9k-10)+81k+81}{81}$

$P(k+1)=\dfrac{10(10^{k+1}-9k-10)}{81}+\dfrac{81(k+1)}{81}$ In RHS of

$P(k+1)=10.\dfrac{10^{k+1}-9k-10}{81}+k+1.....(2)$

eq (2) $\dfrac{10^{k+1}-9k-10}{81}$ is an integer from (1)

So RHS of equation (2) is an integer, therefore $P(k+1)$ is an integer

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q17iii). $\dfrac{3^{2n}-2^{2n}}{5}$ is an integer

Solution; Let $P(n)=\dfrac{3^{2n}-2^{2n}}{5}$

By mathematical induction Step 1; $n=1$

$P(1)=\dfrac{3^2-2^2}{5}=\dfrac{9-4}{5}=\dfrac{5}{5}$

$P(1)=1(\text{an integer}) \quad \therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$P(k)=\dfrac{3^{2k}-2^{2k}}{5}$ (an integer)....(1)

Step 3; To show for $n=k+1$

i.e., To show

$P(k+1)=\dfrac{3^{2(k+1)}-2^{2(k+1)}}{5}$ Is an integer

$P(k+1)=\dfrac{3^{2k}.3^2-2^{2k}.2^2}{5}$

$P(k+1)=\dfrac{9.3^{2k}-4.2^{2k}}{5}$

Subtracting and adding $9.2^{2k}$

$P(k+1)=\dfrac{9.3^{2k}-9.2^{2k}+9.2^{2k}-4.2^{2k}}{5}$

$P(k+1)=\dfrac{9(3^{2k}-2^{2k})+2^{2k}(9-4)}{5}$

$P(k+1)=9.\dfrac{3^{2k}-2^{2k}}{5}+2^{2k}.1.....(2)$

In RHS of eq (2) $\dfrac{3^{2k}-2^{2k}}{5}$ is an integer from (1)

So RHS of eq(2) is an integer, so $P(k+1)$ is an integer

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q18i). $2^n>n \ \ \forall n\in N$ Prove.

Solution; we have $2^n>n \ \ \forall n\in N$

By mathematical induction

Step 1; $n=1$

$2^1>1$

$2>1 \quad \therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$2^k>k$ ....(1)

Step 3; To show for $n=k+1$

i.e., we will to show $2^{k+1}>(k+1)$

Multiply both sides by 2 of equation (1)

$2.2^k>2.k$

$2^{k+1}>k+k$

$2^{k+1}>k+1 \quad as \quad k>1$

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

ii). $n!>n^2$ for every integer $n\ge 4$ and $n!>n^3$ for every integer $n\ge 6$

For $n!>n^2$ for every integer $n\ge 4$

Solution;

By mathematical induction

Step 1; $n=4$

$4!>4^2 \Rightarrow 4\times 3\times 2\times 1>16$

$24>16 \quad \therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$k!>k^2$ ....(1)

Step 3; To show for $n=k+1$

i.e., we will to show $(k+1)!>(k+1)^2$

Multiply both sides by $k+1$ of equation (1)

$(k+1)k!>k^2(k+1)$

$(k+1)!>(k+1)(k+1)$

$as \quad k^2>k+1, \forall n\ge 6$

$(k+1)!>(k+1)^2$

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

$\boxed{\text{Q18: Now}}$ $n!>n^3$ for every integer $n\ge 6$

Solution; By mathematical induction

Step 1; $n=6$

$6!>6^3 \Rightarrow 6\times 5\times 4\times 3\times 2\times 1>216$

$720>216 \quad \therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$k!>k^3$ ....(1)

Step 3; To show for $n=k+1$

i.e., we will to show $(k+1)!>(k+1)^3$

Multiply both sides by $k+1$ of equation (1)

$(k+1)k!>k^3(k+1)$

$(k+1)!>(k+1)^2(k+1)$

$as \quad k^3>(k+1)^2, \forall n\ge 4$

$(k+1)!>(k+1)^3 \qquad \therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q19. i).Show that 5 is a factor of $3^{2n-1}+2^{2n-1}$ when n is any positive integer.

Solution; Let $P(n)=3^{2n-1}+2^{2n-1}$

By mathematical induction Step 1; $n=1$

$P(1)=3^{2-1}+2^{2-1}=3+2$

$P(1)=5(\text{factor of }5) \qquad \therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$P(k)=3^{2k-1}+2^{2k-1}$ (has factor 5)....(1)

Step 3; To show for $n=k+1$

i.e., To show

$P(k+1)=3^{2(k+1)-1}+2^{2(k+1)-1}$ has factor 5

$=3^{2k+2-1}+2^{2k+2-1}=3^2.3^{2k-1}+2^2.2^{2k-1}$

$=9.3^{2k-1}+4.2^{2k-1}$

Subtracting and adding $9.2^{2k-1}$

$P(k+1)=9.3^{2k-1}+9.2^{2k-1}-9.2^{2k-1}+4.2^{2k-1}$

$=9\left(3^{2k-1}+2^{2k-1}\right)-2^{2k-1}(9-4)$

$P(k+1)=9\left(3^{2k-1}+2^{2k-1}\right)-2^{2k-1}.5.....(2)$

In RHS of equation (2), 5 is a factor of $3^{2k-1}+2^{2k-1}$ from (1)

So 5 is a factor of RHS of equation (2), therefore 5 is a factor of $P(k+1)$

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

---

Q19ii).Prove that $2^{2n}-1$ is a multiple of 3 for all positive integers.

Solution; Let $P(n)=2^{2n}-1$

By mathematical induction Step 1; $n=1$

$P(1)=2^2-1=4-1$

$P(1)=3(\text{multiple of }3)$

$\therefore$ true for $n=1$

Step 2; suppose it is true for $n=k$

$P(k)=2^{2k}-1$ (multiple of 3)....(1)

Step 3; To show for $n=k+1$

i.e., To show

$P(k+1)=2^{2(k+1)}-1$ multiple of 3

$=2^{2k+2}-1=2.^2 2^{2k}-1$

$=4.2^{2k}-1$

Subtracting and adding 4

$P(k+1)=4.2^{2k}-4+4-1$

$=4\left(2^{2k}-1\right)+(4-1)$

$P(k+1)=4\left(2^{2k}-1\right)+3.....(2)$

In RHS of equation (2), $2^{2k}-1$ is a multiple of 3 from (1)

So RHS of equation (2) is a multiple of 3, therefore $P(k+1)$ is a multiple of 3

$\therefore$ true for $n=k+1$

Hence by mathematical induction, the given statement is true for all integer values of n.

## Exercise 7.2

Q1. Expand the following

i). $(2x+y)^6$

Solution; we have $(2x+y)^6=\sum_{k=0}^{6}\binom{6}{k}(2x)^{6-k}.y^k$

$$(2x+y)^6=\binom{6}{0}(2x)^{6-0}y^0+\binom{6}{1}(2x)^{6-1}y^1+\binom{6}{2}(2x)^{6-2}y^2$$
$$+\binom{6}{3}(2x)^{6-3}y^3+\binom{6}{4}(2x)^{6-4}y^4$$
$$+\binom{6}{5}(2x)^{6-5}y^5+\binom{6}{6}(2x)^{6-6}y^6$$
$$(2x+y)^6=1.(2x)^6y^0+6.(2x)^5y^1+15.(2x)^4y^2$$
$$+20.(2x)^3y^3+15.(2x)^2y^4+6.(2x)^1y^5$$
$$+1.(2x)^0y^6$$
$$(2x+y)^6=64x^6+6.32x^5y^1+15.16x^4y^2$$
$$+20.8x^3y^3+15.4x^2y^4+6.2x^1y^5+1.y^6$$
$$(2x+y)^6=64x^6+192x^5y^1+240x^4y^2+160x^3y^3$$
$$+60x^2y^4+12x^1y^5+y^6$$

---

ii). $\left(x-\frac{1}{x}\right)^7$

solution; we have $\left(x-\frac{1}{x}\right)^7=\sum_{k=0}^{7}\binom{7}{k}(x)^{7-k}.\left(\frac{-1}{x}\right)^k$

$$\left(x-\frac{1}{x}\right)^7=\binom{7}{0}x^{7-0}\left(\frac{-1}{x}\right)^0+\binom{7}{1}x^{7-1}\left(\frac{-1}{x}\right)^1+\binom{7}{2}x^{7-2}\left(\frac{-1}{x}\right)^2$$
$$+\binom{7}{3}x^{7-3}\left(\frac{-1}{x}\right)^3+\binom{7}{4}x^{7-4}\left(\frac{-1}{x}\right)^4+\binom{7}{5}x^{7-5}\left(\frac{-1}{x}\right)^5$$
$$+\binom{7}{6}x^{7-6}\left(\frac{-1}{x}\right)^6+\binom{7}{7}x^{7-7}\left(\frac{-1}{x}\right)^7$$
$$\left(x-\frac{1}{x}\right)^7=1.x^7-7.x^6.\frac{1}{x^1}+21.x^5.\frac{1}{x^2}-35.x^4.\frac{1}{x^3}+35.x^3.\frac{1}{x^4}$$
$$-21.x^2.\frac{1}{x^5}+7.x^1.\frac{1}{x^6}-1.\frac{1}{x^7}$$
$$\left(x-\frac{1}{x}\right)^7=x^7-7x^5+21x^3-35x^1+\frac{35}{x^1}-\frac{21}{x^3}+\frac{7}{x^5}-\frac{1}{x^7}$$

---

iii). $(3x-2y)^4$

solution; we have

$$(3x-2y)^4=\sum_{k=0}^{4}\binom{4}{k}(3x)^{4-k}.(-2y)^k$$
$$(3x-2y)^4=\binom{4}{0}(3x)^{4-0}(-2y)^0+\binom{4}{1}(3x)^{4-1}(-2y)^1$$
$$+\binom{4}{2}(3x)^{4-2}(-2y)^2+\binom{4}{3}(3x)^{4-3}(-2y)^3$$
$$+\binom{4}{4}(3x)^{4-4}(-2y)^4$$
$$(3x-2y)^4=1.(3x)^4(2y)^0-4(3x)^3(2y)^1+6(3x)^2(2y)^2$$
$$-4(3x)^1(2y)^3+1(3x)^0(2y)^4$$
$$=3^4x^4-4.3^3.2x^3y+6.3^2.2^2x^2y^2-4.3.2^3xy^3+2^4y^4$$
$$(3x-2y)^4=81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$$

---

iv). $\left(\frac{3}{2}x-\frac{3}{x^2}\right)^6$

solution; we have $\left(\frac{3}{2}x-\frac{3}{x^2}\right)^6$

$$3^6\left(\frac{x}{2}-\frac{1}{x^2}\right)^6=3^6\sum_{k=0}^{6}\binom{6}{k}\left(\frac{x}{2}\right)^{6-k}\left(\frac{-1}{x^2}\right)^k$$

$$\left(\frac{3}{2}x-\frac{3}{x^2}\right)^6=3^6\left\{\binom{6}{0}\left(\frac{x}{2}\right)^{6-0}\left(\frac{-1}{x^2}\right)^0+\binom{6}{1}\left(\frac{x}{2}\right)^{6-1}\left(\frac{-1}{x^2}\right)^1\right.$$
$$+\binom{6}{2}\left(\frac{x}{2}\right)^{6-2}\left(\frac{-1}{x^2}\right)^2+\binom{6}{3}\left(\frac{x}{2}\right)^{6-3}\left(\frac{-1}{x^2}\right)^3$$
$$+\binom{6}{4}\left(\frac{x}{2}\right)^{6-4}\left(\frac{-1}{x^2}\right)^4+\binom{6}{5}\left(\frac{x}{2}\right)^{6-5}\left(\frac{-1}{x^2}\right)^5$$
$$\left.+\binom{6}{6}\left(\frac{x}{2}\right)^{6-6}\left(\frac{-1}{x^2}\right)^6\right\}$$
$$\left(\frac{3}{2}x-\frac{3}{x^2}\right)^6=3^6\left\{\left(\frac{x}{2}\right)^6-6\left(\frac{x}{2}\right)^5\left(\frac{1}{x^2}\right)^1+15\left(\frac{x}{2}\right)^4\left(\frac{1}{x^2}\right)^2\right.$$
$$-20\left(\frac{x}{2}\right)^3\left(\frac{1}{x^2}\right)^3+15\left(\frac{x}{2}\right)^2\left(\frac{1}{x^2}\right)^4$$
$$\left.-6\left(\frac{x}{2}\right)^1\left(\frac{1}{x^2}\right)^5+\left(\frac{1}{x^2}\right)^6\right\}$$
$$\left(\frac{3}{2}x-\frac{3}{x^2}\right)^6=729\left\{\frac{x^6}{2^6}-6\frac{x^5}{2^5}.\frac{1}{x^2}+15\frac{x^4}{2^4}.\frac{1}{x^4}-20\frac{x^3}{2^3}.\frac{1}{x^6}\right.$$
$$\left.+15\frac{x^2}{2^2}.\frac{1}{x^8}-6\frac{x^1}{2}.\frac{1}{x^{10}}+\frac{1}{x^{12}}\right\}$$
$$\left(\frac{3}{2}x-\frac{3}{x^2}\right)^6=729\left\{\frac{x^6}{64}-\frac{6x^3}{32}+\frac{15}{16}-\frac{20}{8x^3}+\frac{15}{4x^6}-\frac{6}{2x^9}+\frac{1}{x^{12}}\right\}$$
$$\left(\frac{3}{2}x-\frac{3}{x^2}\right)^6=729\left\{\frac{x^6}{64}-\frac{3x^3}{16}+\frac{15}{16}-\frac{5}{2x^3}+\frac{15}{4x^6}-\frac{3}{x^9}+\frac{1}{x^{12}}\right\}$$

---

Q2. Find the middle term (middle terms in case of two middle terms) in the following expansions.

i). $\left(\frac{a}{3}+9b\right)^8$

Solution; we have $\left(\frac{a}{3}+9b\right)^8$

$n=8$ even and the general term is

$$T_{r+1}=\binom{8}{r}\left(\frac{a}{3}\right)^{8-r}(9b)^r$$

Middle term $=\left(\frac{n+2}{2}\right)^{th}=\left(\frac{8+2}{2}\right)^{th}=5^{th}$ term

To find $T_5=T_{r+1}\Rightarrow r=4$

$$T_{4+1}=\binom{8}{4}\left(\frac{a}{3}\right)^{8-4}(9b)^4$$
$$T_5=\frac{8!}{(8-4)!4!}\frac{a^4}{3^4}\times 9^4b^4$$
$$T_5=\frac{8\times7\times6\times5\times4!}{4\times3\times2\times1.4!}\frac{a^4}{3\times3\times3\times3}\times9\times9\times9\times9.b^4$$
$$T_5=2\times7\times5\times9\times9\times a^4b^4$$
$$T_5=5670a^4b^4$$

---

ii). $\left(3x+\frac{1}{2x}\right)^{10}$

Solution; we have $\left(3x+\frac{1}{2x}\right)^{10}$

$n=10$ even and the general term is

$$T_{r+1}=\binom{10}{r}(3x)^{10-r}\left(\frac{1}{2x}\right)^r$$

Middle term=$\left(\frac{n+2}{2}\right)^{th}=\left(\frac{10+2}{2}\right)^{th}=$ 6th term

To find $T_6=T_{r+1}\Rightarrow r=5$

$$T_{5+1}=\binom{10}{5}(3x)^{10-5}\left(\frac{1}{2x}\right)^5$$

$$T_6=\frac{10!}{(10-5)!5!}\times 3^5x^5\frac{1}{2^5x^5}$$

$$T_6=\frac{10\times9\times8\times7\times6\times5!}{5\times4\times3\times2\times1.5!}3\times3\times3\times3\times3x^5\frac{1}{2\times2\times2\times2\times2.x^5}$$

$$T_6=\frac{9\times7\times3\times3\times3\times3\times3}{2\times2\times2}$$

$$T_6=\frac{15309}{8}$$

$$T_6=1913.625$$

---

iii). $\left(x^4-\frac{1}{x^3}\right)^{11}$

Solution; the general term is

$$T_{r+1}=\binom{11}{r}\left(x^4\right)^{11-r}\left(\frac{-1}{x^3}\right)^r$$

$n=11$ odd there should be two middle terms

First middle term=$\left(\frac{n+1}{2}\right)^{th}=\left(\frac{11+1}{2}\right)^{th}=$ $6^{th}$ term

Second middle term=$\left(\frac{n+3}{2}\right)^{th}=\left(\frac{11+3}{2}\right)^{th}=$ $7^{th}$ term

To find $T_6=T_{r+1}\Rightarrow r=5$

$$T_{5+1}=\binom{11}{5}\left(x^4\right)^{11-5}\left(\frac{-1}{x^3}\right)^5$$

$$T_6=\frac{11!}{(11-5)!5!}.x^{24}.\frac{-1}{x^{15}}$$

$$T_6=\frac{-11\times10\times9\times8\times7\times6!}{6!\times5\times4\times3\times2\times1}.x^{24-15}$$

$$T_6=-11\times3\times2\times7x^9$$

$$T_6=-462x^9$$

To find $T_7=T_{r+1}\Rightarrow r=6$

$$T_{6+1}=\binom{11}{6}\left(x^4\right)^{11-6}\left(\frac{-1}{x^3}\right)^6$$

$$T_7=\frac{11!}{(11-6)!6!}.x^{20}.\frac{1}{x^{18}}$$

$$T_7=\frac{11\times10\times9\times8\times7\times6!}{5\times4\times3\times2\times1\times6!}.x^{20-18}$$

$$T_7=11\times3\times2\times7x^2$$

$$T_7=462x^2$$

---

Q3. Find the coefficient of

i). $x^9$ in $\left(x+\frac{3a}{x^2}\right)^{15}$

Solution; we have $\left(x+\frac{3a}{x^2}\right)^{15}$

The general term is

$$T_{r+1}=\binom{15}{r}(x)^{15-r}\left(\frac{3a}{x^2}\right)^r$$

For the term involving $x^9$, by comparing only powers of x

$$(x)^{15-r}\left(\frac{1}{x^2}\right)^r=x^9$$

$$\frac{x^{15-r}}{x^{2r}}=x^9$$

$$x^{15-3r}=x^9$$

$$\Rightarrow 15-3r=9$$

$$15-9=3r$$

$$3r=6$$

$$r=2$$

Put $r=2$ in the general term

$$T_{2+1}=\binom{15}{2}(x)^{15-2}\left(\frac{3a}{x^2}\right)^2$$

$$T_3=\frac{15!}{(15-2)!2!}.x^{13}.\frac{9a^2}{x^4}$$

$$T_3=\frac{15\times14\times13!}{13!\times2\times1}9a^2x^{13-4}$$

$$T_3=15\times7\times9a^2x^9$$

$$T_3=945a^2x^9$$

Hence the coefficient of $x^9$ is $945a^2$

---

ii). $x^5$ in $\left(2x^2-\frac{1}{3x}\right)^{10}$

Solution; the general term is

$$T_{r+1}=\binom{10}{r}\left(2x^2\right)^{10-r}\left(\frac{-1}{3x}\right)^r$$

For the term involving $x^5$, by comparing only powers of x

$$\left(x^2\right)^{10-r}\left(\frac{1}{x}\right)^r=x^5$$

$$\frac{x^{20-2r}}{x^r}=x^5$$

$$x^{20-3r}=x^5$$

$$\Rightarrow 20-3r=5$$

$$20-5=3r$$

$$3r=15$$

$$r=5$$

Put $r=5$ in the general term

$$T_{5+1}=\binom{10}{5}\left(2x^2\right)^{10-5}\left(\frac{-1}{3x}\right)^5$$

$$T_6=\frac{10!}{(10-5)!5!}.2^5x^{10}.\frac{-1}{3^5x^5}$$

$$T_6=\frac{-10\times9\times8\times7\times6\times5!}{5!\times5\times4\times3\times2\times1}.\frac{32x^{10-5}}{3^5}$$

$$T_6=\frac{-2\times2\times7}{3\times3\times3}\times32x^5$$

$$T_6=-\frac{896}{27}x^5=-33.1852x^5$$

Hence the coefficient of $x^5$ is $-33.1852$

iii). $x$ in $\left(2x^2-\frac{1}{x}\right)^{12}$

Solution; we have $\left(2x^2-\frac{1}{x}\right)^{12}$

The general term is

$$T_{r+1}=\binom{12}{r}\left(2x^2\right)^{12-r}\left(\frac{-1}{x}\right)^r$$

For the term involving $x$, by comparing only powers of x

$$\left(x^2\right)^{12-r}\left(\frac{1}{x}\right)^r=x$$

$$\Rightarrow \frac{x^{24-2r}}{x^r}=x$$

$$x^{24-3r}=x$$

$$\Rightarrow 24-3r=1$$

$$24-1=3r$$

$$3r=23$$

$$r=\tfrac{23}{3}$$

As $r$ is not an integer sothe coefficient of x is zero

Q4. Find term independent of x in following expansions.

i). $\left(x+\frac{1}{2x}\right)^8$

Solution; we have $\left(x+\frac{1}{2x}\right)^8$

The general term is

$$T_{r+1}=\binom{8}{r}(x)^{8-r}\left(\frac{1}{2x}\right)^r$$

For the term independent of x $\left(i.e.,\quad x^0\right)$, by comparing only powers of x

$$(x)^{8-r}\left(\frac{1}{x}\right)^r=x^0$$

$$\frac{x^{8-r}}{x^r}=x^0$$

$$x^{8-2r}=x^0$$

$$8-2r=0$$

$$\Rightarrow r=4$$

Put $r=4$ in the general term

$$T_{4+1}=\binom{8}{4}(x)^{8-4}\left(\frac{1}{2x}\right)^4$$

$$T_5=\frac{8!}{(8-4)!4!}.x^4.\frac{1}{2^4x^4}$$

$$T_5=\frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1}.\frac{x^{4-4}}{2\times2\times2\times2}$$

$$T_5=\frac{7\times5x^0}{2\times2\times2}$$

$$T_5=\frac{35}{8}$$

ii). $\left(2x^2-\frac{1}{x^3}\right)^{10}$

Solution; we have $\left(2x^2-\frac{1}{x^3}\right)^{10}$

The general term is

$$T_{r+1}=\binom{10}{r}\left(2x^2\right)^{10-r}\left(\frac{-1}{x^3}\right)^r$$

For term independent of x $x^0$, by comparing only powers of x

$$\left(x^2\right)^{10-r}\left(\frac{1}{x^3}\right)^r=x^0$$

$$\frac{x^{20-2r}}{x^{3r}}=x^0$$

$$x^{20-5r}=x^0$$

$$20-5r=0$$

$$20=5r$$

$$\Rightarrow r=4$$

Put $r=4$ in the general term

$$T_{4+1}=\binom{10}{4}\left(2x^2\right)^{10-4}\left(\frac{-1}{x^3}\right)^4$$

$$T_5=\frac{10!}{(10-4)!4!}.2^6x^{12}.\frac{1}{x^{12}}$$

$$T_5=\frac{10\times9\times8\times7\times6!}{6!\times4\times3\times2\times1}.2\times2\times2\times2\times2\times2x^{12-12}$$

$$T_5=10\times3\times7\times64x^0$$

$$T_5=13440$$

iii). $\left(2x^2+\frac{1}{x}\right)^9$

Solution; we have $\left(2x^2+\frac{1}{x}\right)^9$

the general term is

$$T_{r+1}=\binom{9}{r}\left(2x^2\right)^{9-r}\left(\frac{1}{x}\right)^r$$

For the term independent of x $\left(i.e.,\quad x^0\right)$, by comparing only powers of x

$$\left(x^2\right)^{9-r}\left(\frac{1}{x}\right)^r=x^0$$

$$\frac{x^{18-2r}}{x^r}=x^0$$

$$x^{18-3r}=x^0$$

$$\Rightarrow 18-3r=0$$

$$18=3r$$

$$r=6$$

Put $r=6$ in the general term

$$T_{6+1}=\binom{9}{6}\left(2x^2\right)^{9-6}\left(\frac{1}{x}\right)^6$$

$$T_7=\frac{9!}{(9-6)!6!}.2^3x^6.\frac{1}{x^6}$$

$$T_7=\frac{9\times8\times7\times6!}{6!\times3\times2\times1}.2\times2\times2x^{6-6}$$

$$T_7=3\times4\times7\times8x^0=672$$

---

Q5. Find (expand)

i). $\left(1+2x-x^2\right)^4$

Solution; we have $\left(1+2x-x^2\right)^4$

$$\left(1+2x-x^2\right)^4=\sum_{k=0}^{4}\binom{4}{k}(1+2x)^{4-k}.\left(-x^2\right)^k$$

$$=\binom{4}{0}(1+2x)^{4-0}\left(-x^2\right)^0+\binom{4}{1}(1+2x)^{4-1}\left(-x^2\right)^1$$

$$+\binom{4}{2}(1+2x)^{4-2}\left(-x^2\right)^2+\binom{4}{3}(1+2x)^{4-3}\left(-x^2\right)^3$$

$$+\binom{4}{4}(1+2x)^{4-4}\left(-x^2\right)^4$$

$$=1.(1+2x)^4-4(1+2x)^3\left(x^2\right)^1+6(1+2x)^2\left(x^2\right)^2$$

$$-4(1+2x)^1\left(x^2\right)^3+\left(x^2\right)^4$$

$$=(1+2x)^4-4x^2(1+2x)^3+6x^4(1+2x)^2$$

$$-4x^6(1+2x)^1+x^8$$

$$=\left\{1+4(2x)^1+6(2x)^2+4(2x)^3+(2x)^4\right\}$$

$$-4x^2\left\{1+3(2x)^1+3(2x)^2+(2x)^3\right\}$$

$$+6x^4\left\{1+2(2x)^1+(2x)^2\right\}-4x^6(1+2x)+x^8$$

$$=\left\{1+8x+24x^2+32x^3+16x^4\right\}$$

$$-4x^2\left\{1+6x+12x^2+8x^3\right\}$$

$$+6x^4\left\{1+4x+4x^2\right\}-4x^6(1+2x)+x^8$$

$$=1+8x+24x^2+32x^3+16x^4-4x^2-24x^3-48x^4$$

$$-32x^5+6x^4+24x^5+24x^6-4x^6-8x^7+x^8$$

$$=1+8x+24x^2-4x^2+32x^3-24x^3+16x^4-48x^4$$

$$+6x^4-32x^5+24x^5+24x^6-4x^6-8x^7+x^8$$

$$=1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8$$

---

ii). $\left(\sqrt{2}+1\right)^5-\left(\sqrt{2}-1\right)^5$

Solution; we have $\left(\sqrt{2}+1\right)^5-\left(\sqrt{2}-1\right)^5$

$$\left(\sqrt{2}+1\right)^5-\left(\sqrt{2}-1\right)^5=\sum_{k=0}^{5}\binom{5}{k}\left(\sqrt{2}\right)^{5-k}-\sum_{k=0}^{5}\binom{5}{k}\left(\sqrt{2}\right)^{5-k}(-1)^k$$

Using Pascal's triangle for n=5

The coefficients are 1 5 10 10 5 1

$$=\left\{\left(\sqrt{2}\right)^5+5\left(\sqrt{2}\right)^4+10\left(\sqrt{2}\right)^3+10\left(\sqrt{2}\right)^2+5\left(\sqrt{2}\right)^1+1\right\}$$

$$-\left\{\left(\sqrt{2}\right)^5-5\left(\sqrt{2}\right)^4+10\left(\sqrt{2}\right)^3-10\left(\sqrt{2}\right)^2+5\left(\sqrt{2}\right)^1-1\right\}$$

$$=\left(\sqrt{2}\right)^5+5\left(\sqrt{2}\right)^4+10\left(\sqrt{2}\right)^3+10\left(\sqrt{2}\right)^2+5\left(\sqrt{2}\right)^1+1$$

$$-\left(\sqrt{2}\right)^5+5\left(\sqrt{2}\right)^4-10\left(\sqrt{2}\right)^3+10\left(\sqrt{2}\right)^2-5\left(\sqrt{2}\right)^1+1$$

$$=10\left(\sqrt{2}\right)^4+20\left(\sqrt{2}\right)^2+2=10(4)+20(2)+2=82$$

---

iii). $(a+b)^5-(a-b)^5$

Solution; we have $(a+b)^5+(a-b)^5$

$$=\sum_{k=0}^{5}\binom{5}{k}a^{5-k}b^k+\sum_{k=0}^{5}\binom{5}{k}a^{5-k}(-b)^k$$

Using Pascal's triangle for n=5

The coefficients are 1 5 10 10 5 1

$$=\left\{a^5+5a^4b+10a^3b^2+10a^2b^3+5a^1b^4+b^5\right\}$$

$$+\left\{a^5-5a^4b+10a^3b^2-10a^2b^3+5a^1b^4-b^5\right\}$$

$$=a^5+5a^4b+10a^3b^2+10a^2b^3+5a^1b^4+b^5$$

$$+a^5-5a^4b+10a^3b^2-10a^2b^3+5a^1b^4-b^5$$

$$=2a^5+20a^3b^2+10a^1b^4$$

---

Q6 Find numerically greatest term in $(3-2x)^{10}$ when $x=\frac{3}{4}$

Solution; we have $(3-2x)^{10}$

When $x=\frac{3}{4}$ put in $(3-2x)^{10}$

$$(3-2x)^{10}\Big|_{x=\frac{3}{4}}=\left(3-2\left(\frac{3}{4}\right)\right)^{10}$$

$$(3-2x)^{10}\Big|_{x=\frac{3}{4}}=\left(3-\frac{3}{2}\right)^{10}$$

$$(3-2x)^{10}\Big|_{x=\frac{3}{4}}=3^{10}\left(1-\frac{1}{2}\right)^{10}$$

$$(3-2x)^{10}\Big|_{x=\frac{3}{4}}=3^{10}\sum_{i=0}^{10}\binom{10}{i}\left(\frac{-1}{2}\right)^i$$

$$(3-2x)^{10}\Big|_{x=\frac{3}{4}}=3^{10}\left\{\binom{10}{0}+\binom{10}{1}\left(\frac{-1}{2}\right)^1\right.$$

$$+\binom{10}{2}\left(\frac{-1}{2}\right)^2+\binom{10}{3}\left(\frac{-1}{2}\right)^3+\binom{10}{4}\left(\frac{-1}{2}\right)^4$$

$$+\binom{10}{5}\left(\frac{-1}{2}\right)^5+\binom{10}{6}\left(\frac{-1}{2}\right)^6+\binom{10}{7}\left(\frac{-1}{2}\right)^7$$

$$\left.+\binom{10}{8}\left(\frac{-1}{2}\right)^8+\binom{10}{9}\left(\frac{-1}{2}\right)^9+\binom{10}{10}\left(\frac{-1}{2}\right)^{10}\right\}$$

$$(3-2x)^{10}\Big|_{x=\frac{3}{4}}=3^{10}\left\{1-\frac{10}{2}+\frac{45}{4}-\frac{120}{8}+\frac{210}{16}-\frac{252}{32}+\frac{210}{64}\right.$$

$$\left.-\frac{120}{128}+\frac{45}{256}-\frac{10}{512}+\frac{1}{1024}\right\}$$

$$(3-2x)^{10}\Big|_{x=\frac{3}{4}}=3^{10}\{1-5+11.25-15+13.125-7.875$$

$$+3.281-0.936+0.176-0.020+0.00098\}$$

Greatest numerical term is at number 4th which is $3^{10}(15)$

---

Q7. Find the greatest term in $\left(1+\frac{x}{2}\right)^{12}$ when $x=\frac{1}{2}$

Solution; when $x=\frac{1}{2}$ put in $\left(1+\frac{x}{2}\right)^{12}$

$$\left(1+\tfrac{x}{2}\right)^{12}\Big|_{x=\frac{1}{2}}=\left(1+\frac{1}{2}\left(\frac{1}{2}\right)\right)^{12}$$

$$\left(1+\tfrac{x}{2}\right)^{12}\Big|_{x=\frac{1}{2}}=\left(1+\frac{1}{4}\right)^{12}$$

$$\left(1+\tfrac{x}{2}\right)^{12}\Big|_{x=\frac{1}{2}}=\sum_{i=0}^{12}\binom{12}{i}\left(\frac{1}{4}\right)^i$$

$$=\binom{12}{0}+\binom{12}{1}\left(\frac{1}{4}\right)^1+\binom{12}{2}\left(\frac{1}{4}\right)^2+\binom{12}{3}\left(\frac{1}{4}\right)^3$$
$$+\binom{12}{4}\left(\frac{1}{4}\right)^4+\binom{12}{5}\left(\frac{1}{4}\right)^5+\binom{12}{6}\left(\frac{1}{4}\right)^6$$
$$+\binom{12}{7}\left(\frac{1}{4}\right)^7+\binom{12}{8}\left(\frac{1}{4}\right)^8+\binom{12}{9}\left(\frac{1}{4}\right)^9$$
$$+\binom{12}{10}\left(\frac{1}{4}\right)^{10}+\binom{12}{11}\left(\frac{1}{4}\right)^{11}+\binom{12}{12}\left(\frac{1}{4}\right)^{12}$$
$$\left(1+\tfrac{x}{2}\right)^{12}\Big|_{x=\frac{1}{2}}=1+\frac{12}{4^1}+\frac{66}{4^2}+\frac{220}{4^3}+\frac{495}{4^4}+\frac{792}{4^5}+\frac{924}{4^6}+\frac{792}{4^7}$$
$$+\frac{495}{4^8}+\frac{220}{4^9}+\frac{66}{4^{10}}+\frac{12}{4^{11}}+\frac{1}{4^{12}}$$
$$\left(1+\tfrac{x}{2}\right)^{12}\Big|_{x=\frac{1}{2}}=1+3+4.125+3.438+1.934+0.773+0.226$$
$$+0.048+0.008+0.0008+0.00006+0.000003$$
$$+0.00000006$$

Greatest term is at the number 3rd which is 4.125

Q8. Find what is mentioned?

i). $\left(\sqrt{x}+y^2\right)^8$ term containing $x^{\frac{5}{2}}$

Solution; the general term is

$$T_{r+1}=\binom{8}{r}\left(\sqrt{x}\right)^{8-r}\left(y^2\right)^r$$

For term involving $x^{\frac{5}{2}}$, by comparing only powers of x

$$(x)^{\frac{8-r}{2}}=x^{\frac{5}{2}}$$
$$\Rightarrow \frac{8-r}{2}=\frac{5}{2}$$
$$8-r=5$$
$$\Rightarrow 8-5=r$$

Put $r=3$ in the general term

$$T_{3+1}=\binom{8}{3}\left(\sqrt{x}\right)^{8-3}\left(y^2\right)^3$$
$$T_4=\frac{8!}{(8-3)!3!}.x^{\frac{5}{2}}y^6$$
$$T_4=\frac{8\times7\times6\times5!}{5!\times3\times2\times1}y^6x^{\frac{5}{2}}$$
$$T_4=8\times7y^6x^{\frac{5}{2}}$$
$$T_4=56y^6x^{\frac{5}{2}}$$

ii). $\left(x^2-\frac{y^3}{2}\right)^{17}$ term containing $y^{15}$

Solution; the general term is

$$T_{r+1}=\binom{17}{r}\left(x^2\right)^{15-r}\left(\frac{-y^3}{2}\right)^r$$

For term involving $y^{15}$, by comparing only powers of y

$$\left(\frac{-y^3}{2}\right)^r=y^{15}$$
$$\Rightarrow 3r=15$$

Put $r=5$ in the general term

$$T_{5+1}=\binom{17}{5}\left(x^2\right)^{17-5}\left(\frac{-y^3}{2}\right)^5$$
$$T_6=-\frac{17!}{(17-5)!5!}.x^{24}\frac{y^{15}}{2^5}$$
$$T_6=-\frac{17\times16\times15\times14\times13\times12!}{12!\times5\times4\times3\times2\times1}x^{24}\frac{y^{15}}{2\times2\times2\times2\times2}$$
$$T_6=\frac{-17\times7\times13}{8}x^{24}y^{15}$$
$$T_6=-\frac{1547}{8}x^{24}y^{15}$$
$$T_6=-193.375x^{24}y^{15}$$

iii). $\left(3x-\frac{5}{x^3}\right)^8$ term independent of $x$

Solution; the general term is

$$T_{r+1}=\binom{8}{r}(3x)^{8-r}\left(\frac{-5}{x^3}\right)^r$$ For term independent of $x$, by comparing only powers of x

$$\frac{x^{8-r}}{x^{3r}}=x^0$$
$$\Rightarrow x^{8-4r}=x^0$$
$$\Rightarrow 8-4r=0$$

Put $r=2$ in the general term

$$T_{2+1}=\binom{8}{2}(3x)^{8-2}\left(\frac{-5}{x^3}\right)^2$$
$$T_3=-\frac{8!}{(8-2)!2!}.3^6x^6\frac{5^2}{x^6}$$
$$T_3=\frac{8\times7\times6!}{6!\times2\times1}3^6\times5^2x^{6-6}$$
$$T_3=4\times7\times3^6\times5^2x^0$$
$$T_3=510300$$

Q9. Expand $(1.04)^5$ up to four places of decimal by use of binomial formula

Solution; $(1.04)^5$

$$(1.04)^5=(1+0.04)^5$$
$$(1.04)^5=\sum_{i=1}^{5}\binom{5}{i}(0.04)^i$$
$$=\binom{5}{0}(0.04)^0+\binom{5}{1}(0.04)^1+\binom{5}{2}(0.04)^2$$
$$+\binom{5}{3}(0.04)^3+\binom{5}{4}(0.04)^4+\binom{5}{5}(0.04)^5$$
$$=1+5\times0.04+10\times0.0016+10\times0.000064$$
$$+5\times0.00000256+1\times0.0000001024$$

Taking up to four decimal places

$$=1+0.20+0.016+0.0006+\cdots$$
$$=1.2166$$

Q10. Show that the sum of the coefficients in the expansion of $(1+x)^n$ is $2^n$ where $n\in N$ and hence show that the sum of the coefficients in the expansion of $(1+x)^7$ is 128.

Solution; $(1+x)^7$ Since we know that

$$(1+x)^n=\sum_{i=0}^{n}\binom{n}{i}(x)^i=\binom{n}{0}+\binom{n}{1}(x)^1+\binom{n}{2}(x)^2+\binom{n}{3}(x)^3+\cdots\binom{n}{n}(x)^n \quad\text{.........(1)}$$

Put $x=1$ in equation (1)

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$$

$$\Rightarrow 2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$$

$\Rightarrow 2^n=$ Sum of coefficients of $(1+x)^n$

Similarly Sum of coefficients of $(1+x)^7=2^7=128$

Q11. Show that sum of odd coefficients is equal to sum of even coefficients in the binomial expansion of $(1+x)^n$ and each of them is equal to $2^{n-1}$

Solution; we have $(1+x)^7$

$$(1+x)^n=\sum_{i=0}^{n}\binom{n}{i}(x)^i=\binom{n}{0}+\binom{n}{1}(x)^1+\binom{n}{2}(x)^2+\binom{n}{3}(x)^3+\cdots\binom{n}{n}(x)^n \quad\text{.........(1)}$$

Put $x=-1$ and take n is even

$$(1+(-1))^n=\binom{n}{0}+\binom{n}{1}(-1)^1+\binom{n}{2}(-1)^2+\binom{n}{3}(-1)^3+\cdots+\binom{n}{n}(-1)^n$$

$$\Rightarrow\quad 0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+\binom{n}{n}$$

$$\Rightarrow\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1}=\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$$

Sum of odd coefficients= sum of even coefficients

Put $x=1$ in equation (1)

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$$

$$\Rightarrow 2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}$$

$\Rightarrow 2^n=$ (Sum of odd coefficients)+(sum of even coefficients)

$\Rightarrow 2^n=2$ (Sum of odd or even coefficients)

$\Rightarrow 2^{n-1}=$ (Sum of odd or even coefficients)

Q12 Consider $(1+x)^n$ and take $\binom{n}{r}=C_r$, show

$$C_1+2C_2x+3C_3x^2+\cdots+nC_nx^{n-1}=n(1+x)^{n-1}$$

Sol; Take LHS $C_1+2C_2x+3C_3x^2+\cdots+nC_nx^{n-1}$

$$=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\cdots+n\binom{n}{n}x^{n-1}$$

$$=\frac{n!}{(n-1)!1!}+2\frac{n!}{(n-2)!2!}x+3\frac{n!}{(n-3)!3!}x^2+\cdots+n\frac{n!}{(n-n)!n!}x^{n-1}$$

$$=\frac{n(n-1)!}{(n-1)!}+2\frac{n(n-1)(n-2)!}{(n-2)!2}x+3\frac{n(n-1)(n-2)(n-3)!}{(n-3)!3\times 2!}x^2+\cdots+n\frac{n!}{(0)!n!}x^{n-1}$$

$$=n+n(n-1)x+\frac{n(n-1)(n-2)}{2!}x^2+\cdots+nx^{n-1}$$

$$=n\left\{1+(n-1)x+\frac{(n-1)(n-2)}{2!}x^2+\cdots+x^{n-1}\right\}$$

$$=n(1+x)^{n-1}=RHS$$

Hence proved

If $n\notin N$ or n is a rational or negative, then

$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$ it is valid only if $|x|<1$, But if $|x|>1$ then it will not exists.

## Exercise 7.3

Q1. Find the first four terms in the expansions of

i). $(1-x)^{-1/2}$

Solution; Take $(1-x)^{-1/2}=[1+(-x)]^{-1/2}$

$$=1+\left(\frac{-1}{2}\right)(-x)+\frac{1}{2!}\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)(-x)^2+\frac{1}{3!}\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)(-x)^3+\cdots$$

$$=1+\frac{x}{2}+\frac{1}{2!}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^2+\frac{-1}{3!}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)x^3+\cdots$$

$$=1+\frac{x}{2}+\frac{3}{8}x^2+\frac{5}{16}x^3+\cdots$$

ii). $(1-x)^{3/2}$

Solution; Take $(1-x)^{3/2}=[1+(-x)]^{3/2}$

$$=1+\left(\frac{3}{2}\right)(-x)+\frac{1}{2!}\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)(-x)^2+\frac{1}{3!}\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)(-x)^3+\cdots$$

$$=1-\frac{3x}{2}+\frac{1}{2!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)x^2-\frac{1}{3!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)x^3+\cdots$$

$$=1-\frac{3x}{2}+\frac{3}{8}x^2+\frac{1}{16}x^3+\cdots$$

iii). $(8+12x)^{2/3}$

Solution; Take

$$(8+12x)^{2/3}=8^{2/3}\left[1+\left(\frac{12x}{8}\right)\right]^{2/3}=2^2\left[1+\left(\frac{3x}{2}\right)\right]^{2/3}$$

$$=4\left\{1+\left(\frac{2}{3}\right)\left(\frac{3x}{2}\right)+\frac{1}{2!}\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{3x}{2}\right)^2\right.$$
$$\left.+\frac{1}{3!}\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\left(\frac{3x}{2}\right)^3+\cdots\right\}$$
$$=4\left\{1+x+\frac{1}{2!}\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\frac{9x^2}{4}\right.$$
$$\left.+\frac{1}{3!}\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\frac{27x^3}{8}+\cdots\right\}$$
$$=4\left\{1+x-\frac{x^2}{4}+\frac{x^3}{6}+\cdots\right\}$$

iv). $(4-8x)^{\frac{-3}{2}}$

Solution; Take $(4-8x)^{\frac{-3}{2}}$

$$(4-8x)^{\frac{-3}{2}}=4^{\frac{-3}{2}}\left[1+\left(\frac{-8x}{4}\right)\right]^{\frac{-3}{2}}$$
$$(4-8x)^{\frac{-3}{2}}=2^{-3}\left[1+(2x)\right]^{\frac{-3}{2}}$$
$$=\frac{1}{2^3}\left\{1+\left(\frac{-3}{2}\right)(-2x)+\frac{1}{2!}\left(\frac{-3}{2}\right)\left(\frac{-3}{2}-1\right)(-2x)^2\right.$$
$$\left.+\frac{1}{3!}\left(\frac{-3}{2}\right)\left(\frac{-3}{2}-1\right)\left(\frac{-3}{2}-2\right)(-2x)^3+\cdots\right\}$$
$$=\frac{1}{8}\left\{1+3x+\frac{1}{2!}\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(4x^2)\right.$$
$$\left.+\frac{1}{3!}\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)(-8x^3)+\cdots\right\}$$
$$=\frac{1}{8}\left\{1+3x+\frac{15x^2}{2}+\frac{35x^3}{2}+\cdots\right\}$$

v). $(1-x)^{-3}$

Solution; we have $(1-x)^{-3}=\left[1+(-x)\right]^{-3}$

$$=1+(-3)(-x)+\frac{1}{2!}(-3)(-3-1)(-x)^2$$
$$+\frac{1}{3!}(-3)(-3-1)(-3-2)(-x)^3+\cdots$$
$$=1+3x+\frac{1}{2!}(-3)(-4)x^2$$
$$-\frac{1}{3!}(-3)(-4)(-5)(-x^3)+\cdots$$
$$=1+3x+6x^2+10x^3+\cdots$$

vi). $\sqrt[3]{1+x}$

Solution; we have $\sqrt[3]{1+x}=[1+x]^{1/3}$

$$=1+\left(\frac{1}{3}\right)(x)+\frac{1}{2!}\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)(x)^2$$
$$+\frac{1}{3!}\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)(x)^3+\cdots$$
$$=1+\frac{x}{3}+\frac{1}{2!}\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)x^2$$
$$-\frac{1}{3!}\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(x^3)+\cdots$$
$$=1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}+\cdots$$

Q2.

i). Find $\sqrt{26}$ correct to 3 decimal places.

Solution; we have $\sqrt{26}$

$$\sqrt{26}=(25+1)^{\frac{1}{2}}$$
$$\sqrt{26}=25^{\frac{1}{2}}\left(1+\frac{1}{25}\right)^{0.5}$$
$$\sqrt{26}=5(1+0.04)^{0.5}$$
$$\sqrt{26}=5\left\{1+(0.5)(0.04)+\frac{(0.5)(0.5-1)}{2!}(0.04)^2+\cdots\right\}$$
$$\sqrt{26}=5\left\{1+0.02+\frac{(0.5)(-0.5)}{2!}(0.0016)+\cdots\right\}$$
$$\sqrt{26}=5\{1+0.02-0.0002+\cdots\}$$
$$\sqrt{26}=5\{1.0198\}$$
$$\sqrt{26}=5.009$$

ii). Find $\frac{1}{\sqrt{0.998}}$ correct to 4 significant figures.

Solution; we have $\frac{1}{\sqrt{0.998}}$

$$\frac{1}{\sqrt{0.998}}=(0.998)^{\frac{-1}{2}}$$
$$\frac{1}{\sqrt{0.998}}=(1-0.002)^{-0.5}$$
$$\frac{1}{\sqrt{0.998}}=\{1+(-0.002)\}^{-0.5}$$
$$\frac{1}{\sqrt{0.998}}=\{1+(-0.5)(-0.002)$$
$$+\frac{(-0.5)(-0.5-1)}{2!}(-0.002)^2+\cdots\}$$
$$\frac{1}{\sqrt{0.998}}=\{1+0.001+0.0000015+\cdots\}$$
$$\frac{1}{\sqrt{0.998}}=1.0010$$

iii). Find cube root of 126 correct to five decimal places.

Solution; we have $\sqrt[3]{126}$

$$\sqrt[3]{126}=(125+1)^{1/3}$$
$$\sqrt[3]{126}=125^{1/3}\left(1+1/125\right)^{1/3}$$
$$\sqrt[3]{126}=5(1+0.008)^{1/3}$$
$$\sqrt[3]{126}=5\left\{1+\left(1/3\right)(0.008)\right.$$
$$\left.+\left(1/3\right)\left(1/3-1\right)(0.008)^2+\cdots\right\}$$
$$\sqrt[3]{126}=5\left\{1+0.00266+\left(1/3\right)\left(-2/3\right)(0.000064)+\cdots\right\}$$
$$\sqrt[3]{126}=5\{1+0.00266666-0.000014222\}$$
$$\sqrt[3]{126}=5\{1.00265\}$$
$$\sqrt[3]{126}=5.01325$$

iv). Evaluate $\sqrt[4]{65}$ to four places f decimals.

Solution; we have $\sqrt[4]{65}$

$$\sqrt[4]{65}=(81-16)^{1/4}$$
$$\sqrt[4]{65}=81^{1/4}\left(1-16/81\right)^{1/4}$$

$$\sqrt[4]{65}=3\left\{1+\left(\frac{-16}{81}\right)\right\}^{1/4}$$

$$\sqrt[4]{65}=3\left\{1+\left(\frac{1}{4}\right)\left(\frac{-16}{81}\right)+\frac{1}{2!}\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)\left(\frac{-16}{81}\right)^2+\frac{1}{3!}\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\left(\frac{-16}{81}\right)^3+\cdots\right.$$

$$\sqrt[4]{65}=3\left\{1-\frac{4}{81}+\frac{1}{2!}\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)\left(\frac{256}{6561}\right)+\frac{1}{3!}\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)\left(\frac{-4096}{531441}\right)+\ldots\right.$$

$$\sqrt[4]{65}=3\left\{1-\frac{4}{81}-\frac{8}{2187}-\frac{224}{531441}+\ldots\right.$$

$$\sqrt[4]{65}=3(1-0.0493827-0.0036579-0.0004214+\ldots)$$

$$\sqrt[4]{65}=3(0.946538)$$

$$\sqrt[4]{65}=2.839614$$

---

Q3. Expand; $\sqrt{\dfrac{1-x}{1+x}}$ up to $x^3$

Solution; $\sqrt{\dfrac{1-x}{1+x}}$

$$\frac{\sqrt{1-x}}{\sqrt{1+x}}=(1-x)^{1/2}(1+x)^{-1/2}$$

$$\sqrt{\frac{1-x}{1+x}}=\left\{1+\left(\tfrac{1}{2}\right)(-x)+\tfrac{1}{2!}\left(\tfrac{1}{2}\right)\left(\tfrac{-1}{2}\right)(-x)^2+\tfrac{1}{3!}\left(\tfrac{1}{2}\right)\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)(-x)^3+\cdots\right.$$
$$\times\left\{1+\left(\tfrac{-1}{2}\right)(x)+\tfrac{1}{2!}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)(x)^2+\tfrac{1}{3!}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)\left(\tfrac{-5}{2}\right)(x)^3+\cdots\right.$$

$$\sqrt{\frac{1-x}{1+x}}=\left\{1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}+\cdots\right\}\times\left\{1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}+\cdots\right\}$$

by multiply and choosing up to $x^3$

$$\sqrt{\frac{1-x}{1+x}}=1\left(1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}\right)-\frac{x}{2}\left(1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}\right)-\frac{x^2}{8}\left(1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}\right)-\frac{x^3}{16}\left(1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}\right)$$

$$\sqrt{\frac{1-x}{1+x}}=1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}-\frac{x}{2}+\frac{x^2}{4}-\frac{3x^3}{16}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{x^3}{16}$$

$$=1-\frac{x}{2}-\frac{x}{2}+\frac{3x^2}{8}+\frac{x^2}{4}-\frac{x^2}{8}-\frac{5x^3}{16}-\frac{3x^3}{16}$$

$$=1-x+\frac{x^2}{2}-\frac{x^3}{2}$$

---

Q4. If x is such that $x^2$ and higher powers may be neglected, then show that $\sqrt{\dfrac{1-3x}{1+4x}}=1-\dfrac{7x}{2}$

Solution; take LHS $\sqrt{\dfrac{1-3x}{1+4x}}$

$$\sqrt{\frac{1-3x}{1+4x}}=\frac{(1-3x)^{1/2}}{(1+4x)^{1/2}}$$

$$\sqrt{\frac{1-3x}{1+4x}}=(1-3x)^{1/2}(1+4x)^{-1/2}$$

Expanding with neglecting $x^2, x^3, x^4, x^5, \ldots$

$$\approx\left\{1+\left(\frac{1}{2}\right)(-3x)+\cdots\right\}\left\{1+\left(\frac{-1}{2}\right)(4x)+\cdots\right\}$$

$$\approx\left(1-\frac{3x}{2}+\cdots\right)(1-2x+\cdots)$$

$$\approx 1-2x-\frac{3x}{2}+\cdots$$

$$\approx 1-\frac{7x}{2}$$

---

Q5. If x is so small that is square and higher powers can be neglected, then show that

$$\frac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}=1-\frac{5}{8}x$$

Solution; Take LHS $\dfrac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}$

$$\frac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}=\frac{8^{\frac{2}{3}}\left(1+\frac{3}{8}x\right)^{\frac{2}{3}}}{2\left(1+\frac{3}{2}x\right).\sqrt{4}\sqrt{1-\frac{5}{4}x}}$$

$$\frac{4\left(1+\frac{3}{8}x\right)^{\frac{2}{3}}}{2\left(1+\frac{3}{2}x\right).2\sqrt{1-\frac{5}{4}x}}=\left(1+\frac{3}{8}x\right)^{\frac{2}{3}}\left(1+\frac{3}{2}x\right)^{-1}\left(1-\frac{5}{4}x\right)^{-1/2}$$

Expanding with neglecting $x^2, x^3, x^4, x^5, \ldots$

$$\approx\left(1+\left(\frac{2}{3}\right)\left(\frac{3x}{8}\right)\right)\left(1+(-1)\left(\frac{3x}{2}\right)\right)\left(1+\left(\frac{-1}{2}\right)\left(\frac{-5x}{4}\right)\right)$$

$$\approx\left(1+\frac{x}{4}\right)\left(1-\frac{3x}{2}\right)\left(1+\frac{5x}{8}\right)$$

$$\approx\left\{1-\frac{3x}{2}+\frac{x}{4}\right\}\left(1+\frac{5x}{8}\right)$$

$$\approx\left(1-\frac{5x}{4}\right)\left(1+\frac{5x}{8}\right)$$

$$\approx 1+\frac{5x}{8}-\frac{5x}{4}$$

$$\approx 1-\frac{5x}{8}$$

---

Q6. If x is large & if $\dfrac{1}{x^3}$ may be neglected, then find approximated with the value of $\dfrac{x\sqrt{x^2-2x}}{(x+1)^2}$

Solution; we have $\dfrac{x\sqrt{x^2-2x}}{(x+1)^2}$

$$\frac{x\sqrt{x^2}\sqrt{1-\frac{2x}{x^2}}}{x^2\left(1+\frac{1}{x}\right)^2}=\frac{x.x}{x^2}\left(1-\tfrac{2}{x}\right)^{1/2}\left(1+\tfrac{1}{x}\right)^{-2}$$

Expanding with neglecting $\frac{1}{x^3},\frac{1}{x^4},\frac{1}{x^5},\ldots$

$$\approx\left(1+\left(\tfrac{1}{2}\right)\left(\tfrac{-2}{x}\right)+\tfrac{1}{2!}\left(\tfrac{1}{2}\right)\left(\tfrac{-1}{2}\right)\left(\tfrac{-2}{x}\right)^2+\cdots\right)$$
$$\left(1+(-2)\left(\tfrac{1}{x}\right)+\tfrac{1}{2!}(-2)(-3)\left(\tfrac{1}{x}\right)^2+\cdots\right)$$

Neglecting $\dfrac{1}{x^3},\dfrac{1}{x^4},\dfrac{1}{x^5},\ldots$

$$\approx\left(1-\frac{1}{x}+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{4}{x^2}\right)+\cdots\right)\left(1-\frac{2}{x}+\frac{3}{x^2}+\cdots\right)$$

$$\approx\left(1-\frac{1}{x}-\frac{1}{2x^2}+\cdots\right)\left(1-\frac{2}{x}+\frac{3}{x^2}+\cdots\right)$$

$$\approx 1-\frac{1}{x}-\frac{1}{2x^2}-\frac{2}{x}+\frac{2}{x^2}+\frac{3}{x^2}$$

$$\approx 1-\frac{1}{x}-\frac{2}{x}-\frac{1}{2x^2}+\frac{2}{x^2}+\frac{3}{x^2}$$

$$\approx 1-\frac{3}{x}+\frac{9}{2x^2}$$

---

Q7. If $x^4$ and higher power are neglected, than show that $(1+x)^{1/4}+(1-x)^{1/4}=a-bx^2$ and find $a$ and $b$

Solution; we have $(1+x)^{1/4}+(1-x)^{1/4}=a-bx^2$

Take LHS $(1+x)^{1/4}+(1-x)^{1/4}$

$$\approx\left\{1+\left(\frac{1}{4}\right)x+\frac{1}{2!}\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)x^2+\frac{1}{3!}\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)x^3\right\}$$

$$+\left\{1+\left(\frac{1}{4}\right)(-x)+\frac{1}{2!}\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)(-x)^2+\frac{1}{3!}\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)(-x)^3\right\}$$

$$\approx 1+\frac{x}{4}-\frac{3x^2}{32}+\frac{7x^3}{128}+1-\frac{x}{4}-\frac{3x^2}{32}-\frac{7x^3}{128}$$

$$\approx 1+1+\frac{x}{4}-\frac{x}{4}-\frac{3x^2}{32}-\frac{3x^2}{32}+\frac{7x^3}{128}-\frac{7x^3}{128}$$

$$\approx 2-\frac{3x^2}{16}$$

Compare with $a+bx^2$ to get $a=2, b=\frac{3}{16}$

---

Q8. If x is of such a size that its values are considered up to $x^3$. Show that

$$\frac{\left(1+\frac{1}{2}x\right)^3-(1+3x)^{\frac{1}{2}}}{1-\frac{5}{6}x}=\frac{15x^2}{8}$$

Solution; we have $\dfrac{\left(1+\frac{1}{2}x\right)^3-(1+3x)^{\frac{1}{2}}}{1-\frac{5}{6}x}=\dfrac{15x^2}{8}$

Take LHS $\dfrac{\left(1+\frac{1}{2}x\right)^3-(1+3x)^{\frac{1}{2}}}{1-\frac{5}{6}x}$

$$=\left\{\left(1+\frac{1}{2}x\right)^3-(1+3x)^{\frac{1}{2}}\right\}\left(1-\frac{5}{6}x\right)^{-1}$$

Expanding with neglecting $x^4, x^5, \ldots$

$$=\left\{\left(1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}\right)-\left(1+\left(\frac{1}{2}\right)(3x)\right.\right.$$

$$\left.\left.+\frac{1}{2!}\frac{1}{2}\frac{-1}{2}(3x)^2+\frac{1}{3!}\frac{1}{2}\frac{-1}{2}\frac{-3}{2}(3x)^3\right)\right\}\left(1-\frac{5}{6}x\right)^{-1}$$

$$=\left\{1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}-1-\frac{3x}{2}+\frac{3x^2}{8}-\frac{27x^3}{16}\right\}$$

$$\left(1+(-1)\left(\frac{-5x}{6}\right)+\frac{(-1)(-2)}{2!}\left(\frac{-5x}{6}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{-5x}{6}\right)^3\right)$$

$$=\left\{\frac{3x^2}{4}-\frac{3x^2}{8}+\frac{x^3}{8}-\frac{27x^3}{16}\right\}\left(1+\frac{5x}{6}+\frac{25x^2}{36}+\frac{125x^3}{216}\right)$$

Multiplying with neglecting $x^4, x^5, \ldots$

$$=\left\{\frac{15x^2}{8}-\frac{25x^3}{16}\right\}\left(1+\frac{5x}{6}+\frac{25x^2}{36}+\frac{125x^3}{216}\right)$$

$$=\frac{15x^2}{8}+\frac{75x^3}{48}-\frac{25x^3}{16}=\frac{15x^2}{8}+\frac{25x^3}{16}-\frac{25x^3}{16}$$

$$=\frac{15x^2}{8}=RHS \text{ Hence proved.}$$

---

Q9. Find the coefficients of $x^n$ in $\left(\dfrac{1+x}{1-x}\right)^2$

Solution; we have $\left(\dfrac{1+x}{1-x}\right)^2$

$$\left(\frac{1+x}{1-x}\right)^2=\frac{(1+x)^2}{(1-x)^2}$$

$$\left(\frac{1+x}{1-x}\right)^2=(1+x)^2(1-x)^{-2}$$

$$=\left(1+2x+x^2\right)\left\{1+(-2)(-x)+\frac{(-2)(-2-1)}{2!}(-x)^2+\cdots\right.$$

$$=\left(1+2x+x^2\right)\left(1+2x+3x^2+\cdots\right)$$

Continuing same pattern up to nth power

$$=\left(1+2x+x^2\right)\left(1+2x+3x^2+4x^3+5x^4+\cdots\right.$$

$$\left.+(n-1)x^{n-2}+nx^{n-1}+(n+1)x^n+\cdots\right)$$

Multiply & taking only those terms containing powers of $x^n$

$$(n+1)x^n+2nx^n+(n-1)x^n$$

$$=(n+1+2n+n-1)x^n$$

$$=4nx^n$$

Hence the coefficient of $x^n$ is $4n$

---

Q10. Find the coefficients of $x^{3n}$ in $\dfrac{\left(1+x^2\right)^3}{\left(1-x^3\right)^2}$

Solution; we have $\dfrac{\left(1+x^2\right)^3}{\left(1-x^3\right)^2}=\left(1+x^2\right)^3\left(1-x^3\right)^{-2}$

$$=\left(1+3x^2+3x^4+x^6\right)\left\{1+(-2)\left(-x^3\right)+\frac{(-2)(-2-1)}{2!}\left(-x^3\right)^2+\cdots\right.$$

$$=\left(1+3x^2+3x^4+x^6\right)\left(1+2x^3+3x^6+\cdots\right)$$

Continuing same pattern up to nth term

$$=\left(1+3x^2+3x^4+x^6\right)\left(1+2x^3+3x^6+4x^9+\cdots\right.$$

$$\left.+(n-2)x^{3(n-3)}+(n-1)x^{3(n-2)}+nx^{3(n-1)}+(n+1)x^{3n}\right.$$

Multiply and taking only those terms containing powers of $x^{3n}$

$$(n+1)x^{3n}+(n-1)x^{3n-6}.x^6$$

$$=(n+1+n-1)x^{3n}$$

$$=2nx^{3n}$$

Hence the coefficient of $x^{3n}$ is $2n$

---

Q11. Identify following series and find the sum of each.

i). $1-\dfrac{1}{2^2}+\dfrac{1.3}{2!}.\dfrac{1}{2^4}+\cdots$

Solution; we have $1-\dfrac{1}{2^2}+\dfrac{1.3}{2!}.\dfrac{1}{2^4}+\cdots$

Let $(1+x)^n = 1-\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$

$\therefore (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$

Compare the two series term by term

First terms i.e., 1 = 1 true

Second terms

$$nx = -\frac{1}{2^2}$$

$$\Rightarrow x = -\frac{1}{n.2^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Third terms

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!}.\frac{1}{2^4}$$

Putting the value of x

$$\frac{n(n-1)}{2!}\left(\frac{-1}{n.2^2}\right)^2 = \frac{1.3}{2!}.\frac{1}{2^4}$$

$$\Rightarrow \frac{n(n-1)}{2!}\frac{1}{n.n.2^4} = \frac{1.3}{2!}.\frac{1}{2^4}$$

$$\Rightarrow \frac{(n-1)}{n} = 1.3$$

$$\Rightarrow n-1 = 3n$$

$$\Rightarrow -1 = 3n-n$$

$$\Rightarrow 2n = -1$$

$$\Rightarrow n = \tfrac{-1}{2}$$

Putting the value of n in equation (1)

$$x = -\frac{1}{\left(\frac{-1}{2}\right)2^2}$$

$$\Rightarrow x = \frac{1}{2}$$

Putting the values of x and n in $(1+x)^n$

$$(1+x)^n = \left(1+\frac{1}{2}\right)^{\frac{-1}{2}}$$

$$(1+x)^n = \left(\frac{3}{2}\right)^{\frac{-1}{2}}$$

$$(1+x)^n = \left(\frac{2}{3}\right)^{\frac{1}{2}}$$

$$\Rightarrow (1+x)^n = \sqrt{\frac{2}{3}}$$

---

ii). $1+\frac{1}{3}+\frac{1}{2!}.\frac{1}{3}+\frac{1.5}{3!}.\frac{1}{3^2}+\cdots$

Solution; we have $1+\frac{1}{3}+\frac{1}{2!}.\frac{1}{3}+\frac{1.5}{3!}.\frac{1}{3^2}+\cdots$

Let $(1+x)^n = 1+\frac{1}{3}+\frac{1}{2!}.\frac{1}{3}+\frac{1.5}{3!}.\frac{1}{3^2}+\cdots$

$\therefore (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$

Compare the two series term by term

First terms i.e., 1 = 1 true

Second terms

$$nx = \frac{1}{3}$$

$$\Rightarrow x = \frac{1}{3n}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Third terms

$$\frac{n(n-1)}{2!}x^2 = \frac{1}{2!}.\frac{1}{3}$$

Putting the value of x

$$\frac{n(n-1)}{2!}\left(\frac{1}{3n}\right)^2 = \frac{1}{2!}.\frac{1}{3}$$

$$\Rightarrow \frac{n(n-1)}{2!}\frac{1}{9n.n} = \frac{1}{2!}.\frac{1}{3}$$

$$\Rightarrow \frac{(n-1)}{3n} = 1 \Rightarrow n-1 = 3n$$

$$\Rightarrow -1 = 3n-n \Rightarrow 2n = -1$$

$$\Rightarrow n = \tfrac{-1}{2}$$

Putting the value of n in equation (1)

$$x = \frac{1}{3.\left(\frac{-1}{2}\right)}$$

$$\Rightarrow x = -\frac{2}{3}$$

Putting the values of x and n in $(1+x)^n$

$$(1+x)^n = \left(1-\frac{2}{3}\right)^{\frac{-1}{2}}$$

$$(1+x)^n = \left(\frac{1}{3}\right)^{\frac{-1}{2}}$$

$$(1+x)^n = (3)^{\frac{1}{2}}$$

$$\Rightarrow (1+x)^n = \sqrt{3}$$

---

iii). $1+\frac{2}{9}+\frac{2.5}{9.18}+\frac{2.5.8}{9.18.27}+\cdots$

Solution; we have $1+\frac{2}{9}+\frac{2.5}{9.18}+\frac{2.5.8}{9.18.27}+\cdots$

Let $(1+x)^n = 1+\frac{2}{9}+\frac{2.5}{9.18}+\frac{2.5.8}{9.18.27}+\cdots$

$\therefore (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$

Compare the two series term by term

First terms i.e., 1 = 1 true

Second terms

$$nx = \frac{2}{9}$$

$$\Rightarrow x = \frac{2}{9n}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Third terms

$$\frac{n(n-1)}{2!}x^2 = \frac{2.5}{9.18}$$

Putting the value of x

$$\frac{n(n-1)}{2!}\left(\frac{2}{9n}\right)^2 = \frac{2.5}{9.18}$$

$$\Rightarrow \frac{n(n-1)}{2!}\frac{4}{9\times 9n.n} = \frac{2.5}{9.18}$$

$\Rightarrow \dfrac{2(n-1)}{n}=5$

$\Rightarrow 2n-2=5n$

$\Rightarrow -2=5n-2n$

$\Rightarrow 3n=-2$

$\Rightarrow n=\frac{-2}{3}$

Putting the value of n in equation (1)

$x=\dfrac{2}{9.\left(\frac{-2}{3}\right)}$

$\Rightarrow x=-\frac{1}{3}$

Putting the values of x and n in $(1+x)^n$

$(1+x)^n=\left(1-\frac{1}{3}\right)^{\frac{-2}{3}}$

$(1+x)^n=\left(\frac{2}{3}\right)^{\frac{-2}{3}}$

$(1+x)^n=\left(\frac{3}{2}\right)^{\frac{2}{3}}$

$\Rightarrow (1+x)^n=\sqrt[3]{\frac{9}{4}}$

---

iv). $1+\dfrac{5}{8}+\dfrac{5.8}{8.12}+\dfrac{5.8.11}{8.12.16}+\cdots$

solution; Let $(1+x)^n=1+\dfrac{5}{8}+\dfrac{5.8}{8.12}+\dfrac{5.8.11}{8.12.16}+\cdots$

$\therefore (1+x)^n=1+nx+\dfrac{n(n-1)}{2!}x^2+\cdots$

Compare the two series term by term

First terms i.e., 1 = 1 true

Second terms

$nx=\dfrac{5}{8}$

$\Rightarrow x=\dfrac{5}{8n}$..............................(1)

Third terms

$\dfrac{n(n-1)}{2!}x^2=\dfrac{5.8}{8.12}$

Putting the value of x

$\dfrac{n(n-1)}{2!}\left(\dfrac{5}{8n}\right)^2=\dfrac{5.8}{8.12}$

$\Rightarrow \dfrac{n(n-1)}{2!}\dfrac{25}{64n.n}=\dfrac{5.8}{8.12}$

$\Rightarrow \dfrac{(n-1)}{2}\dfrac{25}{64n}=\dfrac{5}{12}$

$\Rightarrow 12(25)(n-1)=5\times 2\times 64n$

$\Rightarrow 300(n-1)=640n$

$\Rightarrow 300n-300=640n$

$\Rightarrow -300=640n-300n$

$\Rightarrow 340n=-300$

$\Rightarrow n=\dfrac{-300}{340}$

$n=\dfrac{-15}{17}$

Putting the value of n in equation (1)

$x=\dfrac{5}{8.\left(\frac{-15}{17}\right)}$

$x=-\frac{17}{8\times 3}$

$x=-\frac{17}{24}$

Putting the values of x and n in $(1+x)^n$

$(1+x)^n=\left(1-\frac{17}{24}\right)^{\frac{-15}{17}}$

$(1+x)^n=\left(\frac{7}{24}\right)^{\frac{-15}{17}}$

$\Rightarrow (1+x)^n=\left(\frac{24}{7}\right)^{\frac{15}{17}}$

---

Q12. Find the sum to infinity of the following series.

i). $1+2.\dfrac{1}{3}+3.\dfrac{1}{3^2}+4.\dfrac{1}{3^3}+\cdots$

solution; Let $(1+x)^n=1+2.\dfrac{1}{3}+3.\dfrac{1}{3^2}+4.\dfrac{1}{3^3}+\cdots$

$\therefore (1+x)^n=1+nx+\dfrac{n(n-1)}{2!}x^2+\cdots$

Compare the two series term by term

First terms i.e., 1 = 1 true

Second terms

$nx=\frac{2}{3}$

$\Rightarrow x=\frac{2}{3n}$..........................(1)

Third terms

$\dfrac{n(n-1)}{2!}x^2=\dfrac{3}{3^2}$

Putting the value of x

$\dfrac{n(n-1)}{2!}\left(\dfrac{2}{3n}\right)^2=\dfrac{1}{3}$

$\Rightarrow \dfrac{n(n-1)}{2!}\dfrac{4}{9n.n}=\dfrac{1}{3}$

$\Rightarrow \dfrac{2(n-1)}{3n}=1$

$\Rightarrow 2(n-1)=3n$

$\Rightarrow 2n-2=3n$

$\Rightarrow -2=3n-2n$

$\Rightarrow n=-2$

Putting the value of n in equation (1)

$x=\dfrac{2}{3.(-2)}$

$\Rightarrow x=-\frac{1}{3}$

Putting the values of x and n in $(1+x)^n$

$(1+x)^n=\left(1-\dfrac{1}{3}\right)^{-2}$

$(1+x)^n=\left(\dfrac{2}{3}\right)^{-2}$

$(1+x)^n=\left(\dfrac{3}{2}\right)^{2}$

$\Rightarrow (1+x)^n=\dfrac{9}{4}$

---

ii). $1+\frac{1}{6}+\frac{1.3}{1.2}.\frac{1}{6^2}+\frac{1.3.5}{1.2.3}.\frac{1}{6^3}+\cdots$

Solution; we have $1+\frac{1}{6}+\frac{1.3}{1.2}.\frac{1}{6^2}+\frac{1.3.5}{1.2.3}.\frac{1}{6^3}+\cdots$

Let $(1+x)^n=1+\frac{1}{6}+\frac{1.3}{1.2}.\frac{1}{6^2}+\frac{1.3.5}{1.2.3}.\frac{1}{6^3}+\cdots$

$\therefore (1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots$

Compare the two series term by term

First terms i.e., 1 = 1 true

Second terms

$nx=\frac{1}{6}$

$\Rightarrow x=\frac{1}{6n}$..............................(1)

Third terms

$\frac{n(n-1)}{2!}x^2=\frac{1.3}{1.2}.\frac{1}{6^2}$

Putting the value of x

$\frac{n(n-1)}{2!}\left(\frac{1}{6n}\right)^2=\frac{1.3}{1.2}.\frac{1}{6^2}$

$\Rightarrow \frac{n(n-1)}{2!}\frac{1}{36n.n}=\frac{1.3}{1.2}.\frac{1}{6^2}$

$\Rightarrow \frac{(n-1)}{n}=3$

$n-1=3n$

$-1=3n-n$

$\Rightarrow 2n=-1$

$\Rightarrow n=\frac{-1}{2}$

Putting the value of n in equation (1)

$x=\frac{1}{6.\left(\frac{-1}{2}\right)}$

$\Rightarrow x=-\frac{1}{3}$

Putting the values of x and n in $(1+x)^n$

$(1+x)^n=\left(1-\frac{1}{3}\right)^{\frac{-1}{2}}$

$(1+x)^n=\left(\frac{2}{3}\right)^{\frac{-1}{2}}$

$(1+x)^n=\left(\frac{3}{2}\right)^{\frac{1}{2}}$

$\Rightarrow (1+x)^n=\sqrt{\frac{3}{2}}$

---

Q13. If $y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$ then $y^2+2y-1=0$

Solution; we have $y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$

Adding both sides by 1

$y+1=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$............................(A)

Let $(1+x)^n=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$

$\therefore (1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots$

Compare two series term by term

First terms i.e., 1 = 1 true

Second terms $nx=\frac{1}{2^2}$

$\Rightarrow x=\frac{1}{n.2^2}$....(1)

Third terms

$\frac{n(n-1)}{2!}x^2=\frac{1.3}{2!}.\frac{1}{2^4}$

Putting the value of x

$\frac{n(n-1)}{2!}\left(\frac{-1}{n.2^2}\right)^2=\frac{1.3}{2!}.\frac{1}{2^4}$

$\Rightarrow \frac{n(n-1)}{2!}\frac{1}{n.n.2^4}=\frac{1.3}{2!}.\frac{1}{2^4}$

$\Rightarrow \frac{(n-1)}{n}=1.3$

$\Rightarrow n-1=3n$

$\Rightarrow -1=3n-n$

$\Rightarrow 2n=-1$

$\Rightarrow n=\frac{-1}{2}$

Putting the value of n in equation (1)

$x=\frac{1}{\left(\frac{-1}{2}\right)2^2}$

$\Rightarrow x=\frac{-1}{2}$

Putting the values of x and n in $(1+x)^n$

$(1+x)^n=\left(1-\frac{1}{2}\right)^{\frac{-1}{2}}$

$(1+x)^n=\left(\frac{1}{2}\right)^{\frac{-1}{2}}$

$(1+x)^n=(2)^{\frac{1}{2}}$

$\Rightarrow (1+x)^n=\sqrt{2}$

So equation (A) becomes $y+1=\sqrt{2}$

Squaring $(y+1)^2=\left(\sqrt{2}\right)^2$

$y^2+2y+1=2$

$y^2+2y-1=0$

---

Q14. If $2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$ then

$4y^2+4y-1=0$

Solution; we have $2y=\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$

Adding both sides by 1

$2y+1=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$..................................(A)

Let $(1+x)^n=1+\frac{1}{2^2}+\frac{1.3}{2!}.\frac{1}{2^4}+\cdots$

$$\therefore (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$$

Compare the two series term by term

First terms i.e., 1 = 1 true

Second terms $nx = \frac{1}{2^2}$

$$\Rightarrow x = \frac{1}{n.2^2} \text{.........................}(1)$$

Third terms $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!}.\frac{1}{2^4}$

Putting the value of x

$$\frac{n(n-1)}{2!}\left(\frac{-1}{n.2^2}\right)^2 = \frac{1.3}{2!}.\frac{1}{2^4}$$

$$\Rightarrow \frac{n(n-1)}{2!}\frac{1}{n.n.2^4} = \frac{1.3}{2!}.\frac{1}{2^4}$$

$$\Rightarrow \frac{(n-1)}{n} = 1.3$$

$$\Rightarrow n-1 = 3n$$

$$\Rightarrow -1 = 3n-n$$

$$\Rightarrow 2n = -1$$

$$\Rightarrow n = \tfrac{-1}{2}$$

Putting the value of n in equation (1)

$$x = \frac{1}{\left(\frac{-1}{2}\right)2^2}$$

$$\Rightarrow x = \frac{-1}{2}$$

Putting the values of x and n in $(1+x)^n$

$$(1+x)^n = \left(1-\frac{1}{2}\right)^{\frac{-1}{2}}$$

$$(1+x)^n = \left(\frac{1}{2}\right)^{-1/2}$$

$$(1+x)^n = (2)^{1/2}$$

$$\Rightarrow (1+x)^n = \sqrt{2}$$

So equation (A) becomes $2y+1=\sqrt{2}$

Squaring $(2y+1)^2 = \left(\sqrt{2}\right)^2$

$$4y^2+4y+1 = 2$$

$$4y^2+4y-1 = 0$$

Q15. If x is so small that $x^3$ and higher powers of x can be ignored. Show that the nth root of $(1+x)$ is equal to $\frac{2n+(n+1)x}{2n+(n-1)x}$

Solution; we have to show that

$$\frac{2n+(n+1)x}{2n+(n-1)x} = \sqrt[n]{1+x}$$

Take LHS $\frac{2n+(n+1)x}{2n+(n-1)x}$

$$= \frac{2n\left\{1+\frac{(n+1)}{2n}x\right\}}{2n\left\{1+\frac{(n-1)}{2n}x\right\}}$$

$$= \left\{1+\frac{(n+1)}{2n}x\right\}\left\{1+\frac{(n-1)}{2n}x\right\}^{-1}$$

Expanding and neglecting $x^3$ and higher powers

$$= \left\{1+\frac{(n+1)}{2n}x\right\}\left\{1-\frac{(n-1)}{2n}x+\frac{(n-1)^2}{4n^2}x^2+\cdots\right\}$$

Multiplying and neglecting $x^3$ and higher powers

$$= 1-\frac{(n-1)}{2n}x+\frac{(n-1)^2}{4n^2}x^2+\frac{(n+1)}{2n}x-\frac{(n-1)(n+1)}{4n^2}x^2$$

$$= 1+\frac{(n+1)}{2n}x-\frac{(n-1)}{2n}x+\frac{(n-1)^2}{4n^2}x^2-\frac{(n-1)(n+1)}{4n^2}x^2$$

$$= 1+\{(n+1)-(n-1)\}\frac{x}{2n}+\{(n-1)-(n+1)\}\frac{(n-1)x^2}{4n^2}$$

$$= 1+\{n+1-n+1\}\frac{x}{2n}+\{n-1-n-1\}\frac{(n-1)x^2}{4n^2}$$

$$= 1+\frac{x}{n}-\frac{(n-1)x^2}{2n^2}$$

$$= 1+\frac{1}{n}x-\frac{1}{2!}\frac{1}{n}\frac{(n-1)}{n}x^2$$

$$= 1+\frac{1}{n}x+\frac{1}{2!}\frac{1}{n}\left(\frac{1}{n}-1\right)x^2$$

$$= (1+x)^{\frac{1}{n}}$$

$$= \sqrt[n]{1+x}$$

=RHS Hence proved.

Q16. If x is nearly equal to unity then show that $px^p-qx^q = (p-q)x^{p+q}$

Solution; Given that $x \approx 1 \Rightarrow x = 1+h$, where h is so small such that $h^2, h^3, h^4, \ldots$ are neglected

Take LHS $px^p-qx^q = p(1+h)^p-q(1+h)^q$

Expanding and neglecting $h^2, h^3, h^4, \ldots$

$$px^p-qx^q = p(1+ph+\cdots)-q(1+qh+\cdots)$$

$$= p+p^2h-q-q^2h$$

$$= p-q+\left(p^2-q^2\right)h$$

$$= (p-q)+(p-q)(p+q)h$$

$$= (p-q)\{1+(p+q)h\}$$

$$= (p-q)\{1+h\}^{p+q}$$

$$= (p-q)x^{p+q}$$

$$= RHS$$

Hence proved.