Chapter 5

Miscellaneous Series

$a_n = a_1 + (n-1)d$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \sum_{k=1}^{n} k^0 = n$

$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \because S_n = \frac{n}{2}(a_1 + a_n)$

Exercise 5.1

Sum the following series up to n terms

Q1. $1^2 + 3^2 + 5^2 + 7^2 + \cdots$

Sol: Given $1^2 + 3^2 + 5^2 + 7^2 + \cdots$

Consider the series 1,3,5,7,...

Here $a_1 = 1, d = 2$

$a_n = a_1 + (n-1)d$

$a_n = 1 + 2(n-1)$

$a_n = 1 + 2n - 2$

$a_n = 2n - 1$

So $\sum_{k=1}^{n} T_k = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$

$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2n-1)^2$

$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 4k^2 - 4k + 1$

$\sum_{k=1}^{n} T_k = 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + 1\sum_{k=1}^{n} k^0$

$\sum_{k=1}^{n} T_k = 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n$

$\sum_{k=1}^{n} T_k = n\left(\frac{2(n+1)(2n+1)}{3} - \frac{2(n+1)}{1}\times\frac{3}{3} + \frac{1}{1}\times\frac{3}{3}\right)$

$\sum_{k=1}^{n} T_k = \frac{n}{3}\left(2(2n^2+3n+1) - 6(n+1) + 3\right)$

$\sum_{k=1}^{n} T_k = \frac{n}{3}\left(4n^2 + 6n + 2 - 6n - 6 + 3\right)$

$\sum_{k=1}^{n} T_k = \frac{n}{3}\left(4n^2 - 1\right)$

Q2. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots$

Sol: Given $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots$

Consider the nth term 1,2,3,4,... with $a_1 = 1, d = 1$

$a_n = 1^2 + 2^2 + 3^2 + \cdots n^2$

$a_n = \frac{n(n+1)(2n+1)}{6}$

$a_n = \frac{n(2n^2 + 2n + n + 1)}{6}$

$a_n = \frac{2n^3 + 3n^2 + n}{6}$, So

$\sum_{k=1}^{n} T_k = 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots + (1^2 + 2^2 + 3^2 + \cdots + n^2)$

$\sum_{k=1}^{n} T_k = \frac{1}{6}\sum_{k=1}^{n}(2k^3 + 3k^2 + k)$

$\sum_{k=1}^{n} T_k = \frac{1}{6}\left[2\sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right]$

$\sum_{k=1}^{n} T_k = \frac{1}{6}\left[2\left(\frac{n(n+1)}{2}\right)^2 + 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]$

$\sum_{k=1}^{n} T_k = \frac{n(n+1)}{6}\left[\frac{n(n+1)}{2} + \frac{2n+1}{2} + \frac{1}{2}\right]$

$\sum_{k=1}^{n} T_k = \frac{1}{12}n(n+1)\left[n(n+1) + 2n + 1 + 1\right]$

$\sum_{k=1}^{n} T_k = \frac{1}{12}n(n+1)\left[n(n+1) + 2(n+1)\right]$

$\sum_{k=1}^{n} T_k = \frac{1}{12}n(n+1)^2(n+2)$

Q3. $2^2 + 4^2 + 6^2 + \cdots$

Sol: Given $2^2 + 4^2 + 6^2 + \cdots$

Consider the series 2,4,6,...

Here $a_1 = 2, d = 2$

$a_n = a_1 + (n-1)d$

$a_n = 2 + 2(n-1)$

$a_n = 2 + 2n - 2$

$a_n = 2n$

So $\sum_{k=1}^{n} T_k = 2^2 + 4^2 + 6^2 + \cdots + (2n)^2$

$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k)^2$

$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 4k^2$

$\sum_{k=1}^{n} T_k = 4\sum_{k=1}^{n} k^2$

$\sum_{k=1}^{n} T_k = 4\frac{n(n+1)(2n+1)}{6}$

$\sum_{k=1}^{n} T_k = \frac{2n(n+1)(2n+1)}{3}$

Q4. $1^3 + 3^3 + 5^3 + 7^3 + \cdots$

Sol: Given $1^3 + 3^3 + 5^3 + 7^3 + \cdots$

Consider the series 1,3,5,7,...

Here $a_1 = 1, d = 2$

$a_n = a_1 + (n-1)d$

$a_n = 1 + 2(n-1)$

$a_n = 1 + 2n - 2$

$a_n = 2n - 1$

So $\sum_{k=1}^{n} T_k = 1^3+3^3+5^3+7^3+\cdots+(2n-1)^3$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(2k-1)^3 \qquad (a-b)^3 = a^3-b^3-3a^2b+3ab^2$$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(8k^3-1-12k^2+6k\right)$$

$$\sum_{k=1}^{n} T_k = 8\sum_{k=1}^{n}k^3-\sum_{k=1}^{n} \quad -12\sum_{k=1}^{n}k^2+6\sum_{k=1}^{n}k$$

$$\sum_{k=1}^{n} T_k = 8\left(\frac{n(n+1)}{2}\right)^2-n-12\frac{n(n+1)(2n+1)}{6}+6\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} T_k = 2n^2(n+1)^2-n-2n(n+1)(2n+1)+3n(n+1)$$

$$\sum_{k=1}^{n} T_k = n\left(2n\left(n^2+2n+1\right)-1-2\left(2n^2+3n+1\right)+3(n+1)\right)$$

$$\sum_{k=1}^{n} T_k = n\left(2n^3+4n^2+2n-1-4n^2-6n-2+3n+3\right)$$

$$\sum_{k=1}^{n} T_k = n\left(2n^3-n\right)$$

$$\sum_{k=1}^{n} T_k = n^2\left(2n^2-1\right)$$

Q5. $1^3+5^3+9^3+\cdots$

Sol: Given $1^3+5^3+9^3+\cdots$ Consider series 1,5,9,...

Here $a_1=1, d=4$

$$a_n = a_1+(n-1)d$$

$$a_n = 1+4(n-1)$$

$$a_n = 1+4n-4$$

$$a_n = 4n-3$$

So $\sum_{k=1}^{n} T_k = 1^3+5^3+9^3+\cdots+(4n-3)^3$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(4k-3)^3$$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(64k^3-27-144k^2+108k\right)$$

$$\sum_{k=1}^{n} T_k = 64\sum_{k=1}^{n}k^3-27\sum_{k=1}^{n} \quad -144\sum_{k=1}^{n}k^2+108\sum_{k=1}^{n}k$$

$$\sum_{k=1}^{n} T_k = 64\left(\frac{n(n+1)}{2}\right)^2-27n-144\frac{n(n+1)(2n+1)}{6}+108\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} T_k = 16n^2(n+1)^2-27n-24n(n+1)(2n+1)+54n(n+1)$$

$$\sum_{k=1}^{n} T_k = n\left\{16n\left(n^2+2n+1\right)-27-24\left(2n^2+3n+1\right)+54(n+1)\right\}$$

$$\sum_{k=1}^{n} T_k = n\left(16n^3+32n^2+16n-27-48n^2-72n-24+54n+54\right)$$

$$\sum_{k=1}^{n} T_k = n\left(16n^3-16n^2-2n+3\right)$$

Q6. $1.2+2.3+3.4+\cdots+99.100$

Sol; $1.2+2.3+3.4+\cdots+99.100$ In given series

First factor of each term	2nd factor of each term
1,2,...99	2,3,...,100
Here $a_1=1, d=1$	Here $b_1=2, d=1$
$a_n=a_1+(n-1)d$	$b_n=a_1+(n-1)d$
$a_n=1+1(n-1)$	$b_n=2+n-1$
$a_n=n$	$b_n=n+1$

So $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(k^2+k\right)$$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)$$

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(2n+4)}{2\times 3}$$

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{3}$$

Since Number of terms are n=99

$$\sum_{k=1}^{99} T_k = \frac{99(100)(101)}{3}$$

$$\sum_{k=1}^{99} T_k = 33\times 100\times 101$$

$$\sum_{k=1}^{99} T_k = 333300$$

Q7. $1^2+3^2+5^2+\cdots+99^2$

Sol: Given $1^2+3^2+5^2+\cdots+99^2$

Consider the series 1,3,5,...

Here $a_1=1, d=2$

$$a_n = a_1+(n-1)d$$

$$a_n = 1+2(n-1)$$

$$a_n = 2n-1$$

So $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(2k-1)^2$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(4k^2-4k+1\right)$$

$$\sum_{k=1}^{n} T_k = 4\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$\sum_{k=1}^{n} T_k = 4\frac{n(n+1)(2n+1)}{6}-4\frac{n(n+1)}{2}+n$$

$$\sum_{k=1}^{n} T_k = n\left(\frac{2\left(2n^2+3n+1\right)}{3}-2(n+1)+1\right)$$

$$\sum_{k=1}^{n} T_k = n\left(\frac{4n^2+6n+2-6(n+1)+3}{3}\right)$$

$$\sum_{k=1}^{n} T_k = \frac{n}{3}\left(4n^2-1\right)$$

Since Number of terms are

$$a_n = a_1 + (n-1)d$$
$$99 = 1 + 2(n-1)$$
$$99 = 2n - 1$$
$$\Rightarrow 2n = 99 + 1$$
$$n = 50$$
$$\sum_{k=1}^{50} T_k = \frac{50}{3}\left(4.50^2 - 1\right)$$
$$\sum_{k=1}^{50} T_k = 166650$$

Book answer is wrong

Q8. $2 + (2+5) + (2+5+8) + \cdots + n$ terms

Sol: Given $2 + (2+5) + (2+5+8) + \cdots + A_n$

Consider the nth term $A_n = 2+5+8+\ldots a_n$

Here $a_1 = 2, d = 3$

$$a_n = a_1 + (n-1)d$$
$$a_n = 2 + 3(n-1)$$
$$a_n = 3n - 1$$

So nth term is $2+5+8+\ldots+3n-1$

$$A_n = s_n = \frac{n}{2}(2 + 3n - 1) \qquad \because S_n = \frac{n}{2}(a_1 + a_n)$$
$$A_n = \frac{n}{2}(3n+1)$$
$$A_n = s_n = \frac{1}{2}\left(3n^2 + n\right)$$

So $\sum_{k=1}^{n} T_k = \frac{1}{2}\sum_{k=1}^{n}\left(3k^2 + k\right)$

$$\sum_{k=1}^{n} T_k = \frac{1}{2}\left(3\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right)$$
$$\sum_{k=1}^{n} T_k = \frac{1}{2}\left(\frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$
$$\sum_{k=1}^{n} T_k = \frac{n(n+1)}{2\times 2}(2n+1+1)$$
$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(2n+2)}{2\times 2}$$
$$\sum_{k=1}^{n} T_k = \frac{n(n+1)^2}{2}$$

Q9. Sum 2 + 5 + 10 + 17 + … + n terms

Sol: Given 2 + 5 + 10 + 17 + … + n terms

Separate by subtracting and adding 1 from each term

$2 + 5 + 10 + 17 + \cdots + \text{n terms}$

$$= \overbrace{1+1+1+\cdots+1}^{\text{n terms}} + \overbrace{1+4+9+16+\cdots+A_n}^{\text{n terms}}$$
$$= n + \overbrace{1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2}$$
$$= n + \frac{n(n+1)(2n+1)}{6}$$
$$= \tfrac{n}{6}\left\{6 + (n+1)(2n+1)\right\}$$
$$= \tfrac{n}{6}\left\{6 + 2n^2 + n + 2n + 1\right\}$$
$$= \tfrac{n}{6}\left(2n^2 + 3n + 7\right)$$

Q10. Sum 1.3.5 + 2.4.6 + 3.5.7 + … n terms

Sol: Given 1.3.5 + 2.4.6 + 3.5.7 + … n terms

Separate the series and find the nth term

$$a_n = a_1 + (n-1)d$$

1,2,3,… d=1	3,4,5,… d=1	5,6,7,… d=1
$a_n = a_1 + (n-1)d$	$b_n = b_1 + (n-1)d$	$c_n = c_1 + (n-1)d$
$a_n = 1 + (n-1).1$	$b_n = 3 + (n-1).1$	$c_n = 5 + (n-1).1$
$a_n = n$	$b_n = n+2$	$c_n = n+4$

So $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} a_k.b_k.c_k$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+2)(k+4)$$
$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k\left(k^2 + 6k + 8\right)$$
$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + 6k^2 + 8k$$
$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 8\sum_{k=1}^{n} k$$
$$S_n = \left(\frac{n(n+1)}{2}\right)^2 + 6\frac{n(n+1)(2n+1)}{6} + 8\frac{n(n+1)}{2}$$
$$S_n = n(n+1)\left(\frac{n(n+1)}{4} + 2n + 1 + 4\right)$$
$$S_n = n(n+1)\left(\frac{n^2 + n + 8n + 20}{4}\right)$$
$$S_n = \frac{n(n+1)}{4}\left(n^2 + 9n + 20\right)$$
$$S_n = \tfrac{1}{4}n(n+1)(n+4)(n+5)$$

Q11. Sum $1.3.5 + 3.5.7 + 5.7.9 + \cdots$ to n terms.

Sol: We have $1.3.5 + 3.5.7 + 5.7.9 + \cdots$ to n terms.

First factor	Second factor
1,3,5,… $d_1 = 2$	3,5,7,… $d_2 = 2$
$a_n = 1 + 2.(n-1)$	$b_n = 3 + 2.(n-1)$
$a_n = 2n - 1$	$b_n = 2n + 1$

Third factor

5,7,9,… $d_3 = 2$

$$c_n = 5 + 2.(n-1)$$
$$c_n = 2n + 3$$

So $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} a_k.b_k.c_k$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(2k-1)(2k+1)(2k+3)$$
$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(4k^2 - 1\right)(2k+3)$$
$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 8k^3 + 12k^2 - 2k - 3$$
$$\sum_{k=1}^{n} T_k = 8\sum_{k=1}^{n} k^3 + 12\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k - 3\sum_{k=1}^{n}$$

$$S_n=8\left(\frac{n(n+1)}{2}\right)^2+12\frac{n(n+1)(2n+1)}{6}-2\frac{n(n+1)}{2}-3n$$

$$S_n=2n^2(n+1)^2+2n(n+1)(2n+1)-n(n+1)-3n$$

$$S_n=n\left\{2n\left(n^2+2n+1\right)+2\left(2n^2+2n+n+1\right)-(n+1)-3\right\}$$

$$S_n=n\left\{2n^3+4n^2+2n+4n^2+6n+2-n-1-3\right\}$$

$$S_n=n\left\{2n^3+8n^2+7n-2\right\}$$

Q12. Find the sum to 2n terms of the series whose nth term is $4n^2+5n+1$

Sol: Given general term $4n^2+5n+1$

Let the general term is $T_k=4k^2+5k+1$

Sum up to nth term $\sum_{k=1}^{n}T_k=4\sum_{k=1}^{n}k^2+5\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$

$$S_n=4\frac{n(n+1)(2n+1)}{6}+5\frac{n(n+1)}{2}+n$$

$$S_n=n\left(\frac{2(n+1)(2n+1)}{3}+\frac{5(n+1)}{2}+1\right)$$

$$S_n=\frac{n}{6}\left(4\left(2n^2+n+2n+1\right)+3(5n+5)+6\right)$$

$$S_n=\frac{n}{6}\left(4\left(2n^2+3n+1\right)+3(5n+5)+6\right)$$

$$S_n=\tfrac{n}{6}\left(8n^2+12n+4+15n+15+6\right)$$

$$S_n=\tfrac{n}{6}\left(8n^2+27n+25\right)$$

Replacing n by 2n we get

$$S_{2n}=\tfrac{2n}{6}\left(8(2n)^2+27(2n)+25\right)$$

$$S_{2n}=\tfrac{n}{3}\left(32n^2+54n+25\right)$$

Q13. Find sum of n terms of the series whose nth term is

i). $n^2(2n+3)$

Sol: Given the general term $n^2(2n+3)$

Let the general term is

$T_k=k^2(2k+3)=2k^3+3k^2$

Sum up to nth term

$$\sum_{k=1}^{n}T_k=2\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2$$

$$S_n=2\left(\frac{n(n+1)}{2}\right)^2+3\frac{n(n+1)(2n+1)}{6}$$

$$S_n=\frac{n(n+1)}{2}\left(\frac{2n(n+1)}{2}+2n+1\right)$$

$$S_n=\frac{n(n+1)}{2}\left(n^2+n+2n+1\right)$$

$$S_n=\frac{n(n+1)}{2}\left(n^2+3n+1\right)$$

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ii). $3\left(4^n+2n^2\right)-4n^3$

Sol: Given $3\left(4^n+2n^2\right)-4n^3$

Let the general term is $T_k=3\left(4^k+2k^2\right)-4k^3$

Sum up to nth term

$$\sum_{k=1}^{n}T_k=3\sum_{k=1}^{n}4^k+6\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k^3$$

$$S_n=3\left(4+4^2+4^3+\cdots+4^n\right)+6\frac{n(n+1)(2n+1)}{6}-4\left(\frac{n(n+1)}{2}\right)^2$$

$$S_n=3\left(4.\frac{4^n-1}{4-1}\right)+n(n+1)\left(2n+1-n(n+1)\right)$$

$$S_n=4\left(4^n-1\right)+n(n+1)\left(2n+1-n^2-n\right)$$

$$S_n=4^{n+1}-4+n(n+1)\left(-n^2+n+1\right)$$

$$S_n=4^{n+1}-4-n(n+1)\left(n^2-n-1\right)$$

Sum of finite geometric series

$$S_n=\frac{a_1\left(r^n-1\right)}{r-1},r>1$$

$$S_n=\frac{a_1\left(1-r^n\right)}{1-r},r<1$$

Sum of infinite geometric series $S_\infty=\frac{a_1}{1-r}$

Exercise 5.2

Sum to n terms of the following series

Q1. $1.2+2.2^2+3.2^3+\cdots$

Sol: Given $1.2+2.2^2+3.2^3+\cdots$ consider

First factor	Second factor
1,2,3,... $d_1=1$	$2^1+2^2+\cdots$ $r=2$
$a_n=1+1.(n-1)$	$b_n=b_0r^{n-1}$
$a_n=n$	$b_n=2.2^{n-1}=2^n$

Then nth term $a_n.b_n=n.2^n$

Let $S=1.2+2.2^2+3.2^3+\cdots+n.2^n$ then

$$2S=\quad 1.2^2+2.2^3+3.2^4+\cdots+(n-1).2^n+n.2^{n+1}$$

$$S=1.2+2.2^2+3.2^3+4.2^4+\cdots+\quad n.2^n$$

$$S=-2-\ 2^2-\ 2^3-\ 2^4-\cdots\quad -2^n+\ n2^{n+1}$$

$$S=-2-\left(2^2+2^3+2^4+\cdots+2^n\right)+n2^{n+1}$$

$$S=-2-\frac{2^2\left(2^{n-1}-1\right)}{2-1}+n2^{n+1}\quad\because S_n=\frac{a_1\left(r^n-1\right)}{r-1}$$

$$S=-2-2^{n+1}+2^2+n2^{n+1}$$

$$S=4-2+\ n2^{n+1}-2^{n+1}$$

$$S=2+(n-1)2^{n+1}$$

Q2. $1+2.2+3.2^2+4.2^3+\cdots$

Sol: Given $1+2.2+3.2^2+4.2^3+\cdots$ consider

First factor	Second factor
1,2,3,... $d_1=1$	$2^0+2^1+2^2+\cdots$ $r=2$
$a_n=1+1.(n-1)$	$b_n=b_0r^{n-1}$
$a_n=n$	$b_n=2^0.2^{n-1}=2^{n-1}$

Then nth term $a_n.b_n=n.2^{n-1}$

Let $S=1+2.2+3.2^2+4.2^3+\cdots+(n-1).2^{n-2}+n2^{n-1}$ then

$$2S = \quad 1.2+2.2^2+3.2^3+4.2^4+\cdots+(n-1)2^{n-1}+n.2^n$$

$$S = 1+2.2+3.2^2+4.2^3+5.2^4+\cdots+ \qquad n.2^{n-1}$$

$$S=-1-2-\ 2^2-\ 2^3-\ 2^4-\cdots \qquad -2^{n-1}+\ n2^n$$

$$S=-1-\left(2+2^2+2^3+2^4+\cdots+2^{n-1}\right)+n2^n$$

$$S=-1-\frac{2\left(2^{n-1}-1\right)}{2-1}+n2^n \qquad \because S_n=\frac{a_1\left(r^n-1\right)}{r-1}$$

$$S=-1-2^n+2+n2^n$$

$$S=2-1+\ n2^n-2^n$$

$$S=1+(n-1)2^n$$

Q3. $1+4x+7x^2+10x^3+\cdots$

Sol: Given $1+4x+7x^2+10x^3+\cdots$ Consider

1,4,7,... $d_1=3$ $a_n=1+3.(n-1)$ $a_n=3n-2$	Second factor $x^0,x^1,x^2,x^3,\ldots\ r=x$ $b_n=b_0r^{n-1}$ $b_n=x^0.x^{n-1}=x^{n-1}$

Then nth term $a_n.b_n=(3n-2).x^{n-1}$

Let $S=1+4x+7x^2+10x^3+\cdots+(3n-2)x^{n-1}$ then

$$S=1+4x+7x^2+10x^3+\cdots+(3n-2)x^{n-1}$$

$$xS= \quad x+4x^2+\ 7x^3+\cdots+(3n-5)x^{n-1}+(3n-2)x^n$$

$$S-xS=1+3x+3x^2+3x^3+\cdots+3x^{n-1}-(3n-2)x^n$$

$$(1-x)S=1+3x\left(1+x+x^2+\cdots+x^{n-2}\right)-(3n-2)x^n$$

$$(1-x)S=1-(3n-2)x^n+\frac{3x\left(1-x^{n-1}\right)}{1-x} \quad \because S_n=\frac{a_1\left(1-r^n\right)}{1-r},r<1$$

$$S=\frac{1-(3n-2)x^n}{(1-x)}+\frac{3x\left(1-x^{n-1}\right)}{(1-x)^2}$$

Q4. $1+2x+3x^2+4x^3+\cdots$

Sol: Given $1+2x+3x^2+4x^3+\cdots$ consider

First factor 1,2,3,... $d_1=1$ $a_n=1+1.(n-1)$ $a_n=n$	Second factor $x^0,x^1,x^2,x^3,\ldots\ r=x$ $b_n=b_0r^{n-1}$ $b_n=x^0.x^{n-1}=x^{n-1}$

Then nth term $a_n.b_n=n.x^{n-1}$

Let $S=1+2x+3x^2+4x^3+\cdots+nx^{n-1}$ then

$$S=1+2x+3x^2+4x^3+\cdots\ +\ n\ \ x^{n-1}$$

$$xS= \quad x+2x^2+3x^3+\cdots+(n-1)x^{n-1}+nx^n$$

$$S-xS=1+x+x^2+x^3+\cdots+x^{n-1}-nx^n$$

$$(1-x)S=1+x+x^2+\cdots+x^{n-1}-nx^n$$

$$(1-x)S=\frac{1-x^n}{1-x}-nx^n$$

$$S=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{(1-x)}$$

Q5. $\frac{1}{1}+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\cdots$

Sol: Given $\frac{1}{1}+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\cdots$ consider

1,4,7,... $d_1=3$ $a_n=1+3.(n-1)$ $a_n=3n-2$	Second factor $5^0,5^1,5^2,5^3,\ldots\ r=5$ $b_n=b_0r^{n-1}$ $b_n=5^0.5^{n-1}=5^{n-1}$

Then nth term $\frac{a_n}{b_n}=\frac{(3n-2)}{5^{n-1}}$

Let $S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\cdots+\frac{3n-2}{5^{n-1}}$ then

$$S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\cdots+\frac{3n-2}{5^{n-1}}$$

$$\frac{1}{5}S= \quad \frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\cdots+\frac{3n-5}{5^{n-1}}+\frac{3n-2}{5^n}$$

$$S-\frac{1}{5}S=1+\frac{3}{5}+\frac{3}{5^2}+\frac{3}{5^3}+\cdots+\frac{3}{5^{n-1}}-\frac{3n-2}{5^n}$$

$$\left(1-\frac{1}{5}\right)S=1+\frac{3}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+\cdots+\frac{1}{5^{n-2}}\right)-\frac{3n-2}{5^n}$$

$$\left(\frac{4}{5}\right)S=1+\frac{3}{5}\frac{\left(1-\frac{1}{5^{n-1}}\right)}{\left(1-\frac{1}{5}\right)}-\frac{3n-2}{5^n}$$

$$S=\frac{5}{4}\left\{1+\frac{3}{5}\frac{\left(1-\frac{1}{5^{n-1}}\right)}{\left(\frac{4}{5}\right)}-\frac{3n-2}{5^n}\right\}$$

$$S=\frac{5}{4}\left\{1+\frac{3}{4}\left(1-\frac{1}{5^{n-1}}\right)-\frac{3n-2}{5^n}\right\}$$

$$S=\frac{5}{4}\left\{1+\frac{3}{4}-\frac{3}{4}\frac{1}{5^{n-1}}-\frac{3n-2}{5^n}\right\}$$

$$S=\frac{5}{4}\left\{\frac{7}{4}-\frac{3}{4}\frac{1}{5^{n-1}}-\frac{3n-2}{5^n}\right\}$$

$$S=\frac{35}{16}-\frac{3}{16.5^{n-2}}-\frac{3n-2}{4.5^{n-1}}$$

$$S=\frac{35}{16}-\frac{15+4(3n-2)}{16.5^{n-1}}$$

$$S=\frac{35}{16}-\frac{12n+7}{16.5^{n-1}}$$

Q6. $1-7x+13x^2-19x^3+\cdots$

Sol: Given $1-7x+13x^2-19x^3+\cdots$ consider

1,7,13,... $d=6$ $a_n=1+6.(n-1)$ $a_n=6n-5$	Second factor $x^0,-x^1,x^2,-x^3,\ldots\ r=-x$ $b_n=b_0r^{n-1}$ $b_n=x^0.(-x)^{n-1}=(-x)^{n-1}$

Then nth term $a_n.b_n=(6n-5).(-x)^{n-1}$

Let $S=1-7x+13x^2-19x^3+\cdots+(6n-5)(-x)^{n-1}$ then

$$S=1-7x+13x^2-19x^3+\cdots+(6n-5)(-x)^{n-1}$$

$$xS= \quad x-7x^2+\ 13x^3+\cdots-(6n-11)(-x)^{n-1}+(6n-5)(-x)^n$$

$$S+xS=1-6x+6x^2-6x^3+\cdots+6(-x)^{n-1}+(6n-5)(-x)^n$$

$$(1+x)S=1-6x\left(1-x+x^2-\cdots+(-1)^{n-1}(-x)^{n-2}\right)-(6n-5)(-x)^n$$

$$(1+x)S = 1-(6n-5)(-x)^n - \frac{6x\left(1-(-x)^{n-1}\right)}{1-(-x)}$$

$$S = \frac{1-(6n-5)(-x)^n}{(1+x)} - \frac{6x\left(1-(-x)^{n-1}\right)}{(1+x)^2}$$

Find the sum of infinity of the following series

Q7. $1^2+3^2x+5^2x^2+7^2x^3+\cdots$

Sol: Given $1^2+3^2x+5^2x^2+7^2x^3+\cdots$

Let $S=1^2+3^2x+5^2x^2+7^2x^3+\cdots$ then

$$S = 1^2+3^2x+5^2x^2+7^2x^3+\cdots$$

$$xS = \quad 1^2x+3^2x^2+5^2x^3+7^2x^4+\cdots$$

$$S-xS = 1+8x+16x^2+24x^3+\cdots$$

Consider

$$S_1 = 1+8x+16x^2+24x^3+\cdots$$

$$xS_1 = \quad 1x+8x^2+16x^3+\cdots$$

$$S_1 - xS_1 = 1+7x+8x^2+8x^3+\cdots$$

$$(1-x)S_1 = 1+7x+8x^2\left(1+x+x^2+\cdots\right)$$

$$(1-x)S_1 = 1+7x+8x^2\frac{1}{1-x} \qquad \because S_\infty = \frac{a_1}{1-r}$$

$$S_1 = \frac{1+7x}{(1-x)} + \frac{8x^2}{(1-x)^2}$$

So $(1-x)S = 1+8x+16x^2+24x^3+\cdots$

$$(1-x)S = \frac{1+7x}{(1-x)} + \frac{8x^2}{(1-x)^2}$$

$$S = \frac{1+7x}{(1-x)^2} + \frac{8x^2}{(1-x)^3}$$

Q8. $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$

Sol: Given $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$

Let $S=1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$ then

$$S = 1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$$

$$\frac{1}{3}S = \quad \frac{1}{3}+\frac{4}{3^2}+\frac{9}{3^3}+\frac{16}{3^4}+\frac{25}{3^5}+\cdots$$

$$S-\frac{1}{3}S = 1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\frac{9}{3^4}+\cdots$$

$$\left(1-\frac{1}{3}\right)S = 1+1+\frac{1}{3^2}\left(5+\frac{7}{3}+\frac{9}{3^2}+\cdots\right) \quad \ldots\ldots\ldots\ldots(1)$$

Now take

$$S_1 = 5+\frac{7}{3}+\frac{9}{3^2}+\cdots$$

$$\frac{1}{3}S_1 = \quad \frac{5}{3}+\frac{7}{3^2}+\frac{9}{3^3}+\cdots$$

$$\left(1-\frac{1}{3}\right)S_1 = 5+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\cdots$$

$$\frac{2}{3}S_1 = 5+\frac{2}{3}\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right) \qquad \because S_\infty = \frac{a_1}{1-r}$$

$$\frac{2}{3}S_1 = 5+\frac{2}{3}\left(\frac{1}{1-\frac{1}{3}}\right)$$

$$\frac{2}{3}S_1 = 5+\frac{2}{3}\frac{1}{\frac{2}{3}}$$

$$S_1 = \frac{3}{2}(5+1)$$

$$S_1 = \frac{3}{2}.6$$

$$S_1 = 9$$

Put in equation (1) we get

$$\left(1-\frac{1}{3}\right)S = 1+1+\frac{1}{3^2}\left(5+\frac{7}{3}+\frac{9}{3^2}+\cdots\right)$$

$$\frac{2}{3}S = 2+\frac{1}{3^2}.9 = 2+1 = 3$$

$$S = 3.\frac{3}{2} = \frac{9}{2}$$

Q9. Find the nth term of following Arithmetico-geometric sequence $5,\frac{7}{3},1,\frac{11}{27},\ldots$

Sol: Given $5,\frac{7}{3},1,\frac{11}{27},\ldots$ or $\frac{5}{3^0},\frac{7}{3},\frac{9}{3^2},\frac{11}{3^3},\ldots$

5,7,9,11,... d=2 3,9,27,... r=3

$a_n = 5+2(n-1)$ $b_n = a_1r^{n-1} = 1.3^{n-1}$

$a_n = 2n+3$ $b_n = 3^{n-1}$

$$T_n = \frac{a_n}{b_n} =$$

$$T_n = \frac{[5+2(n-1)]}{3^{n-1}}$$

$$T_n = \frac{5+2n-2}{3^{n-1}}$$

$$T_n = \frac{2n+3}{3^{n-1}}$$

Q10. Find the sum of following Arithmetico-geometric sequence $S=5+\frac{7}{3}+1+\frac{11}{27}+\cdots$

Sol: Given $S=5+\frac{7}{3}+1+\frac{11}{27}+\cdots$ then

$$S = 5+\frac{7}{3}+\frac{9}{3^2}+\frac{11}{3^3}+\cdots$$

$$\frac{1}{3}S = \quad \frac{5}{3}+\frac{7}{3^2}+\frac{9}{3^3}+\frac{11}{3^4}+\cdots$$

$$S-\frac{1}{3}S = 5+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\frac{2}{3^4}+\cdots$$

$$\frac{2}{3}S = 5+\frac{2}{3}\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right) \qquad \because S_\infty = \frac{a_1}{1-r}$$

$$\frac{2}{3}S = 5+\frac{2}{3}\left(\frac{1}{1-\frac{1}{3}}\right)$$

$\frac{2}{3}S = 5 + \frac{2}{3}\left(\frac{1}{\frac{2}{3}}\right)$

$\frac{2}{3}S = 5 + 1$

$\frac{2}{3}S = 6$

$S = 6.\frac{3}{2}$

$S = 9$

A series each term having r factors in AP

$$S_n = \sum_{k=1}^{n} U_k = \frac{\text{nth term} \times \text{next factor}}{(\text{number of factors in each term } +1) \times c.d} + C$$

Reciprocal series

$$S_n = \sum_{k=1}^{n} U_k = C - \frac{\text{nth term with a factor from biginning neglected}}{(\text{number of factors in each term } -1) \times c.d}$$

Exercise 5.3

Find the sum of the following series

Q1. 1.3.5+2.4.6+3.5.7+...

Sol: Given 1.3.5+2.4.6+3.5.7+... Consider

First factor 1,2,3,... $d_1 = 1$ $a_n = 1 + 1.(n-1)$ $a_n = n$	Second factor 3,4,5,... $d_2 = 1$ $b_n = 3 + 1.(n-1)$ $b_n = n + 2$
Third factor 5,6,7,... $d_3 = 1$ $c_n = 5 + 1.(n-1)$ $c_n = n + 4$	And common difference between factors of first term is $d = 2 \neq d_1 = d_2 = d_3$ so making the general term as a same c.d

$U_n = n(n+2)(n+4)$

$U_n = n(n+1+1)(n+2+2)$

$U_n = \{n(n+1)+n\}\{(n+2)+2\}$

$U_n = n(n+1)(n+2) + 2n(n+1) + n(n+2) + 2n$

$U_n = n(n+1)(n+2) + 2n(n+1) + n(n+1+1) + 2n$

$U_n = n(n+1)(n+2) + 2n(n+1) + n(n+1) + n + 2n$

$U_n = n(n+1)(n+2) + 3n(n+1) + 3n$

Then

$$S_n = C + \frac{n(n+1)(n+2)(n+3)}{4} + \frac{3n(n+1)(n+2)}{3} + \frac{3n(n+1)}{2}$$

At n=1

$$1.3.5 = C + \frac{1.2.3.4}{4} + \frac{3.1.2.3}{3} + \frac{3.1.2}{2}$$

$15 = C + 6 + 6 + 3 \Rightarrow C = 0$

then

$$S_n = \frac{n(n+1)(n+2)(n+3)}{4} + n(n+1)(n+2) + \frac{3n(n+1)}{2}$$

$$S_n = \frac{n(n+1)}{4}\{(n+2)(n+3) + 4(n+2) + 6\}$$

$$S_n = \frac{n(n+1)}{4}\{n^2 + 5n + 6 + 4n + 8 + 6\}$$

$$S_n = \frac{n(n+1)}{4}\{n^2 + 9n + 20\}$$

$$S_n = \frac{n(n+1)(n+4)(n+5)}{4}$$

Q2. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$ to infinity

Sol: Given $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$ Consider

First factor 1,4,7,... $d_1 = 3$ $a_n = 1 + 3.(n-1)$ $a_n = 3n - 2$	Second factor 4,7,10,... $d_2 = 3$ $b_n = 4 + 3.(n-1)$ $b_n = 3n + 1$

And common difference between factors of first term is

$d = 3 = d_1 = d_2$

$$U_n = \frac{1}{(3n-2)(3n+1)}$$

Then

$$S_n = C - \frac{1}{(3n+1).3}$$

At n=1

$\frac{1}{1.4} = C - \frac{1}{4.3}$ then

$C = \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$

$$S_n = \frac{1}{3} - \frac{1}{3(3n+1)}$$

$$\lim_{n\to\infty} S_n = \frac{1}{3} - \lim_{n\to\infty} \frac{1}{3(3n+1)}$$

$$\lim_{n\to\infty} S_n = \frac{1}{3}$$

Q3. $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \cdots$ to infinity

Sol: Given $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \cdots$ Consider

First factor 1,3,5,... $d_1 = 2$ $a_n = 1 + 2.(n-1)$ $a_n = 2n - 1$	Second factor 3,5,7,... $d_2 = 2$ $b_n = 3 + 2.(n-1)$ $b_n = 2n + 1$
Third factor 5,7,9,... $d_3 = 2$ $c_n = 5 + 2.(n-1)$ $c_n = 2n + 3$	

And common difference between factors of first term is

$d = 2 = d_1 = d_2 = d_3$,

$$U_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$$

Then $S_n = C - \frac{1}{(2n+1)(2n+3).2(3-1)}$

At n=1

$$\frac{1}{1.3.5}=C-\frac{1}{4.3.5} \quad \text{then}$$

$$C=\frac{1}{15}+\frac{1}{60}=\frac{1}{12}$$

$$S_n=\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)}$$

$$\lim_{n\to\infty} S_n=\frac{1}{12}-\lim_{n\to\infty}\frac{1}{4(2n+1)(2n+3)}$$

$$\lim_{n\to\infty} S_n=\frac{1}{12}$$

Q4. 1.4.7+4.7.10+7.10.13+... to n terms

Solution; 1.4.7+4.7.10+7.10.13+... to n terms

Consider

First factor	Second factor
1,4,7,... $d_1=3$ $a_n=1+3.(n-1)$ $a_n=3n-2$	4,7,10,... $d_2=3$ $b_n=4+3(n-1)$ $b_n=3n+1$
Third factor 7,10,13,... $d_3=3$ $c_n=7+3.(n-1)$ $c_n=3n+4$	

And common difference between factors of first term is

$$d=3=d_1=d_2=d_3$$

$$U_n=\frac{(3n-2)(3n+1)(3n+4)(3n+7)}{(3+1).3}$$

Then $S_n=C+\frac{(3n-2)(3n+1)(3n+4)(3n+7)}{12}$

At n=1

$$1.4.7=C+\frac{1.4.7.10}{12}$$

$$28=C+\frac{70}{3}\Rightarrow C=\frac{14}{3} \quad \text{then}$$

$$S_n=\frac{14}{3}+\frac{(3n-2)(3n+1)(3n+4)(3n+7)}{12}$$

Q5. 1.5.9+2.6.10+3.7.11+... to n terms

Sol: Given 1.5.9+2.6.10+3.7.11+... to n terms

Consider

First factor	Second factor
1,2,3,... $d_1=1$ $a_n=1+1.(n-1)$ $a_n=n$	5,6,7,... $d_2=1$ $b_n=5+1.(n-1)$ $b_n=n+4$
Third factor 9,10,11,... $d_3=1$ $c_n=9+1.(n-1)$ $c_n=n+8$	

And common difference between factors of first term is

$d=4\neq d_1=d_2=d_3$ so making general term as a same c.d

$$U_n=n(n+4)(n+8)$$

$$U_n=n(n+1+3)(n+2+6)$$

$$U_n=\{n(n+1)+3n\}(n+2+6)$$

$$U_n=n(n+1)(n+2)+6n(n+1)+3n(n+2)+18n$$

$$U_n=n(n+1)(n+2)+6n(n+1)+3n(n+1+1)+18n$$

$$U_n=n(n+1)(n+2)+6n(n+1)+3n(n+1)+3n+18n$$

$$U_n=n(n+1)(n+2)+9n(n+1)+21n$$

Then

$$S_n=C+\frac{n(n+1)(n+2)(n+3)}{(3+1).1}+\frac{9n(n+1)(n+2)}{(2+1).1}+\frac{21n(n+1)}{(1+1).1}$$

At n=1

$$1.5.9=C+\frac{1.2.3.4}{4}+\frac{9.1.2.3}{3}+\frac{21.1.2}{2} \quad \text{then}$$

$$45=C+6+18+21\Rightarrow C=0$$

$$S_n=\frac{n(n+1)(n+2)(n+3)}{4}+3n(n+1)(n+2)+\frac{21n(n+1)}{2}$$

$$S_n=\frac{n(n+1)}{4}\{n^2+5n+6+12n+24+42\}$$

$$S_n=\frac{n(n+1)}{4}\{n^2+17n+72\}$$

$$S_n=\frac{n(n+1)}{4}\{n^2+9n+8n+72\}$$

$$S_n=\frac{n(n+1)}{4}\{n(n+9)+8(n+9)\}$$

$$S_n=\frac{n(n+1)(n+8)(n+9)}{4}$$

Q6. $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+\cdots$ to n terms

Sol: Given $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+\cdots$ to n terms

Consider

First factor	Second factor
1,2,3,... $d_1=1$ $a_n=1+1.(n-1)$ $a_n=n$	2,3,4,... $d_2=1$ $b_n=2+1.(n-1)$ $b_n=n+1$
Third factor 3,4,5,... $d_3=1$ $c_n=3+1.(n-1)$ $c_n=n+2$	Fourth factor 4,5,6,... $d_4=1$ $e_n=4+1.(n-1)$ $e_n=n+3$

And common difference between factors of first term is

$d=1=d_1=d_2=d_3=d_4$ but common difference of numerator & denominator B/W each factor is not same.

$$U_n=\frac{(n+3)}{n(n+1)(n+2)}$$

$$U_n=\frac{n}{n(n+1)(n+2)}+\frac{3}{n(n+1)(n+2)}$$

$$U_n=\frac{1}{(n+1)(n+2)}+\frac{3}{n(n+1)(n+2)}$$

Then $S_n=C-\frac{1}{(n+2)}-\frac{3}{2(n+1)(n+2)}$

At n=1

$$\frac{4}{1.2.3}=C-\frac{1}{3}-\frac{3}{2.2.3}$$

$$C=\frac{2}{3}+\frac{1}{3}+\frac{1}{4}=\frac{5}{4}$$

then

$$S_n=\frac{5}{4}-\frac{1}{(n+2)}-\frac{3}{2(n+1)(n+2)}$$

$$S_n=\frac{5}{4}-\frac{2(n+1)+3}{2(n+1)(n+2)}$$

$$S_n=\frac{5}{4}-\frac{2n+2+3}{2(n+1)(n+2)}$$

$$S_n=\frac{5}{4}-\frac{2n+5}{2(n+1)(n+2)}$$

Q7. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\cdots$ to infinity

Sol: Given $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\cdots$ Consider

First factor 1,2,3,... $d_1=1$ $a_n=1+1.(n-1)$ $a_n=n$	Second factor 2,3,4,... $d_2=1$ $b_n=2+1.(n-1)$ $b_n=n+1$
Third factor 3,4,5,... $d_3=1$ $c_n=3+1.(n-1)$ $c_n=n+2$	Fourth factor 1,3,5,... $d_4=2$ $e_n=1+2(n-1)$ $e_n=2n-1$

And common difference between factors of first term is $d=1=d_1=d_2=d_3\neq d_4$ so making general term as a same c.d

$$U_n=\frac{2n-1}{n(n+1)(n+2)}$$

$$U_n=\frac{2n}{n(n+1)(n+2)}-\frac{1}{n(n+1)(n+2)}$$

$$U_n=\frac{2}{(n+1)(n+2)}-\frac{1}{n(n+1)(n+2)}$$

Then $S_n=C-\frac{2}{(n+2)}+\frac{1}{2(n+1)(n+2)}$

At n=1

$$\frac{1}{1.2.3}=C-\frac{2}{3}+\frac{1}{2.2.3}$$

$$C=\frac{1}{6}+\frac{2}{3}-\frac{1}{12}=\frac{2+8-1}{12}=\frac{9}{12}$$

$$C=\tfrac{3}{4}$$

then

$$S_n=\frac{3}{4}-\frac{2}{(n+3)}+\frac{1}{2(n+1)(n+2)}$$

$$\lim_{n\to\infty}S_n=\frac{3}{4}-\lim_{n\to\infty}\frac{2}{(n+3)}+\lim_{n\to\infty}\frac{1}{2(n+1)(n+2)}$$

$$\lim_{n\to\infty}S_n=\frac{3}{4}$$

Q8. Find the sum of the n terms, whose general term is $(n^2+5n+4)(n^2+5n+6)$

Sol: Given $(n^2+5n+4)(n^2+5n+6)$ Consider

$$U_n=(n^2+5n+4)(n^2+5n+6)$$

$$U_n=(n^2+4n+n+4)(n^2+3n+2n+6)$$

$$U_n=\{n(n+4)+1(n+4)\}\{n(n+3)+2(n+3)\}$$

$$U_n=(n+1)(n+4)(n+2)(n+3)$$

$$U_n=(n+1)(n+2)(n+3)(n+4)$$

$$S_n=C+\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5}$$

At n=1

$$(1^2+5.1+4)(1^2+5.1+6)=C+\frac{2.3.4.5.6}{5}$$

$$10\times12=C+144$$

$$120-144=C$$

$$C=-24$$

$$S_n=\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5}-24$$

Q9. Find sum of n terms, whose general term is $n^2(n^2-1)$

Sol; Given nth term $n^2(n^2-1)$

$$U_n=n^2(n^2-1)=n.n(n-1)(n+1)$$

$$U_n=(n-1)n(n+1)n=(n-1)n(n+1)(n+2-2)$$

$$U_n=(n-1)n(n+1)(n+2)-2(n-1)n(n+1)$$

$$\therefore S_n=C+\frac{(n-1)n(n+1)(n+2)(n+3)}{(4+1).1}-\frac{2(n-1)n(n+1)(n+2)}{(3+1).1}$$

At n=1 C=0

$$S_n=\frac{(n-1)n(n+1)(n+2)(n+3)}{5}-\frac{(n-1)n(n+1)(n+2)}{2}$$

$$S_n=(n-1)n(n+1)(n+2)\left\{\frac{(n+3)}{5}-\frac{1}{2}\right\}$$

$$S_n=(n-1)n(n+1)(n+2)\left\{\tfrac{2(n+3)-5}{10}\right\}$$

$$S_n=\tfrac{1}{10}(n-1)n(n+1)(n+2)(2n+1)$$

Q10. Sum to infinity, the series whose general term is $\frac{1}{n(n+1)(n+3)}$

Sol: Given $\frac{1}{n(n+1)(n+3)}$ consider

$$U_n=\frac{1}{n(n+1)(n+3)}$$

$$U_n=\frac{1}{n(n+1)(n+3)}\cdot\frac{(n+2)}{(n+2)}$$

$$U_n=\frac{(n+2)}{n(n+1)(n+2)(n+3)}$$

$$U_n=\frac{n}{n(n+1)(n+2)(n+3)}+\frac{2}{n(n+1)(n+2)(n+3)}$$

$$U_n=\frac{1}{(n+1)(n+2)(n+3)}+\frac{2}{n(n+1)(n+2)(n+3)}$$

$$S_n=C-\frac{1}{2(n+2)(n+3)}+\frac{2}{3(n+1)(n+2)(n+3)}$$

At n= 1

$$\frac{1}{1.2.3}=C-\frac{1}{2.3.4}-\frac{2}{3.2.3.4}$$

$$C=\frac{1}{6}+\frac{1}{24}+\frac{1}{36}=\frac{7}{36}$$

$$S_n=\frac{7}{36}-\frac{1}{2(n+2)(n+3)}+\frac{2}{3(n+1)(n+2)(n+3)}$$

$$\lim_{n\to\infty}S_n=\frac{7}{36}-\lim_{n\to\infty}\frac{1}{2(n+2)(n+3)}+\lim_{n\to\infty}\frac{2}{3(n+1)(n+2)(n+3)}$$

$$\lim_{n\to\infty}S_n=\tfrac{7}{36}$$

Exercise 5.4

Find the sum of the following

Q1. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots$ to n terms

Sol: Given $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots$ Consider

First factor	Second factor
1,3,5,... $d_1=2$	3,5,7,... $d_2=2$
$a_n=1+2.(n-1)$	$b_n=3+2.(n-1)$
$a_n=2n-1$	$b_n=2n+1$

So the nth term of the series is

$$T_n=\frac{1}{(2n-1)(2n+1)}$$

Resolving T_n into Partial fractions, we have

$$\frac{1}{(2n-1)(2n+1)}=\frac{A}{(2n-1)}+\frac{B}{(2n+1)}$$

Multiplying both sides by $(2n-1)(2n+1)$ we obtain

$$1=A(2n+1)+B(2n-1)$$

$$1=n.2(A+B)+A-B$$

By comparing the coefficients we get

$$\begin{matrix}A-B=1\\2(A+B)=0\end{matrix}\Rightarrow\begin{cases}A=\frac{1}{2}\\B=\frac{-1}{2}\end{cases}$$

So that $T_n=\frac{1}{2(2n-1)}-\frac{1}{2(2n+1)}$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)$$

$$S_n=\frac{1}{2}\left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right\}$$

$$S_n=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2}\left(\frac{2n+1-1}{2n+1}\right)$$

$$S_n=\frac{1}{2}.\frac{2n}{2n+1}=\frac{n}{2n+1}$$

Q2. $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\cdots$ to infinity

Sol: Given $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\cdots$ Consider

First factor	Second factor
1,3,5,... $d_1=2$	3,5,7,... $d_2=2$
$a_n=1+2.(n-1)$	$b_n=3+2.(n-1)$
$a_n=2n-1$	$b_n=2n+1$
Third factor	
5,7,9,... $d_3=2$	
$c_n=5+2.(n-1)$	
$c_n=2n+3$	

So the nth term of the series is

$$T_n=\frac{1}{(2n-1)(2n+1)(2n+3)}$$

Resolving T_n into Partial fractions, we have

$$\frac{1}{(2n-1)(2n+1)(2n+3)}=\frac{A}{(2n-1)}+\frac{B}{(2n+1)}+\frac{C}{(2n+3)}$$

Multiplying both sides by $(2n-1)(2n+1)(2n+3)$

$$1=A(2n+1)(2n+3)+B(2n-1)(2n+3)$$
$$+C(2n-1)(2n+1)$$

at $n=\frac{1}{2}\Rightarrow A=\frac{1}{8}$

at $n=\frac{-1}{2}\Rightarrow B=\frac{-1}{4}$

at $n=\frac{-3}{2}\Rightarrow C=\frac{1}{8}$

So that

$$T_n=\frac{1}{8(2n-1)}-\frac{1}{4(2n+1)}+\frac{1}{8(2n+3)}$$

$$\sum_{k=1}^{n}T_k=\frac{1}{8}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{2}{2k+1}+\frac{1}{2k+3}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{8}\left\{\left(\frac{1}{1}-\frac{2}{3}+\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}+\frac{1}{2n+3}\right)\right\}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}T_k=\frac{1}{8}\left(1-\frac{2}{3}+\frac{1}{3}-\lim_{n\to\infty}\frac{1}{2n+3}\right)$$

$$S_\infty=\frac{1}{8}\left(\frac{3-2+1}{3}\right)$$

$$S_\infty=\frac{1}{8}\left(\frac{2}{3}\right)$$

$$S_\infty=\frac{1}{12}$$

Q3. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots$ to infinity

Sol: Given $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots$ Consider

First factor	Second factor
1,4,7,... $d_1=3$	4,7,10,... $d_2=3$
$a_n=1+3.(n-1)$	$b_n=4+3.(n-1)$
$a_n=3n-2$	$b_n=3n+1$

So the nth term of the series is

$$T_n=\frac{1}{(3n-2)(3n+1)}$$

Resolving T_n into Partial fractions, we have

$$\frac{1}{(3n-2)(3n+1)}=\frac{A}{(3n-2)}+\frac{B}{(3n+1)}$$

Multiplying both sides by $(3n-2)(3n+1)$ we obtain

$$1=A(3n+1)+B(3n-2)$$

at $n=\frac{2}{3}\Rightarrow A=\frac{1}{3}$

at $n=\frac{-1}{3}\Rightarrow B=\frac{-1}{3}$

So that

$$T_n=\frac{1}{3(3n-2)}-\frac{1}{3(3n+1)}$$

$$\therefore \sum_{k=1}^{n}T_k=\frac{1}{3}\sum_{k=1}^{n}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{3}\left\{\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)\right\}$$

$$\sum_{k=1}^{n}T_k=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}T_k=\frac{1}{3}\left(1-\lim_{n\to\infty}\frac{1}{3n+1}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}T_k=\frac{1}{3}$$

Q4. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots$ to infinity

Sol: Given $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots$ Consider

First factor 1,2,3,... $d_1=1$ $a_n=1+1.(n-1)$ $a_n=n$	Second factor 2,3,4,... $d_2=1$ $b_n=2+1.(n-1)$ $b_n=n+1$
Third factor 3,4,5,... $d_3=1$ $c_n=3+1.(n-1)$ $c_n=n+2$	

So the nth term of the series is

$$T_n=\frac{1}{n(n+1)(n+2)}$$

Resolving T_n into Partial fractions, we have

$$\frac{1}{n(n+1)(n+2)}=\frac{A}{n}+\frac{B}{(n+1)}+\frac{C}{(n+2)}$$

Multiplying both sides by $n(n+1)(n+2)$ we obtain

$$1=A(n+1)(n+2)+B(n)(n+2)+C(n)(n+1)$$

at $n=0\Rightarrow A=\frac{1}{2}$

at $n=-1\Rightarrow B=-1$

at $n=-2\Rightarrow C=\frac{1}{2}$

So that

$$T_n=\frac{1}{2n}-\frac{1}{(n+1)}+\frac{1}{2(n+2)}$$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{k+1}+\frac{1}{k+2}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\left\{\left(1-\frac{2}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{(n+2)}\right)\right\}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}T_k=\frac{1}{2}\left(1-\frac{2}{2}+\frac{1}{2}-\lim_{n\to\infty}\frac{1}{n+2}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}T_k=\frac{1}{2}\left(\frac{2-2+1}{2}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}T_k=\frac{1}{4}$$

Q5. Find sum of the series $\sum_{k=1}^{n}\frac{1}{9k^2+3k-2}$

Sol: Given $\sum_{k=1}^{n}\frac{1}{9k^2+3k-2}$

$$\sum_{k=1}^{n}\frac{1}{9k^2+3k-2}=\sum_{k=1}^{n}\frac{1}{9k^2+6k-3k-2}$$

$$\sum_{k=1}^{n}\frac{1}{3k(3k+2)-1(3k+2)}=\sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)}$$

Breaking into partial fractions, we have

$$\frac{1}{(3k-1)(3k+2)}=\frac{A}{(3k-1)}+\frac{B}{(3k+2)}$$

Multiplying both sides by $(3k-1)(3k+2)$ we obtain

$$1=A(3k+2)+B(3k-1)$$

at $k=\frac{1}{3}\Rightarrow A=\frac{1}{3}$

at $k=-\frac{2}{3}\Rightarrow B=\frac{-1}{3}$

$$\sum_{k=1}^{n}\frac{1}{9k^2+3k-2}=\sum_{k=1}^{n}\left(\frac{1}{3(3k-1)}-\frac{1}{3(3k+2)}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{3}\sum_{k=1}^{n}\left(\frac{1}{(3k-1)}-\frac{1}{(3k+2)}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{3}\left\{\left(\tfrac{1}{2}-\tfrac{1}{5}\right)+\left(\tfrac{1}{5}-\tfrac{1}{8}\right)+\left(\tfrac{1}{8}-\tfrac{1}{11}\right)+\cdots+\left(\tfrac{1}{3n-1}-\tfrac{1}{3n+2}\right)\right\}$$

$$\sum_{k=1}^{n}T_k=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{3}.\frac{3n+2-2}{2(3n+2)}$$

$$\sum_{k=1}^{n}T_k=\frac{n}{2(3n+2)}$$

Q6. Find sum of the series $\sum_{k=1}^{n}\frac{1}{k^2-k}$

Sol: Given $\sum_{k=1}^{n}\frac{1}{k^2-k}$

$$\sum_{k=1}^{n}\frac{1}{k^2-k}=\sum_{k=1}^{n}\frac{1}{k(k-1)}$$

Breaking into partial fractions, we have

$$\frac{1}{k(k-1)}=\frac{A}{k}+\frac{B}{k-1}$$

Multiplying both sides by $k(k-1)$ we obtain

$$1=A(k-1)+Bk$$

at $k=0\Rightarrow A=-1$

at $k=1\Rightarrow B=1$

$$\sum_{k=2}^{n}\frac{1}{k(k-1)}=\sum_{k=2}^{n}\left(\frac{1}{(k-1)}-\frac{1}{k}\right)$$

$$\sum_{k=2}^{n}T_k=\left\{\left(\tfrac{1}{1}-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{n-1}-\tfrac{1}{n}\right)\right\}$$

$$\sum_{k=2}^{n}T_k=\left(1-\frac{1}{n}\right)=\frac{n-1}{n}$$

Q7. Find sum of the series $\sum_{k=1}^{n}\frac{1}{k^2+7k+12}$

Sol: Given $\sum_{k=1}^{n}\frac{1}{k^2+7k+12}$

$$\sum_{k=1}^{n}\frac{1}{k^2+7k+12}=\sum_{k=1}^{n}\frac{1}{k^2+4k+3k+12}$$

$$\sum_{k=1}^{n}\frac{1}{k(k+4)+3(k+4)}=\sum_{k=1}^{n}\frac{1}{(k+3)(k+4)}$$

Breaking into partial fractions, we have

$$\frac{1}{(k+3)(k+4)}=\frac{A}{(k+3)}+\frac{B}{(k+4)}$$

Multiplying both sides by $(k+3)(k+4)$ we obtain

$$1=A(k+4)+B(k+3)$$

at k $=-3\Rightarrow A=1$

at k $=-4\Rightarrow B=-1$

$$\sum_{k=1}^{n}\frac{1}{k^2+7k+12}=\sum_{k=1}^{n}\left(\frac{1}{(k+3)}-\frac{1}{(k+4)}\right)$$

$$\sum_{k=1}^{n}T_k=\left\{\left(\tfrac{1}{4}-\tfrac{1}{5}\right)+\left(\tfrac{1}{5}-\tfrac{1}{6}\right)+\left(\tfrac{1}{6}-\tfrac{1}{7}\right)+\cdots+\left(\tfrac{1}{n+3}-\tfrac{1}{n+4}\right)\right\}$$

$$\sum_{k=1}^{n}T_k=\left(\frac{1}{4}-\frac{1}{n+4}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{n+4-4}{4(n+4)}$$

$$\sum_{k=1}^{n}T_k=\frac{n}{4(n+4)}$$

Q8. Find the sum of the n term of the series

$$\frac{1}{(x+1)(3x+1)}+\frac{1}{(3x+1)(5x+1)}+\frac{1}{(5x+1)(7x+1)}+\cdots$$

Solution

$$\frac{1}{(x+1)(3x+1)}+\frac{1}{(3x+1)(5x+1)}+\frac{1}{(5x+1)(7x+1)}+\cdots$$

Consider

First factor	Second factor
$(x+1),(3x+1),(5x+1),\ldots$	$(3x+1),(5x+1),(7x+1),\ldots$
$d_1=2x$	$d_2=2x$
$a_n=(x+1)+2x(n-1)$	$b_n=(3x+1)+2x(n-1)$
$a_n=x+1+2xn-2x$	$b_n=3x+1+2xn-2x$
$a_n=2xn-x+1$	$b_n=2xn+x+1$

$$T_n=\frac{1}{(2nx-x+1)(2nx+x+1)}$$

Breaking into partial fractions, we have

$$\frac{1}{(2nx-x+1)(2nx+x+1)}=\frac{A}{(2nx-x+1)}+\frac{B}{(2nx+x+1)}$$

Multiplying both sides by

$$(2nx-x+1)(2nx+x+1)$$

$$1=A(2nx+x+1)+B(2nx-x+1)$$

at $x=\frac{1}{1-2n}\Rightarrow A=\frac{1-2n}{2}$

at $x=\frac{-1}{1+2n}\Rightarrow B=\frac{1+2n}{2}$

$$\frac{1}{(2nx-x+1)(2nx+x+1)}=\frac{1-2n}{2(2nx-x+1)}+\frac{1+2n}{2(2nx+x+1)}$$

$$\sum_{k=1}^{n}\frac{1}{(2kx-x+1)(2kx+x+1)}=\sum_{k=1}^{n}\left(\frac{1-2k}{2(2kx-x+1)}+\frac{1+2k}{2(2kx+x+1)}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1-2k}{(2kx-x+1)}+\frac{1+2k}{(2kx+x+1)}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\left\{\begin{array}{l}\left(\frac{-1}{x+1}+\frac{3}{3x+1}\right)+\left(\frac{-3}{3x+1}+\frac{5}{5x+1}\right)+\left(\frac{-5}{5x+1}+\frac{7}{7x+1}\right)+\\ \left(\frac{-7}{7x+1}+\frac{9}{9x+1}\right)+\cdots+\left(\frac{1-2n}{(2nx-x+1)}+\frac{1+2n}{(2nx+x+1)}\right)\end{array}\right\}$$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\left(\frac{-1}{x+1}+\frac{1+2n}{(2nx+x+1)}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\left(\frac{-2nx-x-1+(x+1)(1+2n)}{(x+1)(2nx+x+1)}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{1}{2}\left(\frac{-2nx-x-1+x+1+2nx+2n}{(x+1)(2nx+x+1)}\right)$$

$$\sum_{k=1}^{n}T_k=\frac{n}{(x+1)(2nx+x+1)}$$