Chapter 4

Sequences

Exercise 4.1

Q1. Classify following into finite & infinite sequences.

i). 2,4,86,8,…,50

Answer; finite sequence

ii). 1,0,1,0,1,…

Answer; infinite sequence

iii). …,-4,0,4,8,…,60

Answer; infinite sequence

iv). $1,\frac{-1}{3},\frac{1}{9},\frac{-1}{27},\ldots,\frac{-1}{2187}$

Answer; finite sequence

Q2. Find first four terms of a sequence with given general terms.

i). $\frac{n(n+1)}{2}$

Sol: Given general term $a_n=\frac{n(n+1)}{2}$

At n=1

$a_1=\frac{1(1+1)}{2}$

$a_1=\frac{2}{2}$

$a_1=1$

At n=2

$a_2=\frac{2(2+1)}{2}$

$a_2=\frac{2.3}{2}$

$a_2=3$

At n=3

$a_3=\frac{3(3+1)}{2}$

$a_3=\frac{3.4}{2}$

$a_3=6$

At n=4

$a_4=\frac{4(4+1)}{2}$

$a_4=\frac{4.5}{2}$

$a_4=10$

Therefore first four terms are 1,3,6,10

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ii). $a_n=(-1)^{n-1}2^{n+1}$

Sol: Given general term $a_n=(-1)^{n-1}2^{n+1}$

At n=1

$a_1=(-1)^{1-1}2^{1+1}$

$a_1=1.2^2$

$a_1=4$

At n=2

$a_2=(-1)^{2-1}2^{2+1}$

$a_2=-1.2^3$

$a_2=-8$

At n=3

$a_3=(-1)^{3-1}2^{3+1}$

$a_3=1.2^4$

$a_3=16$

At n=4

$a_4=(-1)^{4-1}2^{4+1}$

$a_4=-1.2^5$

$a_4=-32$

Therefore first four terms are 4,-8,16,-32

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iii). $\left(\frac{1}{3}\right)^n$

Sol: Given general term $a_n=\left(\frac{1}{3}\right)^n$

$a_1=\left(\frac{1}{3}\right)^1=\frac{1}{3},\qquad a_2=\left(\frac{1}{3}\right)^2=\frac{1}{9}$

$a_3=\left(\frac{1}{3}\right)^3=\frac{1}{27},\qquad a_4=\left(\frac{1}{3}\right)^4=\frac{1}{81}$

Therefore first four terms are $\frac{1}{3},\frac{1}{3^2},\frac{1}{3^3},\frac{1}{3^4}$

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iv). $\frac{n(n-1)(n-2)}{6}$

Sol: Given general term $a_n=\frac{n(n-1)(n-2)}{6}$

At n=1

$a_1=\frac{1(1-1)(1-2)}{6}$

$a_1=\frac{1.0.(-1)}{6}$

$a_1=0$

At n=2

$a_2=\frac{2(2-1)(2-2)}{6}$

$a_2=\frac{2.1.0}{6}$

$a_2=0$

At n=3

$a_3=\frac{3(3-1)(3-2)}{6}$

$a_3=\frac{3.2.1}{6}$

$a_3=1$

At n=4

$a_4=\frac{4(4-1)(4-2)}{6}$

$a_4=\frac{4.3.2}{6}$

$a_4=4$

Therefore first four terms are 0,0,1,4

Q3. Write down the nth term of each sequence as suggested by the pattern.

i). $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots$

Sol: Given $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots$ Or $\frac{1}{1+1},\frac{2}{2+1},\frac{3}{3+1},\ldots$

$a_n=\frac{n}{(n+1)},n=1,2,3,\ldots$

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ii). 2,-4,6,-8,10,…

Sol: Given 2,-4,6,-8,10,…

$a_n=(-1)^{n+1}.(2n),n=1,2,3,\ldots$

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iii). 1,-1,1,-1,…

Sol: Given 1,-1,1,-1,…

$a_n=(-1)^{n+1},n=1,2,3,\ldots$

Q4. Write down the first five terms of each sequence defined recursively.

i). $a_1=3,a_{n+1}=5-a_n$

Sol: Given $a_1=3,a_{n+1}=5-a_n$

At n = 1

$a_{1+1}=5-a_1$

$a_2=5-3$

$\Rightarrow a_2=2$

$a_1=3$

At n = 2

$a_{2+1}=5-a_2$

$a_3=5-2$

$\Rightarrow a_3=3$

$a_2=2$

At n = 3

$a_{3+1}=5-a_3$

$a_4=5-3$

$\Rightarrow a_4=2$

$a_3=3$

At n = 4

$a_{4+1}=5-a_4$

$a_5=5-2$

$\Rightarrow a_5=3$

$a_4=2$

At n = 5

$a_{5+1}=5-a_5$

$a_6=5-3$

$\Rightarrow a_6=2$

$a_5=3$

Therefore first six terms are 3,2,3,2,3,2

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ii). $a_1=3,a_{n+1}=\frac{a_n}{n}$

Sol: Given $a_1=3,a_{n+1}=\frac{a_n}{n}$

At n = 1

$a_{1+1}=\frac{a_1}{1}=\frac{3}{1}$

$\Rightarrow a_2=3$

$a_1=3$

At n = 2

$a_{2+1}=\frac{a_2}{2}$

$\Rightarrow a_3=\frac{3}{2}$

$a_2=3$

At n = 3

At n = 4

$a_{3+1} = \frac{a_3}{3}$ $\quad a_3 = \frac{3}{2}$

$a_4 = \frac{\frac{3}{2}}{3}$

$\Rightarrow a_4 = \frac{1}{2}$

$a_{4+1} = \frac{a_4}{4}$

$a_5 = \frac{\frac{1}{2}}{4}$

$\Rightarrow a_5 = \frac{1}{8}$

At n = 5

$a_{5+1} = \frac{a_5}{5}$ $\quad a_5 = \frac{1}{8}$

$a_6 = \frac{\frac{1}{8}}{5}$

$\Rightarrow a_6 = \frac{1}{40}$

Therefore first six terms are $3, 3, \frac{3}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{40}$

Q5. Write each of the following series in expended form.

i). $\sum_{j=1}^{6}(2j-3)$

Sol: Given $\sum_{j=1}^{6}(2j-3)$

$$\sum_{j=1}^{6}(2j-3) = (2.1-3)+(2.2-3)+(2.3-3)$$
$$+(2.4-3)+(2.5-3)+(2.6-3)$$
$$\sum_{j=1}^{6}(2j-3) = -1+1+3+5+7+9$$

ii). $\sum_{k=1}^{5}(-1)^k 2^{k-1}$

Sol: Given $\sum_{k=1}^{5}(-1)^k 2^{k-1}$

$$\sum_{k=1}^{5}(-1)^k 2^{k-1} = (-1)^1 2^{1-1} + (-1)^2 2^{2-1} + (-1)^3 2^{3-1}$$
$$+(-1)^4 2^{4-1} + (-1)^5 2^{5-1}$$
$$= -1.2^0 + 1.2^1 - 1.2^2 + 1.2^3 - 1.2^4$$
$$= -1+2-4+8-16$$

iii). $\sum_{j=1}^{\infty}\frac{1}{2^j}$

Sol: Given $\sum_{j=1}^{\infty}\frac{1}{2^j}$

$$\sum_{j=1}^{\infty}\frac{1}{2^j} = \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\cdots$$

iv). $\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k$

Sol: Given $\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k$

$$\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k = 1+\left(\frac{3}{2}\right)^1+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\left(\frac{3}{2}\right)^4+\cdots$$
$$\left(\frac{3}{2}\right)^0 = 1$$

Q6. Write the following in terms of factorials.

i). $8\times7\times6\times5$

Sol: Given $8\times7\times6\times5$

$$8\times7\times6\times5 = 8\times7\times6\times5\times\frac{4!}{4!}$$
$$8\times7\times6\times5 = \frac{8!}{4!}$$

ii). $n(n-1)(n-2)$

Sol: Given $n(n-1)(n-2)$

$$n(n-1)(n-2) = n(n-1)(n-2).\frac{(n-3)!}{(n-3)!}$$
$$n(n-1)(n-2) = \frac{n!}{(n-3)!}$$

Q7. Find the Pascal sequence by using its general recursive definition. Note that $n \ge r$

i). n=5

Sol: Since $P_{r+1} = \frac{n-r}{r+1}P_r$ with $r = 0,1,2,3,\ldots$

Here $n=5$ we get

$P_{r+1} = \frac{5-r}{r+1}P_r, r = 0,1,2,3,4,\ldots$

At $r=0$ & using $P_o = 1$

$P_{0+1} = \frac{5-0}{0+1}P_0$

$P_1 = 5.1$ $\quad P_o = 1$

$P_1 = 5$

At $r=1$ & using $P_1 = 5$

$P_{1+1} = \frac{5-1}{1+1}P_1$

$P_2 = \frac{4}{2}.5$ $\quad P_1 = 5$

$P_2 = 10$

At $r=2$ & using $P_2 = 10$

$P_{2+1} = \frac{5-2}{2+1}P_2$

$P_3 = \frac{3}{3}.10$ $\quad P_2 = 10$

$P_3 = 10$

At $r=3$ & using $P_3 = 10$

$P_{3+1} = \frac{5-3}{3+1}P_3$

$P_4 = \frac{2}{4}.10$ $\quad P_3 = 10$

$P_4 = 5$

At $r=4$ & using $P_4 = 5$

$P_{4+1} = \frac{5-4}{4+1}P_4$

$P_5 = \frac{1}{5}.5$ $\quad P_4 = 5$

$P_5 = 1$

At $r=5$ & using $P_5 = 1$

$P_{5+1} = \frac{5-5}{5+1}P_5$

$P_6 = \frac{0}{6}.5$ $\quad P_5 = 1$

$P_6 = 0$

Or $\binom{5}{0}=1, \binom{5}{1}=5, \binom{5}{2}=10, \binom{5}{3}=10, \binom{5}{4}=5, \binom{5}{5}=1$

∴Pascal sequence for n = 5 are 1,5,10,10,5,1,0,0,...

ii). n=6

Sol: Since $P_{r+1} = \frac{n-r}{r+1}P_r$ with $p_o = 1, r = 0,1,2,3,\ldots$

Here $n=6$ so we get $P_{r+1} = \frac{6-r}{r+1}P_r$

Put $r = 0 \therefore P_0 = 1$

$P_{0+1} = \frac{6-0}{0+1}P_0$

$P_1 = \frac{6}{1}.1$ $\quad \therefore P_0 = 1$

$P_1 = 6$

Put $r = 1 \therefore P_1 = 6$

$P_{1+1} = \frac{6-1}{1+1}P_1$

$P_2 = \frac{5}{2}(6)$ $\quad \therefore P_1 = 6$

$P_2 = 15$

Put $r=2$ & $P_2 = 15$

$P_{2+1} = \frac{6-2}{2+1}P_2$

$P_3 = \frac{4}{3}(15)$ $\quad P_2 = 15$

$P_3 = 20$

Put $r=3$ & $P_3 = 20$

$P_{3+1} = \frac{6-3}{3+1}P_3$

$P_4 = \frac{3}{4}(20)$ $\quad P_3 = 20$

$P_4 = 15$

Put $r=4$ & $P_4 = 15$

$P_{4+1} = \frac{6-4}{4+1}P_4$

$P_5 = \frac{2}{5}(15)$ $\quad P_4 = 15$

$P_5 = 6$

Put $r=5$ & $P_5 = 6$

$P_{5+1} = \frac{6-5}{5+1}P_5$

$P_6 = \frac{1}{6}(6)$ $\quad P_5 = 6$

$P_6 = 1$

Put $r=6$ & $P_6 = 1$

$P_{6+1}=\frac{6-6}{6+1}P_6$

$P_7=\frac{0}{7}(1)$ $P_6=1$

$P_7=0$

Or we can calculate

$\binom{6}{r}, r=0,1,2,3,4,\ldots$

$\binom{6}{0}=1,\binom{6}{1}=6,\binom{6}{2}=15,\binom{6}{3}=20,\binom{6}{4}=15,\binom{6}{5}=6,\binom{6}{6}=1$

∴Pascal sequence for n = 6 are 1,6,15,20,15,6,1,0,0,0,....

iii). n=8

Sol: Since $P_{r+1}=\frac{n-r}{r+1}P_r$ with $p_o=1, r=0,1,2,3,\ldots$

We have $n=8$ then $P_{r+1}=\frac{8-r}{r+1}P_r$ ....(1)

Put $r=0$ & $P_0=1$

$P_{0+1}=\frac{8-0}{0+1}P_0$

$P_1=\frac{8}{1}(1)$ $P_0=1$

$P_1=8$

Put $r=1$ & $P_1=8$

$P_{1+1}=\frac{8-1}{1+1}P_1$

$P_2=\frac{7}{2}(8)$ $P_1=8$

$P_2=28$

Put $r=2$ & $P_2=28$

$P_{2+1}=\frac{8-2}{2+1}P_2$

$P_3=\frac{6}{3}(28)$ $P_2=28$

$P_3=56$

Put $r=3$ & $P_3=56$

$P_{3+1}=\frac{8-3}{3+1}P_3$

$P_4=\frac{5}{4}(56)$ $P_3=56$

$P_4=70$

Put $r=4$ & $P_4=70$

$P_{4+1}=\frac{8-4}{4+1}P_4$

$P_5=\frac{4}{5}(70)$ $P_4=70$

$P_5=56$

Put $r=5$ & $P_5=56$

$P_{5+1}=\frac{8-5}{5+1}P_5$

$P_6=\frac{3}{6}(56)$ $P_5=56$

$P_6=28$

Put $r=6$ & $P_6=28$

$P_{6+1}=\frac{8-6}{6+1}P_6$

$P_7=\frac{2}{7}(28)$ $P_6=28$

$P_7=8$

Put $r=7$ & $P_7=8$

$P_{7+1}=\frac{8-7}{7+1}P_7$

$P_8=\frac{1}{8}(8)$ $P_7=8$

$P_8=1$

Put $r=8$ & $P_8=1$

$P_{8+1}=\frac{8-8}{8+1}P_8$

$P_9=\frac{0}{9}(1)$ $P_8=1$

$P_9=0$

Or we can calculate

$\binom{8}{r}, r=0,1,2,3,4,\ldots$

$\binom{8}{0}=1,\binom{8}{1}=8,\binom{8}{2}=28,\binom{8}{3}=56,\binom{8}{4}=70,$

$\binom{8}{5}=56,\binom{8}{6}=28,\binom{8}{7}=8,\binom{8}{8}=1$

∴Pascal sequence for n = 8 are 1,8,28,56,70,28,8,1,0,0,...

General term of AP $a_n=a_1+(n-1)d$

Common difference $d=a_n-a_{n-1}$ or $d=a_2-a_1$

Exercise 4.2

Q1. Find common difference, the 5th term, the 10th term and nth term of each of the given arithmetic sequence.

i). 2,7,12,17,...

Sol: Given 2,7,12,17,...

$d=a_2-a_1$

$d=7-2$

$d=5$

And $a_5=a_1+4d$

$a_5=2+4(5)$ $d=5$

$a_5=22$

And $a_{10}=a_1+9d$

$a_{10}=2+9(5)$ $d=5$

$a_{10}=47$

And $a_n=a_1+(n-1)d$

$a_n=2+5(n-1)$

$a_n=2+5n-5$

$a_n=5n-3$

So $d=5, a_5=22, a_{10}=47, a_n=5n-3$

ii). -4,-2,0,2,...

Sol: Given -4,-2,0,2,...

Here $a_1=-4,\ a_2=-2,\ a_3=0$

$d=a_2-a_1$

$d=-2-(-4)$

$d=2$

Now

$a_5=a_1+4d$

$a_5=-4+4(2)$

$a_5=4$

and

$a_{10}=a_1+9d$

$a_{10}=-4+9(2)$

$a_{10}=14$

Now

$a_n=-4+2(n-1)$ $a_n=a_1+(n-1)d$

$a_n=-4+2n-2$

$a_n=2n-6$

So $d=2, a_5=4, a_{10}=14, a_n=2n-6$

Q2. If in an A.P $a_1=43, a_{10}=7$, find a_{25}

Sol: Given $a_1=43, a_{10}=7$

Now

$a_{10}=a_1+9d$

$7=43+9d$

$-36=9d$

$d=-4$

and

$a_{25}=a_1+24d$

$a_{25}=43+24(-4)$

$a_{25}=43-96$

$a_{25}=-53$

Q3 If $a_6+a_4=6$ & $a_6-a_4=\frac{2}{3}$, find arithmetic sequence.

Sol: Given $a_6+a_4=6$ and $a_6-a_4=\frac{2}{3}$

$a_6+a_4=6$

$a_1+5d+a_1+3d=6$

$2a_1+8d=6$

$a_6-a_4=\frac{2}{3}$

$a_1+5d-a_1-3d=\frac{2}{3}$

$2d=\frac{2}{3}$

$d=\frac{1}{3}$

Putting the value of d

$2a_1+8\left(\frac{1}{3}\right)=6$

$2a_1=\frac{3}{3}\times\frac{6}{1}-\frac{8}{3}$

$2a_1=\frac{18-8}{3}$

$2a_1=\frac{10}{3}$

$a_1=\frac{5}{3}$

Now And

$a_2 = a_1 + d$

$a_2 = \frac{5}{3} + \frac{1}{3}$

$a_2 = \frac{6}{3} = 2$

$a_3 = a_1 + 2d$

$a_3 = \frac{5}{3} + 2\left(\frac{1}{3}\right)$

$a_3 = \frac{5+2}{3}$

$a_3 = \frac{7}{3}$

Now

$a_4 = a_1 + 3d$

$a_4 = \frac{5}{3} + 3\left(\frac{1}{3}\right)$

$a_4 = \frac{5+3}{3}$

$a_4 = \frac{8}{3}$

And

$a_5 = a_1 + 4d$

$a_5 = \frac{5}{3} + 4\left(\frac{1}{3}\right)$

$a_5 = \frac{5+4}{3}$

$a_5 = \frac{9}{3} = 3$

Therefore the arithmetic sequence is $\frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \ldots$

Q4 How many terms are there in a arithmetic sequence in which first and last terms are $\frac{33}{4}$ and $\frac{25}{2}$ respectively and the common difference is $\frac{1}{8}$

Sol: Given $a_1 = \frac{33}{4}, a_n = \frac{25}{2}, d = \frac{1}{8}$

Since we know that $a_n = a_1 + (n-1)d$

$\frac{25}{2} = \frac{33}{4} + \frac{1}{8}(n-1)$ $\quad \times by\ 8$

$8\times\frac{25}{2} = 8\times\frac{33}{4} + 8\times\frac{1}{8}(n-1)$

$4(25) = 2(33) + n - 1$

$100 = 66 + n - 1$

$n = 100 - 66 + 1$

$n = 35$

Q5. Which term of arithmetic sequence 4,1,-2,… is -77?

Sol: Given arithmetic sequence 4,1,-2,…

and the last term $a_n = -77$

Here $a_1 = 4, \quad a_2 = 1, \quad a_3 = -2,$

First we find the common difference

$d = a_2 - a_1$

$d = 1 - 4$

$d = -3$

Now using the formula and putting the values

$-77 = 4 - 3(n-1)$ $\quad a_n = a_1 + (n-1)d$

$-77 = 4 - 3n + 3$

$3n = 7 + 77$

$3n = 84$

$n = 28^{th}$

Q6. A ball rolling up an incline covered 24m during first second, 21m during the 2nd second, 18m during third second. Find how many meters it covered in the eighth second?

Sol: Given $a_1 = 24m, a_2 = 21m, a_3 = 18m$

First we find the common difference

$d = a_2 - a_1$

$d = 21m - 24m$

$d = -3m$

$a_8 = a_1 + 7d$ $\quad \because a_n = a_1 + (n-1)d$

$a_8 = 24m + 7(-3m)$

$a_8 = 24m - 21m$

$a_8 = 3m$

Q7. population of town is decreasing by 500 inhabitants each year. If its population at the beginning of 1960 was 20135, what was its population at the beginning of 1970?

Sol: In 1960 population is 20135 so $a_1 = 20135$

Population decreased 500 per year so $d = -500$

1960 to 1970 will have 10 years so $a_{10} = ?$

$a_{10} = a_1 + 9d$ $\quad \because a_n = a_1 + (n-1)d$

$a_{10} = 20135 + 9(-500)$

$a_{10} = 20135 - 4500$

$a_{10} = 15635$

Q8. Ahmad and Akram can climb 1000 feet in the first hour and 100 feet each succeeding hour. When will they reach at the top of 5400 feet hill?

Sol: During first hour $a_1 = 1000\ feet$

Succeeding hour $d = 100\ feet$

Total feet $a_n = 5400\ feets$

Using formula

$a_n = a_1 + (n-1)d$

$5400 = 1000 + 100(n-1)$

$5400 - 1000 = 100(n-1)$

$4400 = 100(n-1)$

$44 = n - 1$

$n = 44 + 1$

$n = 45\ hours$

Q9. A man earned \$3500 the first year he worked. If he received a raise of \$750 at the end of year for 20 years, what was the salary during his twenty first year of work?

Sol: Given Salary during first year $a_1 = \$3500$

Salary raise each year $d = \$750$

After twenty first year $a_{21} = ?$

So $a_{21} = a_1 + (21-1)d$

$a_{21} = \$3500 + 20\times\$750$

$a_{21} = \$3500 + \$15000$

$a_{21} = \$18500$

Arithmetic mean between a and b $\quad AM = \frac{a+b}{2}$

Exercise 4.3

Q1. Find the arithmetic mean between given numbers.

i). 12,18

Solution: Let $a = 12, b = 18$ using $\therefore AM = \frac{a+b}{2}$

$AM = \frac{12+18}{2}$

$AM = \frac{30}{2}$

$AM = 15$

ii). $\frac{1}{3}, \frac{1}{4}$

Sol: Let $a = \frac{1}{3}, b = \frac{1}{4}$ using $\therefore AM = \frac{a+b}{2}$

$AM = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right)$

$AM = \frac{1}{2}\left(\frac{4+3}{12}\right)$

$AM = \frac{7}{24}$

iii). -6, -216

Sol: Let $a = -6, b = -216$ using $\therefore AM = \frac{a+b}{2}$

$AM = \frac{-6-216}{2}$

$AM = \frac{-222}{2}$

$AM = -111$

iv). $(a+b)^2, (a-b)^2$

Sol: $a = (a+b)^2, b = (a-b)^2$ using $\therefore AM = \frac{a+b}{2}$

$AM = \frac{(a+b)^2 + (a-b)^2}{2}$

$AM = \frac{a^2 + b^2 + 2ab + a^2 + b^2 - 2ab}{2} = \frac{2a^2 + 2b^2}{2}$

$$AM = \frac{2\left(a^2+b^2\right)}{2}$$
$$AM = a^2+b^2$$

Q2. Insert: i). Three arithmetic means between 6 and 41.

Sol: Let A_1, A_2, A_3, are arithmetic means between 6 and 41. i.e. $6, A_1, A_2, A_3, 41$ From AP

Let a_1, a_2, a_3, a_4, a_5 Be corresponding terms then

$a_1 = 6, a_2 = A_1, a_3 = A_2, a_4 = A_3, a_5 = 41$

$a_5 = a_1 + 4d \qquad \because a_n = a_1 + (n-1)d$

$41 = 6 + 4d$

$41 - 6 = 4d$

$d = \frac{35}{4}$

And $A_1 = a_2 = a_1 + d$

$A_1 = 6 + \frac{35}{4}$

$A_1 = \frac{59}{4} = 14\frac{3}{4}$

And $A_2 = a_3 = a_1 + 2d$

$A_2 = 6 + 2\left(\frac{35}{4}\right) = \frac{24+70}{4}$

$A_2 = \frac{94}{4} = \frac{47}{2} = 23\frac{1}{2}$

And $A_3 = a_4 = a_1 + 3d$

$A_3 = 6 + 3\left(\frac{35}{4}\right) = \frac{24+105}{4}$

$A_3 = \frac{129}{4} = 32\frac{1}{4}$

Therefore three arithmetic means are

$14\frac{3}{4}, 23\frac{1}{2}, 32\frac{1}{4}$

ii). Four arithmetic means between 17 and 32.

Solution: Let A_1, A_2, A_3, A_4, are the arithmetic means between17 and 32. i.e. $17, A_1, A_2, A_3, A_4, 32$ from an AP

Let $a_1, a_2, a_3, a_4, a_5, a_6$ be corresponding terms

Where $a_1 = 17, a_2 = A_1, a_3 = A_2, a_4 = A_3, a_5 = A_4, a_6 = 32$

$a_6 = a_1 + 5d \qquad \because a_n = a_1 + (n-1)d$

$32 = 17 + 5d$

$32 - 17 = 5d$

$d = \frac{15}{5}$

$d = 3$

Now

$A_1 = a_2 = a_1 + d$

$A_1 = 17 + 3$

$A_1 = 20$

and

$A_2 = a_3 = a_1 + 2d$

$A_2 = 17 + 2(3)$

$A_2 = 23$

And

$A_3 = a_4 = a_1 + 3d$

$A_3 = 17 + 3(3)$

$A_3 = 26$

And

$A_4 = a_5 = a_1 + 4d$

$A_4 = 17 + 4(3)$

$A_4 = 29$

Therefore four arithmetic means are 20,23,26,29

iii). Five arithmetic means between 9 and 33.

Sol: Let A_1, A_2, A_3, A_4, A_5, are the arithmetic means between 9 & 33. i.e. $9, A_1, A_2, A_3, A_4, A_5, 33$ from an AP

Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ be corresponding terms,

Where $a_1 = 9, a_2 = A_1, a_3 = A_2, a_4 = A_3, a_5 = A_4, a_6 = A_5, a_7 = 33$

$a_7 = a_1 + 6d \qquad \because a_n = a_1 + (n-1)d$

$33 = 9 + 6d$

$33 - 9 = 6d$

$d = \frac{24}{6} = 4$

Now

$A_1 = a_2 = a_1 + d$

$A_1 = 9 + 4$

$A_1 = 13$

And

$A_2 = a_3 = a_1 + 2d$

$A_2 = 9 + 2(4)$

$A_2 = 17$

And

$A_3 = a_4 = a_1 + 3d$

$A_3 = 9 + 3(4)$

$A_3 = 21$

and

$A_4 = a_5 = a_1 + 4d$

$A_4 = 21 + 4$

$A_4 = 25$

And

$A_5 = a_6 = a_1 + 5d$

$A_5 = 9 + 5(4)$

$A_5 = 29$

Therefore five arithmetic means are 13,17,21,25,29

Q3. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is an arithmetic mean between a & b.? Where a &b are not zero simultaneously.

Sol: Given that AM= $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ from definition arithmetic mean between two numbers is $\frac{a+b}{2}$ so both will be equal

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2} \left(\text{cross multiplication}\right)$$
$$2\left(a^{n+1}+b^{n+1}\right) = (a+b)\left(a^n+b^n\right)$$
$$2a^{n+1}+2b^{n+1} = a^{n+1}+ab^n+a^nb+b^{n+1}$$
$$2a^{n+1}-a^{n+1}+2b^{n+1}-b^{n+1} = ab^n+a^nb$$
$$a^{n+1}+b^{n+1} = ab^n+a^nb$$
$$a^{n+1}-a^nb = ab^n-b^{n+1}$$
$$a^n(a-b) = b^n(a-b)$$
$$\frac{a^n}{b^n} = \frac{(a-b)}{(a-b)}$$
$$\left(\tfrac{a}{b}\right)^n = 1$$
$$\left(\tfrac{a}{b}\right)^n = \left(\tfrac{a}{b}\right)^0$$
$$\Rightarrow n = 0$$

Q4. Insert five arithmetic means between 5 and 8 and show that their sum is five times the arithmetic means between 5 and 8.

Sol: Let A_1, A_2, A_3, A_4, A_5, are the arithmetic means between 5 and 8. i.e. $5, A_1, A_2, A_3, A_4, A_5, 8$ from an AP

Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ be corresponding terms

Where $a_1 = 5, a_2 = A_1, a_3 = A_2, a_4 = A_3, a_5 = A_4, a_6 = A_5, a_7 = 8$

Now

$\because a_n = a_1 + (n-1)d$

$a_7 = a_1 + 6d$

$8 = 5 + 6d$

$8 - 5 = 6d$

$d = \frac{3}{6} = \frac{1}{2}$

and

$A_1 = a_2 = a_1 + d$

$A_1 = 5 + \frac{1}{2}$

$A_1 = \frac{11}{2} = 5\frac{1}{2}$

And

$A_2 = a_3 = a_1 + 2d$

$A_2 = 5 + 2\left(\frac{1}{2}\right) = 6$

and

$A_3 = a_4 = a_1 + 3d$

$A_3 = 5 + 3\left(\frac{1}{2}\right)$

$A_3 = \frac{13}{2} = 6\frac{1}{2}$

And

$A_4 = a_5 = a_1 + 4d$

$A_4 = 5 + 4\left(\frac{1}{2}\right)$

$A_4 = 5 + 2 = 7$

and

$A_5 = a_6 = a_1 + 5d$

$A_5 = 5 + 5\left(\frac{1}{2}\right)$

$A_5 = \frac{15}{2} = 7\frac{1}{2}$

Therefore five arithmetic means are

$5\frac{1}{2}, 6, 6\frac{1}{2}, 7, 7\frac{1}{2}$

To show that their sum is five times the arithmetic means between 5 and 8.

$$\left(\frac{11}{2}+\frac{12}{2}+\frac{13}{2}+\frac{14}{2}+\frac{15}{2}\right) = 5.\left(\frac{5+8}{2}\right)$$
$$\frac{65}{2} = \frac{5\times13}{2} \text{ Hence proved}$$

Q5. There are n arithmetic means between 5 & 32 such that ratio of 3rd and 7th means is 7:13. Find value of n.

Solution: Let $A_1, A_2, A_3, A_4, \ldots, A_n$, are the arithmetic means between 5 and 32.

i.e. $5, A_1, A_2, A_3, A_4, ..., A_n, 32$ from an AP

Let $a_1, a_2, a_3, a_4, a_5, ..., a_{n+1}, a_{n+2}$ be corresponding terms

Where $a_1 = 5, a_2 = A_1, a_3 = A_2, a_4 = A_3, ..., a_{n+1} = A_n, a_{n+2} = 32$

According to condition of question

i.e. ratio of the 3rd and 7th means is 7:13 so

then $\frac{A_3}{A_7} = \frac{7}{13}$ putting the corresponding terms

$\frac{a_4}{a_8} = \frac{7}{13}$ by cross multiplication

$13a_4 = 7a_8$ $\quad \because a_n = a_1 + (n-1)d$

$13(a_1 + 3d) = 7(a_1 + 7d)$

$13a_1 + 39d = 7a_1 + 49d$ $\quad a_1 = 5$

$13(5) + 39d = 7(5) + 49d$

$65 + 39d = 35 + 49d$

$65 - 35 = 49d - 39d$

$30 = 10d$

$d = 3$

And $a_{n+2} = a_1 + (n+1)d$

$32 = 5 + 3(n+1)$

$32 - 5 = 3(n+1)$

$27 = 3(n+1)$

$n + 1 = 9$

$n = 9 - 1$

$n = 8$

Sum of first n terms of AP $S_n = \frac{n}{2}(a_1 + a_n)$

Sum of first n terms of AP $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Nth term of AP $a_n = a_1 + (n-1)d$

Exercise 4.4

Q1.Find the indicated term and sum of the indicated number of terms in case of each of the following arithmetic sequence:

i). 9,7,5,3,.... ; 20th term; 20 terms

Sol: Here $a_1 = 9, a_2 = 7, a_3 = 5, n = 20$

$d = a_2 - a_1$

$d = 7 - 9$

$d = -2$

And $a_{20} = a_1 + 19d$ $\quad \because a_n = a_1 + (n-1)d$

$a_{20} = 9 + 19(-2) = 9 - 38$

$a_{20} = -29$

Now

$S_{20} = \frac{20}{2}(a_1 + a_{20})$ $\quad \because S_n = \frac{n}{2}(a_1 + a_n)$

$S_{20} = 10(9 - 29)$

$S_{20} = 10(-20)$

$S_{20} = -200$

ii). $3, \frac{8}{3}, \frac{7}{3}, 2, ...$ 11th term; 11 terms

Sol: Here $a_1 = 3, a_2 = \frac{8}{3}, a_3 = \frac{7}{3}, n = 11$

Common difference

$d = a_2 - a_1$

$d = \frac{8}{3} - 3$

$d = -\frac{1}{3}$

nth term

$a_{11} = a_1 + 10d$

$a_{11} = 3 + 10\left(\frac{-1}{3}\right)$

$a_{11} = 3 - \frac{10}{3} = \frac{-1}{3}$

Now $S_{11} = \frac{11}{2}(a_1 + a_{11})$ $\quad \because S_n = \frac{n}{2}(a_1 + a_n)$

$S_{11} = \frac{11}{2}\left(3 - \frac{1}{3}\right)$

$S_{11} = \frac{11}{2}\left(\frac{8}{3}\right)$

$S_{11} = \frac{44}{3}$

Q2. Some of the components a_1, a_n, n, d and S_n are given. Find the ones that are missing:

i). $a_1 = 2, n = 17, d = 3$ $\quad S_{17} = ?, a_{17} = ?$

Sol: Given $a_1 = 2, n = 17, d = 3$

Now

$\because a_n = a_1 + (n-1)d$

$a_{17} = a_1 + 16d$

$a_{17} = 2 + 16 \times 3$

$a_{17} = 2 + 48$

$a_{17} = 50$

And

$\because S_n = \frac{n}{2}(a_1 + a_n)$

$S_{17} = \frac{17}{2}(a_1 + a_{17})$

$S_{17} = \frac{17}{2}(2 + 50)$

$S_{17} = \frac{17}{2}(52)$

$S_{17} = 17 \times 26$

$S_{17} = 442$

ii). $a_1 = -40, S_{21} = 210, \quad a_{21} = ?, d = ?$

Sol: Given $a_1 = -40, S_{21} = 210, \quad a_{21} = ?, d = ?$

Now

$\because S_n = \frac{n}{2}(a_1 + a_n)$

$S_{21} = \frac{21}{2}(a_1 + a_{21})$

$210 = \frac{21}{2}(-40 + a_{21})$

$\frac{2 \times 210}{21} = -40 + a_{21}$

$20 + 40 = a_{21}$

$a_{21} = 60$

And

$\because a_n = a_1 + (n-1)d$

$a_{21} = a_1 + 20d$

$60 = -40 + 20d$

$60 + 40 = 20d$

$20d = 100$

$d = 5$

iii). $a_1 = -7, d = 8, S_n = 225 \quad a_n = ?, n = ?$

Sol: Given $a_1 = -7, d = 8, S_n = 225 \quad a_n = ?, n = ?$

$225 = \frac{n}{2}(-14 + 8(n-1))$ $\quad S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$2 \times 225 = n(8n - 22)$

$450 = 8n^2 - 22n$

$8n^2 - 22n - 450 = 0$

$4n^2 - 11n - 225 = 0$

$4n^2 - 36n + 25n - 225 = 0$

$4n(n-9) + 25(n-9) = 0$

$(n-9)(4n+25) = 0$

$\therefore n - 9 = 0, \quad 4n + 25 = 0$

$n = 9$, $\quad n = \frac{25}{4}$ not possible

so take $n = 9$

And $a_9 = a_1 + 8d$ $\quad \because a_n = a_1 + (n-1)d$

$a_9 = -7 + 8 \times 8$

$a_9 = -7 + 64$

$a_9 = 57$

iv). $a_n = 4, S_{15} = 30 \quad \Rightarrow n = 15, d = ?, a_1 = ?$

Sol: Here $a_n = 4, S_{15} = 30 \Rightarrow n = 15$

Now

$\because S_n = \frac{n}{2}(a_1 + a_n)$

$S_{15} = \frac{15}{2}(a_1 + a_{15})$

$30 = \frac{15}{2}(a_1 + 4)$

$\frac{2 \times 30}{15} = a_1 + 4$

$4 = a_1 + 4$

$a_1 = 0$

And

$\because a_n = a_1 + (n-1)d$

$a_{15} = a_1 + 14d$

$4 = 0 + 14d$

$14d = 4$

$d = \frac{4}{14}$

$d = \frac{2}{7}$

Q3 Find sum of all numbers divided by 5 from 25 through 350

Sol: Numbers divided by 5 from 25 through 350 are 25,30,35,40,...,350

Let $a_1 = 25, a_2 = 30, a_3 = 35, a_n = 350$

$d = a_2 - a_1$

$d = 30 - 25$ And

$d = 5$

$350 = 25 + 5(n-1)$ $\qquad a_n = a_1 + (n-1)d$

$350 - 25 = 5(n-1)$

$325 = 5(n-1)$

$(n-1) = 65$

$n = 65 + 1$

$n = 66$

Now $S_{66} = \frac{66}{2}(25+350)$ $\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$S_{66} = 33 \times 375$

$S_{66} = 12375$

Q4. The sum of three numbers in an arithmetic sequence is 36 and sum of their cubes is 6336. find them [Hint: suppose the number are $a-d, a, a+d$]

Sol: suppose the number are

$a_1 = a - d,\quad a_2 = a,\quad a_3 = a + d$

according to condition of question

$a_1 + a_2 + a_3 = 36$

Putting the values

$a - d + a + a + d = 36$

$3a = 36$

$a = 12$

And $a_1^3 + a_2^3 + a_3^3 = 6336$

$(a-d)^3 + a^3 + (a+d)^3 = 6336$

$a^3 - d^3 - 3a^2d + 3ad^2 + a^3 + a^3 + d^3 + 3a^2d + 3ad^2 = 6336$

$3a^3 + 6ad^2 = 6336$

divided by 3 and put $a = 12$

$12^3 + 2 \times 12d^2 = 2112$

$1728 + 24d^2 = 2112$

$24d^2 = 2112 - 1728$

$24d^2 = 384$

$d^2 = 16$

$d = \pm 4$

When $d = 4$ and $a = 12$

$a_1 = 12 - 4,\quad a_2 = 12,\quad a_3 = 12 + 4$

$a_1 = 8,\quad a_2 = 12,\quad a_3 = 16$

when $d = -4$ and $a = 12$

$a_1 = 12 + 4,\quad a_2 = 12,\quad a_3 = 12 - 4$

$a_1 = 16,\quad a_2 = 12,\quad a_3 = 8$

Q5. Find 1+3-5+7+9-11+13+15-17+... up to 3n terms?

Solution: 1+3-5+7+9-11+13+15-17+... up to 3n terms Consider

First	Second	Third
1,7,13,...	3,9,15,...	5,11,17,...
$d_1 = 6$	$d_2 = 6$	$d_3 = 6$
$a_n = 1 + 6.(n-1)$	$b_n = 3 + 6.(n-1)$	$c_n = 5 + 6.(n-1)$
$a_n = 6n - 5$	$b_n = 6n - 3$	$c_n = 6n - 1$

Now the sum up to 3n terms

$S_{3n} = S_n + S_n + S_n$

$S_{3n} = \frac{n}{2}(a_1 + a_n) + \frac{n}{2}(b_1 + b_n)\frac{n}{2}(c_1 + c_n)$

$S_{3n} = \frac{n}{2}(1 + 6n - 5) + \frac{n}{2}(3 + 6n - 3) - \frac{n}{2}(5 + 6n - 1)$

$S_{3n} = \frac{n}{2}(6n-4) + \frac{n}{2}(6n) - \frac{n}{2}(6n+4)$

$S_{3n} = n(3n-2) + n(3n) - n(3n+2)$

$S_{3n} = n[3n - 2 + 3n - 3n - 2]$

$S_{3n} = n[3n-4]$

Q5. Find 1+3-5+7+9-11+13+15-17+... up to 3n terms?

Solution: 1+3-5+7+9-11+13+15-17+... up to 3n terms

Adding the three terms to get one term

-1 + 5 + 11 + ... n terms

With $a_1 = -1, d = 6$

$S_n = \frac{n}{2}[2(-1) + 6(n-1)]$ $\qquad S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$S_n = \frac{n}{2}(-2 + 6n - 6)$

$S_n = \frac{n}{2}(6n - 8)$

$S_{3n} = n[3n-4]$

Q6. Show that sum of first n positive odd integers is n^2.

Sol: Take first n positive odd integers 1,3,5,...

$a_1 = 1, a_2 = 3, a_3 = 5$

$d = a_2 - a_1 = 3 - 1 = 2$

Now using $S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$S_n = \frac{n}{2}(2 + 2(n-1))$

$S_n = \frac{n}{2}(2 + 2n - 2) = \frac{n}{2}(2n)$

$S_n = n^2$

Q7. Find four numbers in arithmetic sequence, whose sum is 20 and the sum of whose squares is 120.

[Hint: suppose numbers are $a-3b, a-d, a+d, a+3d$]

Solution: Suppose the numbers are

$a_1 = a - 3b,\quad a_2 = a - d,\quad a_3 = a + d,\quad a_4 = a + 3d$

According to question

$a_1 + a_2 + a_3 + a_4 = 20$

$a - 3b + a - d + a + d + a + 3d = 20$

$4a = 20$

$\Rightarrow a = 5$ and

$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 120$

$(a-3b)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 120$

$a^2 + 9d^2 - 6ad + a^2 + d^2 - 2ad + a^2 + d^2 + 2ad + a^2 + 9d^2 + 6ad = 120$

$\Rightarrow 4a^2 + 20d^2 = 120$

$a^2 + 5d^2 = 30$

Putting the value of a

$5^2 + 5d^2 = 30$

$5d^2 = 30 - 25$

$5d^2 = 5$

$d^2 = 1$ $\qquad \Rightarrow d = \pm 1$

When $a = 5, d = 1$

$a_1 = 5 - 3,\quad a_2 = 5 - 1,\quad ,a_3 = 5 + 1,\quad a_4 = 5 + 3$

$a_1 = 2,\quad a_2 = 4,\quad a_3 = 6,\quad a_4 = 8$

When $a = 5, d = -1$

$a_1 = 5 + 3,\quad a_2 = 5 + 1,\quad a_3 = 5 - 1,\quad a_4 = 5 - 3$

$a_1 = 8,\quad a_2 = 6,\quad a_3 = 4,\quad a_4 = 2$

Q8. Sum of Rs. 1000 is distributed among four people so that each person after first receive Rs. 20 less then the preceding person. How much does each person receive?

Sol: According to conditions $d = -20$ and

$a_1 + a_2 + a_3 + a_4 = 1000$

$a_1 + a_1 + d + a_1 + 2d + a_1 + 3d = 1000$

$4a_1 + 6d = 1000$

$2a_1 + 3d = 500$

Putting the value of d

$2a_1 + 3(-20) = 500$

$2a_1 - 60 = 500$

$2a_1 = 500 + 60$

$a_1 = \frac{560}{2}$

$a_1 = 280$

And

$a_2 = a_1 + d$

$a_2 = 280 - 20$

$a_2 = 260$

And

$a_3 = a_1 + 2d$

$a_3 = 280 - 2(20)$

$a_3 = 240$

And

$a_4 = a_1 + 3d$

$a_4 = 280 - 3(20)$

$a_4 = 220$

Therefore four persons take 280,260,240,220

Q9. To dig a well a company charges \$10 for the first foot, \$12.50 for second foot, \$15 for the third foot and so on. What is depth of a well that costs \$2925 to dig?

Sol: According to conditions

$a_1 = \$10, a_2 = \$12.50, a_3 = \$15, d = \$2.5$

Now using $S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

$\$2925 = \frac{n}{2}\{2\times\$10 + \$2.50(n-1)\}$

$2\times\$2925 = \$n\{20 + 2.5(n-1)\}$

$5850 = 20n + 2.5n^2 - 2.5n$

$\Rightarrow 2.5n^2 + 17.5n - 5850 = 0$

Divided by 2.5

$n^2 + 7n - 2340 = 0$

$n^2 + 52n - 45n - 2340 = 0$

$n(n+52) - 45(n+52) = 0$

$(n+52)(n-45) = 0$

$\therefore n + 52 = 0 \quad or \quad n - 45 = 0$

$n = -52 \quad$ or $\quad n = 45$

$n = -52$ is not possible, so Take only n=45

Q10. Distance which an object dropped from a cliff will fall 16 feet the first second, 48 feet the next second, 80 feet the third second and so on. What is the total distance the object will fall in six second?

Sol: $a_1 = 16, a_2 = 48, a_3 = 80, d = 32$

$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

$S_6 = \frac{6}{2}\{2\times16 + 32(6-1)\}$

$S_6 = 3(32 + 32\times5)$

$S_6 = 3(32 + 160)$

$S_6 = 3\times192 = 576\ feet$

Q11. Affan save Rs. 1 the first day, Rs. 2 the second day, Rs. 3 the third day and Rs. N on the nth day for thirty days. How much does he save at the end of the thirtieth day?

Sol: Given $a_1 = 1, a_2 = 2, a_3 = 3,$

$d = a_2 - a_1$

$d = 2 - 1$

$d = 1$

Now using $a_{30} = a_1 + 29d$

$a_{30} = 1 + 29(1)$

$a_{30} = 30$

Now $S_{30} = \frac{30}{2}[1 + 30]$ using $S_n = \frac{n}{2}\{a_1 + a_n\}$

$S_{30} = 15(31)$

$S_{30} = RS.465$

Q12. A contest will have five cash prizes totaling Rs. 5000 and there will be a Rs. 100 difference between successive prizes. Find the first prize.

Sol: $S_5 = 5000, d = 100$

$S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

$S_5 = \frac{5}{2}\{2a_1 + 100.(5-1)\}$

$5000 = \frac{5}{2}\{2a_1 + 100.(4)\}$

$\frac{5000\times2}{5} = 2a_1 + 400$

$2000 = 2a_1 + 400$

$1000 = a_1 + 200$

$a_1 = 1000 - 200$

$a_1 = 800$

Q13 A theater has 40 rows with 20 seats in first row, 23 in second row, 26 in third row an so on. How many seats are in the theater?

Sol: $n = 40, a_1 = 20, a_2 = 23, a_3 = 26, d = 3$

Now $\because S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

$S_{40} = \frac{40}{2}\{2\times20 + 3(40-1)\}$

$S_{40} = 20\{40 + 3\times39\}$

$S_{40} = 20(40 + 117)$

$S_{40} = 20\times157$

$S_{40} = 3140$

General term of GP $\quad a_n = a_1 r^{n-1}$

Exercise 4.5

Q1. Write first five terms of a geometric sequence given that:

i). $a_1 = 5; r = 3$

Sol: Given $a_1 = 5; \quad r = 3$

$a_2 = a_1 r$

$a_2 = 5\times3$

$a_2 = 15$

$a_3 = a_2 r \quad \because a_2 = a_1 r$

$a_3 = 15\times3$

$a_3 = 45$

$a_4 = a_3 r \quad \because a_3 = a_1 r^2$

$a_4 = 45\times3$

$a_4 = 135$

$a_5 = a_4 r \quad \because a_4 = a_1 r^3$

$a_5 = 135\times3$

$a_5 = 405$

First five terms are $a_1, a_2, a_3, a_4, a_5, \ldots$

i.e., $5, 15, 45, 135, 405, \ldots$

ii). $a_1 = 8; \quad r = \frac{-1}{2}$

Sol: Given $a_1 = 8; \quad r = \frac{-1}{2}$

$a_2 = a_1 r$

$a_2 = 8\times\left(\frac{-1}{2}\right)$

$a_2 = -4$

First five terms are $a_1, a_2, a_3, a_4, a_5, \ldots$

i.e., $8, -4, 2, -1, \frac{1}{2}, \ldots$

iii). $a_1 = \frac{-9}{16}; \quad r = \frac{-2}{3}$

Sol: Given $a_1 = \frac{-9}{16}; \quad r = \frac{-2}{3}$

$a_2 = a_1 r$

$a_2 = \left(\frac{-9}{16}\right)\times\left(\frac{-2}{3}\right)$

$a_2 = \frac{3}{8}$

$a_3 = a_2 r \quad \because a_2 = a_1 r$

$a_3 = \left(\frac{3}{8}\right)\times\left(\frac{-2}{3}\right)$

$a_3 = \frac{-1}{4}$

$a_4 = a_3 r \quad \because a_3 = a_1 r^2$ $\quad a_5 = a_4 r \quad \because a_4 = a_1 r^3$

$a_4 = \left(\frac{-1}{4}\right)\times\left(\frac{-2}{3}\right)$ $\quad a_5 = \left(\frac{1}{6}\right)\times\left(\frac{-2}{3}\right)$

$a_4 = \frac{1}{6}$ $\quad a_5 = \frac{-1}{9}$

First five terms are $a_1, a_2, a_3, a_4, a_5, \ldots$

i.e., $\frac{-9}{16}, \frac{3}{8}, \frac{-1}{4}, \frac{1}{6}, \frac{-1}{9}, \ldots$

iv). $a_1 = \frac{x}{y}; \quad r = \frac{-y}{x}$

Sol: Given $a_1 = \frac{x}{y}; \quad r = \frac{-y}{x}$

$a_2 = a_1 r$ $\quad a_3 = a_2 r \because a_2 = a_1 r^1$

$a_2 = \left(\frac{x}{y}\right)\times\left(\frac{-y}{x}\right)$ $\quad a_3 = (-1)\times\left(\frac{-y}{x}\right)$

$a_2 = -1$ $\quad a_3 = \frac{y}{x}$

$a_4 = a_3 r \because a_3 = a_1 r^2$ $\quad a_5 = a_4 r \because a_4 = a_1 r^3$

$a_4 = \left(\frac{y}{x}\right)\times\left(\frac{-y}{x}\right)$ $\quad a_5 = \left(\frac{-y^2}{x^2}\right)\times\left(\frac{-y}{x}\right)$

$a_4 = \frac{-y^2}{x^2}$ $\quad a_5 = \frac{y^3}{x^3}$

First five terms are $a_1, a_2, a_3, a_4, a_5, \ldots$

i.e., $\frac{x}{y}, -1, \frac{y}{x}, \frac{-y^2}{x^2}, \frac{y^3}{x^3}, \ldots$

Q2. Suppose that the third term of a geometric sequence is 27 and the fifth term is 243. Find the first term and common ratio of the sequence.

Sol: Given $a_3 = 27, a_5 = 243$

$a_5 = a_1 r^2 . r^2$ $\quad a_5 = a_1 r^4$

$a_5 = a_3 r^2$

$243 = 27 r^2$

$r^2 = \frac{243}{27}$

$r^2 = 9$

Taking square root on both sides $\quad \Rightarrow r = \pm 3$

Now $\quad a_3 = a_1 r^2$

$27 = 9 a_1$

$a_1 = \frac{27}{9}$

$a_1 = 3$

Q3. Find seventh term of a geometric sequence that has 2 & $-\sqrt{2}$ for its second and third terms respectively.

Sol: Given $a_2 = 2, \quad a_3 = -\sqrt{2}$

$a_3 = a_1 r^2$

$a_3 = a_1 r . r$

$a_3 = a_2 r \qquad \because a_2 = a_1 r$

$-\sqrt{2} = 2r$

$r = \frac{-\sqrt{2}}{2} = \frac{-\sqrt{2}}{2}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{-2}{2\sqrt{2}}$

$r = \frac{-1}{\sqrt{2}}$

$a_7 = a_2 r^5 \qquad \because a_7 = a_1 r^6$

$a_7 = 2\left(\frac{-1}{\sqrt{2}}\right)^5 \qquad \because a_7 = a_1 r . r^5$

$a_7 = 2\frac{(-1)^5}{\left(\sqrt{2}\right)^5} \qquad \because a_2 = a_1 r$

$a_7 = 2 . \frac{-1}{4\sqrt{2}}$

$a_7 = \frac{-1}{2\sqrt{2}}$

$a_7 = -2^{\frac{-3}{2}}$

Q4. How many terms are there in a geometric sequence in which the first and the last terms are 16 and $\frac{1}{64}$ respectively and $r = \frac{1}{2}$

Sol: Given $a_1 = 16, a_n = \frac{1}{64}$

$a_n = a_1 r^{n-1}$

$\frac{1}{64} = 16\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{64\times 16} = \frac{1}{2^{n-1}}$

$\frac{1}{2^6 \times 2^4} = \frac{1}{2^{n-1}}$

$2^{-n+1} = 2^{-10}$

$\Rightarrow -n + 1 = -10$

$-n = -10 - 1$

$-n = -11$

$\Rightarrow n = 11$

Q5. Find x so that x+7,x-3,x-8 form a three geometric sequence in the given order. Also give the sequence.

Sol: Given $a_1 = x+7, a_2 = x-3, a_3 = x-8$ Forms GP then

$\frac{a_2}{a_1} = \frac{a_3}{a_2} \qquad \Rightarrow a_2^2 = a_1 a_3$

$(x-3)^2 = (x+7)(x-8)$

$x^2 - 6x + 9 = x^2 - x - 56$

$-6x + 9 = -x - 56$

$56 + 9 = -x + 6x$

$65 = 5x$

$x = \frac{65}{5} = 13$ so

$a_1 = x+7, \qquad a_2 = x-3, \qquad a_3 = x-8$

$a_1 = 13+7, \qquad a_2 = 13-3, \qquad a_3 = 13-8$

$a_1 = 20, \qquad a_2 = 10, \qquad a_3 = 5$

Q6. $a_{10} = l, a_{13} = m, a_{16} = n$; Show that $l.n = m^2$

Sol: Given $a_{10} = l, a_{13} = m, a_{16} = n$

Taking LHS

$l.n = a_{10} a_{16} \qquad \because a_n = a_1 r^{n-1}$

$l.n = a_1 r^9 a_1 r^{15}$

$l.n = a_1^2 r^{9+15}$

$l.n = a_1^2 r^{24}$

$l.n = \left(a_1 r^{12}\right)^2$

$l.n = \left(a_{13}\right)^2 \qquad \because a_{13} = a_1 r^{12} = m$

$l.n = m^2 = RHS$

Q7. Show that reciprocal of the terms of a geometric sequence also form a geometric sequence.

Sol: Let $a_1, a_2, a_3, \ldots$ form GP than $r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots$

To show that its reciprocal $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are also form GP

Then common ratio of consecutive terms must be equal

$R = \frac{1}{a_2} \div \frac{1}{a_1}, \qquad R = \frac{1}{a_3} \div \frac{1}{a_2}, \qquad \ldots$

$R = \frac{1}{a_2} \times \frac{a_1}{1}, \qquad R = \frac{1}{a_3} \times \frac{a_2}{1}, \qquad \ldots$

$R = \frac{a_1}{a_2}, \qquad R = \frac{a_2}{a_3}, \qquad \ldots$

$R = \frac{a_1}{a_1 r}, \qquad R = \frac{a_1 r}{a_1 r^2}, \qquad \ldots$

$R = \frac{1}{r}, \qquad R = \frac{1}{r}, \qquad \ldots$

Reciprocal of GP form the common ratio $R=\frac{1}{r}$

where GP having common ratio r

Q8. yearly depreciation of a certain machine is 20% of its value at the beginning of the year. If the original cost of the machine is Rs. 5000, find its value after 5 year?

Sol: Given $a_1 = 5000$

Depreciation= 20%

Remaining Ratio = 100%-20%=80%

So $r = \frac{80}{100} = 0.8$

Value after 5 year

$a_6 = a_1 r^5$

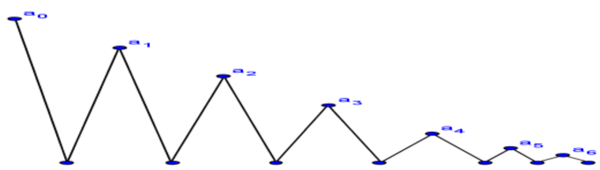
$a_6 = 5000\times(0.8)^5$

$a_6 = 5000\times 0.32768$

$a_6 = Rs.1638.4$

Q9. To test the bounce of a tennis ball, the ball is dropped from a height of 10feet. The ball bounce 75% of its previous height with each bounce. How height does the ball bounce on the sixth bounce?

Sol: Given $a_0 = 10\, feet, r = 75\% = 0.75$



$a_6 = a_1 r^6$

$a_6 = 10(0.75)^6$

$a_6 = 10\times 0.177978$

$a_6 = 1.778\, feet$

$a_6 \approx 1.8\, feet$

Q10. Three numbers, whose sum is 3, form an arithmetic sequence and their squares form a geometric sequence. What are the numbers?

Sol: Let $a_1 = a-d,\quad a_2 = a,\quad a_3 = a+d$

Given that $a_1 + a_2 + a_3 = 3$

By given conditions a_1, a_2, a_3 From an AP so

$a_1 + a_2 + a_3 = 3$

$a - d + a + a + d = 3$

$3a = 3$

$\Rightarrow a = 1$

Since

a_1^2, a_2^2, a_3^2 form G.P so

$\frac{a_2^2}{a_1^2} = \frac{a_3^2}{a_2^2}$

$a_2^4 = a_1^2 \times a_3^2$

putting the values

$a^4 = (a-d)^2 (a+d)^2$

taking square root

$a^2 = \pm(a-d)(a+d)$

Either

$a^2 = (a-d)(a+d)$

$a^2 = a^2 - d^2$

$\Rightarrow d^2 = 0$

$\Rightarrow d = 0$

or

$a^2 = -(a-d)(a+d)$

$a^2 = -a^2 + d^2$

$\Rightarrow d^2 = 2a^2$

$\Rightarrow d = \sqrt{2}a$

$d = \pm\sqrt{2}$

When $d = 0,\quad a = 1$

$a_1 = 1,\quad a_2 = 1,\quad a_3 = 1$

When $d = \sqrt{2},\quad a = 1$

$a_1 = 1-\sqrt{2},\quad a_2 = 1,\quad a_3 = 1+\sqrt{2}$

When $d = -\sqrt{2},\quad a = 1$

$a_1 = 1+\sqrt{2},\quad a_2 = 1,\quad a_3 = 1-\sqrt{2}$

Q11. Rashid borrows Rs. 2000 at 11% interest compound annually. If he pays off the loan in full at the end of four years, how much does he pay?

Sol: Rashid borrows a_0 = 2000

Compound interest $r = 100\% + 11\%$

$r = 111\%$

$r = \frac{111}{100}$

$r = 1.11$

$a_4 = a_0 r^4$

$a_4 = 2000(1.11)^4$

$a_4 = 2000\times 1.5180$

$a_4 = 3036.14$

Geometric mean between two numbers a and b

G.M $= \pm\sqrt{ab}$

Exercise 4.6

Q1. Find the geometric mean of the following:

i). 3.14 and 2.71

Sol: Given $a = 3.14 > 0, b = 2.71 > 0$

$G = \sqrt{ab}$

$G = \sqrt{3.14\times 2.71}$

$G = \sqrt{8.5094}$

$G = 2.91708$

ii). -6 and -216

Sol: Given $a = -6 < 0, b = -216 < 0$

$G = -\sqrt{ab}$

$G = -\sqrt{(-6)(-216)}$

$G = -\sqrt{1296}$

$G = -36$

iii). x+y and x-y

Sol: Given $a = x+y, b = x-y$

$G = \pm\sqrt{ab}$

$G = \pm\sqrt{(x+y)(x-y)}$

$G = \pm\sqrt{x^2 - y^2}$

iv). $\sqrt{2}+3$ and $\sqrt{2}-3$

Sol: Given $a = \sqrt{2}+3 > 0, b = \sqrt{2}-3 < 0$

$G = \sqrt{(\sqrt{2}+3)(\sqrt{2}-3)} \quad \because G = \sqrt{ab}$

$G = \sqrt{(\sqrt{2})^2 - (3)^2}$

$G = \sqrt{2-9}$

$G = \sqrt{-7} = \sqrt{7}\,i$

Not possible

Q2. Insert two geometric means between $\sqrt{3}$ and 3

Solution: Let two geometric means are A_1, A_2

So $\sqrt{3}$, G_1, G_2,3 form geometric sequence

Let a_1, a_2, a_3, a_4 are the corresponding terms

Then $a_1 = \sqrt{3},\ a_2 = G_1,\ a_3 = G_2,\ a_4 = 3$

Now

$\because a_n = a_1 r^{n-1}$

$a_4 = a_1 r^3$

$3 = \sqrt{3} r^3$

And

$a_2 = a_1 r$

$a_2 = 3^{\frac{1}{2}}.3^{\frac{1}{6}}$

$a_2 = 3^{\frac{1}{2}+\frac{1}{6}}$

$r^3 = \frac{3}{\sqrt{3}}$

$r^3 = 3^{1-\frac{1}{2}}$

$r^3 = 3^{\frac{1}{2}}$

$cube-root\ on\ both\ sides$

$r = 3^{\frac{1}{6}}$

$a_2 = 3^{\frac{3+1}{6}}$

$a_2 = 3^{\frac{4}{6}}$

$G_1 = 3^{\frac{2}{3}} \quad G_1 = a_2$

$G_1 = \sqrt[3]{9}$

And

$a_3 = a_1 r^2$

$a_3 = 3^{\frac{1}{2}}\left(3^{\frac{1}{6}}\right)^2$

$a_3 = 3^{\frac{1}{2}+\frac{1}{3}}$

$a_3 = 3^{\frac{3+2}{6}}$

$G_2 = 3^{\frac{5}{6}} \qquad G_2 = a_3$

$G_2 = \sqrt[6]{243}$

Therefore two geometric means are $\sqrt[3]{9}, \sqrt[6]{243}$

Q3. Insert three geometric means between a^4 and b^4

Solution: Let three geometric means are G_1, G_2, G_3

So a^4, G_1, G_2, G_3, b^4 form geometric sequence

Let a_1, a_2, a_3, a_4, a_5 are the corresponding terms

Then $a_1 = a^4,\ a_2 = G_1,\ a_3 = G_2,\ a_4 = G_3,\ a_5 = b^4$

Now

$\because a_n = a_1 r^{n-1}$

$a_5 = a_1 r^4$

$b^4 = a^4 r^4$

$b = ar$

$r = \frac{b}{a}$

and

$\because a_n = a_1 r^{n-1}$

$a_2 = a_1 r$

$a_2 = a^4\left(\frac{b}{a}\right)$

$G_1 = a_2 = a^3 b$

Now

$a_3 = a_1 r^2$

$a_3 = a^4 . \left(\frac{b}{a}\right)^2$

$G_2 = a_3 = a^2 b^2$

and

$a_4 = a_1 r^3$

$a_4 = a^4 . \left(\frac{b}{a}\right)^3$

$G_3 = a_4 = ab^3$

Therefore three geometric means are $a^3 b, a^2 b^2, ab^3$

Q4. Insert four geometric means between -8 & $\frac{1}{4}$

Sol: Let four geometric means are G_1, G_2, G_3, G_4

So -8, G_1, G_2, G_3, ,G_4 $\frac{1}{4}$ form geometric sequence

Let $a_1, a_2, a_3, a_4, a_5, a_6$ are the corresponding terms

Then $a_1 = -8, a_2 = G_1, a_3 = G_2, a_4 = G_3, a_5 = G_4, a_6 = \frac{1}{4}$

$a_6 = a_1 r^5 \qquad \because a_n = a_1 r^{n-1}$

$\frac{1}{4} = -8r^5$

$r^5 = \frac{-1}{32}$

$r^5 = \frac{-1}{2^5}$

$r^5 = \left(\frac{-1}{2}\right)^5$

$\Rightarrow r = \frac{-1}{2}$

Now

$\because a_n = a_1 r^{n-1}$

$a_2 = a_1 r$

$a_2 = -8\left(\frac{-1}{2}\right)$

$G_1 = a_2 = 4$

and

$\because a_n = a_1 r^{n-1}$

$a_3 = a_1 r^2$

$a_3 = -8.\left(\frac{-1}{2}\right)^2$

$G_2 = a_3 = -2$

And

$\because a_n = a_1 r^{n-1}$

$a_4 = a_1 r^3$

$a_4 = -8\left(\frac{-1}{2}\right)^3$

$G_3 = a_4 = 1$

and

$\because a_n = a_1 r^{n-1}$

$a_5 = a_1 r^4$

$a_5 = -8\left(\frac{-1}{2}\right)^4$

$G_4 = a_5 = \frac{-1}{2}$

Therefore four geometric means are 4,-2,1, $\frac{-1}{2}$

Q5. Find two numbers if the difference between them is 48 and their A.M exceeds their G.M by 18.

Solution: $x - y = 48$

$\Rightarrow x = 48 + y$.............................(1)

By given condition

AM – 18 = GM putting the values

$\frac{x+y}{2} - 18 = \sqrt{xy} \qquad \times by\ 2$

$x + y - 36 = 2\sqrt{xy}$

Putting the value of x

$48 + y + y - 36 = 2\sqrt{(48+y)y}$

$2y + 12 = 2\sqrt{48y + y^2} \qquad \div by\ 2$

$y + 6 = \sqrt{48y + y^2}$

Squaring both sides

$(y+6)^2 = \left(\sqrt{48y + y^2}\right)^2$

$y^2 + 12y + 36 = 48y + y^2$

$36 = 48y - 12y + y^2 - y^2$

$36 = 36y$

$y = 1$

Put in equation (1) we get

$x = 48 + 1$

$x = 49$

Therefore the two numbers are 1 & 49

Q6. Prove that the product of n geometric means between a and b is equal to the nth power of a single geometric mean between them.

Solution: Let four geometric means are $G_1, G_2, G_3, \ldots, G_n$

So a , $G_1, G_2, G_3, \ldots, G_n$,b form geometric sequence

Let $a_1, a_2, a_3, a_4, \ldots, a_{n+1}, a_{n+2}$ are corresponding terms

where $a_1 = a, \quad a_2 = G_1, a_3 = G_2, \ldots, a_{n+1} = G_n, a_{n+2} = b$

LHS product of n geometric means

$G_1.G_2.G_3\ldots.G_n = a_2.a_3.a_4\ldots.a_{n+1}$

$= a_1 r^1.a_1 r^2.a_1 r^3\ldots..a_1 r^n$

$= (a_1)^n\, r^{1+2+3+\cdots+n}$

$= (a_1)^n\, r^{\frac{n(n+1)}{2}}$

$= \left(a_1 r^{\frac{(n+1)}{2}}\right)^n$

$= \left(a_1^{\frac{2}{2}} r^{\frac{(n+1)}{2}}\right)^n$

$= \left(a_1^2 r^{n+1}\right)^{\frac{n}{2}}$

$= \left(a_1.a_1 r^{n+1}\right)^{\frac{n}{2}}$

$= (ab)^{\frac{n}{2}}$

$= \sqrt[n]{ab}$

$= RHS$

= nth power of a single geometric mean between a & b.

Q7. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is an geometric mean between a and b.? where a and b are not zero simultaneously.

Sol: Given that GM= $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ from definition geometric mean between two numbers is $\sqrt{ab}$ so both will be equal

$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}\,(\text{cross multiplication})$

$a^{n+1}+b^{n+1}=\sqrt{ab}\left(a^n+b^n\right)$

$a^n.a+b^n.b=a^n\sqrt{ab}+b^n\sqrt{ab}$

$a^n.a+b^n.b=a^na^{\frac{1}{2}}b^{\frac{1}{2}}+b^na^{\frac{1}{2}}b^{\frac{1}{2}}$

$a^n.a-a^na^{\frac{1}{2}}b^{\frac{1}{2}}=b^na^{\frac{1}{2}}b^{\frac{1}{2}}-b^n.b$

$a^na^{\frac{1}{2}}\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)=b^nb^{\frac{1}{2}}\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)$

$\frac{a^na^{\frac{1}{2}}}{b^nb^{\frac{1}{2}}}=\frac{\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)}{\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)}=1$

$\frac{a^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}}=1$

$\left(\frac{a}{b}\right)^{n+\frac{1}{2}}=\left(\frac{a}{b}\right)^0$

$\Rightarrow n+\frac{1}{2}=0$

$n=-\frac{1}{2}$

Sum of n terms of GP $S_n=\frac{a_1\left(r^n-1\right)}{r-1}$ $r>1$

Sum of n terms of GP $S_n=\frac{a_1\left(1-r^n\right)}{1-r}$ $r<1$

Exercise 4.7

Q1. Compute the sum

i). $3+6+12+\cdots+3.2^9$

Sol: $a_1=3, a_2=6, a_3=12, a_n=3.2^9$

$r=\frac{a_2}{a_1}=\frac{6}{3}=2$

Since $a_n=a_1r^{n-1}$ Putting the values

$3.2^9=3.2^{n-1}$

$2^9=2^{n-1}$

$\Rightarrow 9=n-1$

$n=9+1$

$n=10$

Now take $3+6+12+\cdots+3.2^9$

$S_{10}=3.2^0+3.2^1+3.2^2+\cdots+3.2^9$

$S_{10}=3.\left(2^0+2^1+2^2+\cdots+2^9\right)$

$S_{10}=3.\left(\frac{2^{10}-1}{2-1}\right)$

$S_{10}=3\left(2^{10}-1\right)$

$S_{10}=3(1023)$

$S_{10}=3069$

---

ii). $8+4+2+1+\cdots+\frac{1}{16}$

Sol: $a_1=8,\ a_2=4,\ a_3=2,\ a_n=\frac{1}{16}$

$r=\frac{4}{8}$ $\qquad r=\frac{a_2}{a_1}$

$r=\frac{1}{2}$

And $a_n=a_1r^{n-1}$ putting the values

$\frac{1}{16}=8\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{2^4}=\frac{2^3}{2^{n-1}}$

$\frac{1}{2^4}=\frac{1}{2^{n-4}}$

$4=n-4$

$\Rightarrow n=8$

Now take $8+4+2+1+\cdots+\frac{1}{16}$

$S_8=2^3+2^2+2^1+2^0+2^{-1}+2^{-2}+2^{-3}+2^{-4}$

$S_8=\frac{2^3\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}}$

$S_8=\frac{2^3\left(1-\frac{1}{2^8}\right)}{\frac{1}{2}}$

$S_8=\frac{2^3\left(\frac{2^8-1}{2^8}\right)}{\frac{1}{2}}$

$S_8=\left(\frac{2^8-1}{2^{8-3}}\right)\div\frac{1}{2}$

$S_8=\left(\frac{2^8-1}{2^5}\right)\times\frac{2}{1}$

$S_8=\left(\frac{2^8-1}{2^{5-1}}\right)$

$S_8=\frac{256-1}{2^4}$

$S_8=\frac{255}{16}$

---

iii). $2^4+2^5+2^6+2^7+2^8+2^9+2^{10}$

Sol: $a_1=2^4, a_2=2^5, a_3=2^6, a_n=2^{10}$

$r=\frac{a_2}{a_1}$

$r=\frac{2^5}{2^4}$ $\qquad \Rightarrow r=2$

And $a_n=a_1r^{n-1}$ putting the values

$2^{10}=2^42^{n-1}$

$2^{10}=2^{4+n-1}$

$2^{10}=2^{n+3}$

$\Rightarrow 10=n+3$

$n=10-3$

$n=7$

$S_7=2^4+2^5+2^6+2^7+2^8+2^9+2^{10}$

$S_7=\frac{2^4\left(2^7-1\right)}{2-1}$ $\qquad \because S_n=\frac{a_1\left(r^n-1\right)}{r-1}$

$S_7=2^4\left(2^7-1\right)$

$S_7=16(128-1)$

$S_7=16\times127$

$S_7=2032$

---

iv). $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}$

Sol: $a_1=\frac{1}{x},\quad a_2=\frac{1}{x^2},\quad ,a_n=\frac{1}{x^6}$

Now

$r=\frac{a_2}{a_1}=a_2\div a_1$

$r==\frac{1}{x^2}\div\frac{1}{x}$

$r=\frac{1}{x^2}\times\frac{x}{1}$

$r=\frac{1}{x}$

and

$\therefore a_n=a_1r^{n-1}$

$\frac{1}{x^6}=\frac{1}{x}\left(\frac{1}{x}\right)^{n-1}$

$\frac{1}{x^6}=\frac{1}{x.x^{n-1}}$

$\frac{1}{x^6}=\frac{1}{x^{1+n-1}}$

$\frac{1}{x^6}=\frac{1}{x^n}$

Now $\Rightarrow n=6$

$$S_6=\frac{1}{x^1}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}$$

$$S_6=\frac{\frac{1}{x}\left(1-\left(\frac{1}{x}\right)^6\right)}{1-\frac{1}{x}} \qquad \because S_n=\frac{a_1\left(r^n-1\right)}{r-1}$$

$$S_6=\frac{1}{x}\left(\frac{x^6-1}{x^6}\right)\div\frac{x-1}{x}$$

$$S_6=\frac{1}{x}\left(\frac{x^6-1}{x^6}\right)\times\frac{x}{x-1}$$

$$S_6=\frac{x^6-1}{x^6\left(x-1\right)}$$

Q2. Some of components a_1,a_n,n,r & S_n of a geometric sequence are given. Find ones that are missing.

i). $a_1=1,a_n=64,r=-2$

Sol: Given $a_1=1,a_n=64,r=-2$

Since $a_n=a_1r^{n-1}$ putting the values

$$64=1(-2)^{n-1}$$

$$2^6=(-2)^{n-1}$$

$$(-2)^6=(-2)^{n-1}$$

$$\Rightarrow n-1=6$$

$$n=6+1$$

$$n=7$$

And $S_7=\dfrac{1.\left(1-(-2)^7\right)}{1-(-2)} \qquad \because S_n=\dfrac{a_1\left(r^n-1\right)}{r-1}$

$$S_7=\frac{1}{3}\left(1+2^7\right)$$

$$S_7=\frac{1+128}{3}$$

$$S_7=\frac{129}{3}$$

$$S_7=43$$

---

ii). $r=\frac{1}{2},a_9=1$

Sol: Given $r=\frac{1}{2},a_9=1,\Rightarrow n=9$

$$a_9=a_1r^8 \qquad \because a_n=a_1r^{n-1}$$

$$1=a_1\left(\tfrac{1}{2}\right)^8$$

$$1=\tfrac{a_1}{256}$$

$$a_1=256$$

And $S_9=\dfrac{a_1\left(1-r^9\right)}{1-r} \qquad \because S_n=\dfrac{a_1\left(r^n-1\right)}{r-1}$

$$S_9=\frac{256\left(1-\left(\frac{1}{2}\right)^9\right)}{1-\frac{1}{2}}$$

$$S_9=256\left(1-\frac{1}{512}\right)\div\frac{1}{2}$$

$$S_9=256\left(\frac{512-1}{512}\right)\times\frac{2}{1}$$

$$S_9=511$$

---

iii). $r=-2,S_n=-63,a_n=-96$

Sol: Given $r=-2,S_n=-63,a_n=-96$

$$-96=a_1\left(-2\right)^{n-1} \qquad \because a_n=a_1r^{n-1}$$

$$\frac{a_1\left(-2\right)^n}{2}=-96$$

$$a_1\left(-2\right)^n=192$$

$$a_1=\frac{192}{\left(-2\right)^n}\text{..(1)}$$

$$-63=\frac{192}{\left(-2\right)^n}.\frac{1-\left(-2\right)^n}{1-\left(-2\right)} \qquad S_n=\frac{a_1\left(1-r^n\right)}{1-r}$$

$$-63=\frac{192}{\left(-2\right)^n}.\frac{1-\left(-2\right)^n}{3}$$

$$\frac{-63\times3}{192}=\frac{1-\left(-2\right)^n}{\left(-2\right)^n}$$

$$-63\left(-2\right)^n=64\left\{1-\left(-2\right)^n\right\}$$

$$-63\left(-2\right)^n=64-64\left(-2\right)^n$$

$$64\left(-2\right)^n-63\left(-2\right)^n=64$$

$$\left(-2\right)^n=64$$

$$\left(-2\right)^n=2^6$$

$$\left(-2\right)^n=\left(-2\right)^6$$

$$\Rightarrow n=6$$

Put n=6 in eq (1)

$$a_1=\frac{192}{\left(-2\right)^6}=\frac{192}{64}=3$$

Q3. Find r such that $S_{10}=244S_5$

Sol: Given $S_{10}=244S_5$

$$S_{10}=244S_5$$

$$\frac{a_1\left(1-r^{10}\right)}{1-r}=244\frac{a_1\left(1-r^5\right)}{1-r} \qquad \because S_n=\frac{a_1\left(r^n-1\right)}{r-1}$$

$$1-r^{10}=244\left(1-r^5\right)$$

$$1^2-\left(r^5\right)^2=244\left(1-r^5\right)$$

$$\left(1-r^5\right)\left(1+r^5\right)=244\left(1-r^5\right)$$

$$1+r^5=244$$

$$r^5=244-1$$

$$r^5=243=3^5$$

$$\Rightarrow r=3$$

Q4. Prove that $S_n\left(S_{3n}-S_{2n}\right)=\left(S_n-S_{2n}\right)^2$

Solution: LHS when $r>1$

$$S_n\left(S_{3n}-S_{2n}\right)=\frac{a_1\left(r^n-1\right)}{r-1}\left\{\frac{a_1\left(r^{3n}-1\right)}{r-1}-\frac{a_1\left(r^{2n}\right)}{r-1}\right\}$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(r^n-1\right)\left\{\left(r^{3n}-1\right)-\left(r^{2n}-1\right)\right\}$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(r^n-1\right)\left\{r^{3n}-1-r^{2n}+1\right\}$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(r^n-1\right)\left(r^{3n}-r^{2n}\right)$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(r^{4n}-r^{3n}-r^{3n}+r^{2n}\right)$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(r^{4n}-2r^{3n}+r^{2n}\right)$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(\left(r^{2n}\right)^2-2\left(r^{2n}\right)\left(r^n\right)+\left(r^n\right)^2\right)$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(r^{2n}-r^n\right)^2$$

$$=\left(\frac{a_1}{r-1}\right)^2\left(r^{2n}-1-r^n+1\right)^2$$

$$=\left(\frac{a_1}{r-1}\right)^2\left\{\left(r^{2n}-1\right)-\left(r^n-1\right)\right\}^2$$

$$=\left\{\frac{a_1\left(r^{2n}-1\right)}{r-1}-\frac{a_1\left(r^n-1\right)}{r-1}\right\}^2$$

$$=\left(S_n-S_{2n}\right)^2$$

$S_n\left(S_{3n}-S_{2n}\right)=\left(S_n-S_{2n}\right)^2$ Hence proved.

Q5. Find sum S_n of the first n terms of the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$

Sol: It is clear that $a_n=\left(\frac{1}{2}\right)^n, r=\frac{1}{2}, a_1=\frac{1}{2}$

$$S_n=\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} \qquad S_n=\frac{a_1\left(1-r^n\right)}{1-r}$$

$$S_n=\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{\frac{1}{2}}$$

$$S_n=1-\left(\frac{1}{2}\right)^n$$

Q6. A ball rebounds to half the height from which it is dropped. If it is dropped from 10 feet, how far does it travel from the moment it is dropped until the moment of eighth bounce?

Sol: Ball rebound $r=\frac{1}{2}$

A ball dropped from 10feet, First Bounce i.e. $a_1=10$ feet

2nd Bounce and again came down = 5feet +5feet

3rd Bounce and again came down= $\frac{5}{2}$ feet+ $\frac{5}{2}$ feet

Continuing this manner

8th Bounce and again came down= $\frac{5}{2^6}$ feet+ $\frac{5}{2^6}$ feet

Total distance travel

$$S=a_1+2\left(a_2+a_3+a_4+a_5+a_6+a_7+a_8\right)$$

$$=10+2\left(5+\frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\frac{5}{16}+\frac{5}{32}+\frac{5}{64}\right)$$

$$=10+2\times5\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\right)$$

$$=10+10\left(1+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}\right)$$

$$=10+10\frac{1-\frac{1}{2^7}}{1-\frac{1}{2}}$$

$$=10+10\times\frac{2}{1}\left(1-\frac{1}{2^7}\right)=10+10\times\frac{2}{1}\left(1-\frac{1}{128}\right)$$

$$=10+20\left(\frac{128-1}{128}\right)$$

$$=10+\frac{5\times127}{32}$$

$$=10+\frac{635}{32}=10+19\frac{27}{32}$$

$$=29\frac{27}{32}$$

Q7. A man wishes to save money by setting aside RS. 1 the first day, Rs. 2 the second day, Rs. 4 the third day and so on, doubling the amount each day. If this continued, how much be set aside on the 15th day? What is the total amount saved at the end of 30 days?

Sol: Given $a_1=1, a_2=2, a_3=4$

$$r=\frac{a_2}{a_1}=\frac{2}{1}$$

$$r=2$$

And

$$a_{15}=1.2^{14} \qquad a_{15}=a_1r^{14}$$

$$a_{15}=16384$$

Now

$$S_{30}=\frac{1.\left(2^{30}-1\right)}{2-1} \qquad S_{30}=\frac{a_1\left(r^{30}-1\right)}{r-1}$$

$$S_{30}=1073741824-1$$

$$S_{30}=1073741823$$

Q8. Population of an insect is found to triple each week in summer months. If there are twenty insects in colony at he beginning of summer how many are present at the end of 12 weeks assuming no deaths are there.

Sol: Given $a_1=20, r=3, n=12$

$$S_{12}=\frac{20.\left(3^{12}-1\right)}{3-1} \qquad S_{12}=\frac{a_1\left(r^{12}-1\right)}{r-1}$$

$$S_{12}=10\left(3^{12}-1\right)$$

$$S_{12}=10\left(531441-1\right)$$

$$S_{12}=5314400$$

General term of GP $\qquad a_n=a_1r^{n-1}$

Sum of finite series

$$S_n=\frac{a_1\left(r^n-1\right)}{r-1} \qquad r>1$$

Sum of infinite series

$$S_\infty=\frac{a_1}{1-r} \qquad r<1$$

Exercise 4.8

Q1. Find the sum of each geometric sequence:

i). $16, 12, 9, \frac{27}{4}, \ldots$

Sol: Given $a_1=16, a_2=12, a_3=9, a_4=\frac{27}{4}, \ldots$

$$r=\frac{12}{16} \qquad r=\frac{a_2}{a_1}$$

$$r=\frac{3}{4}<1$$

$$S_\infty=\frac{16}{1-\frac{3}{4}} \qquad S_\infty=\frac{a_1}{1-r}$$

$$S_\infty=16\div\frac{1}{4}$$

$$S_\infty=64$$

---

ii). $\frac{1}{25}, \frac{1}{5}, 1, 5, \ldots$

Sol: Given $a_1=\frac{1}{25}, a_2=\frac{1}{5}, a_3=1, a_4=5, \ldots$

$$r=\frac{1}{5}\div\frac{1}{25} \qquad r=\frac{a_2}{a_1}$$

$$r=\frac{1}{5}\times\frac{25}{1}$$

$$r=5>1$$

Sum does not exists

---

iii). $2, \frac{2}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \ldots$

Sol: Given $a_1=2, a_2=\frac{2}{\sqrt{2}}, a_3=1, a_4=\frac{1}{\sqrt{2}}, \ldots$

$$r=\frac{a_2}{a_1}$$

$$r=\frac{2}{\sqrt{2}}\div2$$

$$r=\frac{1}{\sqrt{2}}<1$$

$$S_\infty=\frac{2}{1-\frac{1}{\sqrt{2}}} \qquad \because S_\infty=\frac{a_1}{1-r}$$

$$S_\infty=2\div\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$S_\infty=\frac{2\sqrt{2}}{\sqrt{2}-1}$$

---

iv). $15, 1.5, 0.15, 0.015, \ldots$

Sol: Given $a_1=15, a_2=1.5, a_3=0.15, a_4=0.015, \ldots$

$r=\dfrac{1.5}{15}$ $\qquad r=\dfrac{a_2}{a_1}$

$r=\dfrac{1}{10}$ $\qquad \Rightarrow r=0.1<1$

$S_\infty=\dfrac{15}{1-0.1}$ $\qquad \because S_\infty=\dfrac{a_1}{1-r}$

$S_\infty=\dfrac{15}{0.9}$

$S_\infty=\dfrac{15\times10}{9}$ $\qquad \Rightarrow S_\infty=\dfrac{50}{3}$

Q2. Find the first five terms of the following infinite geometric sequence.

i). $a_1=25, S_\infty=125$

Sol: Given $a_1=25, S_\infty=125$

$125=\dfrac{25}{1-r}$ $\qquad$ Using $S_\infty=\dfrac{a_1}{1-r}$

$\dfrac{125}{25}=\dfrac{1}{1-r}$

$5(1-r)=1$

$5-5r=1$

$5-1=5r$

$r=\frac{4}{5}$

Now

$a_2=a_1r$

$a_2=25\left(\frac{4}{5}\right)$

$a_2=20$

and

$a_3=a_2r \because a_2=a_1r$

$a_3=20\left(\frac{4}{5}\right)$

$a_3=16$

And

$a_4=a_3r \because a_3=a_1r^2$

$a_4=16\left(\frac{4}{5}\right)$

$a_4=\frac{64}{5}$

and

$a_5=a_4r \because a_4=a_1r^3$

$a_5=\frac{64}{5}\left(\frac{4}{5}\right)$

$a_5=\frac{256}{25}$

The first five terms are 25,20,16, $\frac{64}{5}$, $\frac{256}{25}$

ii). $a_1=4, S_\infty=-7$

Sol: Given $a_1=4, S_\infty=-7$

$-7=\dfrac{4}{1-r}$ $\qquad S_\infty=\dfrac{a_1}{1-r}$

$-7(1-r)=4$

$-7+7r=4$

$7r=4+7$

$r=\frac{11}{7}>1$

No such series exists

Q3. Find the first five terms and the sum of an infinite geometric sequence having $a_2=2, a_3=1$

Sol: Given $a_2=2, a_3=1$

Now

$a_3=a_2r \because a_2=a_1r$

$1=2r$

$\Rightarrow r=\frac{1}{2}$

and

$a_2=a_1r \because a_n=a_1r^{n-1}$

$2=a_1\left(\frac{1}{2}\right)$

$\Rightarrow a_1=4$

Now

$a_4=a_3r \because a_3=a_1r^2$

$a_4=1\left(\frac{1}{2}\right)$

$a_4=\frac{1}{2}$

And

$a_5=a_4r \because a_4=a_1r^3$

$a_5=\frac{1}{2}\left(\frac{1}{2}\right)$

$a_5=\frac{1}{4}$

Now $S_\infty=\dfrac{4}{1-\frac{1}{2}}$ $\qquad$ using $S_\infty=\dfrac{a_1}{1-r}$

$S_\infty=4\div\frac{1}{2}$

$S_\infty=8$

Q4. Convert each decimal to a common fraction.

i). $0.\dot{8}$

Sol: Let $S=0.\dot{8}=0.88888888888.....$

$S=0.8+0.08+0.008+0.0008+\cdots$

$S=\frac{8}{10}+\frac{8}{100}+\frac{8}{1000}+\frac{8}{10000}+\cdots$

$S=\frac{8}{10}\left(1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}\cdots\right)$

$S=0.8(1+0.1+0.01+0.001+\cdots)$

Here $a_1=1, a_2=0.1, a_3=0.01$ $\quad r=\dfrac{0.1}{1}=0.1$

$S=0.8(S_\infty)$

$S=0.8\left(\dfrac{1}{1-0.1}\right)$ $\qquad S_\infty=\dfrac{a_1}{1-r}$

$S=\dfrac{0.8}{0.9}$

$S=\dfrac{8}{9}$

ii). $1.\overline{63}$

Sol: Let $S=1.\overline{63}=1.636363636363.....$

$S=1+0.63+0.0063+0.000063+\cdots$

$S=1+\frac{63}{100}+\frac{63}{10000}+\frac{63}{1000000}+\cdots$

$S=1+\frac{63}{100}\left(1+\frac{1}{100}+\frac{1}{10000}+\cdots\right)$

$S=1+0.63(1+0.01+0.0001+\cdots)$

Here $a_1=1, a_2=0.01, a_3=0.0001$

$r=\dfrac{a_2}{a_1}=\dfrac{0.01}{1}=0.01$

$S=1+0.63(S_\infty)$

$S=1+0.63\left(\dfrac{1}{1-0.01}\right)$ $\qquad \because S_\infty=\dfrac{a_1}{1-r}$

$S=1+\dfrac{0.63}{0.99}$

$S=1+\dfrac{7}{11}$ $\qquad \Rightarrow S=\dfrac{18}{11}$

iii). $2.1\dot{5}$

Sol: Let $S=2.1\dot{5}=2.1555555555555.....$

$S=2.1+0.05+0.005+0.0005+\cdots$

$S=2.1+\frac{5}{100}+\frac{5}{1000}+\frac{5}{10000}+\cdots$

$S=2.1+\frac{5}{100}\left(1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)$

$S=2.1+0.05(1+0.1+0.01+0.001+\cdots)$

Here $a_1=1, a_2=0.1, a_3=0.01$ $\quad r=\dfrac{0.1}{1}=0.1$

$S=2.1+0.05(S_\infty)$

$S=2.1+0.05\left(\dfrac{1}{1-0.1}\right)$ $\qquad \because S_\infty=\dfrac{a_1}{1-r}$

$S=2.1+\dfrac{0.05}{0.9}$

$S=\dfrac{21}{10}+\dfrac{1}{18}$

$S=\dfrac{189+5}{90}$

$S=\dfrac{194}{90}$

$S=\dfrac{97}{45}$

iv). $0.\overline{123}$

Sol: Let $S=0.\overline{123}=0.123123123123123.....$

$S=0.123+0.000123+0.000000123+\cdots$

$S=\frac{123}{1000}+\frac{123}{1000000}+\frac{123}{1000000000}+\cdots$

$S=\frac{123}{1000}\left(1+\frac{1}{1000}+\frac{1}{1000000}+\cdots\right)$

$S=0.123(1+0.001+0.000001+\cdots)$

Here $a_1 = 1, a_2 = 0.001, a_3 = 0.000\ 001$

$r = \frac{a_2}{a_1} = \frac{0.001}{1} = 0.001$

Then $S = 0.123(S_\infty)$

$S = 0.123\left(\frac{1}{1-0.001}\right) \qquad \because S_\infty = \frac{a_1}{1-r}$

$S = \frac{0.123}{0.999}$

$S = \frac{123}{999}$

$S = \frac{41}{333}$

Q5. The sum of an infinite geometric series is 15 and the sum of their squares is 45. Find the series.

Sol: According to condition $S = a_1 + a_2 + a_3 + a_4 + \cdots$

$15 = \frac{a_1}{1-r}$

$a_1 = 15(1-r)$....(1)

$45 = a_1^2 + a_2^2 + a_3^2 + a_4^2 + \cdots$

$45 = a_1^2 + a_1^2 r^2 + a_1^2 r^4 + a_1^2 r^6 + \cdots$

$45 = \frac{a_1^2}{1-r^2} \qquad \because S_\infty = \frac{a_1}{1-r}$

$a_1^2 = 45(1-r^2)$

Putting the value of a

$\{15(1-r)\}^2 = 45(1-r^2)$

$225(1-r)^2 = 45(1-r^2)$

$5(1-r)^2 = (1-r)(1+r)$

$5(1-r) = (1+r)$

$5 - 5r = 1 + r$

$5 - 1 = r + 5r$

$4 = 6r$

$r = \frac{2}{3}$

Put in eq (1)

$a_1 = 15(1-\frac{2}{3})$

$a_1 = 15(\frac{1}{3})$

$a_1 = 5$

And

$a_2 = a_1 r$

$a_2 = 5(\frac{2}{3})$

$a_2 = \frac{10}{3}$

Now

$a_3 = a_2 r \because a_2 = a_1 r$

$a_3 = \frac{10}{3}(\frac{2}{3})$

$a_3 = \frac{20}{9}$

and

$a_4 = a_3 r \because a_3 = a_1 r^2$

$a_4 = \frac{20}{9}(\frac{2}{3})$

$a_4 = \frac{40}{27}$

now

$a_5 = a_4 r \qquad \because a_4 = a_1 r^3$

$a_5 = \frac{40}{27}(\frac{2}{3})$

$a_5 = \frac{80}{81}$

$S = 5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \cdots$

Q6. sum of first 6 terms of a geometric series is 9 times the sum of its first three terms. Find the common ratio.

Sol: given $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 9(a_1 + a_2 + a_3)$

$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + a_1 r^5 = 9(a_1 + a_1 r + a_1 r^2)$

$a_1(1 + r + r^2 + r^3 + r^4 + r^5) = 9a_1(1 + r + r^2)$

$1 + r + r^2 + r^3 + r^4 + r^5 = 9(1 + r + r^2)$

$\frac{1-r^6}{1-r} = 9\frac{1-r^3}{1-r} \qquad \because S_n = \frac{a_1(r^n - 1)}{r-1}$

$1 - r^6 = 9(1 - r^3)$

$(1-r^3)(1+r^3) = 9(1-r^3)$

$1 + r^3 = 9$

$r^3 = 9 - 1 = 8$

$r^3 = 2^3 \qquad \Rightarrow r = 2$

But r = 1 is not possible, so we take only $r = 2$

Q7. How many terms of the series: $1 + \sqrt{3} + 3 + \cdots$ be added to get the sum $40 + 13\sqrt{3}$

Sol: Here $a_1 = 1, a_2 = \sqrt{3}, a_3 = 3$

$r = \frac{\sqrt{3}}{1} = \sqrt{3} > 1$

$40 + 13\sqrt{3} = \frac{1.\left\{\left(\sqrt{3}\right)^n - 1\right\}}{\sqrt{3}-1} \qquad \because S_n = \frac{a_1(r^n - 1)}{r-1}$

$(40 + 13\sqrt{3})(\sqrt{3} - 1) = (\sqrt{3})^n - 1$

$40\sqrt{3} - 40 + 13\times 3 - 13\sqrt{3} + 1 = (\sqrt{3})^n$

$(\sqrt{3})^n = 27(\sqrt{3})$

$3^{n/2} = 3^{3+1/2}$

$3^{\frac{n}{2}} = 3^{\frac{7}{2}}$

$\Rightarrow \frac{n}{2} = \frac{7}{2}$

$\Rightarrow n = 7$

Q8 Find an infinite geometric series whose sum is 6 & such that each term is four times sum of all terms that follow it

Sol: Given $6 = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$

$6 = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots$

$6 = a_1(1 + r + r^2 + r^3 + r^4 + \cdots)$

$6 = \frac{a_1}{1-r} \qquad \because S_\infty = \frac{a_1}{1-r}$

$a_1 = 6(1-r)$................................(1)

$a_1 = 4(a_2 + a_3 + a_4 + a_5 + a_6 + \cdots)$

$a_1 = 4(a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + a_1 r^5 + \cdots)$

$a_1 = 4a_1 r(1 + r + r^2 + r^3 + r^4 + \cdots)$

$1 = 4r(1 + r + r^2 + r^3 + r^4 + \cdots)$

$1 = \frac{4r}{1-r} \qquad \because S_\infty = \frac{a_1}{1-r}$

$1 - r = 4r$

$1 = 4r + r$

$1 = 5r$

$r = \frac{1}{5}$

Put in eq (1)

$a_1 = 6(1 - \frac{1}{5})$

$a_1 = 6(\frac{4}{5})$

$a_1 = \frac{24}{5}$

And

$a_2 = a_1 r$

$a_2 = \frac{24}{5}(\frac{1}{5})$

$a_2 = \frac{24}{5^2}$

Now

$a_3 = a_2 r \because a_2 = a_1 r$

$a_3 = \frac{24}{5^2}(\frac{1}{5})$

$a_3 = \frac{24}{5^3}$

and

$a_4 = a_3 r \because a_3 = a_1 r^2$

$a_4 = \frac{24}{5^3}(\frac{1}{5})$

$a_4 = \frac{24}{5^4}$

And $a_5 = a_4 r \because a_4 = a_1 r^3$

$a_5 = \frac{24}{5^4}(\frac{1}{5})$

$a_5 = \frac{24}{5^5}$

So, $S = \frac{24}{5^1} + \frac{24}{5^2} + \frac{24}{5^3} + \frac{24}{5^4} + \cdots$

Q9 If $y = \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$, 0<x<3, then show that $x = \frac{3y}{1+y}$

Sol: Given $y=\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\cdots$ $\therefore r=\frac{x}{3}<1$

$y=\frac{x}{3}\cdot\frac{1}{1-\frac{x}{3}}$ $\because S_n=\frac{a_1\left(r^n-1\right)}{r-1}$

$y=\frac{x}{3}\cdot\frac{1}{\frac{3-x}{3}}$

$y=\frac{x}{3-x}$

$3y-xy=x$

$3y=x+xy$

$3y=x(1+y)$

$x=\frac{3y}{1+y}$

Q10. Find how far a ball travels before coming to rest, if it is dropped from a height of 24 meters and each time it hits the ground, it rebounds one third of the distance from which it fell?

Sol: Given that ball Rebounds one third $r=\frac{1}{3}$

Total distance travels $S=24+\frac{24}{3}+\frac{24}{3^2}+\frac{24}{3^3}+\cdots$

$S=24\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)$

$S=24\left(\frac{1}{1-\frac{1}{3}}\right)$ $\because S_n=\frac{a_1\left(r^n-1\right)}{r-1}$

$S=\frac{24}{\frac{2}{3}}$

$S=12\times 3$

$S=36$

Q11 Number of bacteria in culture increased geometrically from 64000 to 729000 in 6 days. Find daily rate of increase if the rate is assumed to be constant.

Sol: Given $a_0=64000, a_6=729000$

$a_6=a_0r^6$

$729000=64000r^6$

$7290=64r^6$

$r^6=\frac{729}{64}$

$r^6=\left(\frac{3}{2}\right)^6$

$\Rightarrow r=\frac{3}{2}$

$r=1.5=1.5\times 100\%$

$r=150\%$

Already percentage of bacteria =100 %

Increase percentage =50%

Q12. If population of a town increases geometrically at rate of 5% per year and present population is 300,000. What will be the population after 6 years from now?

Sol: Increase rate=5%

Then total population r =105%=1.05

$a_0=300,000$, $r=1.05$

$a_6=a_0r^6$

$a_6=300,000(1.05)^6$

$a_6=402028.6922=402029$

Q13. A machine which costs Rs. 800 depreciates 7 ½ % of its value per year. Show that is value at the end of successive years are in geometrical sequence. Find its value at the end of 9th year.

Sol: Decrease percentage= 7.5%

Remaining percentage =100%-7.5%= 92.5%

i.e., $r=92.5\%=0.925$

Beginning cost a_0 = 800

Then cost of machine at the end of 9th year

$a_9=a_0r^9$

$a_9=800(0.925)^5$

$a_9=396.61$

Q14. A tank contains 16000 liters of water. Each day one half of the water in the tank is removed without replacement. How much water remains in the tank at the end of 8th day?

Sol: Water decrease = $\frac{1}{2}$ i.e., $r=\frac{1}{2}$

Tank contains beginning a_0 = 16000

Tank contains water at 8th day

$a_8=a_0r^8$

$a_8=16000(0.5)^8$

$a_8=62.5$ liters

Harmonic progression(H.P): harmonic progression is a reciprocal of arithmetic progression(A.p)

$a_1,a_2,a_3,a_4,\ldots a_n$ from (AP) therefore

$h_1=\frac{1}{a_1},h_2=\frac{1}{a_2},h_3=\frac{1}{a_3},\ldots,h_n=\frac{1}{a_n}$ from H.P

Exercise 4.9

Q1. Find the indicated term of each of the following harmonic progressions;

i). $\frac{1}{2},\frac{1}{5},\frac{1}{8},\ldots$ 9th term

Sol: Given $\frac{1}{2},\frac{1}{5},\frac{1}{8},\ldots$ its reciprocal

$a_1=2,a_2=5,a_3=8,\ldots$ form an AP, so

$d=5-2=3$

$a_9=a_1+8d$ $a_n=a_1+(n-1)d$

$a_9=2+8\times 3$

$a_9=26$

So $h_9=\frac{1}{26}$ is the required term in HP

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ii). $6,2,\frac{6}{5},\ldots$ 20th term

Sol: Given $6,2,\frac{6}{5},\ldots$ in HP

its reciprocal $a_1=\frac{1}{6},a_2=\frac{1}{2},a_3=\frac{5}{6}$ form an AP

$d=\frac{1}{2}-\frac{1}{6}$ $d=a_2-a_1$

$d=\frac{3-1}{6}$

$d=\frac{2}{6}$

$d=\frac{1}{3}$

$a_{20}=a_1+19d$ $a_n=a_1+(n-1)d$

$a_{20}=\frac{1}{6}+19\times\frac{1}{3}$

$a_{20}=\frac{1+38}{6}$

$a_{20}=\frac{39}{6}$

$a_{20}=\frac{13}{2}$

So $h_{20}=\frac{2}{13}$ is the required term in HP

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iii). $5\frac{2}{3},3\frac{2}{5},2\frac{3}{7},\ldots$ 8th term

Sol: Given $5\frac{2}{3},3\frac{2}{5},2\frac{3}{7},\ldots$ Or $\frac{17}{3},\frac{17}{5},\frac{17}{7},\ldots$ in HP

its reciprocal $a_1=\frac{3}{17},a_2=\frac{5}{17},a_3=\frac{7}{17},\ldots$ Form an AP

$d=\frac{5}{17}-\frac{3}{17}=\frac{2}{17}$

$a_8=a_1+7d \qquad a_n=a_1+(n-1)d$

$a_8=\frac{3}{17}+7\times\frac{2}{17}$

$a_8=\frac{3+14}{17}$

$a_8=\frac{17}{17}=1$

So $h_8=1$ is the required term in HP

Q2. Find five more term of HP, where $\frac{1}{3},1,-1,\ldots$ in HP

Sol: Given $\frac{1}{3},1,-1,\ldots$ in HP

its reciprocal $a_1=3,a_2=1,a_3=-1,\ldots$ form an AP

$d=1-3=-2 \qquad a_n=a_1+(n-1)d$

$a_4=3+3(-2)=-3 \qquad \because a_4=a_1+3d$

$a_5=3+4(-2)=-5 \qquad \because a_5=a_1+4d$

$a_6=3+5(-2)=-7 \qquad \because a_6=a_1+5d$

$a_7=3+6(-2)=-9 \qquad \because a_7=a_1+6d$

$a_8=3+7(-2)=-11 \qquad \because a_8=a_1+7d$

Reciprocal of AP form HP

$h_4=-\frac{1}{3},h_5=-\frac{1}{5},h_6=-\frac{1}{7},h_7=-\frac{1}{9},h_8=-\frac{1}{11}$

Q3 2nd term of an HP is $\frac{1}{2}$ & 5th term is $\frac{-1}{4}$. Find 12th term

Sol: Given that $h_2=\frac{1}{2}\ \&\ h_5=-\frac{1}{4}$ are in HP

its reciprocal $a_2=2,a_5=-4$ form an AP are the corresponding term of HP

$a_5=a_2+3d \qquad a_2=a_1+d$

$-4=2+3d$

$3d=-4-2=-6$

$d=-2$

$a_{12}=a_2+10d \qquad \because a_2=a_1+d$

$a_{12}=2+10\times(-2)=-18$

Reciprocal of AP form HP

So $h_{12}=-\frac{1}{18}$ is the required term in HP

Q4. Find arithmetic, Harmonic and geometric means of each of following. Also verify that $A\times H=G^2$

i). 3.14 and 2.71

Sol: $a=3.14,b=2.71$

$AM=\frac{a+b}{2} \qquad HM=\frac{2ab}{a+b}$

$AM=\frac{3.14+2.71}{2} \qquad HM=\frac{2(3.14)(2.71)}{3.14+2.71}$

$AM=\frac{5.85}{2}=2.925 \qquad HM=\frac{17.0188}{5.85}$

$HM=2.909$

$G=\pm\sqrt{ab}$

$G=\pm\sqrt{3.14\times2.71}$

$G=\pm\sqrt{8.5094}$

$G=\pm2.91708$

Now $A\times H=G^2$

$2.925\times2.909=2.917^2$

8.5 = 8.5 verified

ii). -6 and -216

Sol: $a=-6,b=-216$

$AM=\frac{a+b}{2} \qquad HM=\frac{2ab}{a+b}$

$AM=\frac{-6-216}{2} \qquad HM=\frac{2(-6)(-216)}{-6-216}$

$AM=\frac{-222}{2}=-111 \qquad HM=\frac{2592}{-222}$

$HM=-11.676$

$G=\pm\sqrt{ab}$

$G=-\sqrt{(-6)(-216)}$

$G=-\sqrt{1296}$

$G=-36$

Now $A\times H=G^2$

$-111\times-11.676=36^2$

1296 = 1296 verified

iii). x+y and x-y

Sol: $a=x+y,b=x-y$

$AM=\frac{a+b}{2} \qquad HM=\frac{2ab}{a+b}$

$AM=\frac{x+y+x-y}{2} \qquad HM=\frac{2(x+y)(x-y)}{x+y+x-y}$

$AM=\frac{2x}{2}=x \qquad HM=\frac{2(x^2-y^2)}{2x}$

$HM=\frac{x^2-y^2}{x}$

$G=\pm\sqrt{ab}$

$G=\pm\sqrt{(x+y)(x-y)}$

$G=\pm\sqrt{x^2-y^2}$

Now $A\times H=G^2$

$x\times\frac{(x^2-y^2)}{x}=\left(\sqrt{x^2-y^2}\right)^2$

$x^2-y^2=x^2-y^2$ *verified*

iv). $\sqrt{2}+3$ and $\sqrt{2}-3$

Sol: $a=\sqrt{2}+3,b=\sqrt{2}-3$

$HM=\frac{2ab}{a+b}$

$AM=\frac{a+b}{2} \qquad HM=\frac{2(\sqrt{2}+3)(\sqrt{2}-3)}{\sqrt{2}+3+\sqrt{2}-3}$

$AM=\frac{\sqrt{2}+3+\sqrt{2}-3}{2} \qquad HM=\frac{2(\sqrt{2}^2-3^2)}{2\sqrt{2}}$

$AM=\frac{2\sqrt{2}}{2}=\sqrt{2} \qquad HM=\frac{2(2-9)}{2\sqrt{2}}$

$HM=\frac{-7}{\sqrt{2}}$

$G=\pm\sqrt{(\sqrt{2}+3)(\sqrt{2}-3)} \qquad \therefore G=\pm\sqrt{ab}$

$G=\pm\sqrt{(\sqrt{2})^2-(3)^2}$

$G=\pm\sqrt{2-9}$

$G=\pm\sqrt{-7}$

G does not exists

Q5. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is an Harmonic mean between a and b .?

Sol: Given HM= $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ from definition Harmonic mean b\w two numbers is $\frac{2ab}{a+b}$ so both will be equal

$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{2ab}{a+b}(\text{cross multiplication})$

$(a+b)(a^{n+1}+b^{n+1})=2ab(a^n+b^n)$

$a^na^2+abb^n+aba^n+b^nb^2=2aba^n+2abb^n$

$a^na^2+b^nb^2=2aba^n-aba^n+2abb^n-abb^n$

$a^na^2+b^nb^2=aba^n+abb^n$

$a^na^2-aba^n=abb^n-b^nb^2$

$a^n.a(a-b)=bb^n(a-b)$

$\frac{a^{n+1}}{b^{n+1}}=\frac{(a-b)}{(a-b)}=1$

$$\left(\frac{a}{b}\right)^{n+1}=\left(\frac{a}{b}\right)^{0}=1$$
$\Rightarrow n+1=0$
$n=-1$

Q6. Insert two harmonic means between 12 & 48.
Sol: Let H_1 and H_2 are two harmonic mean. Then 12, H_1 , H_2 , 48 form HP
Its reciprocal $\frac{1}{12},\frac{1}{H_1},\frac{1}{H_2},\frac{1}{48}$ form an AP
Their corresponding terms are a_1,a_2,a_3,a_4 of AP
Then $a_1=\frac{1}{12},a_2=\frac{1}{H_1},a_3=\frac{1}{H_2},a_4=\frac{1}{48}$
Since $a_4=a_1+3d$ and

$\frac{1}{48}=\frac{1}{12}+3d$
$3d=\frac{1}{48}-\frac{1}{12}$
$3d=\frac{1-4}{48}$
$3d=\frac{-3}{48}$
$d=\frac{-1}{48}$

$a_2=a_1+d$
$a_2=\frac{1}{12}-\frac{1}{48}$
$a_2=\frac{4-1}{48}$
$a_2=\frac{3}{48}$ $\Rightarrow a_2=\frac{1}{16}$

Similarly
$a_3=a_2+d$
$a_3=\frac{1}{16}-\frac{1}{48}$
$a_3=\frac{3-1}{48}$
$a_3=\frac{2}{48}=\frac{1}{24}$

so the reciprocals of AP
$H_1=\frac{1}{a_2}$, $H_2=\frac{1}{a_3}$
$H_1=16$, $H_2=24$

Q7. Insert four harmonic means between $\frac{7}{3}$ and $\frac{7}{11}$
Sol: Let H_1 , H_2 , H_3 and H_4 are two harmonic means.
Then $\frac{7}{3}$, H_1 , H_2 , H_3 , H_4, $\frac{7}{11}$ form HP Its reciprocal
$a_1=\frac{3}{7},a_2=\frac{1}{H_1},a_3=\frac{1}{H_2},a_4=\frac{1}{H_3},a_5=\frac{1}{H_4},a_6=\frac{11}{7}$ form AP
Since $a_6=a_1+5d$
$\frac{11}{7}=\frac{3}{7}+5d$
$5d=\frac{11}{7}-\frac{3}{7}=\frac{8}{7}$
$d=\frac{8}{35}$

Now
$a_2=a_1+d$
$a_2=\frac{3}{7}+\frac{8}{35}$
$a_2=\frac{15+8}{35}$
$a_2=\frac{23}{35}$

and
$a_3=a_2+d$
$a_3=\frac{23}{35}+\frac{8}{35}$
$a_3=\frac{31}{35}$

Now
$a_4=a_3+d$
$a_4=\frac{31}{35}+\frac{8}{35}$
$a_4=\frac{39}{35}$

and
$a_5=a_4+d$
$a_5=\frac{39}{35}+\frac{8}{35}$
$a_5=\frac{47}{35}$

So reciprocal of AP
$H_1=\frac{1}{a_2}$, $H_2=\frac{1}{a_3}$, $H_3=\frac{1}{a_4}$, $H_4=\frac{1}{a_5}$
$H_1=\frac{35}{23}$, $H_2=\frac{35}{31}$, $H_3=\frac{35}{39}$, $H_4=\frac{35}{47}$

Q8. Prove that square of the geometric mean of two numbers equals the product of the arithmetic mean and harmonic mean to two numbers.
Sol: Let a and b are any two numbers then geometric mean, arithmetic mean and harmonic mean are
$$G=\pm\sqrt{ab}\quad AM=\frac{a+b}{2}\quad HM=\frac{2ab}{a+b}$$
respectively.
Now $AM\times HM=\frac{a+b}{2}\times\frac{2ab}{a+b}=ab=\left(\sqrt{ab}\right)^2=GM^2$

Q9. The arithmetic mean of two numbers is 8, and harmonic mean is 6. What are the numbers?
Sol: Given AM = 8 and HM = 6

$AM=\frac{a+b}{2}=8$
$a+b=16$.....(1)

$HM=\frac{2ab}{a+b}=6$
$ab=3(a+b)$
$ab=3\times16$
$ab=48$

$\therefore (a-b)^2=(a+b)^2-4ab$
$(a-b)^2=16^2-4\times48$
$(a-b)^2=256-192=64$
$\Rightarrow a-b=8$......(2)
Adding equation (1) and (2)
$a+b=16$
$a-b=8$
$2a=24$
$a=12$
Put in eq (1)
$12+b=16$
$b=16-12=4$

Q10. The harmonic mean of two numbers is $4\frac{4}{5}$, and arithmetic mean is 6. What are the numbers?
Sol: Given $G.M=6$ and $HM=4\frac{4}{5}$

$GM=\sqrt{ab}=6$
$ab=36$.....(1)

$HM=\frac{2ab}{a+b}=\frac{24}{5}$
$5\times36=12(a+b)$
$\frac{5\times36}{12}=(a+b)$
$a+b=15$

$\therefore (a-b)^2=(a+b)^2-4ab$
$(a-b)^2=15^2-4\times36$
$(a-b)^2=225-144=81$
$\Rightarrow a-b=9$......(2)
Adding equation (1) and (2)
$a+b=15$
$a-b=9$
$2a=24$
$a=12$
Put in eq (1)
$12+b=15$
$b=15-12$
$b=3$