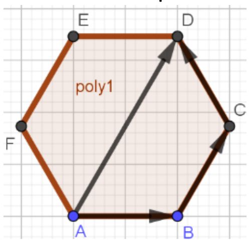
Chapter 3

Vectors

Exercise 3.1

Q1. ABCDEF is a regular hexagon $\overrightarrow{AB}=a, \overrightarrow{BC}=b, \overrightarrow{CD}=c$ state following vectors as scalar multiples of a,b,c

Hint: in a regular hexagon main diagonal $\overrightarrow{AD}$ is double the side $\overrightarrow{BC}$ and parallel to it.



Sol: we have $\overrightarrow{AB}=a, \overrightarrow{BC}=b \text{ and } \overrightarrow{CD}=c$

a). $\overrightarrow{DE}$

Sol: from figure $\overrightarrow{DE}$ and $\overrightarrow{DE}$ are parallel & equal

So $\overrightarrow{DE}=\overrightarrow{DE}$

$\overrightarrow{DE}=-\overrightarrow{ED}$ $\quad\quad \therefore \overrightarrow{DE}=-\overrightarrow{ED}$

$\overrightarrow{DE}=-a$

b). $\overrightarrow{EF}$

Sol: from figure $\overrightarrow{EF}$ and $\overrightarrow{CB}$ are parallel & equal

$\overrightarrow{EF}=\overrightarrow{CB}$

$\overrightarrow{EF}=-\overrightarrow{BC}$

$\overrightarrow{EF}=-b$

c). $\overrightarrow{FA}$

Sol: from figure $\overrightarrow{FA}$ and $\overrightarrow{DC}$ are parallel & equal

$\overrightarrow{FA}=\overrightarrow{DC}$

$\overrightarrow{FA}=-\overrightarrow{CD}$

$\overrightarrow{FA}=-c$

d). $\overrightarrow{AD}$

Sol: diagonal $\overrightarrow{AD}$ is double the side $\overrightarrow{BC}$ and parallel

$\overrightarrow{AD}=2\overrightarrow{BC}$

$\overrightarrow{AD}=2b$

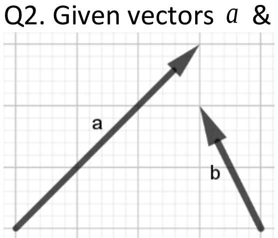
e). $\overrightarrow{BE}$

Sol: diagonal $\overrightarrow{BE}$ is double the side $\overrightarrow{CD}$ and parallel

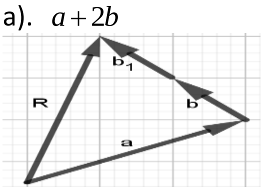
$\overrightarrow{BE}=2\overrightarrow{CD}$

$\overrightarrow{BE}=2c$

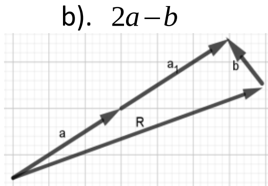
Q2. Given vectors a & b as in figure, draw vectors:



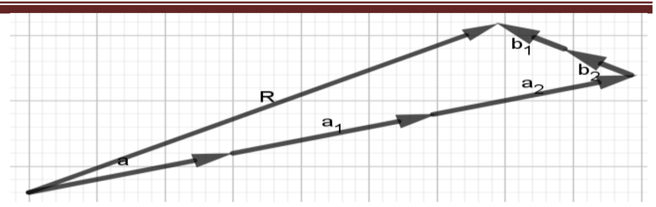
a). $a+2b$



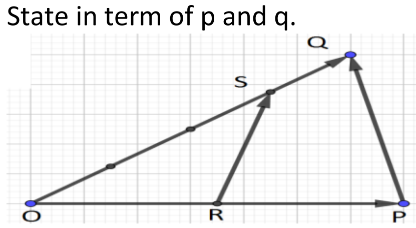
b). $2a-b$



c). $3a-2b$



Q3. In $\Delta OPQ, \overrightarrow{OP}=p, \overrightarrow{OQ}=q$, R is the midpoint of $\overrightarrow{OP}$ and S is the midpoint of $\overrightarrow{OQ}$ such that $|OS|=3|SQ|$. State in term of p and q.



Sol: we have $\Delta OPQ, \overrightarrow{OP}=p, \overrightarrow{OQ}=q$

a). $\overrightarrow{OR}$

Sol: since R is the midpoint of $\overrightarrow{OP}$ i.e. $\overrightarrow{OR}=\overrightarrow{RP}$ so

$\overrightarrow{OP}=\overrightarrow{OR}+\overrightarrow{RP}$

$\overrightarrow{OP}=\overrightarrow{OR}+\overrightarrow{OR}$ $\quad \therefore \overrightarrow{OR}=\overrightarrow{RP}$

$\overrightarrow{OP}=2\overrightarrow{OR}$

$\overrightarrow{OR}=\frac{1}{2}\overrightarrow{OP}$

$\overrightarrow{OR}=\frac{1}{2}p$ $\quad\quad \therefore \overrightarrow{OP}=p$

b). $\overrightarrow{PQ}$

Sol: From figure using Head to tail rule

$\overrightarrow{PQ}=\overrightarrow{PO}+\overrightarrow{OQ}$

$\overrightarrow{PQ}=-\overrightarrow{OP}+\overrightarrow{OQ}$

$\overrightarrow{PQ}=-p+q$

c). $\overrightarrow{OS}$

Sol: From given condition $|OS|=3|SQ|$ or $\frac{1}{3}|OS|=|SQ|$

Or we can write $\frac{|OS|}{|SQ|}=\frac{3}{1}$ or $|OS|:|SQ|=3:1$

Using ratio

$\overrightarrow{OS}=\frac{3q+1.0}{3+1}$

$\overrightarrow{OS}=\frac{3}{4}q$

2nd method

$\overrightarrow{OQ}=\overrightarrow{OS}+\overrightarrow{SQ}$

$\overrightarrow{OQ}=\overrightarrow{OS}+\frac{1}{3}\overrightarrow{OS}$

$\overrightarrow{OQ}=\frac{4}{3}\overrightarrow{OS}$

$\overrightarrow{OS}=\frac{3}{4}\overrightarrow{OQ}$

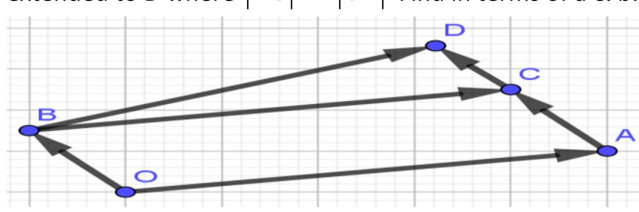
d). $\overrightarrow{RS}$

Sol: From figure using Head to tail rule

$\overrightarrow{RS}=\overrightarrow{OR}+\overrightarrow{OS}$

$\overrightarrow{RS}=-\frac{1}{2}p+\frac{3}{4}q$

Q4. OABC is parallelogram with $\overrightarrow{OA}=a, \overrightarrow{OB}=b$, AC extended to D where $|AC|=2|CD|$ Find in terms of a & b.



a). $\overrightarrow{AD}$

Sol: we have $\overrightarrow{OA}=a, \overrightarrow{OB}=b$ and $|AC|=2|CD|$

Or $|CD|=\frac{1}{2}|AC|$ so

$\overrightarrow{AD}=\overrightarrow{AC}+\overrightarrow{CD}$

$\overrightarrow{AD}=\overrightarrow{AC}+\frac{1}{2}\overrightarrow{AC}$

$\overrightarrow{AD}=\frac{3}{2}\overrightarrow{AC}=\frac{3}{2}b$

b). $\overrightarrow{OD}$

Sol: we have $\overrightarrow{OA}=a,\overrightarrow{OB}=b$

$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}$

$\overrightarrow{OD}=a+\frac{3}{2}b$

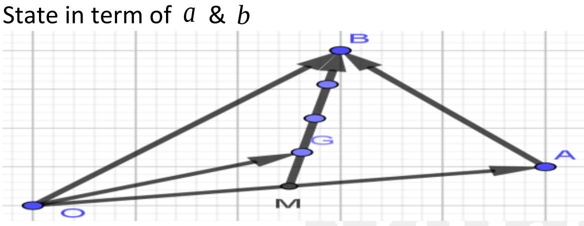
c). $\overrightarrow{BD}$

Sol: we have $\overrightarrow{OA}=a,\overrightarrow{OB}=b$

$\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}$

$\overrightarrow{BD}=a+\frac{1}{2}b$

Q5. OAB is a triangle with $\overrightarrow{OA}=a,\overrightarrow{OB}=b$, M is the midpoint of OA and G lies on MB such that $|MG|=\frac{1}{2}|GB|$. State in term of a & b



a). Find $\overrightarrow{OM}$

Sol: M is midpoint of $\overrightarrow{OA}=a$, So $\overrightarrow{OM}=\overrightarrow{MA}$

$\overrightarrow{OA}=\overrightarrow{OM}+\overrightarrow{MA}$

$\overrightarrow{OA}=\overrightarrow{OM}+\overrightarrow{OM}$

$\overrightarrow{OA}=2\overrightarrow{OM}$

$\overrightarrow{OM}=\frac{1}{2}\overrightarrow{OA}$

$\overrightarrow{OM}=\frac{1}{2}a$

b). $\overrightarrow{MB}$

Sol: we have $\overrightarrow{OA}=a,\overrightarrow{OB}=b$ From figure

$\overrightarrow{MB}=\overrightarrow{MO}+\overrightarrow{OB}$

$\overrightarrow{MB}=-\overrightarrow{OM}+\overrightarrow{OB}$

$\overrightarrow{MB}=-\frac{1}{2}a+b$

c). $\overrightarrow{MG}=\overrightarrow{MO}+\overrightarrow{OB}+\overrightarrow{BG}$

Sol: $\overrightarrow{OA}=a,\overrightarrow{OB}=b$ and $|MG|=\frac{1}{2}|GB|$ or $2|MG|=|GB|$

From figure $\overrightarrow{MG}=\overrightarrow{MO}+\overrightarrow{OB}+\overrightarrow{BG}$

$\overrightarrow{MG}=-\overrightarrow{OM}+b-\overrightarrow{GB}$

$\overrightarrow{MG}=-\frac{1}{2}a+b-2\overrightarrow{MG}$

$3\overrightarrow{MG}=b-\frac{1}{2}a$

$\overrightarrow{MG}=\frac{1}{3}b-\frac{1}{6}a$

d). $\overrightarrow{OG}$

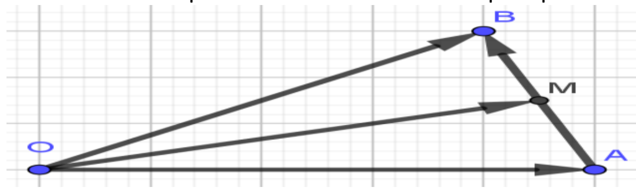
Sol: From figure $\overrightarrow{OG}=\overrightarrow{OM}+\overrightarrow{MG}$

$\overrightarrow{OG}=\frac{1}{2}a+\frac{1}{3}b-\frac{1}{6}a$

$\overrightarrow{OG}=\frac{3a+2b-a}{6}$

$\overrightarrow{OG}=\frac{2a+2b}{6}=\frac{a+b}{3}$

Q6 $\overrightarrow{OA}=p+q,\overrightarrow{OB}=2p-q$ where p & q are two vectors and M is the midpoint of AB. Find in term of p & q.



a). $\overrightarrow{AB}$

Sol: From figure $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$

$\overrightarrow{AB}=-\overrightarrow{OA}+\overrightarrow{OB}$

$\overrightarrow{AB}=-p-q+2p-q$

$\overrightarrow{AB}=p-2q$

b). $\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}$

Sol: M is the midpoint of $\overrightarrow{AB}$, i.e $\overrightarrow{AM}=\overrightarrow{MB}$, so

$\overrightarrow{AB}=\overrightarrow{AM}+\overrightarrow{MB}$

$\overrightarrow{AB}=\overrightarrow{AM}+\overrightarrow{AM}$

$\overrightarrow{AB}=2\overrightarrow{AM}$

$\overrightarrow{AM}=\frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{AM}=\frac{1}{2}(p-2q)$

c). $\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}$

Sol: using $\overrightarrow{AM}=\frac{1}{2}(p-2q)=\frac{1}{2}p-q$

$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}$

$\overrightarrow{OM}=p+q+\frac{1}{2}p-q$

$\overrightarrow{OM}=\frac{3}{2}p$

Q7. Given that $p=3a-b,q=2a-3b$, find number x and y such that $xp+yq=a+9b$

Sol: we have $p=3a-b,q=2a-3b$

$xp+yq=a+9b$ putting the values

$x(3a-b)+y(2a-3b)=a+9b$

$3ax-bx+2ay-3by=a+9b$

$3ax+2ay-bx-3by=a+9b$

$(3x+2y)a+(-x-3y)b=a+9b$

Comparing the coefficients of a and b

$3x+2y=1$...........(1)

$-x-3y=9$ or $-9-3y=x$.....(2)

Putting the value of x in eq (1)

$3(-9-3y)+2y=1$

$-27-9y+2y=1$

$-27-1=7y$

$y=\frac{-28}{7}=-4$

Putting the value of y in eq (2)

$x=-9-3(-4)$

$x=-9+12$

$x=3,y=-4$

Q8. Position vectors of four points A,B,C,D are $a,b,2a+3b,\ a-2b$ respectively. Find $\overrightarrow{AC},\overrightarrow{BD},\overrightarrow{BC},\overrightarrow{CD}$ in term of a & b

Sol: let $\overrightarrow{OA}=a,\overrightarrow{OB}=b,\overrightarrow{OC}=2a+3b,\overrightarrow{OD}=a-2b$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=2a+3b-a=a+3b$

$\overrightarrow{DB}=\overrightarrow{OB}-\overrightarrow{OD}=b-(a-2b)=3b-a$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=2a+3b-b=2a+2b$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=a-2b-2a-3b=-a-5b$

Exercise 3.2

Q1. Find the position vectors of the following points

i). $P(0,0)$

Sol: since $O(0,0)$ so

$OP = P - O = (0,0) - (0,0)$

$OP = (0,0) = 0.i + 0.j$

ii). $Q(3,-2)$

Sol: since $O(0,0)$ so

$OQ = Q - O = (3,-2) - (0,0)$

$OQ = (3,-2) = 3i - 2j$

iii). $R\left(\sqrt{3}, 2\sqrt{2}\right)$

Sol: since $O(0,0)$ so

$OR = R - O = \left(\sqrt{3}, 2\sqrt{2}\right) - (0,0)$

$OR = \left(\sqrt{3}, 2\sqrt{2}\right) = \sqrt{3}i + 2\sqrt{2}j$

Q2. Express the vector $\overrightarrow{PQ}$ in the from of $xi + yj$

i). $P(0,0), Q(4,5)$

Sol: we have $P(0,0), Q(4,5)$

$\overrightarrow{PQ} = OQ - OP = (4,5) - (0,0)$

$\overrightarrow{PQ} = (4,5) = 4i + 5j$

ii). $P(-2,-1), Q(6,-2)$

Sol: we have $P(-2,-1), Q(6,-2)$

$\overrightarrow{PQ} = OQ - OP = (6,-2) - (-2,-1)$

$\overrightarrow{PQ} = (8,-1) = 8i - j$

iii). $P(1,0), Q(0,1)$

Sol: we have $P(1,0), Q(0,1)$

$\overrightarrow{PQ} = OQ - OP = (0,1) - (1,0)$

$\overrightarrow{PQ} = (-1,1) = -i + j$

Q3. If $a = 3i - 5j$ and $b = -2i + 3j$ then find

i). $a + 2b$

Sol: we have $a = 3i - 5j$ and $b = -2i + 3j$

$a + 2b = (3i - 5j) + 2(-2i + 3j)$

$a + 2b = 3i - 5j - 4i + 6j$

$a + 2b = -i + j$

ii). $3a - 2b$

Sol: we have $a = 3i - 5j$ and $b = -2i + 3j$

$3a - 2b = 3(3i - 5j) - 2(-2i + 3j)$

$3a - 2b = 9i - 15j + 4i - 6j$

$3a - 2b = 13i - 21j$

iii). $2(a - b)$

Sol: we have $a = 3i - 5j$ and $b = -2i + 3j$

$2(a - b) = 2\{(3i - 5j) - (-2i + 3j)\}$

$2(a - b) = 2(3i - 5j + 2i - 3j)$

$2(a - b) = 2(5i - 8j)$

$2(a - b) = 10i - 16j$

iv). $|a + b|$

Sol: we have $a = 3i - 5j$ and $b = -2i + 3j$

$|a + b| = |(3i - 5j) + (-2i + 3j)|$

$|a + b| = |3i - 5j - 2i + 3j|$

$|a + b| = |i - 2j|$

$|a + b| = \sqrt{1^2 + (-2)^2}$

$|a + b| = \sqrt{1 + 4} = \sqrt{5}$

v). $|a| - |b|$

Sol: we have $a = 3i - 5j$ and $b = -2i + 3j$

$|a| - |b| = |3i - 5j| - |-2i + 3j|$

$|a| - |b| = \sqrt{3^2 + (-5)^2} - \sqrt{(-2)^2 + 3^2}$

$|a| - |b| = \sqrt{9 + 25} - \sqrt{4 + 9}$

$|a| - |b| = \sqrt{34} - \sqrt{13}$

vi). $\frac{|a|}{|b|}$

Sol: we have $a = 3i - 5j$ and $b = -2i + 3j$

$\frac{|a|}{|b|} = \frac{|3i - 5j|}{|-2i + 3j|}$

$\frac{|a|}{|b|} = \frac{\sqrt{3^2 + (-5)^2}}{\sqrt{(-2)^2 + 3^2}}$

$\frac{|a|}{|b|} = \frac{\sqrt{9 + 25}}{\sqrt{4 + 9}}$

$\frac{|a|}{|b|} = \frac{\sqrt{34}}{\sqrt{13}}$

Q4. Find the unit vector having the same direction as the vector given below

i). $3i$

Sol: Let A= $3i$

$|A| = |3i| = \sqrt{3^2} = 3$

$A = \frac{A}{|A|} = \frac{3i}{3} = i$

ii). $i - j$

Sol: Let $A = i - j$

$|A| = |i - j| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$A = \frac{A}{|A|} = \frac{i - j}{\sqrt{2}} = \frac{i}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

iii). $3i - 4j$

Sol: Let $A = 3i - 4j$

$|A| = |3i - 4j| = \sqrt{3^2 + (-4)^2}$

$|A| = \sqrt{9 + 16} = \sqrt{25} = 5$

$A = \frac{A}{|A|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$

iv). $A = \frac{\sqrt{3}}{2}i - \frac{1}{2}j$

Sol: we have $A = \frac{\sqrt{3}}{2}i - \frac{1}{2}j$

$|A| = \left|\frac{\sqrt{3}}{2}i - \frac{1}{2}j\right| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$

$|A| = \sqrt{\frac{3}{4} + \frac{1}{4}}$

$|A| = \sqrt{\frac{4}{4}} = 1$

$A = \frac{A}{|A|} = \frac{\frac{\sqrt{3}}{2}i - \frac{1}{2}j}{1}$

$A = \frac{\sqrt{3}}{2}i - \frac{1}{2}j$

Q5. If $r = i - 9j, a = i + 2j$ and $b = 5i - j$, determine the real numbers p and q such that $r = pa + qb$

Sol: we have $r = i - 9j, a = i + 2j$ and $b = 5i - j$,

since $r = pa + qb$

Putting the values of r, a and b

$i-9j = p(i+2j)+q(5i-j)$

$i-9j = pi+2Pj+5qi-qj$

$i-9j = pi+5qi+2pj-qj$

$i-9j = (p+5q)i+(2p-q)j$

By comparing the coefficients of i and j

$p+5q=1 \quad ....(1) \quad \text{multiply by 2} \quad \Rightarrow \quad 2p+10q=2$

$2p-q=-9 \qquad\qquad \underline{\pm 2p \mp q = \mp 9}$

$\text{by subtracting} \qquad 11q = 11$

$or \quad q=1 \quad \text{put in (1)}$

$p+5=1$

$p=-4$

Q6. Find length of vector $\overrightarrow{AB}$ from a point $A(-3,5)$ & $B(7,9)$. Also find a unit vector in direction of $\overrightarrow{AB}$

Sol: we have $A(-3,5)$ & $B(7,9)$

$\overrightarrow{AB} = OB - OA = (7,9)-(-3,5)$

$\overrightarrow{AB} = (10,4) = 10i+4j$

$|\overrightarrow{AB}| = |10i+4j| = \sqrt{10^2+4^2}$

$|\overrightarrow{AB}| = \sqrt{100+16} = \sqrt{116} = 2\sqrt{29}$

$$AB = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{10i+4j}{2\sqrt{29}} = \frac{5i}{\sqrt{29}} + \frac{2j}{\sqrt{29}}$$

Q7. If $p=2i+3j$ and $q=i-j$, then find numbers x and y such that $xp+yq=-4i-11j$

Sol: we have $p=2i+3j$ and $q=i-j$

Given that $xp+yq=-4i-11j$ putting the values

$x(2i+3j)+y(i-j)=-4i-11j$

$2xi+3xj+yi-yj=-4i-11j$

$2xi+yi+3xj-yj=-4i-11j$

$(2x+y)i+(3x-y)j=-4i-11j$

Comparing the coefficients we get

$2x+y=-4 \quad ....(1)$

$\underline{3x-y=-11 \quad ....(2)}$

$5x \quad = -15$

$\Rightarrow x=-3 \text{ put in (1)}$

$-6+y=-4$

$y=-4+6$

$y=2$

Q8. If $p=2i-j$ and $q=xi+3j$, then find numbers x such that $|p+q|=5$

Sol: we have $p=2i-j$ and $q=xi+3j$

$p+q=(2i-j)+(xi+3j)$

$p+q=2i+xi-j+3j$

$p+q=(2+x)i+2j$

$|p+q|=|(2+x)i+2j|$

$|p+q|=\sqrt{(2+x)^2+2^2}$

$5=\sqrt{4+x^2+4x+4} \qquad \because |p+q|=5$

$5=\sqrt{x^2+4x+8}$

$25=x^2+4x+8$

$x^2+4x+8-25=0$

$x^2+4x-17=0$

Here

$a=1, b=4, c=-17 \qquad \therefore x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-4\pm\sqrt{4^2-4(1)(-17)}}{2(1)}$$

$$x=\frac{-4\pm\sqrt{16+68}}{2}$$

$$x=\frac{-4\pm\sqrt{84}}{2}$$

$$x=\frac{-4\pm\sqrt{4\times 21}}{2}$$

$$x=\frac{-4\pm 2\sqrt{21}}{2}$$

$x=-2\pm\sqrt{21}$

Q9. If ABCD is a parallelogram such that the coordinates of vertices, A,B and C are respectively given by (-2,3),(1,4) and (0,5). Find the coordinates of the vertex D.

Sol: Given that $A(-2,-3), B(1,4), C(0,5)$ Let $D(x,y)$

Since ABCD is parallelogram so and equal

$\overrightarrow{AB} \,||\, \overrightarrow{CD}$ and $\overrightarrow{BC} \,||\, \overrightarrow{DA}$

Now $\overrightarrow{AB} \,||\, \overrightarrow{CD}$

$B-A=k(D-C)$ for equal sides $k=1$

$B-A+C=D$

$(i+4j)-(-2i-3j)=(xi+yj)-(0i+5j)$

$i+3i+4j+3j=xi+yj-5j$

$3i+7j=xi+(y-5)j$

Comparing the coefficients

$3=x, \qquad 7=y-5$

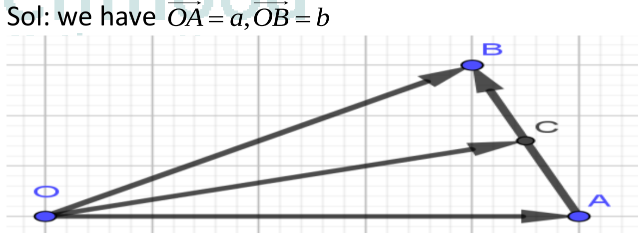
$\qquad\qquad \Rightarrow 7+5=y$

$\qquad\qquad y=12$

Therefore the coordinates of $D(3,6)$

Q10 If a & b are position vectors of A & B respectively, then prove that position vectors of midpoint of line segment joining A and B is $\frac{a+b}{2}$

Sol: we have $\overrightarrow{OA}=a, \overrightarrow{OB}=b$



From figure $\qquad \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$\overrightarrow{AB}=b-a$

Let C in the mid-point of $\overrightarrow{AB}$ which divides internally than $\overrightarrow{AC}=\overrightarrow{CB}$

$\overrightarrow{AB}=\overrightarrow{AC}+\overrightarrow{CB}$

$\overrightarrow{AB}=\overrightarrow{AC}+\overrightarrow{AC} \qquad \therefore \overrightarrow{AC}=\overrightarrow{CB}$

$\overrightarrow{AB}=2\overrightarrow{AC}$

$\overrightarrow{AC}=\frac{1}{2}\overrightarrow{AB}$

Putting the value of $\overrightarrow{AB}=b-a$

$\overrightarrow{AC}=\frac{1}{2}(b-a)$

Now $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$

$\overrightarrow{OC}=a+\frac{1}{2}(b-a)$

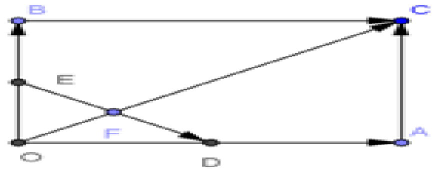
$\overrightarrow{OC}=a+\frac{b}{2}-\frac{a}{2}$

$\overrightarrow{OC}=a-\frac{a}{2}+\frac{b}{2}$

$\overrightarrow{OC}=\frac{a}{2}+\frac{b}{2}=\frac{a+b}{2}$

Q11. Using vectors, Prove that the line passes through the midpoints of adjacent sides of a rectangle divides one of the diagonal in the ratio 1:3

Sol: Let OACB is rectangle with $\overrightarrow{OA}=a, \overrightarrow{OB}=b$



And D and E are the mid points of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, Then

$\overrightarrow{OD}=\overrightarrow{DA}$ $\qquad$ $\overrightarrow{OE}=\overrightarrow{EB}$

$\overrightarrow{OA}=\overrightarrow{OD}+\overrightarrow{DA}$ $\qquad$ $\overrightarrow{OB}=\overrightarrow{OE}+\overrightarrow{EB}$

$\overrightarrow{OA}=\overrightarrow{OD}+\overrightarrow{OD}$ $\qquad$ $\overrightarrow{OB}=\overrightarrow{OE}+\overrightarrow{OE}$

$\overrightarrow{OA}=2\overrightarrow{OD}$ $\qquad$ $\overrightarrow{OB}=2\overrightarrow{OE}$

$\overrightarrow{OD}=\frac{1}{2}\overrightarrow{OA}$ $\qquad$ $\overrightarrow{OE}=\frac{1}{2}\overrightarrow{OB}$

$\overrightarrow{OD}=\frac{1}{2}a$ $\qquad$ $\overrightarrow{OE}=\frac{1}{2}b$

$$\Rightarrow \overrightarrow{ED}=\overrightarrow{EO}+\overrightarrow{OD}$$

$$\overrightarrow{ED}=\overrightarrow{OD}-\overrightarrow{OE}$$

$$\overrightarrow{ED}=\frac{a-b}{2}$$

Here F is the mid point of $\overrightarrow{ED}$ so $\overrightarrow{EF}=\frac{1}{2}\overrightarrow{ED}$

$$\overrightarrow{EF}=\overrightarrow{FD}=\frac{1}{2}\left(\frac{a-b}{2}\right)=\frac{a-b}{4}$$

$$\overrightarrow{OF}=\overrightarrow{OE}+\overrightarrow{EF}$$

$$\overrightarrow{OF}=\frac{b}{2}+\frac{a-b}{4}$$

$$\overrightarrow{OF}=\frac{2}{2}\times\frac{b}{2}+\frac{a-b}{4}$$

$$\overrightarrow{OF}=\frac{2b+a-b}{4}$$

$$\overrightarrow{OF}=\frac{a+b}{4}$$

Now $\quad \overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC} \qquad \because \overrightarrow{AC}=\overrightarrow{OB}$

$$\overrightarrow{OC}=a+b$$

Since

$$\overrightarrow{OC}=\overrightarrow{OF}+\overrightarrow{FC}$$

$$a+b=\frac{a+b}{4}+\overrightarrow{FC}$$

$$\frac{a+b}{1}-\frac{a+b}{4}=\overrightarrow{FC}$$

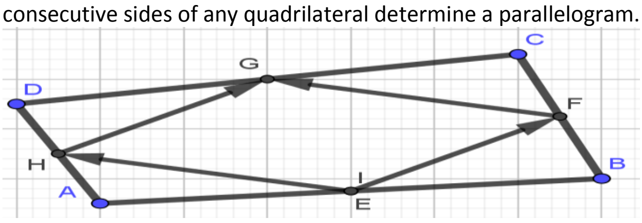
$$3\left(\frac{a+b}{4}\right)=\overrightarrow{FC}$$

$$3\overrightarrow{OF}=\overrightarrow{FC}$$

$$\frac{\overrightarrow{OF}}{\overrightarrow{FC}}=\frac{1}{3}$$

$$\overrightarrow{OF}:\overrightarrow{FC}=1:3$$

Q12. Prove that the line segments joining the midpoints of the consecutive sides of any quadrilateral determine a parallelogram.



Sol: ABCD is a quadrilateral with the position vectors $\overrightarrow{OA}=a, \overrightarrow{OB}=b, \overrightarrow{OC}=c, \overrightarrow{OD}=d$ and E,F,G and H are the mid points of AB, BC, CD and DA respectively.

Using midpoint formula

$$\overrightarrow{OE}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}=\frac{a+b}{2}, \quad \overrightarrow{OF}=\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}=\frac{b+c}{2}$$

$$\overrightarrow{OG}=\frac{\overrightarrow{OC}+\overrightarrow{OD}}{2}=\frac{c+d}{2}, \overrightarrow{OH}=\frac{\overrightarrow{OA}+\overrightarrow{OD}}{2}=\frac{a+d}{2}$$

Now

$$\overrightarrow{EF}=\overrightarrow{OF}-\overrightarrow{OE}=\frac{b+c}{2}-\frac{a+b}{2}=\frac{c-a}{2}.........(1)$$

$$\overrightarrow{FG}=\overrightarrow{OG}-\overrightarrow{OF}=\frac{c+d}{2}-\frac{b+c}{2}=\frac{d-b}{2}........(2)$$

$$\overrightarrow{HG}=\overrightarrow{OG}-\overrightarrow{OH}=\frac{c+d}{2}-\frac{a+d}{2}=\frac{c-a}{2}......(3)$$

$$\overrightarrow{EH}=\overrightarrow{OH}-\overrightarrow{OE}=\frac{a+d}{2}-\frac{a+b}{2}=\frac{d-b}{2}.......(4)$$

from equation (1) and (3) we get

$$\overrightarrow{EF}=1.\overrightarrow{HG} \Leftrightarrow \overrightarrow{EF} \parallel \overrightarrow{HG}$$

from equation (2) and (4) we get

$$\overrightarrow{FG}=1.\overrightarrow{EH} \Leftrightarrow \overrightarrow{FG} \parallel \overrightarrow{EH}$$

So EFGH is a parallelogram.

Exercise 3.3

Q1. Find the components of the vector $\overrightarrow{P_1P_2}$

i). $P_1(5,-2,1), P_2(2,4,2)$

Sol: we have $P_1(5,-2,1), P_2(2,4,2)$

$$\overrightarrow{P_1P_2}=(2,4,2)-(5,-2,1)$$

$$\overrightarrow{P_1P_2}=(-3,6,1)$$

If $\overrightarrow{P_1P_2}=(x,y,z)$ then $(x,y,z)=(-3,6,1)$

$$\Rightarrow x=-3, y=6, z=1$$

---

ii). $P_1(0,0,0), P_2(-2,5,1)$

Sol: we have $P_1(0,0,0), P_2(-2,5,1)$

$$\overrightarrow{P_1P_2}=(-2,5,1)-(0,0,0)$$

$$\overrightarrow{P_1P_2}=(-2,5,1)$$

If $\overrightarrow{P_1P_2}=(x,y,z)$ then $(x,y,z)=(-2,5,1)$

$$\Rightarrow x=-2, y=5, z=1$$

---

iii). $P_1(2,1,-3), P_2(7,1,-3)$

Sol: we have $P_1(2,1,-3), P_2(7,1,-3)$

$$\overrightarrow{P_1P_2}=(7,1,-3)-(2,1,-3)$$

$$\overrightarrow{P_1P_2}=(5,0,0)$$

If $\overrightarrow{P_1P_2}=(x,y,z)$ then $(x,y,z)=(5,0,0)$

$$\Rightarrow x=5, y=0, z=0$$

Q2. Find the initial point of the vector $r=(-2,1,2)$ if the terminal point is $(4,0,-1)$

Sol: Terminal Point B=$(4,0,-1)$ and Initial point A=?

Then $r=AB=OB-OA$

$$(-2,1,2)=(4,0,-1)-OA$$

$$OA=(4,0,-1)-(-2,1,2)$$

$$OA=(6,-1,-3)$$

Q3. Find the terminal point of the vector $r = i + 3j - 3k$ if the initial point is $(-2,1,4)$

Sol: Initial point A= $(-2,1,4)$ Terminal Point B=?

Then $r = AB = OB - OA$

$(1,3,-3) = OB - (-2,1,4)$

$OB = (1,3,-3) + (-2,1,4)$

$OB = (-1,4,1)$

Q4. Let $u = i + 2j - 3k, v = 2i - j + 2k, w = 3i - j + 5k$ Find

i). $u - 2v$

Sol: Since $u = i + 2j - 3k, v = 2i - j + 2k, w = 3i - j + 5k$

$u - 2v = (i + 2j - 3k) - 2(2i - j + 2k)$

$u - 2v = (i + 2j - 3k) - (4i - 2j + 4k)$

$u - 2v = -3i + 4j - 7k$

---

ii). $3v + 2w$

Sol: Since $u = i + 2j - 3k, v = 2i - j + 2k, w = 3i - j + 5k$

$3v + 2w = 3(2i - j + 2k) + 2(3i - j + 5k)$

$3v + 2w = (6i - 3j + 6k) + (6i - 2j + 10k)$

$3v + 2w = 12i - 5j + 16k$

---

iii). $3u - (2v + w)$

Sol: Since $u = i + 2j - 3k, v = 2i - j + 2k, w = 3i - j + 5k$

$3u - (2v + w) = 3(i + 2j - 3k) - \{2(2i - j + 2k) + (3i - j + 5k)\}$

$3u - (2v + w) = 3(i + 2j - 3k) - \{(4i - 2j + 4k) + (3i - j + 5k)\}$

$3u - (2v + w) = (3i + 6j - 9k) - (7i - 3j + 9k)$

$3u - (2v + w) = -4i + 9j - 18k$

Q5. Let $p = i - 3j + 2k, q = i + j$ & $r = 2i + 2j - 4k$ Find

i). $|p + q - r|$

Sol: since $p = i - 3j + 2k, q = i + j$ and $r = 2i + 2j - 4k$

$|p + q - r| = |(i - 3j + 2k) + (i + j) - (2i + 2j - 4k)|$

$|p + q - r| = |2i - 2i - 2j - 2j + 2k + 4k|$

$|p + q - r| = |-4j + 6k|$

$|p + q - r| = \sqrt{(-4)^2 + 6^2} = \sqrt{16 + 36}$

$|p + q - r| = \sqrt{52} = \sqrt{4 \times 13} = 2\sqrt{13}$

---

ii). $|p| + |q|$

Sol: since $p = i - 3j + 2k, q = i + j$

$|p| + |q| = |i - 3j + 2k| + |i + j|$

$|p| + |q| = \sqrt{1^2 + (-3)^2 + 2^2} + \sqrt{1^2 + 1^2}$

$|p| + |q| = \sqrt{1 + 9 + 4} + \sqrt{1 + 1}$

$|p| + |q| = \sqrt{14} + \sqrt{2}$

---

iii). $\left|\frac{1}{|r|}.r\right|$

Sol: since $r = 2i + 2j - 4k$

$|r| = |2i + 2j - 4k| = \sqrt{2^2 + 2^2 + (-4)^2}$

$|r| = \sqrt{4 + 4 + 16} = \sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$

Now $\left|\frac{1}{|r|}.r\right| = \left|\frac{2i + 2j - 4k}{2\sqrt{6}}\right| = \left|\frac{i + j - 2k}{\sqrt{6}}\right|$

$\left|\frac{1}{|r|}.r\right| = \sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-2}{\sqrt{6}}\right)^2}$

$\left|\frac{1}{|r|}.r\right| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} = \sqrt{\frac{6}{6}} = \sqrt{1} = 1$

In questions 6-12 A,B,C,D are the points with position vectors given by

$\overrightarrow{OA} = i + j + k, \overrightarrow{OB} = i - j + 2k, \overrightarrow{OC} = j + k, \overrightarrow{OD} = 2i + j$

Q6. Find $|\overrightarrow{AB}|, |\overrightarrow{BD}|$

Sol: $\overrightarrow{OA} = i + j + k, \overrightarrow{OB} = i - j + 2k, \overrightarrow{OC} = j + k, \overrightarrow{OD} = 2i + j$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (i - j + 2k) - (i + j + k)$

$\overrightarrow{AB} = -2j + k$ Then $|\overrightarrow{AB}| = \sqrt{(-2)^2 + 1^2}$

$|\overrightarrow{AB}| = \sqrt{4 + 1} = \sqrt{5}$

$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = (2i + j) - (i - j + 2k)$

$\overrightarrow{BD} = i + 2j - 2k$ Then $|\overrightarrow{BD}| = \sqrt{1^2 + 2^2 + (-2)^2}$

$|\overrightarrow{BD}| = \sqrt{1 + 4 + 4}$

$|\overrightarrow{BD}| = \sqrt{9} = 3$

Q7. Find the direction cosine of $\overrightarrow{CD}$ and $\overrightarrow{AC}$

Sol: $\overrightarrow{OA} = i + j + k, \overrightarrow{OB} = i - j + 2k, \overrightarrow{OC} = j + k, \overrightarrow{OD} = 2i + j$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (2i + j) - (j + k)$

$\overrightarrow{CD} = 2i - k$ Then $|\overrightarrow{CD}| = |2i - k| = \sqrt{2^2 + (-1)^2}$

$|\overrightarrow{CD}| = \sqrt{4 + 1} = \sqrt{5}$

So the direction of cosine

$\left(\frac{2}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (j + k) - (i + j + k)$

$\overrightarrow{AC} = -i$

$|\overrightarrow{AC}| = |-i| = \sqrt{(-1)^2} = \sqrt{1} = 1$

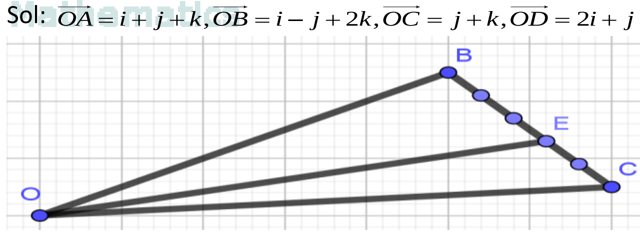
So the direction of cosine

$\left(\frac{-1}{1}, \frac{0}{1}, \frac{0}{1}\right) = (-1, 0, 0)$

Q8. Find the position vector of the point which

i). divides BC internally in the ratio 3:2

Sol: $\overrightarrow{OA} = i + j + k, \overrightarrow{OB} = i - j + 2k, \overrightarrow{OC} = j + k, \overrightarrow{OD} = 2i + j$



$\overrightarrow{OE} = \frac{3\overrightarrow{OC} + 2\overrightarrow{OB}}{3 + 2} = \frac{3(j + k) + 2(i - j + 2k)}{5}$

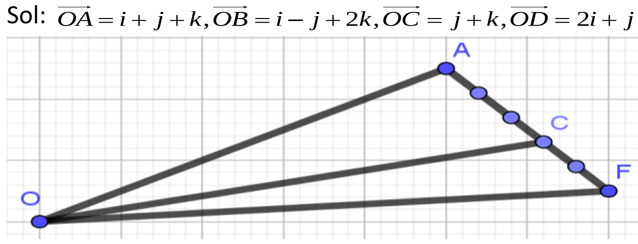
$\overrightarrow{OE} = \frac{3j + 3k + 2i - 2j + 4k}{5}$

$\overrightarrow{OE} = \frac{2i + j + 7k}{5}$

---

ii). Divides AC externally in the ration 3:2

Sol: $\overrightarrow{OA} = i + j + k, \overrightarrow{OB} = i - j + 2k, \overrightarrow{OC} = j + k, \overrightarrow{OD} = 2i + j$



$\overrightarrow{OF} = \frac{3\overrightarrow{OC} - 2\overrightarrow{OA}}{3 - 2} = \frac{3(j + k) - 2(i + j + k)}{3 - 2}$

$\overrightarrow{OF} = \frac{3j + 3k - 2i - 2j - 2k}{1}$

$\overrightarrow{OF} = -2i + j + k$

Q9. Determine whether any of following pairs of lines are parallel.

Sol: $\overrightarrow{OA}=i+j+k, \overrightarrow{OB}=i-j+2k, \overrightarrow{OC}=j+k, \overrightarrow{OD}=2i+j$

i). AB and CD

Sol: $\overrightarrow{OA}=i+j+k, \overrightarrow{OB}=i-j+2k, \overrightarrow{OC}=j+k, \overrightarrow{OD}=2i+j$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(i-j+2k)-(i+j+k)$

$\overrightarrow{AB}=-2j+k$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=(2i+j)-(j+k)$

$\overrightarrow{CD}=2i-k$

$\overrightarrow{AB}$ is not parallel to $\overrightarrow{CD}$

We can not express as $\overrightarrow{AB}=k\ \overrightarrow{CD}$

Where k is some constant

ii). AC and BD

Sol: $\overrightarrow{OA}=i+j+k, \overrightarrow{OB}=i-j+2k, \overrightarrow{OC}=j+k, \overrightarrow{OD}=2i+j$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(j+k)-(i+j+k)$

$\overrightarrow{AC}=-i$

$\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=(2i+j)-(i-j+2k)$

$\overrightarrow{BD}=i+2j-2k$

$\overrightarrow{AC}$ is not parallel to $\overrightarrow{BD}$

We can not express as $\overrightarrow{AC}=k\ \overrightarrow{BD}$

Where k is some constant

iii). AD and BC

Sol: $\overrightarrow{OA}=i+j+k, \overrightarrow{OB}=i-j+2k, \overrightarrow{OC}=j+k, \overrightarrow{OD}=2i+j$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(2i+j)-(i+j+k)$

$\overrightarrow{AD}=i-k$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(j+k)-(i-j+2k)=j+k-i+j-2k$

$\overrightarrow{BC}=-i+2j-k$

$\overrightarrow{AD}$ is not parallel to $\overrightarrow{BC}$

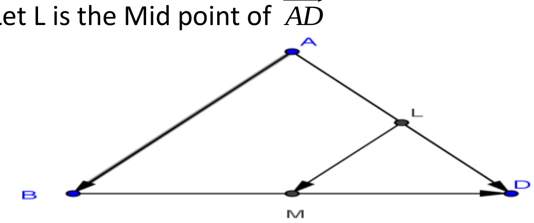
We can not express as $\overrightarrow{AD}=k\ \overrightarrow{BC}$

Where k is some constant

Q10. Let L & M are the position vectors of the mid points of AD & BD respectively. Show that $\overrightarrow{LM}$ is parallel to $\overrightarrow{AB}$

Sol: $\overrightarrow{OA}=i+j+k, \overrightarrow{OB}=i-j+2k, \overrightarrow{OC}=j+k, \overrightarrow{OD}=2i+j$

Let L is the Mid point of $\overrightarrow{AD}$



$\overrightarrow{OL}=\dfrac{\overrightarrow{OD}+\overrightarrow{OA}}{2}=\dfrac{(2i+j)+(i+j+k)}{2}$

$\overrightarrow{OL}=\dfrac{3i+2j+k}{2}$

Let M is the Mid point of $\overrightarrow{BD}$

$\overrightarrow{OM}=\dfrac{\overrightarrow{OD}+\overrightarrow{OB}}{2}=\dfrac{(2i+j)+(i-j+2k)}{2}$

$\overrightarrow{OM}=\dfrac{3i+2k}{2}$

So $\overrightarrow{LM}=\overrightarrow{OM}-\overrightarrow{OL}$

$\overrightarrow{LM}=\dfrac{3i+2k}{2}-\dfrac{3i+2j+k}{2}$

$\overrightarrow{LM}=\dfrac{-2j+k}{2}$

And $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$\overrightarrow{AB}=(i-j+2k)-(i+j+k)$

$\overrightarrow{AB}=-2j+k$

We can express as $\overrightarrow{AB}=k\ \overrightarrow{LM}$

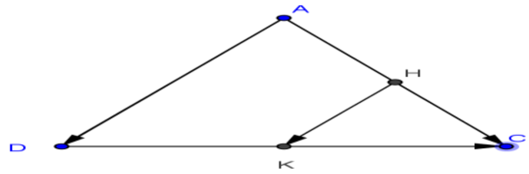
Where k=2 is some constant

i.e., $\overrightarrow{AB}=2.\overrightarrow{LM}$

$-2j+k=2.\left(\dfrac{-2j+k}{2}\right)$

Q11. If H & K are mid points of AC & CD, show that $\overrightarrow{HK}=\frac{1}{2}\overrightarrow{AD}$

Sol: $\overrightarrow{OA}=i+j+k, \overrightarrow{OB}=i-j+2k, \overrightarrow{OC}=j+k, \overrightarrow{OD}=2i+j$



Let H is the Mid point of $\overrightarrow{AC}$

$\overrightarrow{OH}=\dfrac{\overrightarrow{OC}+\overrightarrow{OA}}{2}=\dfrac{(j+k)+(i+j+k)}{2}$

$\overrightarrow{OH}=\dfrac{i+2j+2k}{2}$

Let K is the Midpoint of $\overrightarrow{CD}$

$\overrightarrow{OK}=\dfrac{\overrightarrow{OD}+\overrightarrow{OC}}{2}=\dfrac{(2i+j)+(j+k)}{2}$

$\overrightarrow{OK}=\dfrac{2i+2j+k}{2}$

So $\overrightarrow{HK}=\overrightarrow{OK}-\overrightarrow{OH}$

$\overrightarrow{HK}=\dfrac{2i+2j+k}{2}-\dfrac{i+2j+2k}{2}$

$\overrightarrow{HK}=\dfrac{i-k}{2}$

And $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$

$\overrightarrow{AD}=(2i+j)-(i+j+k)$

$\overrightarrow{AD}=i-k$

We can express as $\overrightarrow{HK}=k\ \overrightarrow{AD}$

Where $k=\frac{1}{2}$ is some constant

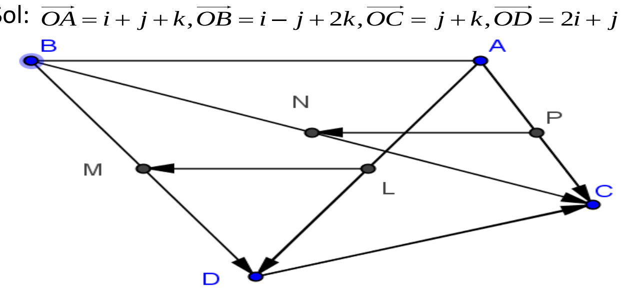
i.e., $\overrightarrow{HK}=\frac{1}{2}\left(\overrightarrow{AD}\right)$

$\dfrac{i-k}{2}=\dfrac{1}{2}.(i-k)$

SO $\overrightarrow{HK}\parallel\overrightarrow{AD}$

Q12. If L,M,N and P are the midpoints of AD, BD, BC and AC respectively, show that $\overrightarrow{LM}$ is parallel to $\overrightarrow{NP}$

Sol: $\overrightarrow{OA}=i+j+k, \overrightarrow{OB}=i-j+2k, \overrightarrow{OC}=j+k, \overrightarrow{OD}=2i+j$



Let L is the Mid point of $\overrightarrow{AD}$

$\overrightarrow{OL}=\dfrac{\overrightarrow{OD}+\overrightarrow{OA}}{2}=\dfrac{(2i+j)+(i+j+k)}{2}$

$\overrightarrow{OL}=\dfrac{3i+2j+k}{2}$

Let M is the Mid point of $\overrightarrow{BD}$

$\overrightarrow{OM}=\dfrac{\overrightarrow{OD}+\overrightarrow{OB}}{2}=\dfrac{(2i+j)+(i-j+2k)}{2}$

$\overrightarrow{OM}=\dfrac{3i+2k}{2}$

So $\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$

$$\overrightarrow{LM} = \frac{3i+2k}{2} - \frac{3i+2j+k}{2}$$

$$\overrightarrow{LM} = \frac{-2j+k}{2}$$

Let N is the Mid point of $\overline{BC}$

$$\overrightarrow{ON} = \frac{\overrightarrow{OC}+\overrightarrow{OB}}{2} = \frac{(j+k)+(i-j+2k)}{2}$$

$$\overrightarrow{ON} = \frac{i+3k}{2}$$

Let P is the Mid point of $\overline{AC}$

$$\overrightarrow{OP} = \frac{\overrightarrow{OC}+\overrightarrow{OA}}{2} = \frac{(j+k)+(i+j+k)}{2}$$

$$\overrightarrow{OP} = \frac{i+2j+2k}{2}$$

So

$$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = \frac{i+2j+2k}{2} - \frac{i+3k}{2}$$

$$\overrightarrow{NP} = \frac{2j-k}{2}$$

We can express as $\overrightarrow{NP} = k\ \overrightarrow{LM}$

Where k= - 1 is some constant

i.e.,

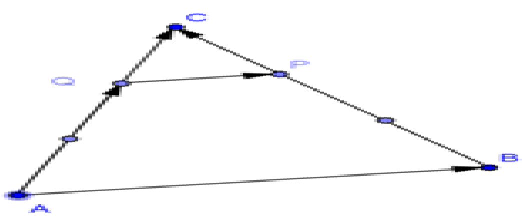
$$\overrightarrow{NP} = -1\left(\overrightarrow{LM}\right)$$

$$\frac{2j-k}{2} = -.\left(\frac{-2j+k}{2}\right)$$

SO $\overrightarrow{NP} \parallel \overrightarrow{LM}$

Q13. Let P & Q divide the sides BC & AC respectively of triangle ΔABC in the ratio 2:1, if $\overrightarrow{AB} = a$ & $\overrightarrow{AC} = b$, then find $\overrightarrow{QP}$ is parallel to $\overrightarrow{AB}$ and is one third of its length.

Sol: Given that $\overrightarrow{AB} = a$ and $\overrightarrow{AC} = b$



$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = b - a$$

Let P & Q divide the sides BC and AC in the ratio 2:1

i.e., $\frac{\overrightarrow{BP}}{\overrightarrow{PC}} = \frac{2}{1}$ $\qquad$ $\frac{\overrightarrow{AQ}}{\overrightarrow{QC}} = \frac{2}{1}$

$\Rightarrow \overrightarrow{PB} = 2\overrightarrow{PC}$ $\qquad$ $\overrightarrow{AQ} = 2\overrightarrow{QC}$

$\therefore \overrightarrow{BC} = \overrightarrow{BP} + \overrightarrow{PC}$ $\qquad$ $\overrightarrow{AC} = \overrightarrow{AQ} + \overrightarrow{QC}$

$\overrightarrow{BC} = \overrightarrow{BP} + \frac{1}{2}\overrightarrow{BP}$ $\qquad$ $\overrightarrow{AC} = \overrightarrow{AQ} + \frac{1}{2}\overrightarrow{AQ}$

$\overrightarrow{BC} = \frac{3}{2}\overrightarrow{BP}$ $\qquad$ $\overrightarrow{AC} = \frac{3}{2}\overrightarrow{AQ}$

$\overrightarrow{BP} = \frac{2}{3}(b-a)$ $\qquad$ $\overrightarrow{AQ} = \frac{2}{3}b$

From figure

$$\overrightarrow{QP} = \overrightarrow{QA} + \overrightarrow{AB} + \overrightarrow{BP}$$

$$\overrightarrow{QP} = -\overrightarrow{AQ} + \overrightarrow{AB} + \overrightarrow{BP}$$

$$\overrightarrow{QP} = -\frac{2}{3}b + a + \frac{2}{3}(b-a)$$

$$\overrightarrow{QP} = -\frac{2}{3}b + a + \frac{2}{3}b - \frac{2}{3}a$$

$$\overrightarrow{QP} = a - \frac{2}{3}a = \frac{a}{3}$$

$\overrightarrow{QP} = \frac{1}{3}\overrightarrow{AB}$ $\qquad$ $\overrightarrow{QP} \parallel \overrightarrow{AB}$

$\overrightarrow{QP}$ is one third of $\overrightarrow{AB}$

Q14. Find the coordinates of P where

a). $\left|\overrightarrow{OP}\right| = 6$ and $\overrightarrow{OP}$ is in direction of $\vec{A} = 2i - 3j + 6k$

Sol: $\left|\overrightarrow{OP}\right| = 6$ and $\overrightarrow{OP}$ is in direction of $\vec{A} = 2i - 3j + 6k$

$$\left|\vec{A}\right| = |2i - 3j + 6k| = \sqrt{2^2 + (-3)^2 + 6^2}$$

$$\left|\vec{A}\right| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$A = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{2i-3j+6k}{7}$$

Then

$$\overrightarrow{OP} = \left|\overrightarrow{OP}\right| OP$$

$$\overrightarrow{OP} = 6.\left(\frac{2i-3j+6k}{7}\right)$$

$$\overrightarrow{OP} = \frac{12i-18j+36k}{7}$$

Then coordinates of P are $P = \left(\frac{12}{7}, \frac{-18}{7}, \frac{36}{7}\right)$

b). $\left|\overrightarrow{OP}\right| = 2$ and $\overrightarrow{OP}$ is in direction of $A = 8i + j - 4k$

Sol: $\left|\overrightarrow{OP}\right| = 2$ and $\overrightarrow{OP}$ is in direction of $A = 8i + j - 4k$

$$\left|\vec{A}\right| = |8i + j - 4k| = \sqrt{8^2 + 1^2 + (-4)^2}$$

$$\left|\vec{A}\right| = \sqrt{64+1+16} = \sqrt{81} = 9$$

$$A = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{8i+j-4k}{9}$$

Then

$$\overrightarrow{OP} = \left|\overrightarrow{OP}\right| OP$$

$$\overrightarrow{OP} = 2.\left(\frac{8i+j-4k}{9}\right)$$

$$\overrightarrow{OP} = \frac{16i+2j-8k}{9}$$

Then coordinates of P are $P = \left(\frac{16}{9}, \frac{2}{9}, \frac{-8}{9}\right)$

c). $\overrightarrow{OP}$ is inclined at equal acute angles to OX, OY and OZ and $\left|\overrightarrow{OP}\right| = 4$

Sol: $\left|\overrightarrow{OP}\right| = 4$ $\overrightarrow{OP}$ is inclined at equal acute angles

so that $\overrightarrow{OP}$ direction of $A = i + j + k$

$$|A| = |i + j + k| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$A = \frac{\vec{A}}{|A|} = \frac{i+j+k}{\sqrt{3}}$$

Then $\overrightarrow{OP} = \left|\overrightarrow{OP}\right| OP$

$$\overrightarrow{OP} = 4.\left(\frac{i+j+k}{\sqrt{3}}\right)$$

$$\overrightarrow{OP} = \frac{4i+4j+4k}{\sqrt{3}}$$

Then coordinates of P are $P = \left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$

Q15. Find the magnitude and inclination to each of the coordinate axes of vector v, if

a). $v = 3i + 4j + 5k$

Sol: we have $v = 3i + 4j + 5k$

$$|v| = |3i + 4j + 5k| = \sqrt{3^2 + 4^2 + 5^2}$$

$$|v| = \sqrt{9+16+25} = \sqrt{50} = \sqrt{25 \times 2}$$

$|v| = 5\sqrt{2}$ then direction of cosines are

$\cos\alpha = \frac{3}{5\sqrt{2}}, \cos\beta = \frac{4}{5\sqrt{2}}, \cos\gamma = \frac{5}{5\sqrt{2}}$

$\alpha = \cos^{-1}\frac{3}{5\sqrt{2}}, \beta = \cos^{-1}\frac{4}{5\sqrt{2}}, \gamma = \cos^{-1}\frac{5}{5\sqrt{2}}$

$\alpha = 64.89^o, \beta = 55.55^o, \gamma = 45^{o/}$

$\alpha = 64^o 54^/, \beta = 55^o 33^/, \gamma = 45^o$

b). $v = -i + j - k$

Sol: we have $v = -i + j - k$

$|v| = |-i + j - k| = \sqrt{(-1)^2 + 1^2 + (-1)^2}$

$|v| = \sqrt{1+1+1} = \sqrt{3}$

$|v| = \sqrt{3}$ then direction of cosine are

$\cos\alpha = \frac{-1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{-1}{\sqrt{3}}$

$\alpha = \cos^{-1}\frac{-1}{\sqrt{3}}, \beta = \cos^{-1}\frac{1}{\sqrt{3}}, \gamma = \cos^{-1}\frac{-1}{\sqrt{3}}$

$\alpha = 125.26^o, \beta = 54.73^o, \gamma = 125.26^o$

$\alpha = 125^o 16^/, \beta = 54^o 44^/, \gamma = 125^o 16^/$

c). v is represented by $\overrightarrow{OP}$ where P is point (5,1,4)

Sol: we have P(5,1,4) and $|v| = \sqrt{5^2 + 1^2 + 4^2}$

$|v| = \sqrt{25+1+16} = \sqrt{42}$

$|v| = \sqrt{42}$ then direction of cosine are

$\cos\alpha = \frac{5}{\sqrt{42}}, \cos\beta = \frac{1}{\sqrt{42}}, \cos\gamma = \frac{4}{\sqrt{42}}$

$\alpha = \cos^{-1}\frac{5}{\sqrt{42}}, \beta = \cos^{-1}\frac{1}{\sqrt{42}}, \gamma = \cos^{-1}\frac{4}{\sqrt{42}}$

$\alpha = 39.5^o, \beta = 81.1^o, \gamma = 51.9^o$

Q16. if $a = 3i - j - k, b = -2i + 4j - 3k, c = i + 2j - k$ then find a unit vector parallel to $3a + 2b + 4c$

Sol: Since $a = 3i - j - k, b = -2i + 4j - 3k, c = i + 2j - k$

$3a + 2b + 4c = 3(3i - j - k) + 2(-2i + 4j - 3k) + 4(i + 2j - k)$

$3a + 2b + 4c = 9i - 3j - 3k - 4i + 8j - 6k + 4i + 8j - 4k$

$3a + 2b + 4c = 9i - 4i + 4i - 3j + 8j + 8j - 3k - 6k - 4k$

$3a + 2b + 4c = 9i + 13j - 13k$

$|3a + 2b + 4c| = |9i + 13j - 13k| = \sqrt{9^2 + 13^2 + (-13)^2}$

$|3a + 2b + 4c| = \sqrt{81 + 169 + 169} = \sqrt{419}$

Then unit vector parallel to $3a + 2b + 4c$ is $\frac{9i + 13j - 13k}{\sqrt{419}}$

Exercise 3.4

Q1. Find the cosine angle between the vectors $a = 2i - 8j + 3k$ and $b = 4j + 3k$

Sol: we have $a = 2i - 8j + 3k$ and $b = 4j + 3k$

$|a| = \sqrt{2^2 + (-8)^2 + 3^2}$ $|b| = \sqrt{4^2 + 3^2}$

$|a| = \sqrt{4 + 64 + 9}$ $|b| = \sqrt{16 + 9}$

$|a| = \sqrt{77}$ $|b| = \sqrt{25} = 5$

$a.b = (2i - 8j + 3k).(4j + 3k)$

$a.b = 0 - 32 + 9$

$a.b = -23$

Let θ be the angle between the vectors

$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|} = \frac{-23}{(\sqrt{77})(5)}$

$\cos\theta = \frac{-23}{5\sqrt{77}}$

Q2. Angle between two vectors V_1 and V_2 is arc $\cos\frac{4}{21}$ if $V_1 = 6i + 3j - 2k$ $V_2 = -2i + \lambda j - 4k$ then find value of λ

Sol: we have $V_1 = 6i + 3j - 2k$ $V_2 = -2i + \lambda j - 4k$

$V_1.V_2 = (6i + 3j - 2k).(-2i + \lambda j - 4k)$

$V_1.V_2 = -12 + 3\lambda + 8 = 3\lambda - 4$

$|V_1| = \sqrt{6^2 + 3^2 + (-2)^2}$ $|V_2| = \sqrt{(-2)^2 + \lambda^2 + (-4)^2}$

$|V_1| = \sqrt{36 + 9 + 4}$ $|V_2| = \sqrt{4 + \lambda^2 + 16}$

$|V_1| = \sqrt{49} = 7$ $|V_2| = \sqrt{\lambda^2 + 20}$

Given that $\theta = \cos^{-1}\frac{4}{21} \Rightarrow \cos\theta = \frac{4}{21}$

$\cos\theta = \frac{V_1.V_2}{|V|_1.|V_2|}$

$\frac{4}{21} = \frac{3\lambda - 4}{7\sqrt{\lambda^2 + 20}}$

$4 \times 7\sqrt{\lambda^2 + 20} = 21(3\lambda - 4)$

Divided by 7 and squaring

$\left(4\sqrt{\lambda^2 + 20}\right)^2 = \left(3(3\lambda - 4)\right)^2$

$16(\lambda^2 + 20) = 9(9\lambda^2 + 16 - 24\lambda)$

$16\lambda^2 + 320 = 81\lambda^2 + 144 - 216\lambda$

$81\lambda^2 - 16\lambda^2 - 216\lambda + 144 - 320 = 0$

$65\lambda^2 - 216\lambda - 176 = 0$

$65\lambda^2 - 260\lambda + 44\lambda - 176 = 0$

$65\lambda(\lambda - 4) + 44(\lambda - 4) = 0$

$(65\lambda + 44)(\lambda - 4) = 0$

$\therefore 65\lambda + 44 = 0$ or $\lambda - 4 = 0$

$\lambda = \frac{-44}{65} = 0$ or $\lambda = 4$

Q3 If $a = 3i + 4j - k,\ b = i - j + 3k\ \&\ c = 2i + j - 5k$ then

a). $a.b$

Sol: Since $a = 3i + 4j - k,\ b = i - j + 3k\ \&\ c = 2i + j - 5k$

$a.b = (3i + 4j - k).(i - j + 3k)$

$a.b = 3 - 4 - 3$

$a.b = -4$

b). $a.c$

Sol: Since $a = 3i + 4j - k,\ b = i - j + 3k\ \&\ c = 2i + j - 5k$

$a.c = (3i + 4j - k)(2i + j - 5k)$

$a.c = 6 + 4 + 5$

$a.c = 15$

c). $a.(b + c)$

Sol: Since $a = 3i + 4j - k,\ b = i - j + 3k\ \&\ c = 2i + j - 5k$

$a.(b + c) = (3i + 4j - k).\{i - j + 3k + 2i + j - 5k\}$

$a.(b + c) = (3i + 4j - k).(3i - 2k)$

$a.(b + c) = 9 + 0 + 2$

$a.(b + c) = 11$

d). $(2a + 3b).c$

Sol: Since $a = 3i + 4j - k,\ b = i - j + 3k\ \&\ c = 2i + j - 5k$

$(2a + 3b).c = \{(6i + 8j - 2k) + (3i - 3j + 9k)\}.(2i + j - 5k)$

$(2a + 3b).c = (9i + 5j + 7k).(2i + j - 5k)$

$(2a + 3b).c = 18 + 5 - 35$

$(2a + 3b).c = -12$

e). $(a - b).c$

Sol: Since $a = 3i + 4j - k,\ b = i - j + 3k\ \&\ c = 2i + j - 5k$

$(a-b).c = \{(3i+4j-k)-(i-j+3k)\}.(2i+j-5k)$

$(a-b).c = (2i+5j-4k).(2i+j-5k)$

$(a-b).c = 4+5+20$

$(a-b).c = 29$

Q4. In ΔABC, $\overrightarrow{AB} = i+2j+3k, \overrightarrow{BC} = -4i+4j$

a). Find the cosine of angle $\angle ABC$

Sol: we have $\overrightarrow{AB} = i+2j+3k, \overrightarrow{BC} = -4i+4j$

$|\overrightarrow{AB}| = \sqrt{1^2+2^2+3^2}, |\overrightarrow{BC}| = \sqrt{(-4)^2+4^2+0^2}$

$|\overrightarrow{AB}| = \sqrt{1+4+9}, \quad |\overrightarrow{BC}| = \sqrt{16+16+0}$

$|\overrightarrow{AB}| = \sqrt{14}, \quad |\overrightarrow{BC}| = \sqrt{32}$

$\overrightarrow{AB}.\overrightarrow{BC} = (i+2j+3k).(-4i+4j)$

$\overrightarrow{AB}.\overrightarrow{BC} = -4+8+0$

$\overrightarrow{AB}.\overrightarrow{BC} = 4$

$$\cos\angle ABC = \frac{\overrightarrow{AB}.\overrightarrow{BC}}{|\overrightarrow{AB}|.|\overrightarrow{BC}|} = \frac{4}{(\sqrt{14})(\sqrt{32})}$$

$$\cos\angle ABC = \frac{4}{\sqrt{7\times2\times16\times2}} = \frac{4}{4\times2\sqrt{7}} = \frac{1}{2\sqrt{7}}$$

---

b). Find vector $\overrightarrow{AC}$ and use it to calculate angle $\angle BAC$

Sol: we have $\overrightarrow{AB} = i+2j+3k, \overrightarrow{BC} = -4i+4j$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{AC} = (i+2j+3k)+(-4i+4j)$

$\overrightarrow{AC} = -3i+6j+3k$

$|\overrightarrow{AC}| = \sqrt{(-3)^2+6^2+3^2}$

$|\overrightarrow{AC}| = \sqrt{9+36+9}$

$|\overrightarrow{AC}| = \sqrt{54}$

$\overrightarrow{AC}.\overrightarrow{AB} = (-3i+6j+3k).(i+2j+3k)$

$\overrightarrow{AC}.\overrightarrow{AB} = -3+12+9$

$\overrightarrow{AC}.\overrightarrow{AB} = 18$

$$\cos\angle BAC = \frac{.\overrightarrow{AC}\overrightarrow{AB}}{|\overrightarrow{AC}|.|\overrightarrow{AB}|}$$

$$\cos\angle BAC = \frac{18}{(\sqrt{54})(\sqrt{14})}$$

$$\cos\angle BAC = \frac{18}{\sqrt{3\times2\times9\times2\times7}}$$

$$\cos\angle BAC = \frac{18}{3\times2\sqrt{21}}$$

$$\angle BAC = \cos^{-1}\frac{3}{\sqrt{21}}$$

$\angle BAC = 49.1066^o$

Q5. A,B,C are points with position vectors a,b,c respectively, relative to an origin O. AB is perpendicular to OC and BC is perpendicular to OA. Show that AC is perpendicular to OB.

Sol: we have $\overrightarrow{OA} = a, \quad \overrightarrow{OB} = b, \quad \overrightarrow{OC} = c$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = b - a$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = c - a$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = c - b$

Since we know that $\overrightarrow{AB} \perp \overrightarrow{OC},$ and $\overrightarrow{BC} \perp \overrightarrow{OA}$

$(b-a).c = 0, \qquad (c-b).a = 0$

$b.c - a.c = 0, \qquad c.a - b.a = 0$

$b.c = a.c, \qquad c.a = b.a$

We have to show that $\overrightarrow{AC} \perp \overrightarrow{OB}$

$(c-a).b = c.b - a.b$

$\therefore a.b = a.c, \quad a.c = b.c$

We get $(c-a).b = c.b - a.b$

$(c-a).b = c.b - a.c$

$(c-a).b = c.b - b.c$

$(c-a).b = 0$

Q6. Given two vectors $a, b(a \neq 0, b \neq 0)$. Show that

a). if $a+b$ and $a-b$ are perpendicular, then $|a| = |b|$

Sol: we have $a+b$ and $a-b$ are perpendicular

$(a+b).(a-b) = 0$

$a^2 - b^2 = 0$

$a^2 = b^2$

$\Rightarrow |a|^2 = |b|^2$

$\Rightarrow |a| = |b|$

---

b). if $|a+b| = |a-b|$ then a and b are perpendicular

Sol: we have $|a+b| = |a-b|$

Squaring both sides we get

$(a+b)^2 = (a-b)^2$

$a^2 + b^2 + 2ab = a^2 + b^2 - 2ab$

$2ab = -2ab$

$2ab + 2ab = 0$

$4ab = 0$

$\Rightarrow a.b = 0$

$\Rightarrow a \perp b$

Q7 Three vectors a,b & c are such that $a \neq b \neq c \neq 0$

a). if $a.(b+c) = b.(a-c)$, prove that $c.(a+b) = 0$

Sol: we have $a.(b+c) = b.(a-c)$

$a.b + a.c = b.a - b.c$

$a.c = -b.c$

$a.c + b.c = 0$

$(a+b)c = 0$

Hence proved

---

b). if $(a.b)c = (b.c)a$, show that a and c are parallel

Sol: Since dot product of two vectors is a scalar

So Let $a.b = k, \quad b.c = t$ then

$(a.b)c = (b.c)a$

$kc = ta$

$c = \frac{t}{k}a$

$\Rightarrow c \parallel a$

Q8. Find angle between following pairs of vectors:

a). $r_1 = i+2j-k, \quad r_2 = i+j-2k$

Sol: we have $r_1 = i+2j-k, \quad r_2 = i+j-2k$

$r_1.r_2 = (i+2j-k).(i+j-2k)$

$r_1.r_2 = (1+2+2)$

$r_1.r_2 = 5$

Now $|r_1| = \sqrt{1^2+2^2+(-1)^2}, \quad |r_2| = \sqrt{1^2+1^2+(-2)^2}$

$|r_1| = \sqrt{1+4+1}, \quad |r_2| = \sqrt{1+1+4}$

$|r_1| = \sqrt{6}, \quad |r_2| = \sqrt{6}$

Let θ be the angle between a and b then

$$\cos\theta = \frac{r_1.r_2}{|r_1||r_2|} = \frac{5}{(\sqrt{6})^2}$$

$$\theta = \cos^{-1}\left(\tfrac{5}{6}\right)$$

b). $r_1 = \lambda(i+2j+2k)$, $r_2 = \mu(3i+2j+6k)$

Sol: we have $r_1 = \lambda(i+2j+2k)$, $r_2 = \mu(3i+2j+6k)$

$r_1.r_2 = \lambda\mu(i+2j+2k).(3i+2j+6k)$

$r_1.r_2 = \lambda\mu(3+4+12)$

$r_1.r_2 = 19\lambda\mu$

Now $|r_1| = \sqrt{\lambda^2}\sqrt{1^2+2^2+2^2}$, $|r_2| = \sqrt{\mu^2}\sqrt{3^2+2^2+6^2}$

$|r_1| = \lambda\sqrt{1+4+4}$, $|r_2| = \mu\sqrt{9+4+36}$

$|r_1| = \lambda\sqrt{9}$, $|r_2| = \mu\sqrt{49}$

$|r_1| = 3\lambda$, $|r_2| = 7\mu$

Let θ be the angle between a and b then

$$\cos\theta = \frac{r_1.r_2}{|r_1||r_2|} = \frac{19\lambda\mu}{21\lambda\mu} = \frac{19}{21}$$

$$\theta = \cos^{-1}\left(\tfrac{19}{21}\right)$$

Q9. Show that $a = i+7j+3k$ is perpendicular to both $b = i-j+2k$ and $c = 2i+j-3k$

Sol: $a = i+7j+3k$, $b = i-j+2k$ and $c = 2i+j-3k$

$a.b = (i+7j+3k).(i-j+2k) = 1-7+6 = 0$

$\Rightarrow a \perp b$

$a.c = (i+7j+3k).(2i+j-3k) = 2+7-9 = 0$

$\Rightarrow a \perp c$

Q10. Show that $a = 13i+23j+7k$ is perpendicular to both $b = 2i+j-7k$ and $c = 3i-2j+k$

Sol: $a = 13i+23j+7k$, $b = 2i+j-7k$ & $c = 3i-2j+k$

$a.b = (13i+23j+7k).(2i+j-7k) = 26+23-49 = 0$

$\Rightarrow a \perp b$

$a.c = (13i+23j+7k).(3i-2j+k) = 39-46+7 = 0$

$\Rightarrow a \perp c$

Q11. Find projection of $a = 3i+j-2k$ on $b = -i-j+5k$

Sol: we have $a = 3i+j-2k$ and $b = -i-j+5k$

$|b| = \sqrt{(-1)^2+(-1)^2+(5)^2}$

$|b| = \sqrt{1+1+25} = \sqrt{27}$

$|b| = \sqrt{9\times 3} = 3\sqrt{3}$

$$\hat{b} = \frac{b}{|b|} = \frac{-i-j+5k}{3\sqrt{3}}$$

Projection of a on b i.e.,

$$a.\hat{b} = (3i+j-2k)\left(\frac{-i-j+5k}{3\sqrt{3}}\right)$$

$$a.\hat{b} = \frac{-3-1-10}{3\sqrt{3}} = \frac{-14}{3\sqrt{3}}$$

Q12. Find work done by force $\vec{F} = 2i+3j+k$ in displacement of an object from a point $A(-2,1,2)$ to point $B(5,0,3)$.

Sol: $\vec{F} = 2i+3j+k$ & points $A(-2,1,2)$ & $B(5,0,3)$

$AB = B - A = (5,0,3)-(-2,1,2)$

$AB = (7,-1,1)$ & $\vec{F} = 2i+3j+k$

$\vec{F}.AB = (2i+3j+k)(7i-j+k)$

$\vec{F}.AB = 14-3+1 = 12$ units

Exercise 3.5

Q1. Find the following cross products

i). $j\times(2j+3k)$

Sol: we have to find $j\times(2j+3k)$

$j\times(2j+3k) = 2j\times j + 3j\times k$

$= 2.0 + 3i = 3i$

ii). $(2i-3j)\times k$

Sol: we have to find $(2i-3j)\times k$

$(2i-3j)\times k = 2i\times k - 3j\times k$

$= 2(-j) - 3i = -3i - 2j$

iii). $(2i-3j+5k)\times(6i+2j-3k)$

Sol: we have to find $(2i-3j+5k)\times(6i+2j-3k)$

$$= \begin{vmatrix} i & j & k \\ 2 & -3 & 5 \\ 6 & 2 & -3 \end{vmatrix}$$

$= i(9-10) - j(-6-30) + k(4+18)$

$= -i + 36j + 22k$

Q2 $a = -2i+6j+3k, b = 3i+3j+6k$ and $c = 2i+7j+4k$, Find $(a-b)\times(c-a)$ and $(a+b)\times(c-a)$

Sol: $a = -2i+6j+3k, b = 3i+3j+6k$ & $c = 2i+7j+4k$

$a-b = (-2i+6j+3k)-(3i+3j+6k)$

$a-b = -5i+3j-3k$

$c-a = (2i+7j+4k)-(-2i+6j+3k)$

$c-a = 4i+j+k$

$a+b = (-2i+6j+3k)+(3i+3j+6k)$

$a+b = i+9j+9k$

Now $(a-b)\times(c-a) = \begin{vmatrix} i & j & k \\ -5 & 3 & -3 \\ 4 & 1 & 1 \end{vmatrix}$

$= i(3+3) - j(-5+12) + k(-5-12)$

$= 6i - 7j - 17k$

$$(a+b)\times(c-a) = \begin{vmatrix} i & j & k \\ 1 & 9 & 9 \\ 4 & 1 & 1 \end{vmatrix}$$

$= i(9-9) - j(1-36) + k(1-36)$

$= 35j - 35k$

Q3. Find a unit vector perpendicular to both $a = i+j+2k$ and $b = -2i+j-3k$

Sol: we have $a = i+j+2k$ and $b = -2i+j-3k$

Let c be the vector perpendicular both a and b

$$a\times b = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ -2 & 1 & -3 \end{vmatrix}$$

$a\times b = i(-3-2) - j(-3+4) + k(1+2)$

$a\times b = -5i - j + 3k$

$|a\times b| = \sqrt{(-5)^2+(-1)^2+3^2}$

$|a\times b| = \sqrt{25+1+9}$

$|a\times b| = \sqrt{35}$

So the unit vector

$$\hat{c} = \frac{a\times b}{|a\times b|} = \frac{-5i-j+3k}{\sqrt{35}}$$

$$\hat{c} = \frac{-5i}{\sqrt{35}} - \frac{j}{\sqrt{35}} + \frac{3k}{\sqrt{35}}$$

Q4. Find a vector of magnitude 10 and perpendicular to $a=2i-3j+4k$ and $b=4i-2j-4k$

Sol: we have $a=2i-3j+4k$ and $b=4i-2j-4k$

Let c be the vector perpendicular both a and b

$$a\times b=\begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 4 & -2 & -4 \end{vmatrix}$$

$$a\times b=i(12+8)-j(-8-16)+k(-4+12)$$

$$a\times b=20i-24j+8k$$

$$|a\times b|=\sqrt{20^2+(-24)^2+8^2}$$

$$|a\times b|=\sqrt{400+576+64}$$

$$|a\times b|=\sqrt{1040}=\sqrt{16\times 65}=4\sqrt{65}$$

And its magnitude $|c|=10$

$$\hat{c}=\frac{a\times b}{|a\times b|}=\frac{20i-24j+8k}{4\sqrt{65}}$$

$$\hat{c}=\frac{5i}{\sqrt{65}}-\frac{6j}{\sqrt{65}}+\frac{2k}{\sqrt{65}}$$

So $\vec{c}=|c|\hat{c}=10.\left(\frac{5i}{\sqrt{65}}-\frac{6j}{\sqrt{65}}+\frac{2k}{\sqrt{65}}\right)$

Q5 For vectors $a=2i-3j-k$ & $b=i+4j-2k$ verify that

a). $a\times b=-b\times a$

Sol: we have $a=2i-3j-k$ and $b=i+4j-2k$

Now $a\times b=\begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$

$$a\times b=i(6+4)-j(-4+1)+k(8+3)$$

$$a\times b=10i+3j+11k \quad ......(1)$$

Now $b\times a=\begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 2 & -3 & -1 \end{vmatrix}$

$$b\times a=i(-4-6)-j(-1+4)+k(-3-8)$$

$b\times a=-10i-3j-11k$ Multiply b.s by -1

$-b\times a=10i+3j+11k$ using eq (1)

$$-b\times a=a\times b$$

---

b). $(a+b)\times(a-b)=-2(a\times b)$

Sol: we have $a=2i-3j-k$ and $b=i+4j-2k$

$$a+b=(2i-3j-k)+(i+4j-2k)$$

$$a+b=3i+j-3k$$

$$a-b=(2i-3j-k)-(i+4j-2k)$$

$$a+b=i-7j+k$$

Taking RHS $a\times b=\begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$

$$a\times b=i(6+4)-j(-4+1)+k(8+3)$$

$$a\times b=10i+3j+11k \quad ......(1)$$

Taking LHS $(a+b)\times(a-b)=\begin{vmatrix} i & j & k \\ 3 & 1 & -3 \\ 1 & -7 & 1 \end{vmatrix}$

$$=i(1-21)-j(3+3)+k(-21-1)$$

$$=-20i-6j-22k$$

$$(a+b)\times(a-b)=-2(10i+3j+11k)$$

$$(a+b)\times(a-b)=-2(a\times b)$$

Q6. Find the area of a triangle ABC whose vertices are $A(0,0,0),B(1,1,1),C(0,2,3)$

Sol: we have $A(0,0,0),B(1,1,1),C(0,2,3)$

$$AB=B-A=(1,1,1)-(0,0,0)=(1,1,1)$$

$$AC=C-A=(0,2,3)-(0,0,0)=(0,2,3)$$

$$AB\times AC=\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$=i(3-2)-j(3-0)+k(2-0)$$

$$=i-3j+2k$$

Area of triangle

$$\tfrac{1}{2}|AB\times AC|=\tfrac{1}{2}\sqrt{1^2+(-3)^2+2^2}$$

$=\frac{1}{2}\sqrt{1+9+4}=\frac{\sqrt{14}}{2}$ square units

Q7. Find the area of a triangle whose vertices are $A(0,0,2),B(-1,3,2),C(1,0,4)$

Sol: $AB=B-A=(-1,3,2)-(0,0,2)=(-1,3,0)$

$$AC=C-A=(1,0,4)-(0,0,2)=(1,0,2)$$

$$AB\times AC=\begin{vmatrix} i & j & k \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$=i(6-0)-j(-2-0)+k(0-3)$$

$$=6i+2j-3k$$

$$\tfrac{1}{2}|AB\times AC|=\tfrac{1}{2}\sqrt{6^2+2^2+(-3)^2}$$

$=\frac{1}{2}\sqrt{36+4+9}=\frac{1}{2}\sqrt{49}=\frac{7}{2}$ square units

---

Q8. Find area of the parallelogram with vertices $A(1,2,-3),B(5,8,1),C(4,-2,2),D(0,-8,-2)$

Sol: we have $A(1,2,-3),B(5,8,1),C(4,-2,2),D(0,-8,-2)$

$$AB=B-A=(5,8,1)-(1,2,-3)$$

$$AB=(4,6,4)$$

$$AD=D-A=(0,-8,-2)-(1,2,-3)$$

$$AD=(-1,-10,1)$$

Now $AB\times AD=\begin{vmatrix} i & j & k \\ 4 & 6 & 4 \\ -1 & -10 & 1 \end{vmatrix}$

$$=i(6+40)-j(4+4)+k(-40+6)$$

$$=46i-8j-34k$$

$$|AB\times AD|=\sqrt{46^2+(-8)^2+(-34)^2}$$

$=\sqrt{2116+64+1156}=\sqrt{3336}$ square units

Q9. A force $F=i+2j-3k$ is applied at $P=(1,2,3)$. Find its moment about $A=(1,1,1)$. What is magnitude of this moment?

Sol: Since $F=i+2j-3k$ $P=(1,2,3)$ & $A=(1,1,1)$

$$r=AP=P-A=(1,2,3)-(1,1,1)$$

$$r=(0,1,2)$$

Moment $M=r\times F$

$$M=\begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{vmatrix}$$

$$M=i(-3-4)-j(0-2)+k(0-1)$$

$$M=-7i+2j-k$$

Magnitude of the Moment

$$|M|=|r\times F|=\sqrt{49+4+1}=\sqrt{54}$$

Q10. Find area of parallelogram whose diagonals are

i). $a=4i+j-2k$ and $b=-2i+3j+4k$

Sol: Area of parallelogram

$$\frac{1}{2}(a\times b)=\frac{1}{2}\begin{vmatrix} i & j & k \\ 4 & 1 & -2 \\ -2 & 3 & 4\end{vmatrix}$$

$$=\tfrac{1}{2}\left[i(4+6)-j(16-4)+k(12+2)\right]$$

$$=\tfrac{1}{2}(10i-12j+14k)$$

$$=5i-6j+7k$$

$$\left|\frac{a\times b}{2}\right|=\sqrt{25+36+49}=\sqrt{110}$$

ii). $a=3i+2j-2k$ and $b=i-3j+4k$

Sol: we have $a=3i+2j-2k$ and $b=i-3j+4k$

$$\frac{1}{2}(a\times b)=\frac{1}{2}\begin{vmatrix} i & j & k \\ 3 & 2 & -2 \\ 1 & -3 & 4\end{vmatrix}$$

$$=\tfrac{1}{2}\left[i(8-6)-j(12+2)+k(-9-2)\right]$$

$$=\tfrac{1}{2}(2i-14j+11k)$$

$$=i-7j+5.5k$$

$$\left|\frac{a\times b}{2}\right|=\sqrt{1+49+30.25}=\sqrt{80.25}$$

Exercise 3.6

Q1 Prove theorem 3 of section 3.27

i.e., Let *i,j* and *k* be the unit vectors.

Prove that

a). $i.j\times k=j.k\times i=k.i\times j=1$

Sol: we have to prove $i.j\times k=j.k\times i=k.i\times j=1$

$i.j\times k=i.i=1$ using $j\times k=i$

$j.k\times i=j.j=1$ using $k\times i=j$

$k.i\times j=k.k=1$ using $i\times j=k$

b). $i.k\times j=j.i\times k=k.j\times i=-1$

Sol: we have to prove $i.k\times j=j.i\times k=k.j\times i=-1$

$i.k\times j=i.(-i)=-1$ using $k\times j=-i$

$j.i\times k=j.(-j)=-1$ using $i\times k=-j$

$k.j\times i=k.(-k)=-1$ using $j\times i=-k$

Q2. Find the volume of the parallelepiped whose edges are represented by

$a=3i+j-k, b=2i-3j+k, c=i-3j-4k$

Sol: volume of the parallelepiped

$$\vec{a}.\vec{b}\times\vec{c}=\begin{vmatrix} 3 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -3 & -4\end{vmatrix}$$

$$=3(12+3)-1(-8-1)-1(-6+3)$$

$$=45+9+3=57$$

Q3. For the vectors

$a=3i+2k, b=i+2j+k, c=-j+4k$

Verify that $a.b\times c=b.c\times a=c.a\times b$ but $a.b\times c=-c\times b.a$

Sol: we have $a=3i+2k, b=i+2j+k, c=-j+4k$

$$\vec{a}.\vec{b}\times\vec{c}=\begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4\end{vmatrix}$$

$$=3(8+1)-0(4-0)+2(-1+0)$$

$$=27-0-2=25\text{..........(1)}$$

$$\vec{b}.\vec{c}\times\vec{a}=\begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \\ 3 & 0 & 2\end{vmatrix}$$

$$=1(-2-0)-2(0-12)+1(0+3)$$

$$=-2+24+3=25\text{..........(2)}$$

$$\vec{a}.\vec{b}\times\vec{c}=\begin{vmatrix} 0 & -1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1\end{vmatrix}$$

$$=0(0-4)+1(3-2)+4(6-0)$$

$$=0+1+24=25\text{..........(3)}$$

From equations (1), (2) and (3) we get

$a.b\times c=b.c\times a=c.a\times b$

Now to show that $a.b\times c=-c\times b.a$

LHS $a.b\times c=b\times c.a$ using dot is commutative

$a.b=b.a$

$a.b\times c=-c\times b.a$ Using $b\times c=-c\times b$

Q4. Verify that the triple product of $\vec{a}=i-j, \vec{b}=j-k$ and $\vec{c}=k-i$ is zero

Sol: we have $\vec{a}=i-j, \vec{b}=j-k$ and $\vec{c}=k-i$

$$\vec{a}.\vec{b}\times\vec{c}=\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1\end{vmatrix}$$

$$=1(1-0)+1(0-1)+0(0+1)$$

$$=1-1+0=0$$

Q5. Find the value of c so that the vectors $\vec{a}=ci+j+k, \vec{b}=i+cj+k, \vec{c}=i+j+ck$ are coplanar.

Sol: If the points are coplanar then $a.b\times c=0$

$$\vec{a}.\vec{b}\times\vec{c}=\begin{vmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c\end{vmatrix}=0$$

$$c(c^2-1)-1(c-1)+1(1-c)=0$$

$$c(c^2-1)-1(c-1)+1(-c+1)=0$$

$$c(c+1)(c-1)-1(c-1)-1(c-1)=0$$

$$(c-1)\{c(c+1)-1-1\}=0$$

$$(c-1)\{c^2+c-2\}=0$$

$$(c-1)(c^2+2c-c-2)=0$$

$$(c-1)(c(c+2)-1(c+2))=0$$

$$(c-1)(c-1)(c+2)=0$$

Either $c-1=0$, $c=1$ or $c+2=0$, $c=-2$

Be the required values of c.

Q6. Let $a=a_1i+a_2j+a_3k$ and $b=b_1i+b_2j+b_3k$ Find $a\times b$ and prove that

Sol: we have $a=a_1i+a_2j+a_3k$ and $b=b_1i+b_2j+b_3k$

Now
$$a\times b=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=i(a_2b_3-a_3b_2)-j(a_1b_3-a_3b_1)+k(a_1b_2-a_2b_1)$$

i). $a\times b$ is orthogonal to both a and b (use dot product)

Sol: we have $a=a_1i+a_2j+a_3k$ and $b=b_1i+b_2j+b_3k$

$a\times b=i\left(a_2b_3-a_3b_2\right)-j\left(a_1b_3-a_3b_1\right)+k\left(a_1b_2-a_2b_1\right)$

Now

$a.\left(a\times b\right)=a_1\left(a_2b_3-a_3b_2\right)-a_2\left(a_1b_3-a_3b_1\right)+a_3\left(a_1b_2-a_2b_1\right)$

$=a_1a_2b_3-a_1a_3b_2-a_2a_1b_3-a_2a_3b_1+a_3a_1b_2-a_3a_2b_1$

$=a_1a_2b_3-a_2a_1b_3-a_1a_3b_2+a_3a_1b_2-a_2a_3b_1-a_3a_2b_1=0$

So a is orthogonal to $a\times b$ Now

$b.\left(a\times b\right)=b_1\left(a_2b_3-a_3b_2\right)-b_2\left(a_1b_3-a_3b_1\right)+b_3\left(a_1b_2-a_2b_1\right)$

$=b_1a_2b_3-b_1a_3b_2-b_2a_1b_3-b_2a_3b_1+b_3a_1b_2-b_3a_2b_1$

$=b_1a_2b_3-b_3a_2b_1-b_1a_3b_2-b_2a_3b_1-b_2a_1b_3+b_3a_1b_2=0$

So b is orthogonal to $a\times b$

ii). Find $|a\times b|^2$

Sol: we have $a=a_1i+a_2j+a_3k$ and $b=b_1i+b_2j+b_3k$

$a\times b=i\left(a_2b_3-a_3b_2\right)-j\left(a_1b_3-a_3b_1\right)+k\left(a_1b_2-a_2b_1\right)$

$|a\times b|^2=\left(a_2b_3-a_3b_2\right)^2+\left(a_1b_3-a_3b_1\right)^2+\left(a_1b_2-a_2b_1\right)^2$

$|a\times b|^2=a_2^2b_3^2+a_3^2b_2^2-2a_2b_3a_3b_2+a_1^2b_3^2+a_3^2b_1^2$

$-2a_1b_3a_3b_1+a_1^2b_2^2+a_2^2b_1^2-2a_1b_2a_2b_1$

iii). Find $|a.b|^2$, $|a|^2$, $|b|^2$

Sol: we have $a=a_1i+a_2j+a_3k$ and $b=b_1i+b_2j+b_3k$

$a.b=a_1b_1+a_2b_2+a_3b_3$

$|a.b|^2=\left(a_1b_1+a_2b_2+a_3b_3\right)^2$

$|a|^2=a_1^2+a_2^2+a_3^2$ and $|b|^2=b_1^2+b_2^2+b_3^2$

iv). Show that $|a\times b|^2=(a.a)(b.b)-(a.b)^2$

Sol: we have $a=a_1i+a_2j+a_3k$ and $b=b_1i+b_2j+b_3k$

taking RHS $(a.a)(b.b)-(a.b)^2$

$=\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)-\left(a_1b_1+a_2b_2+a_3b_3\right)^2$

$=a_1^2b_1^2+a_2^2b_1^2+a_3^2b_1^2+a_1^2b_2^2+a_2^2b_2^2+a_3^2b_2^2+a_1^2b_3^2+a_2^2b_3^2+a_3^2b_3^2$

$-a_1^2b_1^2-a_2^2b_2^2-a_3^2b_3^2-2a_1b_1a_2b_2-2a_2b_2a_3b_3-2a_1b_1a_3b_3$

$=a_2^2b_1^2+a_3^2b_1^2+a_1^2b_2^2+a_3^2b_2^2+a_1^2b_3^2+a_2^2b_3^2$

$-2a_1b_1a_2b_2-2a_2b_2a_3b_3-2a_1b_1a_3b_3$.............(1)

Now taking LHS

$a\times b=i\left(a_2b_3-a_3b_2\right)-j\left(a_1b_3-a_3b_1\right)+k\left(a_1b_2-a_2b_1\right)$

$|a\times b|^2=\left(a_2b_3-a_3b_2\right)^2+\left(a_1b_3-a_3b_1\right)^2+\left(a_1b_2-a_2b_1\right)^2$

$|a\times b|^2=a_2^2b_3^2+a_3^2b_2^2-2a_2b_3a_3b_2+a_1^2b_3^2+a_3^2b_1^2$

$-2a_1b_3a_3b_1+a_1^2b_2^2+a_2^2b_1^2-2a_1b_2a_2b_1$..........(2)

From (1) and (2) we get the required result.

Q7 Do points (4,-2,1), (5,1,6), (2,2,-5) and (3,5,0) lie in a plane?

Sol: Let A(4,-2,1), B(5,1,6), C(2,2,-5) and D(3,5,0)

$\vec{a}=\overrightarrow{AB}=B-A=(5,1,6)-(4,-2,1)=(1,3,5)$

$\vec{b}=\overrightarrow{AC}=C-A=(2,2,-5)-(4,-2,1)=(-2,4,-6)$

$\vec{c}=\overrightarrow{AD}=D-A=(3,5,0)-(4,-2,1)=(-1,7,-1)$

If the points are coplanar then $a.b\times c=0$

$$a.b\times c=\begin{vmatrix}1&3&5\\-2&4&-6\\-1&7&-1\end{vmatrix}$$

$=1(-4+42)-3(2-6)+5(-14+4)$

$=38+12-50$

$=50-50=0$

so all the points are coplanar.

Q8. For what values of c the following vectors are coplanar?

i). $u=i+2j+3k, v=2i-3j+4k, w=3i+j+ck$

Sol: if u,v,w are coplanar then $u.v\times w=0$

$$u.v\times w=\begin{vmatrix}1&2&3\\2&-3&4\\3&1&c\end{vmatrix}=0$$

$\Rightarrow 1(-3c-4)-2(2c-12)+3(2+9)=0$

$\Rightarrow -3c+4-4c+24+33=0$

$\Rightarrow 53-7c=0$

$\Rightarrow c=\frac{53}{7}$

ii). $u=i+j-k, v=i-2j+k, w=ci+j-ck$

Sol: if u,v,w are coplanar then $u.v\times w=0$

$$u.v\times w=\begin{vmatrix}1&1&-1\\1&-2&1\\c&1&-c\end{vmatrix}=0$$

$\Rightarrow 1(2c-1)-1(-c-c)-1(1+2c)=0$

$\Rightarrow 2c-1+2c-1-2c=0$

$\Rightarrow 2c-2=0$

$\Rightarrow c=1$

iii). $u=i+j+2k, v=2i+3j+k, w=ci+2j+6k$

Sol: if u,v,w are coplanar then $u.v\times w=0$

$$u.v\times w=\begin{vmatrix}1&1&2\\2&3&1\\c&2&6\end{vmatrix}=0$$

$\Rightarrow 1(18-2)-1(12-c)+2(4-3c)=0$

$\Rightarrow 16-12+c+8-6c=0$

$\Rightarrow 12-5c=0$

$\Rightarrow c=\frac{12}{5}$

Q9. Find the value of tetrahedron with the following.

a). Vectors as coterminous edges

Sol: $a=i+2j+3k, b=4i+5j+6k, c=7j+8k$

Volume of tetrahedron= $\frac{1}{6}\left(\vec{a}.\vec{b}\times\vec{c}\right)$

$$=\frac{1}{6}\begin{vmatrix}1&2&3\\4&5&6\\0&7&8\end{vmatrix}$$

$=\frac{1}{6}\left[1(40-42)-2(32-0)+3(28-0)\right]$

$=\frac{1}{6}\left[1(-2)-2(32)+3(28)\right]$

$=\frac{1}{6}\left[-2-64+84\right]=\frac{18}{6}=3$

b). Points A(2,3,1), B(-1,-2,0) C(0,2,-5), D(0,1,-2) as vertices

Sol: Let A(2,3,1), B(-1,-2,0) C(0,2,-5), D(0,1,-2)

$\vec{a}=\overrightarrow{AB}=B-A=(-1,-2,0)-(2,3,1)=(-3,-5,-1)$

$\vec{b}=\overrightarrow{AC}=C-A=(0,2,5)-(2,3,1)=(-2,-1,-6)$

$\vec{c}=\overrightarrow{AD}=D-A=(0,1,-2)-(2,3,1)=(-2,-2,-3)$

Volume of tetrahedron= $\frac{1}{6}\left(\vec{a}.\vec{b}\times\vec{c}\right)$

$$=\frac{1}{6}\begin{vmatrix}-3&-5&-1\\-2&-1&-6\\-2&-2&-3\end{vmatrix}$$

$=\frac{1}{6}\left[-3(3-12)+5(6-12)-1(4-2)\right]$

$=\frac{1}{6}\left[-3(-9)+5(-6)-1(2)\right]$

$=\frac{1}{6}\left[27-30-2\right]=\frac{-5}{6}$

Volume can not be negative so

Volume of tetrahedron= $\frac{5}{6}$ cubic units