Integration:

The technique or method to find such a function whose derivative is given involves the inverse process of differentiation, called anti derivative or integration.

Differential of variable:

Let f be a differentiable function defined as

$$y = f(x) \;\Rightarrow y + \delta y = f(x + \delta x)$$

$$\Rightarrow \delta x = f(x+\delta x) - y \Rightarrow \delta y = f(x+\delta x) - f(x)$$

$$Now \;\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x)$$

$\because$ *before the limit is reached.* $\frac{\delta y}{\delta x}$ *differs from*

$f'(x)$ *bysmall real number* $\in$. *i.e* $\frac{\delta y}{\delta x} = f'(x) + \in$

$\Rightarrow f'(x)$ *is called differntial of dependent varibale* y *we denoted differential of* y *as* dy.

so $dy = f'(x)\delta x \Rightarrow dx = \delta y = f'(x)dx$

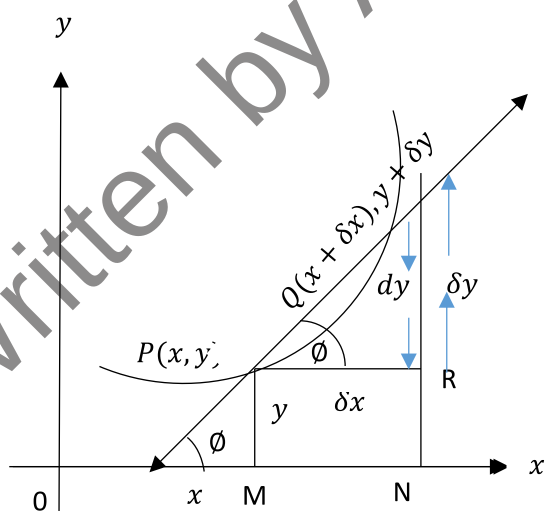
Note: 1.The differential of x is denoted by dx and defined as $dx = \delta x$

$$i.e \; for \; y = x \Rightarrow dy = \frac{d}{dx}(x)\delta x$$

$$\Rightarrow dy = 1.\delta x \Rightarrow dx = \delta x \quad \because y = x$$

2. $f'(x)$ *is used differential coefficeient.*

Distinguishing between dy and δx



Let us draw the graph of curve of a function $y = f(x)$ *Let* $P(x, y)$ *and* $Q(x+\delta x, y+\delta y)$ *be two neighbouring points on the curve at point* $P(x, y)$ *s.that it makes an angle* $\emptyset$ *with* $x-$*axis. Also* Draw $\perp$ PM *and* QN *on* $x-$*axix also draw* $\perp$ PR *on* QN *on* $x-$*axis. in fig.* $|PR| = dx$

$$|QR| = |QT| + |TR|$$

$$\Rightarrow \delta y = |QT| + |TR| \to (i)$$

$$In \; \triangle TPR, \qquad tan\emptyset dx = \frac{|TR|}{|PR|} = \frac{|TR|}{dx}$$

$$\Rightarrow |TR| = tan\emptyset dx$$

So $(i) \Rightarrow \delta y = \; tan\emptyset dx + |QT|$

$$\Rightarrow \delta y = \left(\frac{dy}{dx}\right) dx + |QT| \qquad \because \frac{dy}{dx} = tan\emptyset$$

$$\delta y = dy + |QT| \quad \because |QT| \; is \; very \; small$$

So *by neglecting* $|QT|$

$$\Rightarrow \delta y \approx dy$$

Example:

Find δy and dy of the function defined as

$$f(x) = x^2 \; when \; x = 2 \; and \; dx = 0.01$$

Solution:

$$Let \; y = f(x) \qquad dy = ?$$

⇨ $y = x^2$

⇨ $\frac{dy}{dx} = 2x \Rightarrow dy = 2dx$

Take $x = 2$ *and* $dx = 0.01$

$$dy = 2(2)(0.01) = 0.04$$

Now we find $\delta y, y + \delta y = (x + \delta x)^2$

⇨ $\delta y = (x+\delta x)^2 - y \quad , y = (x)^2 = (2)^2 = 4$

$= (2 + 0.01)^2 - 4 \qquad \because dx = \delta x = 0.01$

$\delta y = 4.041 - 4 = 0.0401$

Example:

Use differentials find $\frac{dy}{dx}$ when $\frac{y}{x} - lnx = lnc$

Solution:

$$\frac{y}{x} - lnx = lnc$$

⇨ $d\left(\frac{y}{x} - lnx\right) = d(lnc)$

⇨ $d\left(\frac{y}{x}\right) - d(lnx) = 0$

⇨ $\frac{xdy - ydx}{x^2} - \frac{1}{x}dx = 0$

⇨ $\frac{xdy - ydx}{x^2} = \frac{1}{x}dx$

⇨ $xdy - ydx = xdx$

⇨ $xdy = xdx + ydx$

⇨ $dy = \frac{x+y}{x}dx$

⇨ $\frac{dy}{dx} = \frac{x+y}{x}$

Exercise 3.1

Q.1: Find δy and dy in the following cases:

i) $y = x^2 - 1$

when x changes from 3 to 3.02

SOLUTION:

$y = x^2 - 1$ *As x changes from 3 to 3.02, so*

$y = x^2 - 1$

$d(y) = d(x^2 - 1)$

$dy = 2x\,dx - 0 = 2x\,dx$

Put value of x and dx

$dy = 2(3)(0.02)$

$dy = 0.12$

Now

$y + \delta y = (x + \delta x)^2 - 1$

$\delta y = (x + \delta x)^2 - 1 - y$

Put value of y

$\delta y = (x + \delta x)^2 - 1 - (x^2 - 1)$

$\delta y = (x + \delta x)^2 - 1 - x^2 + 1$

$\delta y = (x + \delta x)^2 - x^2$

$x = 3$

$\delta x = dx = 3.02 - 3 = 0.02$

Put value of x and δx

$\delta y = (3 + 0.02)^2 - (3)^2$

$\delta y = 0.1204$

ii) $y = x^2 + 2x$

when x changes from 2 to 1.8

SOLUTION:

$y = x^2 + 2x$

Now

$y = x^2 + 2x$

$d(y) = d(x^2 + 2x)$

$dy = 2x\,dx + 2dx$

Put value of x and dx

$dy = 2(2)(-0.2) + 2(-0.2)$

$dy = -1.2$

Now

$y + \delta y = (x + \delta x)^2 + 2(x + \delta x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - y$

Put value of y

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - (x^2 + 2x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - x^2 - 2x$

$\delta y = (x + \delta x)^2 + 2\delta x - x^2$

$x = 2$,

$\delta x = dx = 1.8 - 2 = -0.2$

Put value of x and δx

$\delta y = (2 - 0.2)^2 + 2(-0.2) - (2)^2$

$\delta y = -1.16$

iii) $y = \sqrt{x}$

when x changes from 4 to 4.01

SOLUTION:

$y = \sqrt{x}$

Now

$y = \sqrt{x}$

$d(y) = d(\sqrt{x})$

$dy = \frac{1}{2\sqrt{x}}\,dx$

Put value of x and dx

$dy = \frac{1}{2\sqrt{4}}\,(0.41)$

$dy = 0.1025$

Now.

$y + \delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

Put value of y

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

$x = 4$,

$\delta x = dx = 4.41 - 4 = 0.41$

Put value of x and δx

$\delta y = \sqrt{4 + 0.41} - \sqrt{4}$

$\delta y = 0.1$

Q.2: Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations.

i) $xy + x = 4$

Taking differentials on both sides

$d(xy + x) = d(4)$

$d(xy) + d(x) = 0$

$x\,dy + y\,dx + dx = 0$

$x\,dy + (y + 1)dx = 0$

$x\,dy = -(y + 1)dx$

$\frac{dy}{dx} = -\frac{y+1}{x}$ *and*

$\frac{dx}{dy} = -\frac{x}{y+1}$

ii) $x^2 + 2y^2 = 16$

Taking differentials on both sides

$d(x^2 + 2y^2) = d(16)$

$d(x^2) + 2d(y^2) = 0$

$2x\,dx + 2.2y^{2-1}.dy = 0$

$2x\,dx + 4y\,dy = 0$

$4y\,dy = -2x\,dx$

$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$ *and*

$\frac{dx}{dy} = -\frac{2y}{x}$

iii) $x^4 + y^2 = xy^2$

Taking differentials on both sides

$d(x^4 + y^2) = d(xy^2)$

$d(x^4) + d(y^2) = (x)'(y^2) + (y^2)'x$

$4x^3\,dx + 2y\,dy = dx.y^2 + (2y\,dy)x$

$4x^3\,dx + 2y\,dy = y^2\,dx + 2xy\,dy$

$2y\,dy - 2xy\,dy = y^2\,dx - 4x^3\,dx$

$(2y - 2xy)\,dy = (y^2 - 4x^3)dx$

$\frac{Dy}{dx} = \frac{y^2 - 4x^3}{2y - 2xy}$ *taking reciprocal*

$\frac{dx}{dy} = \frac{2y - 2xy}{y^2 - 4x^3}$

iv) $xy - \ln x = c$

Taking differentials on both sides

$d(xy - \ln x) = d(c)$

$d(xy) - d(\ln x) = 0$

$x\,dy + y\,dx - \frac{1}{x}dx = 0$

$x\,dy = -y\,dx + \frac{1}{x}dx$

$x\,dy = -\left(y - \frac{1}{x}\right)dx$

$x\,dy = -\left(\frac{xy-1}{x}\right)dx$

$\frac{dy}{dx} = \frac{1-xy}{x^2}$ *and*

$\frac{dx}{dy} = \frac{x^2}{1-xy}$

Q.3: Use differentials to approximate the values of:

i) $\sqrt[4]{17}$

SOLUTION:

Let $y = \sqrt[4]{x} = x^{\frac{1}{4}}$

We take $x = 16$,

$\delta x = dx = 17 - 16 = 1$

$y = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$

Now $y = x^{\frac{1}{4}}$

$d(y) = d\left(x^{\frac{1}{4}}\right)$

$dy = \frac{1}{4}x^{\frac{1}{4}-1}\,dx$

$dy = \frac{1}{4}x^{-\frac{3}{4}}\,dx$

Put $x = 16$, $dx = 1$

$dy = \frac{1}{4}(16)^{-\frac{3}{4}}\,(1) = \frac{1}{4}(2^4)^{-\frac{3}{4}}$

$dy = \frac{1}{4}(2)^{-3} = \frac{1}{4}.\frac{1}{8} = \frac{1}{32}$

$dy = 0.03125$

Thus $\sqrt[4]{17} \approx y + dy$

$= 2 + 0.03125$

$= 2.03125$

ii) $(31)^{\frac{1}{5}}$

SOLUTION:

Let $y = x^{\frac{1}{5}}$

We take $x = 32$,

$\delta x = dx = 31 - 32 = -1$

$y = (32)^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2$

Now $y = x^{\frac{1}{5}}$

$d(y) = d\left(x^{\frac{1}{5}}\right)$

$dy = \frac{1}{5}x^{\frac{1}{5}-1}\,dx$

$dy = \frac{1}{5}x^{-\frac{4}{5}}\,dx$

Put $x = 32$, $dx = -1$

$dy = \frac{1}{5}(32)^{-\frac{4}{5}}\,(-1) = -\frac{1}{5}(2^5)^{-\frac{4}{5}}$

$dy = \frac{1}{5}(2)^{-4} = \frac{1}{5}.\frac{1}{16} = \frac{1}{80}$

$dy = -0.0125$

Thus $(31)^{\frac{1}{5}} \approx y + dy$

$= 2 - 0.0125$

$= 1.9875$

iii) $\cos 29°$

SOLUTION:

Let $y = \cos x$

We take $x = 30°$,

$\delta x = dx = 29° - 30° = -1° = -0.01745$

$y = \cos 30° = 0.866$

Now $y = \cos x$

$d(y) = d(\cos x)$

$dy = -\sin x\;dx$

$dy = -\sin 30°\,(-0.01745)$

$dy = -(0.5)\,(-0.01745)$

$dy = 0.0087$

Thus $\cos 29° \approx y + dy$

$= 0.866 + 0.0087$

$= 0.8747$

iv) $\sin 61°$

SOLUTION:

Let $y = \sin x$

We take $x = 60°$,

$\delta x = dx = 61° - 60° = 1° = 0.01745$

$y = \sin 60° = 0.866$

Now $y = \sin x$

$d(y) = d(\sin x)$

$dy = \cos x\;dx$

$dy = \cos 60°\,(0.01745)$

$dy = (0.5)\,(0.01745)$

$dy = 0.0087$

Thus $\sin 61° \approx y + dy$

$= 0.866 + 0.0087$

$= 0.8747$

Q.4: Find the approximate increase in the volume of a cube if the lenght of each edge changes from 5 to 5.02.

SOLUTION:

Lenght of each edge of cube $= x$ *unit*

Volume of a cube $= L.W.H$

$V = x.x.x$

$V = x^3$

$d(V) = (x^3)$

$dV = 3x^2\,dx$

when x changes from 5 to 5.02, so

$x = 5$, $dx = 5.02 - 5 = 0.02$

$dV = 3(5)^2\,(0.02) = 1.5$ *cubic units*

Q.5: Find the approximate increase in the area of a circular disc if its diameter is increased from 44 cm to 44.4 cm.

SOLUTION:

Let radius of circular disc $= x$ *cm*

Area of a disc $= \pi r^2$

$A = \pi x^2$

$d(A) = d(\pi x^2)$

$dA = \pi.2x\,dx$

As diameter changes from 44 to 44.4,

so radius changes from 22 to 22.4, so

$x = 22$, $dx = 22.2 - 22 = 0.2$

$dA = \pi(2)(22)(0.2)$

$dA = 27.646\;\;cm^2$

Integration as anti-derivative (inverse of derivative)

$Integration: \quad v.v.v.important\ defination(***)$
$The\ inverse\ process\ of\ differentiation\ is\ called$
$anti-differentiation\ or\ integration.$

Consider $F(x) is\ antidervative\ of\ a\ function\ if$

$$F'(x) = f(x)\ then \int f(x)dx = \int F'^{(x)}dx$$

$$= \int \frac{d}{dx}F(x)dx$$

$$\int f(x)dx = F(x) + c$$

$\because \frac{d}{dx} and \int dx\ are\ inverse\ operations\ of\ each\ other.$

*The symbol $\int ...dx\ indicates\ that\ integrand\ is\ two\ integrated\ w.r.t\ "x"$

*The anti- derivative of a function is also called integrated is called integrand of the integral.

*The function which is to be integrated is called integrand of the integral.

Some standard formulae for Anti-derivatives

$$\int 1dx = x + c\ , \int x^n dx = \frac{x^{n+1}}{n+1} + c(n \neq -1)$$

$$\int sindx = -cosx + c\ , \int cosxdx = sinx + c$$

$$\int \sec^2 x\, dx = tanx + c, \quad \int cosec^2 dx = -cotx + c$$

$$\int secxtanxdx = secx + c, \int cosecxcotxdx$$

$$= -cosecx$$

$$\int e^x dx = e^x + c \quad , \int a^x dx = \frac{1}{lna}.a^x + c$$

$$\int \frac{1}{x}dx = ln|x| + c, x \neq 0\ , \int tanxdx = lnsec|x| + c$$

$$= -ln|cosx| + c$$

$$\int cotxdx = ln|sinx| + c$$

$$\int secxdx = \ln|secx + tanx| + c$$

$$\int cosecxdx = \ln|cosecx - cots| + c$$

Here c is constant of integration. These formulae can be verified by showing that the derivatives of right hand side of each w.r.t "x" is equal to the corresponding integral

Examples:

1. $\int x^5 dx \qquad \because \int x^n dx = \frac{x^{(n+1)}}{n+1} + c$

$$= \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$$

2. $\int \frac{1}{\sqrt{x^3}}dx$

$$= \int \frac{1}{x^{\left(\frac{2}{3}\right)}}dx = \int x^{-\frac{3}{2}}dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = -\frac{2}{\sqrt{x}} + c$$

3. $\int \frac{1}{(2x+3)^4}dx =$

$$\int (2x+3)^{-4}dx = \frac{1}{2}.\frac{(2x+3)^{-4+1}}{-4+1} + c$$

$$= -\frac{1}{6(2x+3)^3} + c$$

4. $\int cos2xdx \qquad \because \int cosaxdx = \frac{sinax}{a} + c$

$$= \frac{sin2x}{2} + c$$

5. $\int sin3xdx \qquad \because \int sinaxdx = -\frac{cosax}{a} + c$

$$= -\frac{cos3x}{3} + c$$

6. $\int cosec^2 xdx$

$$= -cotx + c$$

7. $\int sec5xtan5xdx$

$$= \frac{sec5x}{5} + c \qquad \because \int secaxtanaxdx = \frac{secax}{a} + c$$

$$= \frac{se5x}{5} + c$$

8. $\int e^{ax+b}dx \qquad \because \int e^{ax}dx = \frac{e^{ax}}{a} + c$

$$\frac{e^{ax+b}}{a} + c$$

9. $\int 3^{\lambda x}dx = \quad \because \int e^{ax}dx = \frac{e^{ax}}{a} + c$

$$= \frac{3^{\lambda x}}{\lambda ln3}$$

10. $\int \frac{1}{ax+b}dx$

$$\int (ax+b)^{-1}dx = \frac{1}{a}\ln(ax+b) + c$$

1. $\int af(x)dx = a\int f(x)dx$
2. $\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

Prove that

$$\int [\boldsymbol{f(x)}]^{\boldsymbol{n}}\boldsymbol{f'(x)dx} = \frac{[\boldsymbol{f(x)}]^{\boldsymbol{n+1}}}{\boldsymbol{n+1}} + \boldsymbol{c}, (\boldsymbol{n \neq -1}$$

Proof:

We know that $\frac{d}{dx}\left(f^{n+1}(x)\right)$

$= (n+1)f^n(x).\frac{d}{dx}f(x)$

⇨ $\frac{d}{dx}(f^n(n+1) = (n+1)f^n(x).f'(x)dx$

Taking integration

$$\int \frac{d}{dx} f^{(n+1)}(x)dx = (n+1)\int f^n(x).f'(x)dx$$

⇨ $f^{n+1}(x) = (n+1)\int f^n(x)f'(x)dx$

⇨ $\int f^n(x)\, f'(x)dx = \frac{f^{n+1}(x)}{n+1} + c \quad by\ def.$

Hence proved.

Prove that $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$

Proof:

We know that

$$\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)}.f'(x)$$

Taking integration both sides

$$\int \frac{d}{dx}[\ln f(x)]dx = \int \frac{1}{f(x)}.f'(x)dx$$

⇨ $\ln f(x) = \int \frac{f'(x)}{f(x)} dx$

⇨ $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c \quad by\ defination$

$(\int f(x)dx = F(x) + c)$

Hence proved.

Exercise 3.2

$Q.1$: ***Evaluate the following indefinate integrals:***

$i)\ \int(3x^2 - 2x + 1)\ dx$

SOLUTION:

$= \int 3x^2\,dx - \int 2x\,dx + \int 1\,dx$

$= 3\int x^2\,dx - 2\int x\,dx + \int 1\,dx$

$= 3.\frac{x^{2+1}}{2+1} - 2.\frac{x^{1+1}}{1+1} + x + c$

$= 3.\frac{x^3}{3} - 2.\frac{x^2}{2} + x + c$

$= x^3 - x^2 + x + c$

$ii)\ \int(\sqrt{x} + \frac{1}{\sqrt{x}})\ dx$

SOLUTION:

$= \int \sqrt{x}\,dx + \int \frac{1}{\sqrt{x}}dx$

$= \int x^{\frac{1}{2}}\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x + c$

$iii)\ \int x(\sqrt{x} + 1)\ dx$

SOLUTION:

$= \int x(\sqrt{x} + 1)\ dx$

$= \int x\sqrt{x}\,dx + \int x\,dx$

$= \int x^{1+\frac{1}{2}}\,dx + \int x\,dx$

$= \int x^{\frac{3}{2}}\,dx + \int x\,dx$

$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c$

$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c$

$= \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$

$iv)\ \int(2x+3)^{\frac{1}{2}}\ dx$

SOLUTION:

$= \int(2x+3)^{\frac{1}{2}}\ dx$

$\times\ and\ \div\ by\ 2\ to\ make\ derivative$

$= \frac{1}{2}\int(2x+3)^{\frac{1}{2}}.2\,dx$

$= \frac{1}{2}\frac{(2x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{1}{2}\frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{2}.\frac{2}{3}\ (2x+3)^{\frac{3}{2}} + c$

$= \frac{1}{3}\ (2x+3)^{\frac{3}{2}} + c$

$v)\ \int(\sqrt{x} + 1)^2\ dx$

SOLUTION:

$= \int(\sqrt{x} + 1)^2\ dx$

$= \int((\sqrt{x})^2 + 2\sqrt{x}.1 + (1)^2)\ dx$

$= \int[x + 2\sqrt{x} + 1]$

$= \int x\,dx + 2\int x^{\frac{1}{2}}\,dx + \int 1\,dx$

$= \frac{x^{1+1}}{1+1} + 2.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + x + c$

$= \frac{x^2}{2} + 2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x + c$

$= \frac{1}{2}x^2 + 2.\frac{2}{3}x^{\frac{3}{2}} + x + c$

$= \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} + x + c$

$vi)\ \int(\sqrt{x} - \frac{1}{\sqrt{x}})^2\ dx$

SOLUTION:

$= \int(\sqrt{x} - \frac{1}{\sqrt{x}})^2\ dx$

$= \int\left[(\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x}.\frac{1}{\sqrt{x}}\right]\ dx$

$= \int\left[x + \frac{1}{x} - 2\right]\ dx$

$= \int x\,dx + \int \frac{1}{x}dx - 2\int 1\,dx$

$= \frac{x^{1+1}}{1+1} + \ln x - 2\,x + c$

$= \frac{1}{2}x^2 + \ln x - 2\,x + c$

NOTE: FOR Q. (vi)

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FOR EXAMPLE:

$\int \frac{1}{x}\ dx = \int x^{-1}\ dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} = \infty$

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$vii)\ \int \frac{3x+2}{\sqrt{x}}\ dx$

SOLUTION:

$\int \frac{3x+2}{\sqrt{x}}\, dx$

$= \int \left[\frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right] dx$

$= \int \left[\frac{3\sqrt{x}\sqrt{x}}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right] dx \quad \because x = \sqrt{x}.\sqrt{x}$

$= \int \left[3\sqrt{x} + \frac{2}{\sqrt{x}}\right] dx$

$= \int \left[3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\right] dx$

$= 3\int x^{\frac{1}{2}}\, dx + 2\int x^{-\frac{1}{2}}\, dx$

$= 3\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 3\frac{2}{3}\, x^{\frac{3}{2}} + 2\,.2x^{\frac{1}{2}} + c$

$= 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + c$

viii) $\int \frac{\sqrt{y}(y+1)}{y}\, dx$

SOLUTION:

$\int \frac{\sqrt{y}(y+1)}{y}\, dx$

$= \int \frac{\sqrt{y}(y+1)}{\sqrt{y}\,\sqrt{y}}\, dx$

$= \int \frac{y+1}{\sqrt{y}}\, dx$

$= \int \left[\frac{y}{\sqrt{y}}dx + \frac{1}{\sqrt{y}}\, dx\right]$

$= \int \left[\sqrt{y}dx + \frac{1}{\sqrt{y}}\, dx\right]$

$= \int \left[y^{\frac{1}{2}}\, dx + y^{-\frac{1}{2}}\, dx\right]$

$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + x + c$

ix) $\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}}\, d\theta$

SOLUTION:

$\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}}\, d\theta$

$= \int \frac{(\sqrt{\theta})^2+(1)^2-2\sqrt{\theta}}{\sqrt{\theta}}\, d\theta$

$= \int \frac{\theta+1-2\sqrt{\theta}}{\sqrt{\theta}}\, d\theta$

$= \int \left[\frac{\theta}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}}\right] d\theta$

$= \int \left[\sqrt{\theta} + \frac{1}{\sqrt{\theta}} - 2\right] d\theta$

$= \int \theta^{\frac{1}{2}}\, d\theta + \int \theta^{-\frac{1}{2}}\, d\theta - 2\int 1\, d\theta$

$= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\theta + c$

$= \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} - 2\theta + c$

$= \frac{2}{3}\theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} - 2\theta + c$

x) $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\, dx$

SOLUTION:

$\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\, dx$

$= \int \frac{(\sqrt{x})^2+(1)^2-2\sqrt{x}}{\sqrt{x}}\, dx$

$= \int \frac{x+1-2\sqrt{x}}{\sqrt{x}}\, dx$

$= \int \left[\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}}\right] dx$

$= \int \left[\sqrt{x} + \frac{1}{\sqrt{x}} - 2\right] dx$

$= \int x^{\frac{1}{2}}\, dx + \int x^{-\frac{1}{2}}\, dx - 2\int 1\, dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2x + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2x + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2x + c$

xi) $\int \frac{e^{2x}+e^x}{e^x}\, dx$

SOLUTION:

$\int \frac{e^{2x}+e^x}{e^x}\, dx$

$= \int \left[\frac{e^{2x}}{e^x} + \frac{e^x}{e^x}\right] dx$

$= \int [e^x + 1]\, dx$

$= \int e^x\, dx + \int 1\, dx$

$= \frac{e^x}{1} + x + c$

$= e^x + x + c$

NOTE: ***DERIVATION*** *M*

EXPONENTIAL FUNCTION KA JAB DERIVATIVE LATY H T FUNCTION AS IT AUR POWER KA DERIVATIVE MULTIPLY KARTY H. LAKIN INTEGRATION M DIVIDE KARE GAI.

Q. 2: ***Evaluate***:

i) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

SOLUTION:

$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \cdot \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\, dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}\, dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\, dx$

$= \frac{1}{a-b}\int\left(\sqrt{x+a} - \sqrt{x+b}\right)dx$

$= \frac{1}{a-b}\left\{\int (x+a)^{\frac{1}{2}}\, dx + \int (x+b)^{\frac{1}{2}}\, dx\right\}$

using $\int [f(x)]^n . f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$

$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+b)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + c$

$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$

$= \frac{1}{a-b}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(x+b)^{\frac{3}{2}}\right\} + c$

$= \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} + (x+b)^{\frac{3}{2}}\right\} + c$

ii) $\int \frac{1-x^2}{1+x^2}\, dx$

SOLUTION:

$\int \frac{1-x^2}{1+x^2}\ dx$

$= \int \frac{2-1-x^2}{1+x^2}\ dx$

$= \int \frac{2-(1+x^2)}{1+x^2}\ dx$

$= \int \frac{2}{1+x^2}\ dx - \int \frac{1+x^2}{1+x^2}\ dx$

$= 2\int \frac{1}{1+x^2}\ dx - \int 1\ dx$

$= 2\tan^{-1} x - x + c$

iii) $\int \frac{\boldsymbol{dx}}{\sqrt{\boldsymbol{x+a}}+\sqrt{\boldsymbol{x}}}$

SOLUTION:

$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\ \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}\ dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2-(\sqrt{x})^2}\ dx = \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\ dx$

$= \frac{1}{a}\int\left(\sqrt{x+a} - \sqrt{x+}\ \right)dx$

$= \frac{1}{a}\left\{\int (x+a)^{\frac{1}{2}}\ dx + \int (x)^{\frac{1}{2}}\ dx\right\}$

using $\int [\boldsymbol{f(x)}]^{\boldsymbol{n}}.\boldsymbol{f'(x)} = \frac{[\boldsymbol{f(x)}]^{\boldsymbol{n+1}}}{\boldsymbol{n+1}} + \boldsymbol{c}$

$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + c$

$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$

$= \frac{1}{a}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(x)^{\frac{3}{2}}\right\} + c$

$= \frac{2}{3a}\left\{(x+a)^{\frac{3}{2}} + (x)^{\frac{3}{2}}\right\} + c$

iv) $\int (\boldsymbol{a - 2x})^{\frac{3}{2}}\ \boldsymbol{dx}$

SOLUTION:

$\int (a-2x)^{\frac{3}{2}}\ dx$

$\times$ *and* $\div$ *by* 2

$= \frac{1}{-2}\int (a-2x)^{\frac{3}{2}}.(-2)\ dx$

$= -\frac{1}{2}\frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

$= -\frac{1}{2}\frac{(a-2x)^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= -\frac{1}{2}.\frac{2}{5}(a-2x)^{\frac{5}{2}} + c$

$= -\frac{1}{5}(a-2x)^{\frac{5}{2}} + c$

FUNCTION AS IT AUR POWER KE DERIVATIVE S DIVIDE KARNA H.

$\int e^x\, dx = \frac{e^x}{1} + c = e^x + c$

v) $\int \frac{(\boldsymbol{1+e^x})^{\boldsymbol{3}}}{\boldsymbol{e^x}}\ \boldsymbol{dx}$

SOLUTION:

$\int \frac{(1+e^x)^3}{e^x}\ dx$

$\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)$

$= \int \frac{1^3+(e^x)^3+3(1)(e^x)(1+e^x)}{e^x}\ dx$

$= \int \frac{1+e^{3x}+3e^x(1+e^x)}{e^x}\ dx$

$= \int \left[\frac{1}{e^x} + \frac{e^{3x}}{e^x} + \frac{3e^x(1+e^x)}{e^x}\right]\ dx$

$= \int [e^{-x} + e^{2x} + 3 + 3e^x]\ dx$

$= \int e^{-x}\ dx + \int e^{2x}\ dx + 3\int 1\ dx + 3\int e^x\ dx$

$= \frac{e^{-x}}{-1} + \frac{e^{2x}}{2} + 3x + 3\frac{e^x}{1} + c$

$= -e^{-x} + \frac{1}{2}e^{2x} + 3x + 3e^x + c$

vi) $\int \mathbf{sin}(\boldsymbol{a+b})\boldsymbol{x}\ \boldsymbol{dx}$

SOLUTION:

$\int \sin(a+b)x\ dx$

$= \frac{-\cos(a+b)x}{a+b} + c$

$= -\frac{1}{a+b}\cos(a+b)x + c$

DERIVATION M FUNCTION KA DERIVATIVE LENA HOTA H AUR SATH ANGLE KE DERIVATIVE KO MULTIPLY KARTY H. BUT INTEGRATION M ANGLE KE DERIVATIVE K DIVIDE KARE GAI.

vii) $\int \sqrt{\mathbf{1} - \mathbf{cos}\,\mathbf{2}\boldsymbol{x}}\ \boldsymbol{dx}$

SOLUTION:

$\int \sqrt{1-\cos 2x}\ dx$

As $\mathbf{sin^2}\,\boldsymbol{x} = \frac{\mathbf{1-cos\,2}\boldsymbol{x}}{\mathbf{2}}$

So $\mathbf{1 - cos\,2}\boldsymbol{x} = \mathbf{2sin^2}\,\boldsymbol{x}$

$= \int \sqrt{2\sin^2 x}\ dx$

$= \int \sqrt{2}\sqrt{\sin^2 x}\, dx$

$= \sqrt{2}\int \sin x\ dx$

$= \sqrt{2}(-\cos x) + c$

$= -\sqrt{2}\cos x + c$

viii) $\int \ln x\ \frac{1}{x}\, dx$

SOLUTION:

$\int \ln x\ .\ \frac{1}{x}\ dx$

As $\boldsymbol{f(x) = \ln x}$

And $\boldsymbol{f'(x)} = \frac{\mathbf{1}}{\boldsymbol{x}}$, ***so***

using $\int [\boldsymbol{f(x)}]^{\boldsymbol{n}} = \frac{[\boldsymbol{f(x)}]^{\boldsymbol{n+1}}}{\boldsymbol{n+1}}$

$= \frac{(\ln x)^{\mathbf{1+1}}}{\mathbf{1+1}} + \boldsymbol{c}$

$= \frac{(\ln x)^{\mathbf{2}}}{\mathbf{2}} + \boldsymbol{c}$

ix) $\int \mathbf{sin^2}\,\boldsymbol{x}\,\boldsymbol{dx}$

SOLUTION:

$\int \sin^2 x\, dx$

As $\mathbf{sin^2}\,\boldsymbol{x} = \frac{\mathbf{1-cos\,2}\boldsymbol{x}}{\mathbf{2}}$

$= \int \frac{1-\cos 2x}{2}\ dx$

$= \frac{1}{2}\int (1-\cos 2x)dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

x) $\int \frac{\mathbf{1}}{\mathbf{1 + cos}\,\boldsymbol{x}}\ \boldsymbol{dx}$

SOLUTION:

$\int \frac{1}{1+\cos x}\ dx$

As $\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$

So $1+\cos x = 2\cos^2\frac{x}{2}$

$= \int \frac{1}{2\cos^2\frac{x}{2}}\,dx$

$= \frac{1}{2}\int \sec^2\frac{x}{2}\,dx$

$= \frac{1}{2}\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c = \tan\frac{x}{2} + c$

$\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$ *in functions k derivative exist ni karty jab b ye function a jay t ap ye formula use kare.*

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\tan^2 x = \sec^2 x - 1$

$\cot^2 x = \csc^2 x - 1$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRTION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$ *IN KI INTEGRATION NI HOTI.*

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x$

$(\operatorname{cosec} x)' = -\csc x \cot x$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRTION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$ *IN KI INTEGRATION NI HOTI.*

xi) $\int \frac{ax+b}{ax^2+2bx+c}\,dx$

***SOLUTION*:**

$\int \frac{ax+b}{ax^2+2bx+c}\,dx$

$\times$ & $\div$ *by 2 to make derivative uper*

$= \frac{1}{2}\int \frac{2(ax+b)}{ax^2+2bx+c}\,dx$

$= \frac{1}{2}\int \frac{2ax+2b}{ax^2+2bx+c}\,dx$

Using $\int \frac{f'(x)}{[f(x)]} = \ln[f(x)]$

$= \frac{1}{2}\ln(ax^2+2bx+c) + c$

xii) $\int \cos 3x \sin 2x\,dx$

***SOLUTION*:**

$\int \cos 3x \sin 2x\,dx$

$\times$ & $\div$ *by 2 to make formula*

$= \frac{1}{2}\int 2\cos 3x \sin 2x\,dx$

As $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$= \frac{1}{2}\int[\sin(3x+2x) - \sin(3x-2x)]\,dx$

$= \frac{1}{2}\int[\sin(5x) - \sin(x)]\,dx$

$= \frac{1}{2}\{\int \sin 5x\,dx - \int \sin x\,dx\}$

$= \frac{1}{2}\left\{\frac{-\cos 5x}{5} - \frac{-\cos x}{1}\right\} + c$

$= -\frac{1}{2}\left\{\frac{\cos 5x}{5} - \cos x\right\} + c$

xiii) $\int \frac{\cos 2x - 1}{1+\cos 2x}\,dx$

***SOLUTION*:**

$= \int \frac{\cos 2x - 1}{1+\cos 2x}\,dx$

$= -\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$

$\because \sin^2 x = \frac{1-\cos 2x}{2} \Rightarrow 2\sin^2 x = 1 - \cos 2x$

$\because \cos^2 x = \frac{1+\cos 2x}{2} \Rightarrow 2\cos^2 x = 1 + \cos 2x$

$= -\int \frac{2\sin^2 x}{2\cos^2 x}\,dx = -\int \tan^2 x\,dx$

$= -\int(\sec^2 x - 1)\,dx \quad \because 1+\tan^2 x = \sec^2 x$

$= -\int \sec^2 x\,dx + \int 1\,dx$

$= -\tan x + x + c$

xiv) $\int \tan^2 x\,dx$

***SOLUTION*:**

$\int \tan^2 x\,dx$

$= \int(\sec^2 x - 1)\,dx \quad \because 1+\tan^2\theta = \sec^2\theta$

$= \int \sec^2 x\,dx - \int 1\,dx = \tan x - x + c$

Integration by method of substitution

Sometimes it is possible to convert an integral into standard form by a suitable change of a variable. This is called substitution method.

i.e Evaluate $\int f(x)dx$ *by method of subsitution*

Let $x = \emptyset(t) \Rightarrow dx = \emptyset''(t)dt$

So $\int f(x)dx = \int f(\emptyset(t))\emptyset'(t)dt$

Some useful substitutions:

1. $\sqrt{a^2 - x^2}$ $put x = a\sin\theta$ $(\because 1 - \sin^2\theta = \cos^2\theta)$
2. $\sqrt{x^2 - a^2}$ $put\ x = a\sec\theta$ $(\because \sec^2\theta - 1 = \tan^2\theta$
3. $\sqrt{a^2 + x^2}$ $put\ x = a\tan\theta$ $(\because \sec^2\theta = 1 + \tan^2\theta$
4. $\sqrt{x+a}(0r)\sqrt{x-a}$ $put\ \sqrt{x+a} = t$ $or(\sqrt{x-a}) = t$
5. $\sqrt{2ax - x^2}$ $put\ x - a = a\sin\theta$
6. $\sqrt{2ax + x^2}$ $put\ x + a = a\sec\theta$

Exercise 3.3

Evaluate the following integrals:

$\boldsymbol{Q.1}$: $\int \frac{-2x}{\sqrt{4-x^2}}\ dx$

SOLUTION:

$\int \frac{-2x}{\sqrt{4-x^2}}\ dx$

$= \int (4-x^2)^{-\frac{1}{2}}\ (-2x)\ dx$

$Here\ f(x) = 4-x^2$

$f'(x) = -2x$

$= \frac{(4-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

$= \frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{4-x^2} + c \quad \because t = 4-x^2$

$\boldsymbol{Q.2}$: $\int \frac{dx}{x^2+4x+13}$

SOLUTION:

By completing square

$= \int \frac{dx}{x^2+4x+4-4+13}$

$= \int \frac{dx}{(x+2)^2+9}$

$= \int \frac{1}{(x+2)^2+(3)^2}\ dx$

$\because \int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

$\boldsymbol{Q.3}$: $\int \frac{x^2}{4+x^2}\ dx$

SOLUTION:

$(+)\ and\ (-)\ 4$

$= \int \frac{4+x^2-4}{4+x^2}\ dx$

$= \int \left(\frac{4+x^2}{4+x^2} - \frac{4}{4+x^2}\right)\ dx$

$= \int 1\ dx - \int \frac{4}{4+x^2}\ dx$

$= \int 1\ dx - 4\int \frac{1}{2^2+x^2}\ dx$

$= x - 4.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + c$

$\boldsymbol{Q.4}$: $\int \frac{1}{x\ln x}\ dx$

SOLUTION:

$\int \frac{1}{\ln x}.\frac{1}{x}\ dx$

$As\ f(x) = ln\ x$

$And\ f'(x) = \frac{1}{x}$, so

$using\ \int \frac{f'(x)}{[f(x)]} = ln[f(x)]$

$= ln[\ln x] + c$

$\boldsymbol{Q.5}$: $\int \frac{e^x}{e^x+3}\ dx$

SOLUTION:

$\int \frac{e^x}{e^x+3}\ dx$

$Here\ f(x) = e^x$

$And\ f'(x) = e^x$, so

$using\ \int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$= ln(e^x+3) + c$

$\boldsymbol{Q.6}$: $\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}\ dx$

SOLUTION:

$\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}\ dx$

$\int (x^2+2bx+c)^{-\frac{1}{2}}.(x+b)\ dx$

$Here\ f(x) = x^2+2bx+c$

$Here\ f'(x) = 2x+2b = 2(x+b)$

$\times\ and\ \div\ by\ 2$

$= \frac{1}{2}\int (x^2+2bx+c)^{-\frac{1}{2}}.2(x+b)\ dx$

$= \frac{1}{2}\ \frac{(x^2+2bx+c)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{1}{2}\ \frac{(x^2+2bx+c)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \sqrt{x^2+2bx+c} + c$

$\boldsymbol{Q.7}$: $\int \frac{\sec^2 x}{\sqrt{\tan x}}\ dx$

SOLUTION:

$\int \frac{\sec^2 x}{\sqrt{\tan x}}\ dx$

$= \int (\tan x)^{-\frac{1}{2}}\ \sec^2 x$

$Here\ f(x) = \tan x$

$Here\ f'(x) = \sec^2 x$

$= \frac{(\tan x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{(\tan x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{\tan x} + c$

$\boldsymbol{Q.8}$: **(a) Show that**

$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right) + c$

SOLUTION:

$L.H.S = \int \frac{dx}{\sqrt{x^2-a^2}}$

$Put\ x = a\sec\theta \implies dx = a\sec\theta\ \tan\theta\ d\theta$

$= \int \frac{a\sec\theta\ \tan\theta\ d\theta}{\sqrt{(a\sec\theta)^2-a^2}} = \int \frac{a\sec\theta\ \tan\theta}{\sqrt{a^2sec^2\ \theta-a^2}}\ d\theta$

$= \int \frac{a\sec\theta\ \tan\theta}{\sqrt{a^2(sec^2\ \theta-1)}}\ d\theta = \int \frac{\sec\theta\ \tan\theta}{\sqrt{tan^2\ \theta}}\ d\theta$

$= \int \frac{\sec\theta\ \tan\theta}{\tan\theta}\ d\theta = \int \sec\theta\ d\theta$

$= ln|\sec\theta + \tan\theta| + c_1$

Then back substitution:

$x = a\sec\theta \implies \frac{x}{a} = \sec\theta$

$And\ 1+tan^2 = sec^2\ \theta$

$tan^2\ \theta = sec^2\ \theta - 1$

$\tan\theta = \sqrt{sec^2\ \theta - 1}$

$\tan\theta = \sqrt{\left(\frac{x}{a}\right)^2 - 1}$

$\tan\theta = \sqrt{\frac{x^2-a^2}{a^2}}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

Now put values

$$= ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c_1$$

$$= ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1$$

Using $\ln\frac{A}{B} = lnA - lnB$

$$= ln|\ x + \sqrt{x^2 - a^2}| - lna + c_1$$

Where $c = -lna + c_1$

$$= ln|\ x + \sqrt{x^2 - a^2}| + c$$

Q.9: $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

SOLUTION:

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$

Put $x = \tan\theta$

$$\Longrightarrow \quad d(x) = \sec^2\theta\, d\theta$$

$$= \int \frac{\sec^2\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}\ d\theta$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}\ d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta}\ d\theta$$

$$= \int \frac{1}{\sec\theta}\ d\theta$$

$$= \int \cos\theta$$

$$= \frac{\sin\theta}{1} + c$$

$$= \frac{\sin\theta}{\cos\theta}.\cos\theta + c$$

$$= \tan\theta\, .\cos\theta + c$$

$$= \frac{\tan\theta}{\sec\theta} + c$$

$$= \frac{\tan\theta}{\sqrt{\sec^2\theta}} + c$$

$$= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + c$$

Put $\tan\theta = x$

$$= \frac{x}{\sqrt{1+x^2}} + c$$

Q.10: $\int \frac{1}{(1+x^2)Tan^{-1}x}\ dx$

SOLUTION:

$$\int \frac{1}{(1+x^2)Tan^{-1}x}\ dx$$

$$\int \frac{1}{Tan^{-1}x} \cdot \frac{1}{(1+x^2)}\ dx$$

Here $f(x) = Tan^{-1}x$

Here $f'(x) = \frac{1}{(1+x^2)}$

using $\int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$$= ln|Tan^{-1}x| + c$$

Q.11: $\int \sqrt{\frac{1+x}{1-x}}\ dx$

SOLUTION:

By rationalizing

$$= \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}}\ dx$$

$$= \int \sqrt{\frac{1+x}{1-x} \times \frac{1+x}{1+x}}\ dx$$

$$= \int \sqrt{\frac{(1+x)^2}{1-x^2}}\ dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\ dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\ dx + \int \frac{x}{\sqrt{1-x^2}}\ dx$$

$$= \sin^{-1}x + \frac{1}{-2}\int(1-x^2)^{-\frac{1}{2}}(-2x)\ dx$$

$$= \sin^{-1}x - \frac{1}{2}.\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \sin^{-1}x - \frac{1}{2}.\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \sin^{-1}x - \sqrt{1-x^2} + c$$

Q.12: $\int \frac{\sin\theta}{1+\cos^2\theta}\ d\theta$

SOLUTION:

$$\int \frac{\sin\theta}{1+\cos^2\theta}\ d\theta = \int \frac{1}{1+\cos^2\theta}\ \sin\theta\ d\theta$$

Put $\cos\theta = t \Longrightarrow -\sin\theta\ d\theta = dt$

$$\int \frac{1}{1+t^2}\ .-dt = -\tan^{-1}t + c$$

Put $t = \cos\theta$

$$= -\tan^{-1}(\cos\theta) + c$$

Q.13: $\int \frac{ax}{\sqrt{a^2-x^4}}\ dx$

SOLUTION:

$$\int \frac{ax}{\sqrt{a^2-x^4}}\ dx = a\int \frac{x}{\sqrt{a^2-(x^2)^2}}\ dx$$

Put $x^2 = t \quad \Longrightarrow 2x\ dx = dt \quad \Longrightarrow x\ dx = \frac{1}{2}\ dt$

$$= \frac{a}{2}\int \frac{1}{\sqrt{a^2-t^2}}\ dt = \frac{a}{2}\ \sin^{-1}\frac{t}{a} + c$$

using $\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\frac{x}{a}$

$$= \frac{a}{2}\ \sin^{-1}\frac{x^2}{a} + c \qquad \because\ x^2 = t$$

Q.14: $\int \frac{dx}{\sqrt{7-6x-x^2}}$

SOLUTION:

$$\int \frac{dx}{\sqrt{7-6x-x^2}}$$

By completing square

$$= \int \frac{dx}{\sqrt{7-x^2-6x-9+9}}$$

$$= \int \frac{dx}{\sqrt{7-(x^2+6x+9)+9}}$$

$$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$$

Using $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$

$$= \sin^{-1}\frac{x+3}{a} + c$$

Q.15: $\int \frac{\cos x}{\sin x ln \sin x}\ dx$

SOLUTION:

$$\int \frac{1}{ln\sin x}\ \frac{\cos x}{\sin x}\ dx$$

Here $f(x) = ln\sin x$

And $f'(x) = \frac{\cos x}{\sin x}$, *so*

using $\int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$$= ln[ln\sin x] + c$$

Q.16: $\int \cos x\frac{ln\sin x}{\sin x}\ dx$

SOLUTION:

$$\int ln\sin x\, .\frac{\cos x}{\sin x}\ dx$$

Here $f(x) = ln\sin x$

And $f'(x) = \frac{\cos x}{\sin x}$, *so*

$= \frac{[\ln \sin x]^{1+1}}{1+1} + c$

$= \frac{1}{2}[\ln \sin x]^2 + c$

Q.17: $\int \frac{x\,dx}{4+2x+x^2}$

SOLUTION:

$\int \frac{x\,dx}{4+2x+x^2}$

$= \frac{1}{2}\int \frac{2x}{4+2x+x^2}\,dx$

$= \frac{1}{2}\int \frac{2x+2-2}{4+2x+x^2}\,dx$

$= \frac{1}{2}\left\{\int \frac{2x+2}{4+2x+x^2}dx - \int \frac{2}{4+2x+x^2}\,dx\right\}$

$= \frac{1}{2}\left\{\ln(4+2x+x^2) - \int \frac{2}{x^2+2x+1^2+4-1^2}\,dx\right\}$

$= \frac{1}{2}\ln(4+2x+x^2) - \frac{1}{2}\int \frac{2}{(x+1)^2+(\sqrt{3})^2}\,dx$

using $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$= \frac{1}{2}\ln(x^2+2x+4) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{(x+1)}{\sqrt{3}} + c$

Q.18: $\int \frac{x}{x^4+2x^2+5}\,dx$

SOLUTION:

$= \int \frac{x}{(x^2)^2+2x^2+5}\,dx$

Put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2}\,dt$

$= \int \frac{\frac{1}{2}}{t^2+2t+5}\,dt$

$= \frac{1}{2}\int \frac{1}{t^2+2t+1+5-1}\,dt$

$= \frac{1}{2}\int \frac{1}{(t+1)^2+2^2}\,dt$

$= \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\frac{t+1}{2} + c$

Put $x^2 = t$

$= \frac{1}{4}\tan^{-1}\frac{x^2+1}{2} + c$

Q.19: $\int \left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\times\left(\frac{1}{\sqrt{x}}-1\right)\right]dx$

SOLUTION:

$\int \left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\times\left(\frac{1}{\sqrt{x}}-1\right)\right]dx$

Put $\sqrt{x}-\frac{x}{2} = t$

$\Longrightarrow\ d\left(\sqrt{x}-\frac{x}{2}\right) = d(t)$

$\frac{1}{2\sqrt{x}} - \frac{1}{2} = dt$

$\frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right) = dt$

$\left(\frac{1}{\sqrt{x}}-1\right) = 2dt$

$= \int[\cos t \times 2\,dt]$

$= 2\int[\cos t\,dt]$

$= 2\,\frac{\sin t}{1} + c$

Put value of t

$= 2\,\sin\left(\sqrt{x}-\frac{x}{2}\right) + c$

Q.20: $\int \frac{x+2}{\sqrt{x+3}}\,dx$

[**Q.19: solve on page 9**]

SOLUTION:

$\int \frac{x+2}{\sqrt{x+3}}\,dx = \int \frac{x+2+1-1}{\sqrt{x+3}}\,dx = \int \frac{x+3}{\sqrt{x+3}}\,dx - \int \frac{1}{\sqrt{x+3}}\,dx = \int \sqrt{x+3}\,dx - \int \frac{1}{\sqrt{x+3}}\,dx = \int (x+3)^{\frac{1}{2}}.1\,dx - \int (x+3)^{-\frac{1}{2}}.1\,dx$

Now integrate

$\frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+3)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}(x+3)^{\frac{3}{2}} + 2\sqrt{x+3} + c$

Q.21: $\int \frac{\sqrt{2}}{\sin x+\cos x}\,dx$

SOLUTION:

$\int \frac{1}{\frac{1}{\sqrt{2}}(\cos x+\sin x)}\,dx$

$= \int \frac{1}{\cos x\cdot\frac{1}{\sqrt{2}}+\sin x\cdot\frac{1}{\sqrt{2}}}\,dx$

$= \int \frac{1}{\cos x\ \cos 45^\circ+\sin x\ \sin 45^\circ}\,dx$

using $\cos(\alpha-\beta) = \cos\alpha\ \sin\beta + \sin\alpha\ \sin\beta$

$= \int \frac{1}{\cos(x-45^\circ)}\,dx$

$= \int \sec(x-45^\circ)\,dx$

using $\int \sec x\ dx = \ln|\sec x + \tan x| + c$

$\ln|\sec(x-45^\circ) + \tan(x-45^\circ)| + c$

Q.22: $\int \frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}$

SOLUTION:

$= \int \frac{1}{\sin x\,.\,\frac{1}{2}+\cos x\,.\,\frac{\sqrt{3}}{2}}\,dx$

$= \int \frac{1}{\sin x\,.\ \cos 60^\circ+\cos x\,.\ \sin 60^\circ}\,dx$

using $\sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta$

$= \int \frac{1}{\sin(x+60^\circ)}\,dx$

$= \int \text{cosec}(x+60^\circ)\,dx$

using $\int \text{cosec}\,x\ dx = \ln|\text{cosec}\,x - \cot x| + c$

$= \ln|\text{cosec}(x-60^\circ) + \cot(x-60^\circ)| + c$

Integration by parts.

We know that for two functions f and g

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

$$\Rightarrow\ f(x)g'(x) = \frac{d}{dx}\big((f(x)g(x)\big) = f'(x)g(x)$$

Taking integrations w.r.t x *we get*

$$\int f(x)g'(x)dx = \int\left[\frac{d}{dx}f(x)g(x) - f'(x)g(x)\right]dx$$

$$= \int\left(\frac{d}{dx}\big(f(x)g(x)\big) - \int f'(x)g(x)dx\right.$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Or $\int f(x)g'(x)dx = f(x)\int g'(x)\,dx - \int (g'(x)dx)f'(x)dx$

In other words.

$$\int (1st\ function)(2nd\ function)dx$$

$$= (1st\ funct.).\int (2nd\ funct.)\,dx$$

$$- \int (integrated\ funct.)\frac{d}{dx}(istfunction)dx$$

This is called "integrations by parts"

Some basic rules for Integration by parts.

*some the function as 2nd function whose integration is known or possible.
*if integration of both given functions are known but one of the given function is polynomial functions then whose polynomial function as first function.
*if integration of both given function are known but no one is polynomial function. Then we may choose any function as 1st.
*if we are given only one function whose integration is unknown or cannot be easily find.

$$i.e, \sin^{-1} x, \cos^{-1} x, \sqrt{a^2 - x^2}, \frac{1}{\sqrt{x^2 - a^2}} e.t.c$$

Then we take 1 as 2nd function.

"Review above Rules"

	1st function	2nd function
$\int x^n cosdx$	x^n	$cosx$
$\int x^n sinxdx$	x^n	$sinx$
$\int x^n \sin^{-1} xdx$	$\sin^{-1} x$	x^n
$\int x^n \tan^{-1} xdx$	$\tan^{-1} x$	x^n
$\int e^x sinxdx$	e^x or $sinx$	$sinx$ e^x
$\int lnxx^n dx$	x^n	lnx
$\int \tan^{-1} xdx$	$\tan^{-1} x$	1
$\int \sqrt{a^2 + x^2} dx$	$\sqrt{a^2 + x^2}$	1

You may remember the word "ILATE"
I=inverse function
L=logarithmic function
A=algebraic function
T=trigonometric functions
E=exponential functions.

Remember useful formulas

1. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^1\left(\frac{x}{a}\right) + c$
2. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$
3. $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left|x + \sqrt{x^2 + a^2} + c\right|$

Prove that $\int e^x\left(f(x) + f'(x)\right)dx + e^x f(x) + c$

Prove: $\int e^x f(x)dx = f(x)e^x - \int e^x f'(x)dx$

⇨ $\int e^x f(x)dx + \int e^x f'(x)dx = e^x f(x)$

⇨ $\int e^x\left(f(x) + f'(x)\right)dx = e^x f(x) + c$

Hence proved.

Exercise 3.4

i) $\int x \sin x \, dx$

SOLUTION:

$\int x \sin x \, dx$

$Here \; U = x \;, \; V = \sin x$

$Using \int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= x . \int \sin x \, dx - \int [(x)' . \int \sin x \, dx]\, dx$

$= x . (-\cos x) - \int [1 . (-\cos x)]\, dx$

$= -x \cos x - \int [-\cos x]\, dx$

$= -x \cos x + \int [\cos x]\, dx$

$= -x \cos x + \sin x + c$

$= \sin x - x \cos x + c$

ii) $\int \ln x \, dx$

SOLUTION:

$\int \ln x . 1\, dx$

$Here \; U = \ln x \;, \; V = 1$

$Using \int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \ln x . \int 1\, dx - \int [(\ln x)' . \int 1\, dx]\, dx$

$= \ln x . x - \int \left[\frac{1}{x} . x\right] dx$

$= \ln x . x - \int 1\, dx$

$= x \ln x - x + c$

iii) $\int x \ln x \, dx$

SOLUTION:

$\int x \ln x \, dx$

$Here \; U = \ln x \;, \; V = x$

$Using \int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \ln x . \int x\, dx - \int [(\ln x)' . \int x\, dx]\, dx$

$= \ln x . \frac{x^2}{2} - \int \left[\frac{1}{x} . \frac{x^2}{2}\right] dx$

$= \ln x . \frac{x^2}{2} - \frac{1}{2}\int x\, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} . \frac{x^2}{2} + c$

$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$

iv) $\int x^2 \ln x \, dx$

SOLUTION:

$\int x^2 \ln x \, dx$

$Here \; U = \ln x \;, \; V = x^2$

$Using \int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \ln x . \int x^2\, dx - \int [(\ln x)' . \int x^2\, dx]\, dx$

$= \ln x . \frac{x^3}{3} - \int \left[\frac{1}{x} . \frac{x^3}{3}\right] dx$

$= \ln x . \frac{x^3}{3} - \frac{1}{3}\int x^2\, dx$

$= \frac{x^3}{3} \ln x - \frac{1}{3} . \frac{x^3}{3} + c$

$= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + c$

v) $\int x^3 \ln x \, dx$

SOLUTION:

$\int x^2 \ln x \, dx$

$Here \; U = \ln x \;, \; V = x^3$

$Using \int U.V = U . \int V\, dx - \int [U' . \int V\, dx]\, dx$

$= \ln x . \int x^3\, dx - \int [(\ln x)' . \int x^3\, dx]\, dx$

$= \ln x . \frac{x^4}{4} - \int \left[\frac{1}{x} . \frac{x^4}{4}\right] dx$

$= \ln x . \frac{x^4}{4} - \frac{1}{4}\int x^3\, dx$

$= \frac{x^4}{4} \ln x \; - \frac{1}{4} . \frac{x^4}{4} + c$

$= \frac{x^4}{4}\left(\ln x \; - \frac{1}{4}\right) + c$

vi) $\int x^4 \ln x \, dx$

SOLUTION:

$\int x^4 \; \ln x \; dx$

Here $U = \ln x \;\; , V = x^4$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \ln x \, . \int x^4 \,dx - \int [(\ln x)' . \int x^4 \,dx]\,dx$

$= \ln x \, . \frac{x^5}{5} - \int \left[\frac{1}{x} . \frac{x^5}{5}\right] dx$

$= \ln x \, . \frac{x^5}{5} - \frac{1}{5}\int x^4 \,dx$

$= \frac{x^5}{5} \ln x \; - \frac{1}{5} . \frac{x^5}{5} + c$

$= \frac{x^5}{5}\left(\ln x \; - \frac{1}{5}\right) + c$

vii) $\int \tan^{-1} x \, dx$

SOLUTION:

$\int 1 . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\; , V = 1$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x \, . \int 1\,dx - \int [(\tan^{-1} x)' . \int 1\,dx]\,dx$

$= \tan^{-1} x \, . x \; - \int \left[\frac{1}{1+x^2} . x\right] dx$

$= x \tan^{-1} x \; - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$= x \tan^{-1} x \; - \frac{1}{2}\ln|1 + x^2| +$

viii) $\int x^2 \sin x \, dx$

SOLUTION:

$\int x^2 \; \sin x \; dxn$

Here $U = x^2 \;\; , V = \sin x$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= x^2 . \int \sin x\,dx - \int [(x^2)' . \int \sin x \; dx]\,dx$

$= x^2 . (-\cos x) - \int [2x . (-\cos x)] \; dx$

$= -x^2 \; \cos x + 2\int [x . \cos x] \; dx$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= -x^2 \; \cos x + 2\{x . \int \cos x\,dx - \int [(x)' . \int \cos x \; dx]\,dx\}$

$= -x^2 \; \cos x + 2\{x . \sin x - \int [1 . \sin x\,]\}$

$= -x^2 \; \cos x + 2x . \sin x - 2\int \sin x \; dx$

$= -x^2 \; \cos x + 2x . \sin x - 2(-\cos x) + c$

$= -x^2 \; \cos x + 2x \; \sin x + 2\cos x + c$

ix) $\int x^2 \tan^{-1} x \, dx$

SOLUTION:

$\int x^2 . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\; , V = x^2$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x \, . \int x^2 \,dx - \int [(\tan^{-1} x)' . \int x^2 \,dx]\,dx$

$= \tan^{-1} x \, . \frac{x^3}{3} - \int \left[\frac{1}{1+x^2} . \frac{x^3}{3}\right] dx$

$= \frac{x^3}{3}\tan^{-1} x \; - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx \quad ,$

x

$1 + x^2 \overline{)x^3}$

$\pm x^3 \pm 1$

-1

$= \frac{x^3}{3}\tan^{-1} x \; - \frac{1}{3}\int \left(x - \frac{x}{1+x^2}\right) dx$

$= \frac{x^3}{3}\tan^{-1} x \; - \frac{1}{3}\int x\,dx + \frac{1}{3.2}\int \frac{2x}{1+x^2}\,dx$

$= \frac{x^3}{3}\tan^{-1} x \; - \frac{1}{3}\frac{x^2}{2} + \frac{1}{6}\ln|1 + x^2| + c$

$\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

x) $\int x \tan^{-1} x \, dx$

SOLUTION:

$\int x \, . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\; , V = x$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x \, . \int x\,dx - \int [(\tan^{-1} x)' . \int x\,dx]\,dx$

$= \tan^{-1} x \, . \frac{x^2}{2} \; - \int \left[\frac{1}{1+x^2} . \frac{x^2}{2}\right] dx$

$= \frac{x^2}{2}\tan^{-1} x \; - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$

1

$1 + x^2 \overline{)x^2}$

$\pm x^2 \pm 1$

-1

$= \frac{x^2}{2}\tan^{-1} x \; - \frac{1}{2}\int \left(\frac{1+x^2-1}{1+x^2}\right) dx$

$= \frac{x^2}{2}\tan^{-1} x \; - \frac{1}{2}\int \frac{1+x^2}{1+x^2}\,dx - \frac{1}{2}\int \frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1} x \; - \frac{1}{2}\int 1\,dx - \frac{1}{2}\tan^{-1} x$

$= \frac{x^2}{2}\tan^{-1} x \; - \frac{1}{2}x - \frac{1}{2}\tan^{-1} x + c$

$= \left(\frac{1}{2}\tan^{-1} x\right)(x^2 + 1) - \frac{1}{2}x + c$

xi) $\int x^3 \tan^{-1} x \, dx$

SOLUTION:

$\int x^3 . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\; , V = x^3$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x \, . \int x^3 \,dx - \int [(\tan^{-1} x)' . \int x^3 \,dx]\,dx$

$= \tan^{-1} x \, . \frac{x^4}{4} \; - \int \left[\frac{1}{1+x^2} . \frac{x^4}{4}\right] dx$

$= \frac{x^4}{4}\tan^{-1} x \; - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx$

$x^2 - 1$

$1 + x^2 \overline{)x^4}$

$\pm x^4 \pm x^2$

$-x^2$

$\mp x^2 \mp 1$

1

$= \frac{x^4}{4}\tan^{-1} x \; - \frac{1}{4}\int \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx$

$= \frac{x^4}{4}\tan^{-1} x \; - \frac{1}{4}\int x^2\,dx + \frac{1}{4}\int 1 - \frac{1}{4}\int \frac{1}{1+x^2}\,dx$

$= \frac{x^4}{4}\tan^{-1} x \; - \frac{1}{4}\frac{x^3}{3} + \frac{1}{4}x - \frac{1}{4}\tan^{-1} x + c$

$= \frac{1}{4}\left[x^4 \; \tan^{-1} x \; - \frac{x^3}{3} + x - \; \tan^{-1} x\right] + c$

$= \frac{1}{4}\left[(x^4 - 1) \; \tan^{-1} x \; - \frac{x^3}{3} + x\right] + c$

$= \frac{1}{4}\left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x\right] + c$

xii) $\int x^3 \cos x \, dx$

SOLUTION:

$\int x^3 \; \cos x \; dxn$

Here $U = x^3 \;\; , V = \cos x$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= x^3 . \int \cos x\,dx - \int [(x^3)' . \int \cos x \; dx]\,dx$

$= x^3 . (\sin x) - \int [3x^2 . (\sin x)] \; dx$

$= x^3 \; \sin x - 3\int [x^2 \; \sin x] \; dx$

Using $\int U.V = U \; \int V\,dx - \int [U' \int V\,dx]\,dx$

$= x^3 \; \sin x - 3\{x^2 . \int \sin x\,dx - \int [(x^2)' . \int \sin x \; dx]\,dx\}$

$= x^3 \; \sin x - 3\{x^2 . (-\cos x) - \int [2x\,(-\cos x)\,]\}$

$= x^3 \; \sin x + 3x^2 \; \cos x - 6\int \; x \cos x \; dx$

$= x^3 \; \sin x + 3x^2 \; \cos x - 6\{x \int \cos x \; dx - \int [(x)' \int \cos x\,dx]\,dx\}$

$= x^3 \; \sin x + 3x^2 \; \cos x - 6\{x \; \sin x - \int [1 . \sin x]\,dx\}$

$= x^3 \; \sin x + 3x^2 \; \cos x - 6\{x \; \sin x - \int \sin x \; dx\}$

$= x^3 \; \sin x + 3x^2 \; \cos x - 6\{x \; \sin x - (-\cos x)\} + c$

$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + c$
$= (x^3 - 6x)\sin x + (3x^2 - 6)\cos x + c$

$$\textbf{xiii)} \int \sin^{-1} x \, dx$$

SOLUTION:

$\int 1 . \sin^{-1} x \, dx$
Here $U = \sin^{-1} x \; , \; V = 1$
Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$
$= \sin^{-1} x . \int 1\,dx - \int \left[(\sin^{-1} x)' . \int 1\,dx\right] dx$
$= \sin^{-1} x . x - \int \left[\frac{1}{\sqrt{1-x^2}} . x\right] dx$ *skip*
$= x\sin^{-1} x - \frac{1}{-2}\int (1-x^2)^{-\frac{1}{2}}(-2x)\,dx$
$= x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$
$= x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= x\sin^{-1} x + \sqrt{1-x^2} + c$

$$\textbf{xiv)} \int x \sin^{-1} x \, dx$$

SOLUTION:

$\int x . \sin^{-1} x \, dx$
Here $U = \sin^{-1} x \; , \; V = x$
Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$
$= \sin^{-1} x . \int x\,dx - \int \left[(\sin^{-1} x)' . \int x\,dx\right] dx$
$= \sin^{-1} x . \frac{x^2}{2} - \int \left[\frac{1}{\sqrt{1-x^2}} . \frac{x^2}{2}\right] dx$
$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}\,dx + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx + c$
Using $\sqrt{a^2-x^2}dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left\{\frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}\right\} - \frac{1}{2}\sin^{-1} x + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x + c$
$= \frac{x^2}{2}\sin^{-1} x + \left(\frac{1}{4} - \frac{1}{2}\right)\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$
$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$

$\textbf{xv)} \int e^x \sin x \cos x \, dx$

SOLUTION:

Let $I = \int e^x \sin x \cos x \, dx$
Multiply and divide by 2
$I = \frac{1}{2}\int e^x 2\sin x \cos x \, dx$
$I = \frac{1}{2}\int e^x \sin 2x \, dx$
Here $U = \sin 2x \; , \; V = e^x$
Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$
$I = \sin 2x \int e^x \, dx - \int [(\sin 2x)' . \int e^x\,dx]\,dx$
$I = \sin 2x \; e^x - \int [\cos 2x . 2 . e^x]\,dx$
$I = \sin 2x \; e^x - 2\int \cos 2x \; e^x \, dx$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$
$I = \sin 2x \; e^x - 2\left\{\cos 2x \int e^x \, dx - \int \left[(\cos 2x)' . \int e^x\,dx\right] dx\right\}$
$I = \sin 2x \, e^x - 2\{\cos 2x \; e^x - \int (-\sin 2x)\; e^x \, dx\}$
$I = e^x \sin 2x - 2\cos 2x \; e^x - 2\int 2\sin x \cos x \; e^x \, dx$
$I = e^x \sin 2x - 2\cos 2x \; e^x - 4\int \sin x \cos x \; e^x \, dx$
Put $I = \int e^x \sin x \cos x \, dx$
$I = e^x \sin 2x - 2\cos 2x \; e^x - 4I$
$5I = e^x(\sin 2x - 2\cos 2x)$
$I = \frac{e^x}{5}(\sin 2x - 2\cos 2x)$

$$\textbf{xvi)} \int x \sin x \cos x \, dx$$

SOLUTION:

$\int x \sin x \cos x \, dx = \frac{1}{2}\int x . 2\sin x \cos x \, dx = \frac{1}{2}\int x \sin 2x \, dx$
Here $U = x \; , \; V = \sin 2x$
Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$
$= \frac{1}{2}[x . \int \sin 2x \, dx - \int [(x)' . \int \sin 2x \, dx]\,dx]$
$= \frac{1}{2}x . \left(-\frac{\cos 2x}{2}\right) - \frac{1}{2}\int \left[1\left(-\frac{\cos 2x}{2}\right)\right] dx =$
$-\frac{1}{4}x \cos 2x + \frac{1}{4}\int \cos 2x \, dx$
$= -\frac{1}{4}x\cos 2x + \frac{1}{4}\frac{\sin 2x}{2} + c = \frac{1}{4}\left[-x\cos 2x + \frac{\sin 2x}{2}\right] + c = \frac{1}{4}\left[-x\cos 2x + \frac{2\sin x \cos x}{2}\right] + c$
$= \frac{1}{4}[-x\cos 2x + \sin x \cos x] + c = \frac{1}{4}[\sin x \cos x - x\cos 2x] + c$

$$\textbf{xvii)} \int x \cos^2 x \, dx$$

SOLUTION:

$\int x \cos^2 x \, dx = \int x . \frac{1+\cos 2x}{2}\,dx$ ***As*** $\boldsymbol{\cos^2 x = \frac{1+\cos 2x}{2}}$
$= \frac{1}{2}\int x . (1+\cos 2x)\,dx = \frac{1}{2}\int (x + x\cos 2x)\,dx$
$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx = \frac{1}{2}\frac{x^2}{2} + \frac{1}{2}\int x\cos 2x\,dx$
Here $U = x \; , \; V = \cos 2x$
Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$
$= \frac{x^2}{4} + \frac{1}{2}[x . \int \cos 2x\,dx - \int [x' . \int \cos 2x\,dx]\,dx]$
$= \frac{x^2}{4} + \frac{1}{2}\left[x . \frac{\sin 2x}{2} - \int \left[1 . \frac{\sin 2x}{2}\right] dx\right]$
$= \frac{x^2}{4} + \frac{1}{2}x . \frac{\sin 2x}{2} - \frac{1}{4}\int \sin 2x\,dx$
$= \frac{x^2}{4} + \frac{x\sin 2x}{4} - \frac{1}{4} . \frac{-\cos 2x}{2}$
$= \frac{1}{4}\left(x^2 + x\sin 2x + \frac{1}{2}\cos 2x\right) + c$

$$\textbf{xviii)} \int x \sin^2 x \, dx$$

SOLUTION:

$\int x \sin^2 x \, dx = \int x . \frac{1-\cos 2x}{2}\,dx$ *As* $\sin^2 x = \frac{1-\cos 2x}{2}$
$= \frac{1}{2}\int x . (1-\cos 2x)\,dx = \frac{1}{2}\int (x - x\cos 2x)\,dx$
$= \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx = \frac{1}{2}\frac{x^2}{2} - \frac{1}{2}\int x\cos 2x\,dx$
Here $U = x \; , \; V = \cos 2x$
Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$
$= \frac{x^2}{4} - \frac{1}{2}[x . \int \cos 2x\,dx - \int [x' . \int \cos 2x\,dx]\,dx]$

$= \frac{x^2}{4} - \frac{1}{2}\left[x.\frac{\sin 2x}{2} - \int\left[1.\frac{\sin 2x}{2}\right]dx\right]$

$= \frac{x^2}{4} - \frac{1}{2}x.\frac{\sin 2x}{2} + \frac{1}{4}\int \sin 2x\, dx$

$= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4}\left(\frac{-\cos 2x}{2}\right) + c$

$= \frac{1}{4}\left(x^2 - x\sin 2x - \frac{1}{2}\cos 2x\right) + c$

xix) $\int(\ln x)^2\, dx$

SOLUTION:

$\int(\ln x)^2 . 1\, dx$

Here $U = (\ln x)^2 \;\; , V = 1$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= (\ln x)^2 . \int 1\,dx - \int[((\ln x)^2)'.\int 1\,dx]\,dx$

$= (\ln x)^2 . x - \int\left[2(\ln x)\frac{1}{x}.x\right]dx$

$= x(\ln x)^2 - 2\int \ln x\, dx$

$= x(\ln x)^2 - 2[\int(\ln x).1\, dx]$

Here $U = \ln x \;\; , V = 1$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x(\ln x)^2 - 2[\ln x . \int 1\,dx - \int[(\ln x)'.\int 1\,dx]\,dx]$

$= x(\ln x)^2 - 2\left[\ln x . x - \int\left[\frac{1}{x}.x\right]dx\right]$

$= x(\ln x)^2 - 2[\ln x . x - \int 1\,dx]$

$= x(\ln x)^2 - 2[x \ln x - x] + c$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

$= x\ln x\,(\ln x - 2) + 2x + c$

xx) $\int \ln(\tan x)\, \sec^2 x\, dx$

SOLUTION:

$\int \ln(\tan x)\, \sec^2 x\, dx$

Here $U = \ln(\tan x) \;\; , V = \sec^2 x$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln(\tan x) \; .\int \sec^2 x\, dx -$

$\int[[\ln(\tan x)]'.\int \sec^2 x\, dx]\,dx$

$= \ln(\tan x) . \tan x - \int\left[\frac{\sec^2 x}{\tan x}.\tan x\right]dx$

$= \tan x . \ln(\tan x) - \int \sec^2 x\, dx$

$= \tan x . \ln(\tan x) - \tan x + c$

xxi) $\int \frac{x .\, \sin^{-1} x}{\sqrt{1-x^2}}\, dx$

SOLUTION:

$\int \frac{x .\, \sin^{-1} x}{\sqrt{1-x^2}}\, dx = \frac{1}{-2}\int \sin^{-1} x\left[(1-x^2)^{-\frac{1}{2}}(-2x)\right]$

Here $U = \sin^{-1} x \;\; , V = (1-x^2)^{-\frac{1}{2}}(-2x)$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= -\frac{1}{2}\{\sin^{-1} x \int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx -$

$\int\left[(\sin^{-1} x)'.\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx\right]dx\}$

$= -\frac{1}{2}\left\{\sin^{-1} x \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int\left[\frac{1}{\sqrt{1-x^2}}.\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]dx\right\} =$

$-\frac{1}{2}\left\{\sin^{-1} x \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \int\left[\frac{1}{\sqrt{1-x^2}}.\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]dx\right\}$

$= -\frac{1}{2}\{2\,\sin^{-1} x\sqrt{1-x^2} - \int[2]\,dx\} =$

$-\frac{1}{2}\{2\,\sin^{-1} x\sqrt{1-x^2} - 2\int 1\,dx\} =$

$-\sin^{-1} x\sqrt{1-x^2} + x + c$

$= x - \sqrt{1-x^2}\,\sin^{-1} x + c$

Q. 2: ***Evaluate the following integrals***:

i) $\int \tan^4 x\, dx$

SOLUTION:

$\int \tan^4 x\, dx$

$= \int \tan^2 x . \tan^2 x\, dx$

$= \int \tan^2 x\, (\sec^2 x - 1)\,dx$

$= \int \tan^2 x \sec^2 x\,dx - \int(\sec^2 x - 1)\, dx$

$= \frac{\tan^3 x}{3} - \int \sec^2 x\,dx + \int 1\, dx$

$= \frac{\tan^3 x}{3} - \tan x + x + c$

ii) $\int \sec^4 x\, dx$

SOLUTION:

$\int \sec^4 x\, dx$

$= \int \sec^2 x . \sec^2 x\, dx$

$= \int \sec^2 x\, (1 + \tan^2 x)\,dx$

$= \int \sec^2 x\, dx + \int \sec^2 x \tan^2 x\, dx$

$= \tan x + \frac{\tan^3 x}{3} + c$

$= \tan x + \frac{1}{3}\tan^3 x + c$

iv) $\int \tan^3 x \sec x\, dx$

SOLUTION:

$\int \tan^3 x \sec x\, dx$

$\int \tan^2 x\, \tan x \sec x\, dx$

$= \int(\sec^2 x - 1)\, \tan x \sec x\, dx$

$= \int \sec^2 x \sec x \tan x\, dx - \int \sec x \tan x\, dx$

$= \frac{1}{3}\sec^3 x - \sec x + c$

v) $\int x^3 e^{5x}\, dx$

SOLUTION:

$\int x^3 e^{5x}\, dx$

Here $U = x^3 \;\; , V = e^{5x}$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x^3 \int e^{5x}\,dx - \int[(x^3)'.\int e^{5x}\,dx]\,dx$

$= x^3\frac{e^{5x}}{5} - \int 3x^2\frac{e^{5x}}{5}\, dx$

$= x^3\frac{e^{5x}}{5} - \frac{3}{5}\{\int x^2 e^{5x}\, dx\}$

Again integrating by parts

$= x^3\frac{e^{5x}}{5} - \frac{3}{5}\{x^2 \int e^{5x}\,dx - \int[(x^2)'.\int e^{5x}\,dx]\,dx\}$

$= x^3\frac{e^{5x}}{5} - \frac{3}{5}\left\{x^2\frac{e^{5x}}{5} - \int 2x\frac{e^{5x}}{5}\, dx\right\}$

$= x^3\frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\int x\, e^{5x}\, dx$

Again integrating by parts

$= x^3\frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\{x \int e^{5x}\,dx -$

$\int[(x)'.\int e^{5x}\,dx]\,dx\}$

$= x^3\frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\left\{x\frac{e^{5x}}{5} - \int 1.\frac{e^{5x}}{5}\, dx\right\}$

$= x^3\frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x\, e^{5x} - \frac{6}{125}\int e^{5x}\, dx$

$= x^3\frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x\, e^{5x} - \frac{6}{125}\frac{e^{5x}}{5} + c$

$= \frac{e^{5x}}{5}\left(x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125}\right) + c$

vi) $\int e^{-x} \sin 2x \, dx$

***SOLUTION*:**

Let $I = \int \sin 2x \, e^{-x} \, dx$

Here $U = \sin 2x$, $V = e^{-x}$

Using $\int U.V = U.\int V\,dx - \int [U'.\int V\,dx]\,dx$

$I = \sin 2x \int e^{-x}\,dx - \int [(\sin 2x)' \int e^{-x}\,dx]\,dx$

$I = \sin 2x \frac{e^{-x}}{-1} - \int \left[(\cos 2x \,.\, 2)\frac{e^{-x}}{-1}\right] dx$

$I = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x \, dx$

Again integrating by parts

$I = -e^{-x}\sin 2x + 2\{\cos 2x \int e^{-x}\,dx - \int [(\cos 2x)' \int e^{-x}\,dx]\,dx\}$

$I = -e^{-x}\sin 2x + 2\left\{\cos 2x \frac{e^{-x}}{-1} - \int \left[(-\sin 2x.\,2)\frac{e^{-x}}{-1}\right] dx\right\}$

$I = -e^{-x}\sin 2x - 2\cos 2x \; e^{-x} - 4\int e^{-x}\sin 2x \, dx$

$I = -e^{-x}\sin 2x - 2\cos 2x \; e^{-x} - 4I + c_1$

$5I = -e^{-x}\sin 2x - 2\cos 2x \; e^{-x} + c_1$

$I = -\frac{2}{5}\cos 2x \; e^{-x} - \frac{1}{5}e^{-x}\sin 2x + \frac{c_1}{5}$

$I = -\frac{2}{5}e^{-x}(\cos 2x + \frac{1}{2}e^{-x}\sin 2x) + c$ *where* $c = \frac{c_1}{5}$

vii) $\int e^{2x} \cos 3x \, dx$

***SOLUTION*:**

Let $I = \int e^{2x} \cos 3x \, dx$

Here $U = \cos 3x$, $V = e^{2x}$

Using $\int U.V = U.\int V\,dx - \int [U'.\int V\,dx]\,dx$

$I = \cos 3x \int e^{2x}\,dx - \int [(\cos 3x)' \int e^{2x}\,dx]\,dx$

$I = \cos 3x \frac{e^{2x}}{2} - \int \left[(-\sin 3x \,.\, 3)\frac{e^{2x}}{2}\right] dx$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2}\int [\sin 3x \, e^{2x}]\,dx$

Again integrating by parts

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2}\{\sin 3x \int e^{2x}\,dx - \int [(\sin 3x)' \int e^{2x}\,dx]\,dx\}$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2}\left\{\sin 3x \frac{e^{2x}}{2} - \int \left[\cos 3x.\,3\,.\frac{e^{2x}}{2}\right] dx\right\}$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x \, e^{2x} - \frac{9}{4}\int \cos 3x \; e^{2x} \, dx$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x \, e^{2x} - \frac{9}{4}I + c_1$

$I + \frac{9}{4}I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x \, e^{2x} + c_1$

$\frac{13}{4}I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x \, e^{2x} + c_1$

$I = \frac{4}{13}\frac{e^{2x}}{2}(\cos 3x + \frac{3}{2}\sin 3x) + \frac{4}{13}c_1$

$I = \frac{2}{13}e^{2x}\left(\cos 3x + \frac{3}{2}\sin 3x\right) + c$ *where* $\frac{4}{13}c_1 = c$

$I = \frac{3}{13}e^{2x}(\sin 3x + \frac{2}{3}\cos 3x\,) + c$

viii) $\int \operatorname{cosec}^3 x \, dx$

***SOLUTION*:**

Let $I = \int \operatorname{cosec}^2 x \operatorname{cosec} x \, dx$

Here $U = \operatorname{cosec} x$, $V = \operatorname{cosec}^2 x$

Using $\int U.V = U.\int V\,dx - \int [U'.\int V\,dx]\,dx$

$= \operatorname{cosec} x \int \operatorname{cosec}^2 x \; dx - \int [(\operatorname{cosec} x)'.\int \operatorname{cosec}^2 x \, dx]\,dx$

$I = \operatorname{cosec} x \; (-\cot x) - \int [(-\operatorname{cosec} x \cot x)(-\cot x)]\,dx$

$I = -\operatorname{cosec} x \; \cot x - \int \operatorname{cosec} x \cot^2 x \; dx$

$I = -\operatorname{cosec} x \; \cot x - \int \operatorname{cosec} x \, (\operatorname{cosec}^2 x - 1)\,dx$

$I = -\operatorname{cosec} x \; \cot x - \int \operatorname{cosec}^3 x\,dx + \int \operatorname{cosec} x\,dx$

$I = -\operatorname{cosec} x \; \cot x - I + \int \operatorname{cosec} x\,dx$

$2I = -\operatorname{cosec} x \; \cot x + \ln|\operatorname{cosec} x - \cot x| + c_1$

$I = -\frac{1}{2}[\cot x \operatorname{cosec} x - \ln|\operatorname{cosec} x - \cot x|] + \frac{1}{2}c_1$

$I = -\frac{1}{2}[\cot x \operatorname{cosec} x - \ln|\operatorname{cosec} x - \cot x|] + c$

***TIT BIT*:**

Jab pure quadratic equation h aur us ka derivative b majood na h t substituition s solve karty h aur substitution m trigonometry functions hi let karty lakin j c s start hu w let nai karny nai t book answer ni aye ga ut jin pure quadratic equation walay questions ki power $\frac{1}{2}$ h t un k ap by parts integration k method s b kar saktay h.

Q.3: Show that $\int e^{ax} \sin bx \, dx$

$$= \frac{1}{\sqrt{a^2+b^2}} e^{ax} \sin\left(bx - \tan^{-1}\left(\frac{b}{a}\right)\right) + c$$

***SOLUTION*:**

Let $I = \int e^{ax} \sin bx \, dx$

Here $U = \sin bx$, $V = e^{ax}$

Using $\int U.V = U.\int V\,dx - \int [U'.\int V\,dx]\,dx$

$I = \sin bx \int e^{ax}\,dx - \int [(\sin bx)'.\int e^{ax}\,dx]\,dx$

$I = \sin bx \frac{e^{ax}}{a} - \int \left[\cos bx \;.\; b.\frac{e^{ax}}{a}\right] dx$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a}\int \cos bx \; e^{ax}\,dx$

Again integrating by parts

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a}\{\cos bx \int e^{ax}\,dx - \int [(\cos bx)'.\int e^{ax}\,dx]\,dx\}$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a}\left\{\cos bx \frac{e^{ax}}{a} - \int \left[-\sin bx \,.\, b.\frac{e^{ax}}{a}\right] dx\right\}$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx \, e^{ax} - \frac{b^2}{a^2}\int \sin bx \; e^{ax}\,dx$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx \, e^{ax} - \frac{b^2}{a^2}I + c_1$

$I + \frac{b^2}{a^2}I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx \, e^{ax} + c_1$

$\left(\frac{a^2+b^2}{a^2}\right)I = e^{ax}\left(\sin bx \frac{1}{a} - \frac{b}{a^2}\cos bx\right) + c_1$

$I = \frac{a^2}{a^2+b^2}e^{ax}\left(\sin bx \frac{1}{a} - \frac{b}{a^2}\cos bx\right) + \frac{a^2}{a^2+b^2}c_1$

$I = \frac{1}{a^2+b^2}e^{ax}(a \; \sin bx - b\cos bx) + c$ *(A)* *where* $\frac{a^2}{a^2+b^2}c_1 = c$

Let $a = r\cos\theta$ (1) , $b = r\sin\theta$ (2)

Squaring and adding (1) and (2)

$a^2 + b^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$a^2 + b^2 = r^2(\cos^2\theta + \sin^2\theta)$

$a^2 + b^2 = r^2 \quad \Rightarrow \quad r = \sqrt{a^2+b^2}$

dividing (1) and (2)

$\frac{r\sin\theta}{r\cos\theta} = \frac{b}{a}$

$\tan\theta = \frac{b}{a}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Now Put values in (A)

$I = \frac{1}{a^2+b^2}e^{ax}(r\cos\theta \;\; \sin bx - r\sin\theta\cos bx) + c$

$I = \frac{r}{a^2+b^2}e^{ax}(\cos\theta \;\; \sin bx - \sin\theta\cos bx) + c$ *(Take r common)*

$I = \frac{\sqrt{a^2+b^2}}{a^2+b^2}e^{ax}(\sin bx\cos\theta - \cos bx\sin\theta) + c$ *(Put value r)*

$I = \frac{1}{\sqrt{a^2+b^2}}e^{ax}(\sin(bx-\theta))c$

Using $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$I = \frac{1}{\sqrt{a^2+b^2}} e^{ax} \sin(bx - \tan^{-1}\left(\frac{b}{a}\right) + c$

Put $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ have proved.

Q.4: Evaluate the following indefinate integrals:

i) $\int \sqrt{a^2 - x^2}\, dx$

SOLUTION:

Let $I = \int \sqrt{a^2 - x^2} . 1\, dx$

Here $U = \sqrt{a^2 - x^2}$, $V = 1$

Using $\int U.V = U . \int V\, dx - \int [U'. \int V\, dx]\, dx$

$I = \sqrt{a^2 - x^2} \int 1\, dx - \int \left[\left(\sqrt{a^2 - x^2} \right)' . \int 1\, dx \right] dx$

$I = \sqrt{a^2 - x^2} . x - \int \left[\frac{-2x}{2\sqrt{a^2-x^2}} . x \right] dx$

$I = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\, dx$

$I = x\sqrt{a^2 - x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\, dx$

$I = x\sqrt{a^2 - x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\, dx + \int \frac{a^2}{\sqrt{a^2-x^2}}\, dx$

$I = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{1}{\sqrt{a^2-x^2}}\, dx$

Using $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + c$

$I = x\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\frac{x}{a} + c_1$

$I + I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} + c_1$

$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} + c_1$

$I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$ where $\frac{c_1}{2} = c$

ii) $\int \sqrt{x^2 - a^2}\, dx$

SOLUTION:

Let $I = \int \sqrt{x^2 - a^2} . 1\, dx$

Here $U = \sqrt{x^2 - a^2}$, $V = 1$

Using $\int U.V = U . \int V\, dx - \int [U'. \int V\, dx]\, dx$

$I = \sqrt{x^2 - a^2} \int 1\, dx - \int \left[\left(\sqrt{x^2 - a^2} \right)' . \int 1\, dx \right] dx$

$I = \sqrt{x^2 - a^2} . x - \int \left[\frac{2x}{2\sqrt{x^2-a^2}} . x \right] dx$

$I = x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\, dx$

$I = x\sqrt{x^2 - a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\, dx$

$I = x\sqrt{x^2 - a^2} - \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}\, dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\, dx$

$I = x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{1}{\sqrt{x^2-a^2}}\, dx$

Using $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = ln\left|x + \sqrt{x^2 - a^2}\right| + c$

$I = x\sqrt{x^2 - a^2} - I - a^2 ln\left|x + \sqrt{x^2 - a^2}\right| + c_1$

$I + I = x\sqrt{x^2 - a^2} - a^2 ln\left|x + \sqrt{x^2 - a^2}\right| + c_1$

$2I = x\sqrt{x^2 - a^2} - a^2 ln\left|x + \sqrt{x^2 - a^2}\right| + c_1$

$I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} ln\left|x + \sqrt{x^2 - a^2}\right| + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} ln\left|x + \sqrt{x^2 - a^2}\right| + c$

where $\frac{c_1}{2} = c$

iii) $\int \sqrt{4 - 5x^2}\, dx$

SOLUTION:

Let $I = \int \sqrt{4 - 5x^2} . 1\, dx$

Here $U = \sqrt{4 - 5x^2}$, $V = 1$

Using $\int U.V = U . \int V\, dx - \int [U'. \int V\, dx]\, dx$

$I = \sqrt{4 - 5x^2} \int 1\, dx - \int \left[\left(\sqrt{4 - 5x^2} \right)' . \int 1\, dx \right] dx$

$I = \sqrt{4 - 5x^2} . x - \int \left[\frac{-10x}{2\sqrt{4-5x^2}} . x \right] dx$

$I = x\sqrt{4 - 5x^2} - \int \frac{-5x^2}{\sqrt{4-5x^2}}\, dx$

$I = x\sqrt{4 - 5x^2} - \int \frac{4-5x^2-4}{\sqrt{4-5x^2}}\, dx$

$I = x\sqrt{4 - 5x^2} - \int \frac{4-5x^2}{\sqrt{4-5x^2}}\, dx + \int \frac{4}{\sqrt{4-5x^2}}\, dx$

$I = x\sqrt{4 - 5x^2} - \int \sqrt{4 - 5x^2}\, dx + 4 \int \frac{1}{\sqrt{5\left(\frac{4}{5} - x^2\right)}}\, dx$

$I = x\sqrt{4 - 5x^2} - \int \sqrt{4 - 5x^2}\, dx + \frac{4}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - x^2}}\, dx$

Using $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + c$

$I = x\sqrt{4 - 5x^2} - I + \frac{4}{\sqrt{5}} \sin^{-1}\left(\frac{x}{\frac{2}{\sqrt{5}}}\right) + c_1$

$2I = x\sqrt{4 - 5x^2} + \frac{4}{\sqrt{5}} \sin^{-1}\left(\frac{\sqrt{5}x}{2}\right) + c_1$

$I = \frac{x}{2}\sqrt{4 - 5x^2} + \frac{2}{\sqrt{5}} \sin^{-1}\left(\frac{\sqrt{5}x}{2}\right) + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{4 - 5x^2} + \frac{2}{\sqrt{5}} \sin^{-1}\left(\frac{\sqrt{5}x}{2}\right) + c$ where $\frac{c_1}{2} = c$

iv) $\int \sqrt{3 - 4x^2}\, dx$

SOLUTION:

Let $I = \int \sqrt{3 - 4x^2} . 1\, dx$

Here $U = \sqrt{3 - 4x^2}$, $V = 1$

Using $\int U.V = U . \int V\, dx - \int [U'. \int V\, dx]\, dx$

$I = \sqrt{3 - 4x^2} \int 1\, dx - \int \left[\left(\sqrt{3 - 4x^2} \right)' . \int 1\, dx \right] dx$

$I = \sqrt{3 - 4x^2} . x - \int \left[\frac{-8x}{2\sqrt{3-4x^2}} . x \right] dx$

$I = x\sqrt{3 - 4x^2} - \int \frac{-4x^2}{\sqrt{3-4x^2}}\, dx$

$I = x\sqrt{3 - 4x^2} - \int \frac{3-4x^2-3}{\sqrt{3-4x^2}}\, dx$

$I = x\sqrt{3 - 4x^2} - \int \frac{3-4x^2}{\sqrt{3-4x^2}}\, dx + \int \frac{3}{\sqrt{3-4x^2}}\, dx$

$I = x\sqrt{3 - 4x^2} - \int \sqrt{3 - 4x^2}\, dx + 3 \int \frac{1}{\sqrt{4\left(\frac{3}{4} - x^2\right)}}\, dx$

$I = x\sqrt{3 - 4x^2} - \int \sqrt{3 - 4x^2}\, dx + \frac{3}{2} \int \frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - x^2}}\, dx$

Using $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + c$

$I = x\sqrt{3 - 4x^2} - I + \frac{3}{2} \sin^{-1}\left(\frac{x}{\frac{\sqrt{3}}{2}}\right) + c_1$

$2I = x\sqrt{3 - 4x^2} + \frac{3}{2} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c_1$

$I = \frac{x}{2}\sqrt{3 - 4x^2} + \frac{3}{4} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{3 - 4x^2} + \frac{3}{4} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c$ where $\frac{c_1}{2} =$

v) $\int \sqrt{x^2 + 4}\, dx$

SOLUTION:

Let $I = \int \sqrt{x^2 + 4} . 1\, dx$

Here $U=\sqrt{x^2+4}$, $V=1$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I=\sqrt{x^2+4}\int 1\,dx - \int\left[\left(\sqrt{x^2+4}\right)'.\int 1\,dx\right]dx$

$I=\sqrt{x^2+4}.x - \int\left[\frac{2x}{2\sqrt{x^2+4}}.x\right]dx$

$I = x\sqrt{x^2+4} - \int\frac{x^2}{\sqrt{x^2+4}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\frac{x^2+4-4}{\sqrt{x^2+4}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\frac{x^2+4}{\sqrt{x^2+4}}\,dx + \int\frac{4}{\sqrt{\sqrt{x^2+4}}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\ \sqrt{x^2+4}\,dx + 4\int\frac{1}{\sqrt{x^2+4}}\,dx$

Using $\int\frac{1}{\sqrt{x^2+a^2}}\,dx = ln\left|x+\sqrt{x^2+a^2}\right|+c$

$I = x\sqrt{x^2+4} - I + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$I + I = x\sqrt{x^2+4} + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$2I = x\sqrt{x^2+4} + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$I = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}ln\left|x+\sqrt{x^2+2^2}\right| + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{x^2+4} + 2\ ln\left|x+\sqrt{x^2+2^2}\right| + c$ where $\frac{c_1}{2} = c$

vi) $\int \mathbf{x^2\, e^{ax}\, dx}$

SOLUTION:

$\int x^2\, e^{ax}\, dx$

Here $U = x^2$, $V = e^{ax}$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x^2\int e^{ax}\,dx - \int[(x^2)'.\int e^{ax}\,dx]\,dx$

$= x^2\,\frac{e^{ax}}{a} - \int 2x\,\frac{e^{ax}}{a}\ dx$

$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a}\{\int x\ e^{ax}\ dx\}$

Again integrating by parts

$$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a}\left\{x\,\frac{e^{ax}}{a} - \int 1.\,\frac{e^{ax}}{a}dx\right\}$$

$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a^2}x\,e^{ax} + \frac{2}{a^2}\int\ e^{ax}\ dx$

$= x^2\,\frac{e^{ax}}{a} - \frac{2}{a^2}x\,e^{ax} + \frac{2}{a^2}\,\frac{e^{ax}}{a} + c$

Take common $\frac{e^{ax}}{a}$

$= \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + c$

Q. 5: *Evaluate the following integrals*:

i) $\int \mathbf{e^x\left(\frac{1}{x} + \ln x\right)dx}$

SOLUTION:

$= \int e^{1.x}\left(1.\ln x + \frac{1}{x}\right)dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{1.x}\ \ln x + c$

$= e^x\ \ln x + c$

ii) $\int \mathbf{e^x(\cos x + \sin x)dx}$

SOLUTION:

$= \int e^{1\,.\,x}(1.\sin x + \cos x)dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{1.x}\sin x + c$

$= e^x\sin x + c$

iii) $\int \mathbf{e^{ax}\left(a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right)dx}$

SOLUTION:

$\int e^{ax}\left(a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right)dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{ax}\sec^{-1}x + c$

iv) $\int \mathbf{e^{3x}\left(\frac{3\sin x - \cos x}{\sin^2 x}\right)dx}$

SOLUTION:

$= \int e^{3x}\left(\frac{3\sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx$

$= \int e^{3x}\left(\frac{3}{\sin x} - \frac{\cos x}{\sin x.\sin x}\right)dx$

$= \int e^{3x}(3\,\mathrm{cosec}\,x - \cot x\,\mathrm{cosec}\,x)dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{3x}\,\mathrm{cosec}\,x + c$

v) $\int \mathbf{e^{2x}(-\sin x + 2\cos x)dx}$

SOLUTION:

$= \int e^{2x}(2\cos x - \sin x)dx$

$= \int e^{2x}(2\cos x + (-\sin x))dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{2x}\cos x + c$

vi) $\int \mathbf{\frac{x\,e^x}{(1+x)^2}dx}$

SOLUTION:

$= \int e^x\left[\frac{1+x-1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right]dx$

$= e^x.\frac{1}{1+x} + c$

vii) $\int \mathbf{e^{-x}(\cos x - \sin x)dx}$

SOLUTION:

$= \int e^{-x}(-\sin x + \cos x)dx$

$= \int e^{-1\,.\,x}(-1.\sin x + \cos x)dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{-1.x}\sin x + c$

$= e^{-x}\sin x + c$

viii) $\int \mathbf{\frac{e^{m\tan^{-1}x}}{1+x^2}dx}$

SOLUTION:

$= \int e^{m\tan^{-1}x}\,\frac{1}{1+x^2}dx$

Put $y = \tan^{-1}x$

$dy = \frac{1}{1+x^2}\ dx$

$= \int e^{my}dy = \frac{e^{my}}{m} + c = \frac{e^{m\tan^{-1}x}}{m} + c$ Put $y = \tan^{-1}x$

ix) $\int \mathbf{\frac{2x}{1-\sin x}dx}$

SOLUTION:

$\int\frac{2x}{1-\sin x}\times\frac{1+\sin x}{1+\sin x}\ dx = \int\frac{2x(1+\sin x)}{1-\sin^2 x}dx = \int\frac{2x(1-\sin x)}{\cos^2 x}dx = \int 2x\left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x\,\cos x}\right)dx$

$= \int 2x(\sec^2 x + \tan x\sec x)dx = \int 2x\ \sec^2 x\ dx - \int 2x\ \tan x\sec x\ dx$

Here $U = 2x$, $V = \sec^2 x$ and $U = 2x$, $V = \tan x\sec x$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= [2x\ \tan x - \int 2(1)\tan x\ dx] + [2x\ \sec x - \int 2(1)\sec x\ dx]$

$= 2x\,.\tan x - 2\ln|\sec x| + 2x\,.\sec x - 2\,ln|\sec x + \tan x| + c$

x) $\int \frac{e^x(1+x)}{(2+x)^2}dx$

SOLUTION:

$= \int e^x\left[\frac{2-1+x}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{(2+x)-1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{2+x} - \frac{1}{(2+x)^2}\right]dx$

$\int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^x.\frac{1}{2+x} + c$

xi) $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x dx$

SOLUTION:

$= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$

$= \int e^x\left(-\cot\frac{x}{2} + \frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx$

$\int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^x\left(-\cot\frac{x}{2}\right) + c = -e^x\cot\frac{x}{2} + c$

Integration involving Partial Fraction

If $P(x), Q(x)$ *are two polynomial function and* $Q(x) \neq 0$

In rational fraction $\frac{P(x)}{Q(x)}$ *can be factorized into linear and* Quadratic (irreducible) factors then the rational function is written as as a sum of simpler rational functions, each of which can be integrated by methods already known.

Here we discuss examples of the three cases of partial fraction and then apply integrated.

Case1.

when $Q(x)$ *contain non − repeated linear factors. e. g;*

$$\frac{P(x)}{(x-a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Or $\frac{-x+6}{(x-2)(x-3)(x-4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x-4}$ *e.tc*

Case2.

when $Q(x)$ *contain non repeated and repeates linear factors.*

$$\frac{P(x)}{(x-a)(x+b)^2} = \frac{A}{x-a} + \frac{B}{x+b} + \frac{C}{(x+b)^2}$$

$$\frac{2x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\ e.t.c$$

Case3.

When $Q(x)$ *contain non repeated irreducible quadratic factors.*

$$\frac{P(x)}{(x+b)(x^2+c)} = \frac{A}{x+b} + \frac{Bx+C}{x^2+c}$$

$$\frac{1}{(x-1)(x^2+1+2x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1+2x}$$

Exercise 3.5

Evaluate the following integrals.

Q1. $\int \frac{3x+1}{x^2-x-6}dx$

Solution: $\int \frac{3x+1}{x^2-x-6}dx$

Now

$\frac{3x+1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$

⇨ $3x + 1 = A(x+2) + B(x-3) \to (i)$

$Put\ x - 3 = 0 \Rightarrow x = 3\ in(i)$

$3(3) + 1 = A(3+2) + B(0) \Rightarrow 5A = 10 \Rightarrow A = 2$

$Put x + 2 = 0 \Rightarrow x = -2\ in(i)$

$3(-2) + 1 = A(0) + B(-2-3) \Rightarrow -5B = -6 + 1$

$\Rightarrow -5B = -5 \Rightarrow B = 1$

$so\ \frac{3x+1}{x^2-x-6} = \frac{2}{x-3} + \frac{1}{x+2}$

$\Rightarrow \int \frac{3x+1}{x^2-x-6}dx = 2\int \frac{1}{x-3} + \int \frac{1}{x+2}dx$

$= 2\ln|x-3| + \ln|x+2| + c$

Q2. $\int \frac{5x+8}{(x+3)(2x-1)}dx$

Solution: $\int \frac{5x+8}{(x+3)(2x-1)}dx$

Now.

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$$

⇨ $5x + 8 = A(2x-1) + B(x+3) \to (i)$

$Put\ 2x - 1 = 0 \Rightarrow x = \frac{1}{2}\ in(i)$

⇨ $5\left(\frac{1}{2}\right) + 8 = A(0) + B\left(\frac{1}{2} + 3\right)$

⇨ $\frac{5+16}{2} = B\left(\frac{1+6}{2}\right) \Rightarrow 7B = 21 \Rightarrow B = 3$

Put $x + 3 = 0 \Rightarrow x = -3\ in\ (i)$

⇨ $5(-3) + 8 = A(2(-3) - 1) + B(0)$

⇨ $-15 + 8 = -7A \Rightarrow -7 = -7A \Rightarrow A = 1$

So $\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$

$$\int \frac{5x+8}{(x+3)(2x-1)}dx = \int \frac{1}{x+3}dx + 3\int \frac{1}{2x-1}dx$$

$$= \ln|x+3| + \frac{3}{2}\int \frac{2}{2x-1}dx$$

$$\int \frac{5x+8}{(x+3)(2x-1)}dx = \ln|x+3| + \frac{3}{2}\ln|2x-1| + c$$

Q3. $\int \frac{x^2+3x-34}{x^2+2x-15}dx$

Solution: $\int \frac{x^2+3x-34}{x^2+2x-15}dx$

So $\int \left(1+\frac{x-19}{x^2+2x-15}\right)dx$

$= \int 1dx + \int \frac{x-19}{x^2+2x-15}$

$$\begin{array}{r} 1 \\ x^2+2x-15\overline{)x^2+3x-34} \\ \pm x^2 \pm 2x \mp 15 \\ \hline x-19 \end{array}$$

$Now\ \frac{x-19}{x^2+2x-15} = \frac{A}{x-3}+\frac{B}{x+5} \to (i)$

$\Rightarrow x-19 = A(x+5)+B(x-3) \to (ii)$

$\because x^2+2x-15$
$= x^2+5x-3x-15$
$x(x+5)-3(x+5)$
$(x-3)(x+5)$

$put\ x-3=0 \Rightarrow x=3\ in(ii)$

$\Rightarrow 3-19 = A(3+5)+B(0)$

$\Rightarrow -16 = 8A \Rightarrow A=-2$

$put\ x+5=0 \Rightarrow x=-5\ in(ii)$

$\Rightarrow -5-19 = A(0)+B(-5-3) \Rightarrow -24=-8B$

$\Rightarrow B=3$

$$(i) \Rightarrow \frac{x-19}{x^2+2x-15} = -\frac{2}{x-3}+\frac{3}{x+5}$$

Thus,

$. \int \frac{x^2+3x-34}{x^2+2x-15}dx = \int 1dx + \int \frac{-2}{x-3}dx + \int \frac{3}{x+5}dx$

$$= x - 2\ln|x-3| + 3\ln|x+5| + c$$

Q4. $\int \frac{(a-b)x}{(x-a)(x-b)}dx, (a>b)$

Solution: $\int \frac{(a-b)x}{(x-a)(x-b)}dx$

Now

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a}+\frac{B}{x-b}$$

$$\Rightarrow (a-b)x = A(x-b)+B(x-a) \to (i)$$

$$Put\ x-a=0 \Rightarrow x=a\ in\ (i)$$

$$\Rightarrow (a-b).a = A(a-b)+B(a-a)$$

$$(a-b).b = A(0)+B(a-b) \Rightarrow A=a$$

$$Put\ x-b=0 \Rightarrow x=b\ in\ (i)$$

$$\Rightarrow (a-b).b = A(0)+B(b-a)$$

$$(a-b).b = -B(a-b)$$

$$b=-B$$

$$B=-b$$

Thus

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a}+\frac{-b}{x-b}$$

$$\int \frac{(a-b)x}{(x-a)(x-b)}dx = \int \frac{a}{x-a}dx - \int \frac{b}{x-b}dx$$

$$= a\ln|x-a| - b\ln|x-b| + c$$

Q5. $\int \frac{3-x}{1-x-6x^2}dx$

Solution: $\int \frac{3-x}{1-x-6x^2}dx$

$\because 1-x-6x^2$
$= -6x^2-x+1$
$-3x(2x+1)+1(2x+1)$
$(2x+1)(1-3x)$

Now

$\frac{3-x}{1-x-6x^2} = \frac{A}{2x+1}+\frac{B}{1-3x}$

$\Rightarrow 3-x = A(1-3x)+B(2x+1) \to (i)$

$$Put\ 2x+1=0 \Rightarrow x=-\frac{1}{2}in\ (i)$$

$\Rightarrow 3-\left(-\frac{1}{2}\right) = A\left(1-3\left(-\frac{1}{2}\right)+B(0)\right)$

$\Rightarrow 3+\frac{1}{2} = A\left(1+\frac{3}{2}\right) \Rightarrow \frac{7}{2} = A\left(\frac{5}{2}\right)$

$\Rightarrow A = \frac{7}{2}$

$Put\ 1-3x=0 \Rightarrow 1=3x \Rightarrow x=\frac{1}{3}in\ (i)$

$$\Rightarrow 3-\frac{1}{3} = A(0)+B\left(2\left(\frac{1}{3}\right)+1\right)$$

$$\frac{9-1}{3} = B\left(\frac{2+3}{3}\right) \Rightarrow 8=5B \Rightarrow \frac{8}{5}$$

So

$$\frac{3-x}{1-x-6x^2} = \frac{7/5}{2x+1}+\frac{8/5}{1-3x}$$

$: \int \frac{3-x}{1-x-6x^2}dx = \frac{7}{5}\int \frac{1}{2x+1}dx + \frac{8}{5}\int \frac{1}{1-3x}dx$

$$= \frac{7}{10}\int \frac{2}{2x+1}dx - \frac{8}{15}\int \frac{-3}{1-3x}dx$$

$$= \frac{7}{10}\ln|2x+1| - \frac{8}{5}\ln|1-3x| + C$$

Q.6 $\int \frac{2x}{x^2-a^2}dx$

Solution: $\int \frac{2x}{x^2-a^2}dx$

Now

$\frac{2x}{x^2-a^2} = \frac{A}{x-a}+\frac{B}{x+a}$ $\quad \because x^2-a^2 = (x-a)(x+a)$

$$\Rightarrow 2x = A(x+a)+B(x-a) \to (i)$$

Put $x-a=0 \Rightarrow x=a\ in\ (i)$

$\Rightarrow 2a = A(a+a)+B(0) \Rightarrow 2a=2A \Rightarrow A=1$

Put $x+a=0 \Rightarrow x=-a\ in\ (i)$

$\Rightarrow 2(-a) = A(0)+B(-a-a) \Rightarrow -2a=-2aB$

$$\Rightarrow B=1$$

So $\frac{2x}{x^2-a^2} = \frac{1}{x-a}+\frac{1}{x+a}$

$$\int \frac{2x}{x^2-a^2}dx = \int \frac{1}{x-a}dx + \int \frac{1}{x+a}dx$$

$$= \ln|x-a| + \ln|x+a| + c$$

$$= \ln|(x-a)(x+a)| + c$$

$$= \ln|x^2-a^2| + c$$

Q.7 $\int \frac{1}{6x^2+5x-4}dx$

Solution: $\int \frac{1}{6x^2+5x-4}dx$

Now

$\because 6x^2+5x-6$
$= 6x^2-3x+8x-4$
$3x(2x-1)+4(2x-1)$
$(2x-1)(3x+4)$

$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1}+\frac{B}{3x+4}$

$\Rightarrow 1 = A(3x+4)+B(2x-1) \to (i)$

Put$2x-1=0 \Rightarrow x=\frac{1}{2}in(i)$

$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right)+4\right)+B(0) \Rightarrow 1 = A\left(\frac{3+8}{2}\right)$

$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$

$$Put\ 3x+4=0 \Rightarrow x=-\frac{4}{3}in\ (i)$$

$\Rightarrow 1 = A(0)+B\left(2\left(-\frac{4}{3}\right)-1\right) \Rightarrow 1 = B\left(\frac{-8-3}{3}\right)$

$$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$$

So

$$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1}+\frac{B}{3x+4}$$

$$1 = A(3x+4)+B(2x-1) \to (i)$$

$$Put\ 2x-1=0 \Rightarrow x=\frac{1}{2}\ put\ in\ (i)$$

$$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right)+4\right) \Rightarrow 1 = A\left(\frac{3}{2}+4\right)$$

$$\Rightarrow 1 = A\left(\frac{3+8}{2}\right)\Rightarrow 1 = A\left(\frac{11}{2}\right)\Rightarrow A=\frac{2}{11}$$

Put $3x+4=0 \Rightarrow x=-\frac{4}{3}$ put in (i)

$$\Rightarrow 1 = A(0)+B\left(2\left(-\frac{4}{3}\right)-1\right)\Rightarrow 1 = B\left(\frac{-8-3}{3}\right)$$

$$\Rightarrow 3=-11B\Rightarrow B=-\frac{3}{11}$$

So, $\frac{1}{6x^2+5x-4} = \frac{\frac{2}{11}}{2x-1}+\frac{-3/11}{3x+4}$

$\Rightarrow \int \frac{1}{6x^2+5x-4}dx = \frac{1}{11}\int\frac{2}{2x-1}dx - \frac{1}{11}\int\frac{3}{3x+4}dx$

$$= \frac{1}{11}\ln|2x-1| - \frac{1}{11}ln|3x+4|+c$$

$$=\frac{1}{11}\ln\left|\frac{2x-1}{3x+4}\right|+c$$

Q.8 $\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2}dx$

$$\begin{array}{r|l} & x \\ 2x^2-3x-2 & 2x^3-3x^2-x-7 \\ & \underline{\pm 2x^3 \pm 3x^2 \mp 2x} \\ & x-7 \end{array}$$

$$\int\frac{2x^2-3x^2-x-7}{2x^2-3x-2}dx = \int\left(x+\frac{x-7}{2x^2-3x-2}\right)dx$$

$$=\int xdx + \int\frac{x-7}{2x^2-3x-2}dx$$

NOw

$\frac{x-7}{(x-2)(2x+1)} = \frac{A}{x-2}+\frac{B}{2x+1}$

$\because 2x^2-3x-2$
$= 2x^2-4x+x-2$
$3x(x-2)+1(x-2))$
$(x-2)(3x+1)$

$x-7 = A(2x+1)+B(x-2) \to (i)$

$\Rightarrow Put\ x-2=0 \Rightarrow x=2\ in\ (i)$

$\Rightarrow 2-7 = A(2(2)+1)+B(0)\Rightarrow -5=5A\Rightarrow A=-\frac{5}{5}=-1$

$$A=-1$$

$$Put\ 2x+1=0\Rightarrow x=-\frac{1}{2}\ in\ (i)$$

$\Rightarrow -\frac{1}{2}-7 = A(0)+B\left(-\frac{1}{2}-2\right)\Rightarrow\frac{-1-14}{2}=B\left(\frac{-1-4}{2}\right)$

$\Rightarrow -15=-5B\Rightarrow B=3$

So

$$\frac{x-7}{2x^2-3x-2}=\frac{-1}{x-2}+\frac{3}{2x+1}$$

Thus $\int\frac{2x^2-3x^2-x-7}{2x^2-3x-2}dx = \int xdx = \int\frac{1}{x-2}dx + 3\int\frac{1}{2x+!}dx$

$$=\frac{x^2}{2}-\ln|x-2|+\frac{3}{2}\int\frac{2}{2x+1}dx$$

$$=\frac{x^2}{2}-\ln|x-2|+\frac{3}{2}\ln|2x+1|+c$$

Q.9 $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

Solution: $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

Now

$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$

$$3x^2-12x+11 = A(x-2)(x-3)+B(x-1)(x-3) + C(x-1)(x-2) \to (i)$$

Put $x-1=0\Rightarrow x=1\ in\ (i)$

$\Rightarrow 3(1)^2-12(1)+11 = A(1-2)(1-3)+B(0)+C(0)$

$$=3-12+11 = A(-1)(-2)$$

$$\Rightarrow 2=2A\Rightarrow A=1$$

$$Put\ x-2=0\Rightarrow x=2in(i)$$

$\Rightarrow 3(2)^2-12(2)+11 = A(0)+B(2-1)(2-3)+C(0)$

$$\Rightarrow 12-24+11=-B$$

$$\Rightarrow -1=-B\Rightarrow B=1$$

$$Put\ x-3=0\Rightarrow x=3\ in(i)$$

$\Rightarrow 3(3)^2-12(3)+11 = A(0)+B(0)+C(3-1)(3-2)$

$$\Rightarrow 3(9)-36+11=C(2)(1)$$

$$\Rightarrow 27-36+11=2C$$

$$\Rightarrow 2=2C\Rightarrow C=1$$

So

$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$

$\int\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

$$=\int\frac{1}{x-1}dx+\int\frac{1}{x-2}dx+\int\frac{1}{x-3}dx$$

$$=\ln|x-1|+\ln|x-2|+\ln|x-3|+c$$

Q10. $\int\frac{2x-1}{x(x-1)(x-3)}dx$

Solution: $\int\frac{2x-1}{x(x-1)(x-3)}dx$

Now

$$\frac{2x-1}{x(x-1)(x-3)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-3}$$

$$2x-1 = A(x-1)(x-3)+B(x)(x-3) + C(x)(x-1)\to(i)$$

$$Putx=0\ in\ (i)$$

$$2(0)-1=A(0-1)(0-3)+B(0)(C(0)$$

$$\Rightarrow -1=A(-1)(-3)\Rightarrow -1=3A\Rightarrow A=-\frac{1}{3}$$

Put $x-1=0\Rightarrow x=1\ in\ (i)$

$\Rightarrow 2(1)-1=A(0)+B(1)(1-3)+C(0)$

$$\Rightarrow 1=B(-2)\Rightarrow B=-\frac{1}{2}$$

Put $x-3=0\Rightarrow x=3\ in\ (i)$

$$\Rightarrow 2(3)-1=A(0)+B(0)+C(3)(3-1)$$

$$\Rightarrow 5=6C\Rightarrow C=\frac{5}{6}$$

So

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{-1}{3x} + \frac{-1}{2(x-1)} + \frac{5}{6(x-3)}$$

$$\frac{2x-1}{x(x-1)(x-3)} = -\frac{1}{3x} - \frac{1}{2(x-1)} + \frac{5}{6(x-3)}$$

$$\int \frac{x-1}{x(x-1)(x-3)}dx$$

$$= -\frac{1}{3}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{x-1} + \frac{5}{6}\int \frac{1}{x-3}$$

$$= -\frac{1}{3}\ln|x| + \frac{1}{2}\ln|x-1| + \frac{5}{6}\ln|x-3| + c$$

Q.11 $\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Solution: $\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Now

$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)} = \frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$$

$$5x^2+9x+6 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x+1)(x-1)$$

Put $x+1=0 \Rightarrow x=-1\ in(i)$

$$5(-1)^2+9(-1)+6 = A(0) + B(-1-1)(2(-1)+3) + C(0)$$

$$\Rightarrow 5-9+6 = B(-2)(1)$$

$$\Rightarrow 2 = -2B \Rightarrow B = -1$$

Put $x-1=0 \Rightarrow x=1\ in(i)$

$$\Rightarrow 5(1)^2+9(1)+6 = A(1+1)(2(1)+3) + B(0) + C(0)$$

$$\Rightarrow 5+9+6 = A(2)(5) \Rightarrow 20 = A10 \Rightarrow A = 2$$

Put $2x+3=0 \Rightarrow x=-\frac{3}{2}\ in\ (i)$

$$\Rightarrow 5\left(-\frac{3}{2}\right)^2 + 9\left(-\frac{3}{2}\right) + 6 = A(0) + B(0) + C\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}-1\right)$$

$$5\left(\frac{9}{4}\right) + \left(-\frac{27}{2}\right) + 6 = C\left(\frac{-3+2}{2}\right)\left(\frac{-3-2}{2}\right)$$

$$\frac{45}{4} - \frac{27}{2} + 6 = C\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)$$

$$\frac{45-54+24}{4} = C\frac{5}{4}$$

$$\Rightarrow 15 = 5C \Rightarrow C = 3$$

$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)} = \frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}$$

$$= \frac{2}{x-1} + \frac{-1}{x+1} + \frac{3}{2x+3}$$

$$:\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx = 2\int \frac{1}{x-1}dx - 1\int \frac{1}{x+1}dx + \frac{3}{2}\int \frac{2}{2x+3}dx$$

$$= 2\ln|x-1| - ln|x+1| + \frac{3}{2}|2x+3| + c$$

Q12. $\int \frac{4+7x}{(1+x)^2(2+3x)}dx$

Solution:

$$\int \frac{4+7x}{(1+x)^2(2+3x)}dx$$

Now

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{2x+3}$$

$$\Rightarrow 4+7x = A(1+x)(2x+3) + B(2x+3) + C(1+x)^2 \rightarrow (i)$$

Put $1+x=0 \Rightarrow x=-1\ in\ (i)$

$$\Rightarrow 4+7(-1) = A(0) + B(-2+3) + C(0)$$

$$\Rightarrow -3 = B \Rightarrow B = -3$$

$$Put\ 2+3x=0 \Rightarrow x = -\frac{2}{3}\ in(i)$$

$$4+7\left(-\frac{2}{3}\right) = A(0) + B(0) + C\left(1-\frac{2}{3}\right)^2$$

$$4-\frac{14}{3} = C\left(\frac{3-2}{3}\right)^2$$

$$\frac{12-14}{3} = C\left(\frac{1}{9}\right)$$

$$\Rightarrow -\frac{2}{3} = \frac{1}{9}C \Rightarrow -\frac{2}{3}\times\frac{9}{1} = C \Rightarrow C = -6$$

From (i)

$$4+7x = A(2+3x+2x+3x^2) + 2B + 3Bx + C(1+2x+x^2)$$

$$\Rightarrow 4+7x = 2A + 5Ax + 3x^2A + 2B + 3Bx + C + 2Cx + cx^2$$

Equating coefficient of x^2

$$0 = 3A + C \Rightarrow 3A = -C \Rightarrow 3A = -(-6)$$

$$\Rightarrow 3A = 6 \Rightarrow A = \frac{6}{3} = 2 \Rightarrow A = 2$$

So,

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{2}{1+x} + \frac{3}{(1+x)^2} - \frac{6}{2x+3}$$

$$\int \frac{4+7x}{(1+x)^2(2+3x)}dx = 2\int \frac{1}{1+x}dx + 3\int (1+x^2)^{-2}dx + \frac{6}{3}\int \frac{3}{2+3x}dx$$

$$= 2\ln|1+x| + \frac{3(1+x)^{-1}}{-1} - 2\ln|2+3x| + c$$

$$\ln|1+x|^2 - \frac{3}{1+x} - \ln|2+3x|^2 + c$$

Q.13 $\int \frac{2x^2}{(x-1)^2(x+1)}dx$

Solution:

Now

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$\Rightarrow 2x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Put$x-1=0 \Rightarrow x=1\ in(i)$

$$\Rightarrow 2(1)^2 = A(0) + B(1+1) + C(0)$$

$$\Rightarrow 2 = 2B \Rightarrow B = 1$$

Put $x+1=0 \Rightarrow x=-1\ in\ (i)$

$$\Rightarrow 2(-1)^2 = A(0) + B(o) + C(-1-1)^2$$

$$2 = 4C \Rightarrow C = \frac{1}{2}$$

From (i)

$$2x^2 = A(x^2-1) + Bx + B + C(x^2+1-2x)$$

$$\Rightarrow 2x^2 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$$

Equating coefficients of x^2, *we have*

$$\Rightarrow 2 = A + C \Rightarrow 2 = A + \tfrac{1}{2} \Rightarrow A = 2 - \tfrac{1}{2}$$

$$\Rightarrow A = \frac{3}{2}$$

So,

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{3/2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{1/2}{(x+1)}$$

$$\int \frac{2x^2}{(x-1)^2(x+1)}dx = \frac{3}{2}\int \frac{1}{x-1}dx + \int (x-1)^{-2}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$= \frac{3}{2}\ln|x-1| + \frac{(x-1)^{-1}}{-1} + \frac{1}{2}\ln|x+1| + c$$

$$= \frac{3}{2}\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x+1| + c$$

Q.14 $\int \frac{1}{(x-1)(x+1)^2}dx$

Solution: $\int \frac{1}{(x-1)(x+1)^2}dx$

Now

$$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \to (i)$$

Put $x - 1 = 0 \Rightarrow x = 1$ *in* (i)

$$1 = A(1+1)^2 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

Put $x + 1 = 0 \Rightarrow x = -1$ *in* (i)

$$1 = C(-1-1)$$

$$\Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$$

From (i)

$$\Rightarrow 1 = A(x^2+2x+1) + B(x^2-1) + Cx - C$$

$$\Rightarrow 1 = Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$$

Equating coefficient of x^2, *we have*

$$0 = A + B \Rightarrow 0 = \frac{1}{4} + B \Rightarrow B = -\frac{1}{4}$$

$$\frac{1}{(x-1)(x+1)^2} = \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{(x+1)^2}$$

$$\int \frac{1}{(x-1)(x+1)^2}dx = \frac{1}{4}\int \frac{1}{x-1}dx - \frac{1}{4}\int \frac{1}{x+1}dx - \frac{1}{2}\int (x+1)^{-2}dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\frac{(x+1)^{-1}}{-1} + C$$

$$= \{\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1|\} + \frac{1}{2(x+1)} + C$$

Q.15 $\int \frac{x+4}{x^3-3x^2+4}dx$

Solution: $\int \frac{x+4}{x^3-3x^2+4}dx$

Now

$$\because x^3 - 3x^2 + 4 = x^3 + x^2 - 4x^2 + 4$$

$$= x^2(x+1) - 1(x^2-1)$$

$$= x^2(x+1) - 4(x-1)(x+1)$$

$$= (x+1)(x^2-4x+4)$$

$$\Rightarrow x^3 - 3x^2 + 4 = (x+1)(x-2)^2$$

Now

$$\frac{x+4}{x^3-3x^2+4} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\Rightarrow x + 4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1) \to (i)$$

Put $x + 1 = 0 \Rightarrow x = -1$ $in (i)$

$$\Rightarrow -1 + 4 = A(-1-2)^2 + B(0) + C(0)$$

$$\Rightarrow 3 = 9A \Rightarrow A = \frac{1}{3}$$

Put $x - 2 = 0 \Rightarrow x = 2$ *in* (i)

$$\Rightarrow 2 + 4 = A(0) + B(0) + C(2+1)$$

$$\Rightarrow 6 = 3C \Rightarrow C = 2$$

From (i)

$$x + 4 = A(x^2-4x+4) + B(x^2-2x+x-2) + Cx + C$$

$$\Rightarrow x + 4 = Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C$$

Equating coefficients of x^2

$$\Rightarrow 0 = A + B \Rightarrow 0 = \frac{1}{3} + B \Rightarrow B = -\frac{1}{3}$$

$$\frac{x+4}{x^3-3x^2+4} = \frac{1/3}{x+1} + \frac{-1/3}{x-2} + \frac{2}{(x-2)^2}$$

$$\int \frac{x+4}{x^3-3x^2+4}dx = \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{1}{x-2} + 2\int (x-2)^{-2}dx$$

$$= \frac{1}{3}ln|x+1| - \frac{1}{3}\ln|x-2| + 2\frac{(x-2)^{-1}}{-1} + c$$

$$= \frac{1}{3}ln|x+1| - \frac{1}{3}\ln|x-2| - \frac{2}{x-2} + c$$

$$= \frac{1}{3}\{\ln|x+1| - \ln|x-2|\} - \frac{2}{x-2} + c$$

Q16. $\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$

Solution:

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-2)} + \frac{D}{(x-2)^2}$$

$$\Rightarrow x^3 - 6x^2 + 25 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2 \to (i)$$

Put $x + 1 = 0 \Rightarrow x = -1$ *in* (i)

$$\Rightarrow (-1)^3 - 6(-1)^2 + 25 = A(0) + B(-1-2)^2 + C(0) + D(0)$$

$$-1 - 6 + 25 = 9B$$

$$9B = 18 \Rightarrow B = 2$$

Put $x - 2 = 0 \Rightarrow x = 2$ *in* (i)

$$\Rightarrow (2)^3 - 6(2)^2 + 25 = D(2+1)^2$$

$$\Rightarrow 8 - 24 + 25 = 9D$$

$$9 = 9D \Rightarrow D = 1$$

From (i)

$$x^3-6x^2+25=A(x+1)(x^2-4x+4)+B(x^2-4x+4)$$
$$+C(x^2+1+2x)(x-2)+D(x^2+1+2x)$$
$$=A(x^3-4x62+4x+x62-4x+4)+Bx^2-4Bx+4B$$
$$+C(x^3-2x^2+x-2+2x^2-4x)+Dx^2+D+2Dx$$
$$=Ax^3-3Ax^2+4A+Bx^2-4Bx+4B+Cx^3$$
$$-3Cx+Dx^2+D+2Dx$$

Equating coefficients of x^3 and x^2

$FOr\ x^3$

$$1=A+C\rightarrow(ii)$$

For x^2 $\quad -6=-3A+B+D$

$$-6=-3A+2+1$$
$$-6-3=-3A\Rightarrow -9=-3A\Rightarrow A=3\ put\ in\ (ii)$$
$$1=3+C\Rightarrow C=1-3=-2\Rightarrow C=-2$$
$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}=\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{2}{(x-2)}+\frac{1}{(x-2)^2}$$
$$\int\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$$
$$=3\int\frac{1}{x+1}dx+2\int(x+1)^{-2}dx-2\int\frac{1}{x-2}dx+\int(x-2)^{-2}dx$$
$$=3ln|x+1|+2\frac{(x+1)^{-1}}{-1}-2ln|x-2|+\frac{(x-2)^{-1}}{-1}+C$$
$$=3ln|x+1|-\frac{2}{x+1}-2ln|x-2|-\frac{1}{x-2}+C$$

Q.17 $\int\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$

Solution:

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}$$
$$=\frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2}+\frac{D}{(x+2)^3}$$

$\Rightarrow x^3+22x^2+14x-17=A(x+2)^3+B(x-3)(x+2)^2+C(x-3)(x+2)+D(x-3)\rightarrow(i)$

$$Put\ x-3=0\Rightarrow x=3\ in\ (i)$$
$$\Rightarrow(3)^3+22(3)^2+14(3)-17=A(3+2)^3$$
$$\Rightarrow 27+198+42-17=125A$$
$$\Rightarrow 250=125A\Rightarrow A=2$$
$$Put\ x+2=0\rightarrow x=-2\ in(i)$$
$$\Rightarrow(-2)^3+22(-2)^2+14(-2)-17=D(-2-3)$$
$$-8+88-28-17=-5D\Rightarrow 35=-5D$$
$$D=-7$$

From (i)

$$x^3+22x^2+14x-17$$
$$=A[x^3+6x^2+12x+8]+B(2-3)(x^2+4x+4)$$
$$+C(x^2+2x-3x-6)+Dx-3D$$
$$=Ax^3+6Ax^2+12Ax+8A$$
$$+B(x^3+4x^2+4x-3x^2-12x-12)$$
$$+Cx^2-Cx-6c+Dx-3D$$

Equating coefficients of $x^2 and\ x^3$

$$For\ x^3;1=A+B\Rightarrow 1=2+B\Rightarrow B=-1$$
$$For\ x^2;\quad 22=6A+B+C\Rightarrow 22=6(2)-1+C$$
$$\Rightarrow C=22-12+1=11\Rightarrow C=11$$

So

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}$$
$$=\frac{2}{x-3}+\frac{1}{x+2}+\frac{11}{(x+2)^2}-\frac{7}{(x+2)^3}$$

$$\int\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$$
$$=2\int\frac{1}{x-3}-\int\frac{1}{x+3}dx+\int(x+2)^{-2}dx-7\int(x+2)^{-3}dx$$
$$=2ln|x-3|-ln|x+2|+11\frac{(x+2)^{-1}}{-1}-7\frac{(x+2)^{-2}}{-2}+c$$
$$=2ln|x-3|-ln|x+2|-\frac{11}{x+2}+\frac{7}{2}\frac{1}{(x+2)^2}+c$$

Q.18 $\int\frac{x-2}{(x+1)(x^2+1)}dx$

Solution:

$$\frac{x-2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$
$$\Rightarrow x-2=A(x^2+1)+(Bx+C)(x+1)\rightarrow(i)$$
$$Put\ x+1=0\Rightarrow x=-1\ in(i)$$
$$\Rightarrow -1-2=A((-1)^2+1)$$
$$-3=2A\Rightarrow A=-\frac{3}{2}$$

From (i)

$$\Rightarrow x-2=Ax^2+A+Bx^2+Bx+Cx+C$$

Equating coefficients of x^2 $and\ x$

$$For\ x^2;\quad 0=A+B\Rightarrow 0=-\frac{3}{2}+B\Rightarrow B=\frac{3}{2}$$
$$For\ x;1=B+C\Rightarrow 1=\frac{3}{2}+C$$
$$\Rightarrow C=1-\frac{3}{2}=-\frac{1}{2}\Rightarrow C=-\frac{1}{2}$$

So

$$\frac{x-2}{(x+1)(x^2+1)}=\frac{-3/2}{x+1}+\frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}$$

$$\int\frac{x-2}{(x+1)(x^2+1)}dx=-\frac{3}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{3x-1}{x^2+1}dx$$
$$=-\frac{3}{2}ln|x+1|+\frac{1}{2}\int\frac{3x}{x^2+1}dx-\frac{1}{2}\int\frac{1}{x^2+1}dx$$
$$=-\frac{3}{2}\ln|x+1|+\frac{3}{2}.\frac{1}{2}\ln|x^2+1|-\frac{1}{2}\tan^{-1}x$$
$$=-\frac{3}{2}\ln|x+!|+\frac{3}{4}\ln|x^2+1|-\frac{1}{2}\tan^{-1}x+c$$

Q.19 $\int\frac{x}{(x-1)(x^2+1)}dx$

Solution: $\int\frac{x}{(x-1)(x^2+1)}dx$

Now

$$\frac{x}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$
$$\Rightarrow x=A(x^2+1)+(Bx+C)(x-1)\rightarrow(i)$$

Put$x-1=0\Rightarrow x=1\ in(i)$

$$\Rightarrow 1=A((1)^1+1)$$
$$\Rightarrow A=\frac{1}{2}$$

From (i)

$$\Rightarrow x=Ax^2+A+Bx^2-Bx+Cx-C$$

Equating coefficients of $x^2 and\ x\ we\ have$

$For\ x^2\ ;\quad \Rightarrow 0 = A + B \Rightarrow 0 = \frac{1}{2} + B \Rightarrow B = -\frac{1}{2}$

$$For\ x;\quad 1 = -B + C \Rightarrow 1 = -\left(-\frac{1}{2}\right) + C$$

$$\Rightarrow 1 = \frac{1}{2} + C \Rightarrow 1 - \frac{1}{2} = C \Rightarrow C = \frac{1}{2}$$

So,

$$\frac{x}{(x-1)(x^2+1)} = \frac{1/2}{x-1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}$$

$$\int \frac{x}{(x-1)(x^2+1)} dx = \frac{1}{2}\int \frac{1}{x-1} dx - \frac{1}{2}\int \frac{x-1}{x^2+!} dx$$

$$= \frac{1}{2} ln|x-1| - \frac{1}{4}\int \frac{2x-2}{x^2+1} dx$$

$$\frac{1}{2} ln|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1} dx + \frac{1}{2}\int \frac{1}{x^2+1} dx$$

$$= \frac{1}{2} ln|x-1| - \frac{1}{4} ln|x^2+1| + \frac{1}{2} tan^{-1} x + c$$

Q.20 $\int \frac{9x-7}{(x+3)(x^2+1)} dx$

Solution: $\int \frac{9x-7}{(x+3)(x^2+1)} dx$

Now

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 9x - 7 = A(x^2+1) + (Bx+C)(x+3) \to (i)$$

Put$x + 3 = 0 \Rightarrow x = -3\ in\ (i)$

$$\Rightarrow 9(-3) - 7 = A((-3)^2 + 1)$$

$$-27 - 7 = 10A \Rightarrow -34 = 10A \Rightarrow A = -\frac{34}{10}$$

$$\Rightarrow A = -\frac{17}{5}$$

From (i)

$$9x - 7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$$

Equating coefficients of $x^3\ and\ x$

$$For\ x^2\ ,\Rightarrow 0 = A + B \Rightarrow 0 = -\frac{17}{5} + B \Rightarrow B = \frac{17}{5}$$

$$and for\ x; 3B + C = 9 \Rightarrow 3\left(\frac{17}{5}\right) + C = 9 \Rightarrow \frac{51}{5} + C = 9$$

$$C = 9 - \frac{51}{5} = \frac{45-51}{5} \Rightarrow C = -\frac{6}{5}$$

So

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17/5}{x+3} + \frac{\frac{17}{5}x - \frac{6}{5}}{x^2+1}$$

$\int \frac{9x-7}{(x+3)(x^2+1)} dx$

$$= -\frac{17}{5}\int \frac{1}{x+3} dx + \frac{17}{5}\int \frac{x}{x^2+1} dx - \frac{6}{5}\int \frac{1}{x^2+1} dx$$

$$= -\frac{17}{5}\ln|x+3| + \frac{17}{10}\ln|x^2+1| - \frac{6}{5}\tan^{-1} x + c$$

Q21. $\int \frac{1+4x}{(x-3)(x^2+4)} dx$

Solution: $\int \frac{1+4x}{(x-3)(x^2+4)} dx$

Now

$$\frac{\mathbf{1+4x}}{\mathbf{(x-3)(x^2+4)}} = \frac{\mathbf{A}}{\mathbf{x-3}} + \frac{\mathbf{Bx+C}}{\mathbf{x^2+4}}$$

$$\Rightarrow \mathbf{1 + 4x = A(x^2+4) + (Bx+C)(x-3) \to (i)}$$

$$\mathbf{Put\ x - 3 = 0 \Rightarrow x = 3\ in\ (i)}$$

$$\mathbf{1 + 4(3) = A((3)^2 + 4) + B(3) + C(0)}$$

$$\Rightarrow 13 = A(9+4) \Rightarrow 13 = 13A \Rightarrow A = 1$$

From (i)

$$1 + 4x = Ax^2 + 4a + Bx^2 - 3Bx + Cx - 3C$$

Equating Coefficients of $x^2\ and\ x$

$$\Rightarrow 0 = A + B\quad for\ x^2$$

$$0 = 1 + B \Rightarrow B = -1$$

$$\Rightarrow 4 = -3B + C \Rightarrow 4 - 3 = C \Rightarrow C = 1$$

So

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3} + \frac{(-)x+1}{x^2+4}$$

$$\int \frac{1+4x}{(x-3)(x^2+4)} dx$$

$$= \int \frac{1}{x-3} dx - \int \frac{x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$$

$$= \ln|x-3| - \frac{1}{2}\int \frac{2x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$$

$$= \ln|x-3| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

Q.22

$$\int \frac{\mathbf{12}}{\mathbf{x^3+8}} \boldsymbol{dx}$$

Solution:

$\int \frac{12}{x^3+8} dx\quad \because a^3 - b^3 = (a+b)(a^2 - ab + b^2)$

Now

$$\frac{12}{x^3+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$

$$\Rightarrow 12 = A(x^2 - 2x + 4) + (Bx+C)(x+2) \to (i)$$

Put $x + 2 = 0 \Rightarrow x = -2\ in\ (i)$

$\Rightarrow$12$= A(4+4+4) \Rightarrow 12 = 12A \Rightarrow A = 1$

From (i)

12$= Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$

Equating coefficients of $x^2\ and\ x$ we have

$$for\ x^2\ ;\ 0 = A + B \Rightarrow 0 = 1 + B \Rightarrow B = -1$$

$$for\ x;\ 0 = -2(1) + 2(-1) + C \Rightarrow 0 = -2 - 2 + C$$

$\Rightarrow$$C = 4$

So

$$\frac{12}{x^3+8} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}$$

$$\int \frac{12}{x^3+8} dx = \int \frac{1}{x+2} dx - \int \frac{x-4}{x^2-2x+4} dx$$

$$= \int \frac{1}{x+2} dx - \frac{1}{2}\int \frac{2x-8}{x^2-2x+4} dx$$

$$= ln|x+2| - \frac{1}{2}\int \frac{2x-2-6}{x^2-2x+4} dx$$

$$= ln\ |x+2| - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4} dx + \frac{6}{2}\int \frac{1}{x^2-2x+4} dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{1}{x^2-2x+1+3} dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{1}{(x-1)^2 + \sqrt{3}} dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

Q23. $\int \frac{9x^2+6}{x^3-8}dx$

Solution:

$$\int \frac{9x^2+6}{x^3-8}dx$$

Now

$$\frac{9x^2+6}{x^3-8}=\frac{9x^2+6}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

$\Rightarrow 9x+6=A(x^3+2x+4)+(Bx+C)(x-2)\to(i)$

Put $=x-2=0\Rightarrow x=2$ in (i)

$9(2)+6=A[(2)^2+2(x)+4]+B(2)+C(0)$

$\Rightarrow 24=12A\Rightarrow A=2$

From (i)

$9x+6=Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$

Equating coefficient of x^2 and x

For x^2; $0=A+B\Rightarrow 0=2+B\Rightarrow B=-2$

FOr x; $9=2A-2B+C\Rightarrow 9=2(2)-2(-2)+C$

$\Rightarrow 9=4+4+C\Rightarrow 9-8=C\Rightarrow C=1$

So

$$\frac{9x^2+6}{(x-2)(x^2+2x+4)}=\frac{2}{x-2}+\frac{-2x+1}{x^2+2x+4}$$

$$\int\frac{9x^2+6}{x^3-8}dx=2\int\frac{1}{x-2}dx-\int\frac{2x-1}{x^2+2x+4}dx$$

$$=-2\int\frac{1}{x-2}dx-\int\frac{2x+2-2-1}{x^2+2x+4}dx$$

$$=-2\int\frac{1}{x-2}dx-\int\frac{2x+2}{x^2+2x+4}dx+3\int\frac{1}{x^2+2x+4}dx$$

$$=2\ln|x-2|-\ln|x^2+2x+4|+3\int\frac{1}{(x+1)^2+(\sqrt{3})^2}dx$$

$$2\ln|x-2|-\ln|x^2+2x+4|+\frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

$$2\ln|x-2|-\ln|x^2+2x+4|+\sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$$

Q.24 $\int\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Solution: $\int\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Now

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+4}$$

$\Rightarrow 2x^2+5x+3=A(x-1)(x^2+4)+B(x^2+4)+(Cx+D)(x-1)^2\to(i)$

Put $x-1=0\Rightarrow x=1$ in (i)

$\Rightarrow 2(1)^2+5(1)+3=B(1+4)$

$2+5+3=5B\Rightarrow 10=5B\Rightarrow B=2$

From (i)

$$2x^2+5x+3=A(x^3+4x-x^2-4)+Bx^2+4B+(Cx+D)(x^2+1+2x)$$

$$Ax^3+4Ax-Ax^2-4A+Bx^2+4B+Cx^3+Cx-2Cx^2+Dx^2+D-2Dx$$

Equating coefficient of x^3, x^2 and x we get

For x^3 $\Rightarrow 0=A+C\Rightarrow C=-A\to(ii)$

For x^2; $2=-A+B-2C+D$

Put $B=2$ and $C=-A$

$2=-A+2-2(-A)+D$

$\Rightarrow 2-2=-A+2A+D\Rightarrow 0=A+D$

$\Rightarrow D=-A\to(iii)$

For x; $5=4A+C-2D$ put $C=-A$ and $D=-A$

$\Rightarrow 5=4A-A-2(-A)$

$5=3A+2A\Rightarrow 5=5A\Rightarrow A=1$

So (ii)$\Rightarrow C=-1$ and $(iii)\Rightarrow D=-1$

Thus

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}=\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{(-1)x\pm1}{x^2+4}$$

$\Rightarrow \int\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx=\int\frac{1}{x-1}dx+2\int(x-1)^{-2}-\int\frac{x+1}{x^2+4}dx$

$$=\int\frac{1}{x-1}dx+2\int(x-1)^{-2}dx-\int\frac{x}{x^2+4}dx-\int\frac{1}{x^2+4}dx$$

$$=\ln|x-1|+\frac{2(x-1)^{-1}}{-1}-\frac{1}{2}\int\frac{2x}{x^2+4}dx-\int\frac{1}{(x)^2+(2)^2}dx$$

$$=\ln|x-1|-\frac{2}{x-1}-\frac{1}{2}\ln|x^2+4|-\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

Q25.

$$\int\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$$

Solution:

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{Cx+D}{x^2+x+1}$$

$$\Rightarrow 2x^2-x-7=A(x+2)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x+2)^2$$

Put $x+2=0\Rightarrow x=-2$

$\Rightarrow 2(-2)^2-2(-2)-7=B((-2)^2+(-2)+1)$

$\Rightarrow 8+2-7=B(4-2+1)$

$\Rightarrow 3=3B\Rightarrow B=1$

From (i)

$$2x^2-x-7=A(x^3+x^2+x+2x^2+2)+Bx^2+Bx+B+C(Cx+D)(x^2+4x+4)$$

$$=Ax^3+3Ax^2+3A+Bx^2+Bx+B+Cx^3+4Cx^2+4Cx+Dx^2+4Dx+4D$$

Equating coefficients of x^3, x^2 and x

for x^3; $2+3A+B+4C+D$

Put $B=1$, $C=-A\to(ii)$

For x^2; $2=3A+B+4C+D\Rightarrow 2-1=-A+D$

$\Rightarrow D=A+1\to(iii)$

For x; $-1=3A+B+4C+4D$

Put $B=1, C=-A, D=A+1$

$\Rightarrow -1=3A+1-4A+4A+4$

$-1-1-4=3A\Rightarrow -6=3A\Rightarrow A=-2$

So $(ii)\Rightarrow C=2$ and $(iii)\Rightarrow B=-1$

Thus,

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}=\frac{-2}{x+2}+\frac{1}{(x+2)^2}+\frac{2x-1}{x^2+x+1}$$

$\int\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$

$$=-2\int\frac{1}{x+2}dx+\int(x+2)^{-2}dx+\int\frac{2x+1-2}{x^2+x+1}dx$$

$$=-2\ln|x+2|+\frac{(x+2)^{-1}}{-1}+\int\frac{2x+1}{x^2+x+1}dx-2\int\frac{1}{x^2+x+1}dx$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$

$$-2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -2\ln|x+2| - \frac{1}{x+2}\ln|x^2+x+1| - \frac{2}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^3+x+1| - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Q.26 $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

Solution: $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

$$\because \frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(x^2-x+1)}$$

$$\Rightarrow 3x+1 = (Ax+B)(x^2-x+1) + (Cx+D)(4x^2+1)$$

$$3x+1 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + 4Cx^3 + Cx + 4Dx^2 + D$$

Equating coefficients of x^3, x^2, x and constants terms.

For x^3; $0 = A + 4C \to (i)$

For x^2; $0 = -A + B + 4D \to (ii)$

for x; $3 = A + B + C \to (iii)$

For constant term; $1 = B + D \to (iv)$

From (i) $A = -4C$ and $(iv) \Rightarrow B = 1 - D$

Put in (ii) and (iii)

$\Rightarrow 0 = -(-4C) + (1-D) + 4D$ and $3 = -4C - (1-D) + C$

$0 = 4C + 1 + 4D$ $\quad 3 = -4C - 1 + D + C$

$0 = 4C + 3D + 1 \to (v)$ $\quad 0 = -3C + D - 4$

$\Rightarrow D = 3C + 4$ put in (iv)

$\Rightarrow 0 = 4C + 3(3C+4) + 1$

$0 = 4C + 3(3C+4) + 1$

$0 = 4C + 9C + 12 + 1 \Rightarrow 0 = 13C + 13$

$\Rightarrow -13C = 12 \Rightarrow C = -1$

As $A = -4C \Rightarrow A = -4(-1) \Rightarrow A = 4 \because C = -1$

As $D = 3C + 4 \Rightarrow D = 3(-1) + 4 = -3 + 4$

$\Rightarrow D = 1$

As $B = 1 - D = 1 - 1 = 0 \Rightarrow B = 0$

Thus

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1} + \frac{(-1)x+1}{(x^2-x+1)}$$

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{1}{2}\frac{8x}{4x^2+1} + \frac{(-1)(x-1)}{x^2-x+1}$$

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx = \frac{1}{2}\int \frac{8x}{4x^3+1} - \frac{1}{2}\int \frac{2x-2}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\int \frac{2x-1-1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|x^2+1| - \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| = \frac{1}{2}\int \frac{1}{x^2-x+\frac{1}{4}+\frac{3}{4}}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^3}$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c$$

$$\frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

Q27. $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

Solution:

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$$

$$\because \frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+4x+5}$$

$$\Rightarrow 4x+1 = (Ax+B)(x^2+4x+5) + (Cx+D)(x^2+4)$$

$$\Rightarrow 4x+1 = Ax^3 + 4Ax^2 + 5Ax + Bx^2 + 4Bx$$
$$+ 5B + Cx^3 + 4cx + Dx^2 + 4D$$

Equating coefficients of x^3, x^2, x and constant term.

Put x^3; $0 = A + C \to (i)$

for x^2; $0 = 4A + B + D \to (ii)$

for x; $4 = 5A + 4B + 4C \to (iiii)$

For constant term $1 = 5B + 4D \to (iv)$

From $(i) \Rightarrow A = -C$ and $(iv) \Rightarrow 5B = 1 - 4D$

$B = \frac{1-4D}{5}$ put in (ii) and (iii)

so $(ii) \Rightarrow 0 = 4(-C) + \frac{1-4D}{5} + D$ and $(iii) \Rightarrow 4 = 5(-C) + 4\left(\frac{-4D}{5}\right) + 4C$

$\Rightarrow 0 = -4C + \frac{1-4D}{5} + D$ $\quad \Rightarrow 20 = -25C + 4 - 16D + 20C$

$0 = -20C + 1 - 4D + 5D$ $\quad \Rightarrow 16D = -5C + 4 - 20$

$\Rightarrow 0 = -20C + D + 1$ $\quad \Rightarrow D = \frac{-5C-16}{16} \to (vi)$

$\Rightarrow D = 20C - 1 \to (v)$

By (v) and $(vi) \Rightarrow 20C - 1 = \frac{-5C-16}{16}$

$\Rightarrow 320C - 16 = -5C - 16 \Rightarrow 320C + 5C = 0$

$\Rightarrow 320C = 0 \Rightarrow C = 0$

As $a = -C \Rightarrow A = 0$

As $D = 20C - 1 \Rightarrow D = 20(0) - 1 \Rightarrow D = -1$

As $B = \frac{1-4D}{5} \Rightarrow B = \frac{1-4(-1)}{5} = \frac{5}{3} = 1$

$B = 1$

So

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{0x+1}{x^2+4} + \frac{0x+(-1)}{x^2+4x+5}$$

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}$$

$$= \int \frac{1}{x^2+4}dx - \int \frac{1}{x^2+4x+4+1}dx$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} - \int \frac{1}{(x-2)^2+(1)^2}dx$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} - \tan^{-1}(x-2) + c$$

Q28. $\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx$

$$\because \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+4a^2}$$

$$\Rightarrow 6a^2 = (Ax+B)(x^2+4a^2) + (Cx+D)(x^2+a^2)$$

$$\Rightarrow 6a^2 = Ax^3 + 4a^2Ax + Bx^2 + 4Ba^2 + Cx^3 + Ca^2x + Dx^2 + Da^2$$

Equating coefficients of $x^3 + x^2, x$ *and constants term.*

$$Put\ x^3; 0 = A + C \to (i)$$

$$for\ x^2; 0 = B + D \to (ii)$$

$$for\ x; 0 = 4a^2A + a^2C \Rightarrow 0 = (4A+C)a^2$$

$$\Rightarrow 4A + C \to (iiii)$$

For constant term $1 = 5B + 4D \to (iv)$

From $(i) \Rightarrow A = -C$ *and* $(iv) \Rightarrow B = -D$

$$Put\ in\ (iii)\ and\ (iv) so$$

$$(iii)\ 4(-C) + C = 0 \Rightarrow -4C + c = 0 \Rightarrow -3C = 0$$

$$\Rightarrow C = 0$$

$$(iv) 4(-D) + D = 6 \Rightarrow -4D + d \Rightarrow -3D = 6$$

$$D = -2$$

As $A = -C \ \Rightarrow A = 0 \ \because C = 0$

$As\ B = -D \Rightarrow B = -(-2) \Rightarrow B = 2 \ \because D = -2$

So

$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{0x+2}{x^2+a^2} + \frac{0x+(-2)}{x^2+4a^2}$$

$$\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx = 2\int \frac{1}{x^2+a^2}dx - 2\int \frac{1}{x^2+(2a)^2}dx$$

$$= \frac{2}{a}tan^{-1}\frac{x}{a} - \frac{1}{a}tan^{-1}\frac{x}{2a} + c$$

$$= \frac{2}{a}tan^{-1}\frac{x}{a} - \frac{1}{a}tan^{-1}\frac{x}{2a} + c$$

Q29. $\int \frac{2x^2-2}{(x^4+x^2+1)(x^2-x+1)}dx$

Solution:

$$\int \frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$$

$$2x^2 - 2 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1)$$

$$= Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D$$

Equating coefficients of x^3, x^2, x *and constant term.*

$$for\ x^3;\ \ 0 = A + C \to (i)$$

$$for\ x^2; 0 = A - B + C + D \to (ii)$$

$$for\ x; 2 = -A + B + C + D \to (iii)$$

$$for\ constant\ term;\ -2 = -A + C - 2$$

$$\Rightarrow 2 + 2 = -A + C \Rightarrow -A + C = 4 \to (v)$$

Put $A + C = 0\ in (ii) \Rightarrow 0 = -B + D \to (vi)$

$$Now\ by\ (i) + (v) \Rightarrow 2C = 4 \Rightarrow C = 2$$

$$as\ A + C = 0 \Rightarrow A + 2 = 0 \Rightarrow A = -2$$

Now by (iv)+$(vi) \Rightarrow 2D = -D \Rightarrow D = -1$

$As\ B + D = -2 \ \Rightarrow B - 1 = -2 \Rightarrow B = -1$

So;

$$\frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)} = \frac{-2x-1}{x^2+x+1} + \frac{2x-1}{x^2-x+1}$$

$$\int \frac{2x^2-2}{(x^2+x+1)(x^2-x+1)}$$

$$= -\int \frac{2x+1}{x^2+x+1}dx + \int \frac{2x-1}{x^2-x+1}dx$$

$$= -ln|x^2+x+1| + ln|x^2-x+1| + c$$

$$= ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + \boldsymbol{c}$$

Q 30. $\int \frac{3x-8}{(x^2-x+2)(x62+x+2)}dx$

Solution: $\int \frac{3x-8}{(x^2-x+2)(x62+x+2)}dx$

$$\because \frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{Ax+B}{x^2-x+2} + \frac{Cx+D}{x^2+x+2}$$

$$3x - 8 = (Ax+B)(x^2+x+2) + (Cx+D)(x^2-x+2)$$

$$= Ax^3 + Ax^2 + 2Ax + Bx^2 + Bx + 2B + Cx^3 - Cx^2 + 2Cx + Dx^2 - Dx + 2D$$

Equating coefficients of x^3, x^2, x *and constant term.*

$$for\ x^3;\ \ 0 = A + C \to (i)$$

$$for\ x^2; 0 = A + B - C + D \to (ii)$$

$$for\ x; 3 = 2A + B + 2C - D \to (iii)$$

$$for\ constant\ term;\ -8 = 2B + 2D \Rightarrow B + D = -4 \to (iv)$$

From $(i) \Rightarrow A = -C$ *and from*$(iv) \Rightarrow B = -4 - D$

Put *in* (ii)*and* (iii)*so*

$(ii) \Rightarrow 0 = -C + (-4 - D) - C + D$

$$0 = -C - 4 - B - C + D$$

$$0 = -2C - 4$$

$$\Rightarrow 2C = -4 \Rightarrow C = -2 \ \ as\ \ A = -C \Rightarrow A = 2$$

$$(ii) \Rightarrow 3 = 2(-C) - 4 - D + 2C - D$$

$$3 = -2C - 4 + 2C - 2D$$

$$\Rightarrow 3 + 4 = -2D \Rightarrow D = -\frac{7}{2}$$

As $B = -4 - D = -4\left(-\frac{7}{2}\right) = -4 + \frac{7}{2} = \frac{-8+7}{2}$=-$1/2 \Rightarrow B = -\frac{1}{2}$

So

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{2x-1/2}{x^2-x+2} + \frac{-2x+(-7/2)}{x^2+x+2}$$

$$\int \frac{3x-8}{(x^2-x+2)(x^2-x+2)}$$

$$= -\int \frac{2x+1-1-1/2}{x^2-x+2}dx + \int \frac{2x+1-1+7/2}{x^2+x+2}dx$$

$$= \int \frac{2x-1+\frac{1}{2}}{x^2-x+2}dx + \int \frac{2x+1+\frac{5}{2}}{x^2+x+2}dx$$

$$= \int \frac{2x-1}{x^2-x+2}dx + \frac{1}{2}\int \frac{1}{x^2-x+2}dx - \int \frac{2x+1}{x^2+x+2}dx - \int \frac{5/2}{x^2+x+2}dx$$

$$= ln|x^2-x+2| + \frac{1}{2}\int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+2} - ln|x^2+x+2| - \frac{5}{2}\int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+2}dx$$

$$= ln|x^2-x+2| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}} - ln|x^2+x+2| - \frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx$$

$$= \ln|x^2 - x + 1| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{7}{4}}dx - \ln|x^2+x+2|$$
$$-\frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx$$
$$= \ln|x^2-x+1| + \frac{1}{2}.\int \frac{1}{\left(x-\frac12\right)^2+\left(\frac{\sqrt7}{2}\right)^2}dx - \ln|x^2+x+1|$$
$$-\frac52\int\frac{1}{\left(x+\frac12\right)^2+\left(\frac{\sqrt7}{2}\right)^2}$$
$$= \ln|x^2-x+1| + \frac12.\frac{1}{\frac{\sqrt7}{2}}\tan^{-1}\left(\frac{x-\frac12}{\frac{\sqrt7}{2}}\right) - \ln|x^2+x+1|$$
$$-\frac52.\frac{1}{\frac{\sqrt7}{2}}\tan^{-1}\left(\frac{x+1/2}{\frac{\sqrt7}{2}}\right)$$
$$\ln|x^2-x+1| + \frac{1}{\sqrt7}\tan^{-1}\left(\frac{2x-1}{\sqrt7}\right) - \ln|x^2+x+1|$$
$$-\frac{5}{\sqrt7}\tan^{-1}\left(\frac{2x+1}{\sqrt7}\right)+c$$

Q31. $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$

Solution: $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$

$$\because \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+2x+3}$$
$$\Rightarrow 3x^3+4x^2+9x+5 = (Ax+B)(x^2+2x+3) + (Cx+D)(x^2+x+1)$$
$$ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$$

Equation coefficients of x^3, x^2, x *and constant term.*

$$\text{for } x^3;\ 3 = A + C \rightarrow (i)$$

For x^2; $4 = 2A + B + C + D \rightarrow (ii)$

For x; $9 = 3A + 2B + C + D \rightarrow (iii)$

For constant term; $5 = 3B + 2B + C + D \rightarrow (iv)$

From $(i) \Rightarrow A = 3 - C$ *and from* $(iv) \Rightarrow D = 5 - 3B$

Put in (ii) and (iii)

$$(ii) \Rightarrow 4 = 2(3-C) + B + C + 5 - 3B$$
$$4 = 6 - 2C + B + C + 5 - 3B$$
$$4 - 6 - 5 = -C - 2B \Rightarrow -7 = -(C+2B)$$
$$\Rightarrow C + 2B = 7 \rightarrow (v)$$
$$(iii) \Rightarrow 9 = 3(3-C) + 2B + C + 5 - 3B$$
$$\Rightarrow 9 = 9 - 3C + 2B + C + 5 - 3B$$
$$\Rightarrow 9 - 9 - 5 = -2C - B \Rightarrow B = -2C + 5 \text{ put in}(v)$$
$$\Rightarrow C + 2(-2C+5) = 7 \Rightarrow C - 4C + 10 = 7$$
$$\Rightarrow -3C + 10 + 7 \Rightarrow -3C = 7 - 10$$
$$\Rightarrow -3C = -3 \Rightarrow C = 1$$

As $B = 5 - 2C = 5 - 2(1) = 3 \Rightarrow B = 3$

As $D = 5 - 3B = 5 - 3(3) = 5 - 9 = -4 \Rightarrow D = -4$

As $A = 3 - C = 3 - 1 = 2 \Rightarrow A = 2$

So

$$\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{2x+3}{x^2+x+1} + \frac{x-4}{x^2+2x+3}$$

$$\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$$
$$= \int\frac{2x+1+2}{x^2+x+1}dx + \frac12\int\frac{2x-8}{x^2+2x+3}dx$$
$$= \int\frac{2x+1}{x^2+x+1}dx + 2\int\frac{1}{x^2+x+1}dx + \frac12\int\frac{2x+2-10}{x^2+2x+3}dx$$
$$= \ln|x^2+x+1| + 2\int\frac{1}{x^2+x+\frac14+\frac34}dx + \frac12\int\frac{2x+2}{x^2+2x+3}dx$$
$$-5\int\frac{1}{x^2+2x+3}dx$$
$$= \ln|x^2+x+1| + 2\int\frac{1}{\left(x+\frac12\right)^2+\left(\frac{\sqrt3}{2}\right)^2}dx + \frac12\ln|x^2+2x+3|$$
$$-5\int\frac{1}{x^2+2x+1+2}dx$$
$$= \ln|x^2+x+1| + 2.\frac{1}{\frac{\sqrt3}{2}}\tan^{-1}\left(\frac{x+\frac12}{\frac{\sqrt3}{2}}\right) + \frac12\ln|x^2+2x+3|$$
$$-5\int\frac{1}{(x+1)^2+(\sqrt2)^2}dx$$
$$= \ln|x^2+x+1| + \frac{4}{\sqrt3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right) + \frac12\ln|x^2+2x+3|$$
$$-\frac{5}{\sqrt2}\tan^{-1}\left(\frac{x+1}{\sqrt2}\right)+c$$
$$= \ln|x^2+x+1| + \ln|x^2+2x+3|^{\frac12} + \frac{4}{\sqrt3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)$$
$$-\frac52\tan^{-1}\left(\frac{x+1}{\sqrt2}\right)+c$$
$$= \ln|x^2+x+1|\sqrt{x^2+2x+3} + \frac{4}{\sqrt3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)$$
$$-\frac52\tan^{-1}\left(\frac{x+1}{\sqrt2}\right)+c$$

The Definite integrals:

If $\int f(x)dx = \emptyset(x) + c$, *then the integral of* $f(x)$ *from a to b is denoted by* $\int_a^b f(x)dx$

And read a+cs definite integral of $f(x)$ here a is called lower limit and b is called upper limit.

*the interval $[a, b]$ is called range of integration.

We evaluate $\int_a^b f(x)dx$ as;

Consider $\int f(x)dx = \emptyset(x) + c$

$$\Rightarrow \int_a^b f(x)dx = |\emptyset(x) + c|_a^b$$
$$= [\emptyset(b) + c] - [\emptyset(a) + c]$$
$$= \emptyset(b) + c - \emptyset(a) - c$$
$$\Rightarrow \int_a^b f(x)dx = \emptyset(b) - \emptyset(a)$$

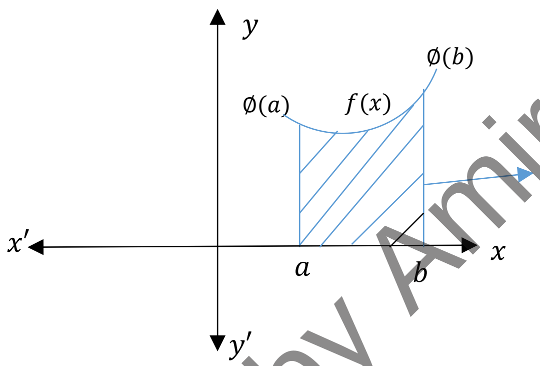
Note: if the lower limit is a constant and upper limit is a variable, then the integral is a function of the upper limit.

$$\int_a^x \boldsymbol{f(t)dt} = |\emptyset(\boldsymbol{t})|_a^x = \emptyset(\boldsymbol{x}) - \emptyset(\boldsymbol{a})$$

The area under the curve

$$\int_a^b f(x)dx = \emptyset(b) - \emptyset(a)$$

Represented the "area of region" bounded under the curve of function $f(x)$ the $x - axis$ and between two ordinates $x = a, x = b$ *as shown in figure*.



Fundamental theorem of calculus:

If $f(x)$is continuous $\forall\ x \in [a, b]$ *and* $\emptyset'(x) = f(x)$

$$\int_a^b f(x)dx = \emptyset(b) - \emptyset(a)$$

Is called fundamental theorem of integral calculus.

Properties of Definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
$$= \emptyset(b) - \emptyset(a)$$
$$= -[\emptyset(a) - \emptyset(b)]$$
$$= -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\textbf{(b)} \int_a^b \boldsymbol{f(x)dx} = \int_a^c \boldsymbol{f(x)dx} + \int_c^b \boldsymbol{f(x)dx}\ \ \boldsymbol{a < c < b}$$

Proof:

$$\int_a^c f(x)dx = \emptyset(c) - \emptyset(a)$$
$$\int_c^b f(x)dx = \emptyset(b) - \emptyset(c)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx =$$
$$= \emptyset(c) - \emptyset(a) + \emptyset(b) - \emptyset(c)$$
$$= \emptyset(b) - \emptyset(a)$$
$$\int_a^b f(x)dx$$
$$\Rightarrow \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\textbf{(c)} \int_a^a \boldsymbol{f(x)dx} = \boldsymbol{0}$$

Proof:

$$\int_a^a f(x)dx = \emptyset(a) - \emptyset(a)$$
$$= 0$$
$$\Rightarrow \int_a^a f(x)dx = 0$$

Also member $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

and $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Exercise 3.6

Evaluate the following indefinate integrls:

$Q.1$: $\int_1^2 (x^2+1)\,dx$

SOLUTION:

$\int_1^2 (x^2+1)\,dx$

$= \left|\frac{x^3}{3}+x\right|_1^2$

$= \left(\frac{2^3}{3}+2\right)-\left(\frac{1^3}{3}+1\right)$

$= \left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)$

$= \left(\frac{8+6}{3}\right)-\left(\frac{1+3}{3}\right)$

$= \frac{14}{3}-\frac{4}{3}$

$= \frac{14-4}{3}=\frac{10}{3}$

$Q.2$: $\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)\,dx$

SOLUTION:

$= \int_{-1}^1 \left(x^{\frac{1}{3}}.1+1\right)\,dx$

$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}+x\right|_{-1}^1$

$= \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}}+x\right|_{-1}^1$

$= \left|\frac{3}{4}x^{\frac{4}{3}}+x\right|_{-1}^1$

$= \left(\frac{3}{4}(1)^{\frac{4}{3}}+1\right)-\left(\frac{3}{4}(-1)^{\frac{4}{3}}+(-1)\right)$

$= \left(\frac{3}{4}.1+1\right)-\left(\frac{3}{4}.1-1\right)$

$= \left(\frac{3+4}{4}\right)-\left(\frac{3-4}{4}\right)$

$= \frac{7}{4}-\frac{-1}{4}=\frac{7}{4}+\frac{1}{4}$

$= \frac{7+1}{4}=\frac{8}{4}=2$

$Q.3$: $\int_{-2}^0 \frac{1}{(2x-1)^2}\,dx$

SOLUTION:

$= \int_{-2}^0 (2x-1)^{-2}\,dx$

$= \frac{1}{2}\int_{-2}^0 (2x-1)^{-2}\,.2\,dx$

$= \frac{1}{2}\left|\frac{(2x-1)^{-2+1}}{-2+1}\right|_{-2}^0$

$= \frac{1}{2}\left|\frac{(2x-1)^{-1}}{-1}\right|_{-2}^0$

$= -\frac{1}{2}\left|\frac{1}{2x-1}\right|_{-2}^0$

$= -\frac{1}{2}\left[\left(\frac{1}{2(0)-1}\right)-\left(\frac{1}{2(-2)-1}\right)\right]$

$= -\frac{1}{2}\left[\left(\frac{1}{-1}\right)-\left(\frac{1}{-5}\right)\right]$

$= -\frac{1}{2}\left[-1+\frac{1}{5}\right]=-\frac{1}{2}\left[\frac{-5+1}{5}\right]$

$= -\frac{1}{2}\left[\frac{-4}{5}\right]=\frac{2}{5}$

$Q.4$: $\int_{-6}^2 \sqrt{3-x}\,dx$

SOLUTION:

$= \int_{-6}^2 (3-x)^{\frac{1}{2}}dx$

$= (-1)\int_{-6}^2 (3-x)^{\frac{1}{2}}(-1)dx$

$= -\left|\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-6}^2$

$= -\left|\frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-6}^2$

$= -\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-6}^2$

$= -\frac{2}{3}\left[\left((3-2)^{\frac{3}{2}}\right)-\left((3-(-6))^{\frac{3}{2}}\right)\right]$

$= -\frac{2}{3}\left[\left((1)^{\frac{3}{2}}\right)-\left((9)^{\frac{3}{2}}\right)\right]$

$= -\frac{2}{3}\left[1-\left((3^2)^{\frac{3}{2}}\right)\right]$

$= -\frac{2}{3}[1-27]=\frac{52}{3}$

$Q.5$: $\int_1^{\sqrt{5}} \sqrt{(2t-1)^3}\,dt$

SOLUTION:

$= \int_1^{\sqrt{5}}(2t-1)^{\frac{3}{2}}\,dt$

$= \frac{1}{2}\int_1^{\sqrt{5}}(2t-1)^{\frac{3}{2}}.2\,dt$

$= \frac{1}{2}\left|\frac{(2t-1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right|_1^{\sqrt{5}} = \frac{1}{2}\left|\frac{(2t-1)^{\frac{5}{2}}}{\frac{5}{2}}\right|_1^{\sqrt{5}}$

$= \frac{1}{2}.\frac{2}{5}\left[\left((2\sqrt{5}-1)^{\frac{5}{2}}\right)-\left((2(1)-1)^{\frac{5}{2}}\right)\right]$

$$= \frac{1}{5}\left[(2\sqrt{5}-1)^{\frac{5}{2}}-1\right]$$

Q.6: $\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx$

SOLUTION:

$$= \int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\,x\,dx$$

$$= \frac{1}{2}\int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\,.2x\,dx$$

$$= \frac{1}{2}\left|\frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_2^{\sqrt{5}} = \frac{1}{2}\left|\frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}}\right|_2^{\sqrt{5}}$$

$$= \frac{1}{2}.\frac{2}{3}\left[\left(\left((\sqrt{5})^2-1\right)^{\frac{3}{2}}\right)-\left(\left((2)^2-1\right)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}\left[\left((4)^{\frac{3}{2}}\right)-\left(3^{\frac{3}{2}}\right)\right] = \frac{1}{3}\left[\left((2^2)^{\frac{3}{2}}\right)-\left(3^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}[8-3\sqrt{3}] = \frac{8}{3}-\sqrt{3}$$

Q.7: $\int_1^2 \frac{x}{x^2+2}\,dx$

SOLUTION:

$$= \frac{1}{2}\int_1^2 \frac{2x}{x^2+2}\,dx$$

$$= \frac{1}{2}|\ln(x^2+2)|_1^2$$

$$= \frac{1}{2}[\ln(2^2+2)-\ln(1^2+2)]$$

$$= \frac{1}{2}[\ln(6)-\ln(3)]$$

$$= \frac{1}{2}\left[\ln\left(\frac{6}{3}\right)\right] = \frac{1}{2}\ln 2$$

$$= \ln 2^{\frac{1}{2}} = \ln\sqrt{2}$$

Properties of natural log

$\ln(AB) = \ln A + \ln B$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$\ln A^B = B\ln A$

Q.8: $\int_2^3 \left(x-\frac{1}{x}\right)^2\,dx$

SOLUTION:

$$= \int_2^3 \left(x^2+\frac{1}{x^2}-2x.\frac{1}{x}\right)\,dx$$

$$= \int_2^3 (x^2+x^{-2}-2)\,dx$$

$$= \left|\frac{x^{2+1}}{2+1}+\frac{x^{-2+1}}{-2+1}-2x\right|_2^3$$

$$= \left|\frac{x^3}{3}+\frac{x^{-1}}{-1}-2x\right|_2^3 = \left|\frac{x^3}{3}-\frac{1}{x}-2x\right|_2^3$$

$$= \left(\frac{3^3}{3}-\frac{1}{3}-2(3)\right)-\left(\frac{2^3}{3}-\frac{1}{2}-2(2)\right)$$

$$= \left(\frac{27}{3}-\frac{1}{3}-6\right)-\left(\frac{8}{3}-\frac{1}{2}-4\right)$$

$$= \left(\frac{27-1-18}{3}\right)-\left(\frac{16-3-24}{6}\right)$$

$$= \left(\frac{8}{3}\right)-\left(\frac{-11}{6}\right) = \frac{16+11}{6} = \frac{27}{6} = \frac{9}{2}$$

Q.9: $\int_{-1}^{1}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\,dx$

SOLUTION:

$$= \int_{-1}^{1}(x^2+x+1)^{\frac{1}{2}}\left(\frac{2x+1}{2}\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1}(x^2+x+1)^{\frac{1}{2}}(2x+1)\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2+x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{(x^2+x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{1}{2}.\frac{2}{3}\left[\left((1^2+1+1)^{\frac{3}{2}}\right)-\left(((-1)^2+(-1)+1)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}\left[\left((3)^{\frac{3}{2}}\right)-\left((1-1+1)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}\left[\left((3)^{\frac{3}{2}}\right)-\left((1)^{\frac{3}{2}}\right)\right]$$

$$= \frac{1}{3}[3\sqrt{3}-1] = \sqrt{3}-\frac{1}{3} \quad ANS.$$

$$3^{\frac{3}{2}} = \left(3^{\frac{1}{2}}\right)^3 = (\sqrt{3})^3 = (\sqrt{3})^2(\sqrt{3})^1 = 3\sqrt{3}$$

Q.10: $\int_0^3 \frac{dx}{x^2+9}$

SOLUTION:

$$= \int_0^3 \frac{1}{3^2+x^2}\,dx = \left|\frac{1}{3}\tan^{-1}\frac{x}{3}\right|_0^3$$

$$= \frac{1}{3}\left[\left(\tan^{-1}\frac{3}{3}\right)-\left(\tan^{-1}\frac{0}{3}\right)\right]$$

$$= \frac{1}{3}[(\tan^{-1}1)-(\tan^{-1}0)]$$

$$= \frac{1}{3}\left[\frac{\pi}{4}-0\right] = \frac{\pi}{12}$$

Q.11: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t\,dt$

SOLUTION:

$$= |\sin t|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(\sin\frac{\pi}{3}\right)-\left(\sin\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2}-\frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

Q.12: $\int_1^2 \left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right) dx$

SOLUTION:

Here $f(x) = x+\frac{1}{x}$

$f'(x) = 1-\frac{1}{x^2}$

$$\left|\frac{\left(x+\frac{1}{x}\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^2 = \left|\frac{\left(x+\frac{1}{x}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^2$$

$$=\frac{2}{3}\left[\left(\left(2+\frac{1}{2}\right)^{\frac{3}{2}}\right)-\left(\left(1+\frac{1}{1}\right)^{\frac{3}{2}}\right)\right]$$

$$=\frac{2}{3}\left[\left(\left(\frac{5}{2}\right)^{\frac{3}{2}}\right)-\left((2)^{\frac{3}{2}}\right)\right]$$

$$=\frac{2}{3}\left[\frac{5}{2}\sqrt{\frac{5}{2}}-2\sqrt{2}\right]$$

$$=\frac{2}{3}.\frac{5}{2}\frac{\sqrt{5}}{\sqrt{2}}-\frac{2}{3}.2\sqrt{2}$$

$$=\frac{5\sqrt{5}}{3\sqrt{2}}-\frac{4\sqrt{2}}{3}=\frac{5\sqrt{5}-4(2)}{3\sqrt{2}}$$

$$=\frac{5\sqrt{5}-8}{3\sqrt{2}}$$

Q.13: $\int_1^2 \ln x \; dx$

SOLUTION:

Consider

$\int \ln x \; dx = \int \ln x \,.1\, dx$

Here $U = \ln x$, $V = 1$

Using $\int U.V = U.\int V\,dx - \int [U'.\int V\,dx]\,dx$

$= \ln x \,.\int 1\,dx - \int[(\ln x)'.\int 1\,dx]\,dx$

$= \ln x \,.x - \int\left[\frac{1}{x}.x\right]dx$

$= \ln x\,.x - \int 1\,dx$

$= x\ln x - x + c$

Taking limits

$\int_1^2 \ln x\,dx = |x\ln x - x|_1^2$

$= (2\ln 2 - 2) - (1\ln 1 - 1)$

$= (2\ln 2 - 2) - (1(0) - 1)$

$= (2\ln 2 - 2)$

Q.14: $\int_0^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)dx$

SOLUTION:

$$\int_0^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)dx = \left|\frac{e^{\frac{x}{2}}}{\frac{1}{2}}-\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right|_0^2 = \left|2e^{\frac{x}{2}}+2e^{-\frac{x}{2}}\right|_0^2$$

$$=2\left[\left(e^{\frac{2}{2}}+e^{-\frac{2}{2}}\right)-\left(e^{\frac{0}{2}}+e^{-\frac{0}{2}}\right)\right]$$

$$=2[(e^1+e^{-1})-(e^0+e^0)]$$

$$=2\left[e+\frac{1}{e}-1-1\right]=2\left[e+\frac{1}{e}-2\right]$$

$$=2\left[\frac{e^2+1-2e}{e}\right]=\frac{2}{e}(e^2+1^2-2e)$$

Q.15: $\int_0^{\frac{\pi}{4}}\frac{\cos\theta+\sin\theta}{\cos 2\theta+1}\,d\theta$

SOLUTION:

$$=\int_0^{\frac{\pi}{4}}\frac{\cos\theta+\sin\theta}{1+\cos 2\theta}\,dx=\int_0^{\frac{\pi}{4}}\frac{\cos\theta+\sin\theta}{2\cos^2\theta}\,d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\frac{\cos\theta}{\cos^2\theta}+\frac{\sin\theta}{\cos^2\theta}\right]dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta.\cos\theta}\right]d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}[\sec\theta+\sec\theta\tan\theta]\,d\theta$$

$$=\frac{1}{2}|\ln|\sec\theta+\tan\theta|+\sec\theta|_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left[\left(\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|+\sec\frac{\pi}{4}\right)-(\ln|\sec 0+\tan 0|+\sec 0)\right]$$

$$=\frac{1}{2}\left[\left(\ln\left|\sqrt{2}+1\right|+\sqrt{2}\right)-(\ln|1+0|+1)\right]$$

$$=\frac{1}{2}\left[\left(\ln\left|\sqrt{2}+1\right|+\sqrt{2}\right)-(0+1)\right]$$

$$=\frac{1}{2}\left[\ln\left|\sqrt{2}+1\right|+\sqrt{2}-1\right]$$

Q.16: $\int_0^{\frac{\pi}{6}}\cos^3\theta\;d\theta$

SOLUTION:

$$=\int_0^{\frac{\pi}{6}}\cos\theta\,\cos^2\theta\;d\theta=\int_0^{\frac{\pi}{6}}\cos\theta\,(1-\sin^2\theta)\,d\theta$$

$$=\int_0^{\frac{\pi}{6}}(\cos\theta-\sin^2\theta\cos\theta)\;d\theta=\left|\sin\theta-\frac{\sin^3\theta}{3}\right|_0^{\frac{\pi}{6}}$$

$$=\left(\sin\frac{\pi}{6}-\frac{\sin^3\frac{\pi}{6}}{3}\right)-\left(\sin 0-\frac{\sin^2 0}{3}\right)$$

$$=\left(\frac{1}{2}-\frac{\left(\frac{1}{2}\right)^3}{3}\right)-\left(0-\frac{0}{3}\right)=\left(\frac{1}{2}-\frac{1}{24}\right)-(0)=\frac{12-1}{24}=\frac{11}{24}$$

Q.17: $\int_0^{\frac{\pi}{6}}\cos^2\theta\;\cot^2\theta\,d\theta$

SOLUTION:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos^2\theta\;(\mathrm{cosec}^2\theta-1)\,d\theta=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cos^2\theta\,\mathrm{cosec}^2\theta-\cos^2\theta)\,d\theta=$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{\cos^2\theta}{\sin^2\theta}-\cos^2\theta\right)d\theta=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cot^2\theta-\cos^2\theta)\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\operatorname{cosec}^2\theta - 1 - \frac{1+\cos 2\theta}{2})\, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\frac{2\operatorname{cosec}^2\theta - 2 - 1 - \cos 2\theta}{2})\, d\theta$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\operatorname{cosec}^2\theta - 3 - \cos 2\theta)\, d\theta$

$= \frac{1}{2}\left[-2\cot\theta - 3\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\left(-2\cot\frac{\pi}{4} - 3\frac{\pi}{4} - \frac{\sin 2(\frac{\pi}{4})}{2}\right) — 2\cot\frac{\pi}{6} - 3\frac{\pi}{6} - \frac{\sin 2(\frac{\pi}{6})}{2}\right)$

$= \frac{1}{2}\left[\left(-2(1) - \frac{3\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(-2\sqrt{3} - \frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left[\left(-2 - \frac{3\pi}{4} - \frac{1}{2}.1\right) — 2\sqrt{3} - \frac{\pi}{2} - \frac{1}{2}\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2}\left[-2 - \frac{3\pi}{4} - \frac{1}{2} + 2\sqrt{3} + \frac{\pi}{2} + \frac{\sqrt{3}}{4}\right]$

$= \frac{1}{2}\left[-2 - \frac{1}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{\pi}{2}\right]$

$= \frac{1}{2}\left[-2 - \frac{1}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{\pi}{2}\right]$

$= \frac{1}{2}\left[\frac{-8-2+8\sqrt{3}+\sqrt{3}-3\pi+2\pi}{4}\right]$

$= \frac{1}{2}\left[\frac{-10+9\sqrt{3}-\pi}{4}\right]$

$= \frac{-10+9\sqrt{3}-\pi}{8}$

***Q*. 18**: $\int_0^{\frac{\pi}{4}} \cos^4 t\, dt$

SOLUTION:

$\int_0^{\frac{\pi}{4}} \cos^4 t\, dt = \int_0^{\frac{\pi}{4}} (\cos^2 t)^2\, dt = \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos 2t}{2}\right)^2\, dt$

$= \int_0^{\frac{\pi}{4}} \frac{1+\cos^2 2t + 2\cos 2t}{4}\, dt$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(1 + \frac{1+\cos 4t}{2} + 2\cos 2t\right)\, dt$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\frac{2+1+\cos 4t + 4\cos 2t}{2}\right)\, dt = \frac{1}{8}\int_0^{\frac{\pi}{4}} (3 + \cos 4t + 4\cos 2t)\, dt$

$= \frac{1}{8}\left[3t + \frac{\sin 4t}{4} + 4\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{8}\left[\left(3.\frac{\pi}{4} + \frac{\sin 4(\frac{\pi}{4})}{4} + 4\frac{\sin 2(\frac{\pi}{4})}{2}\right) - \left(3(0) + \frac{\sin 4(0)}{4} + 4\frac{\sin 2(0)}{2}\right)\right]$

$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + \frac{\sin\pi}{4} + 2\sin\left(\frac{\pi}{2}\right)\right) - \left(0 + \frac{\sin 0}{4} + 4\frac{\sin 0}{2}\right)\right]$

$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + \frac{0}{4} + 2.1\right) - \left(0 + \frac{0}{4} + 4.\frac{0}{2}\right)\right] = \frac{1}{8}\left[\left(\frac{3\pi}{4} + 0 + 2\right) - 0\right]$

$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + 2\right)\right] = \frac{1}{8}\left[\left(\frac{3\pi+8}{4}\right)\right] = \frac{3\pi+8}{32}$

***Q*. 19**: $\int_0^{\frac{\pi}{3}} \cos^2\theta\ \sin\theta\, d\theta$

SOLUTION:

$\int_0^{\frac{\pi}{3}} \cos^2\theta\ \sin\theta\, d\theta = -\int_0^{\frac{\pi}{3}} \cos^2\theta\,(-\sin\theta)\, d\theta = -\left|\frac{\cos^3\theta}{3}\right|_0^{\frac{\pi}{3}} = -\frac{1}{3}\left[\left(\cos^3\frac{\pi}{3}\right) - (\cos^3 0)\right] = -\frac{1}{3}\left[\cos^3\frac{\pi}{3} - \cos^3 0\right]$

$= -\frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - (1)^3\right] = -\frac{1}{3}\left[\frac{1}{8} - 1\right] = -\frac{1}{3}\left[\frac{1-8}{8}\right] = -\frac{1}{3}\left[\frac{-7}{8}\right] = \frac{7}{24}$

***Q*. 20**: $\int_0^{\frac{\pi}{4}} (1 + \cos^2\theta)\tan^2\theta\ d\theta$

SOLUTION:

$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta\cos^2\theta)\, d\theta$

$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta\cos^2\theta)\, d\theta = \int_0^{\frac{\pi}{4}} (\tan^2\theta + \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta)\, d\theta$

$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \sin^2\theta)\, dt = \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - 1 + \frac{1-\cos 2\theta}{2}\right] d\theta$

$= \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - 1 + \frac{1}{2} - \frac{\cos 2\theta}{2}\right] d\theta = \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right] d\theta$

$= \left[\tan\theta - \frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \left[\tan\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \left[\left(\tan\frac{\pi}{4} - \frac{1}{2}\frac{\pi}{4} - \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right)\right) - \left(\tan 0 - \frac{1}{2}0 - \frac{1}{4}\sin 2(0)\right)\right]$

$= \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - \left(0 - 0 - \frac{1}{4}\sin(0)\right)\right]$

$= \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - (0)\right] = \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}(1)\right) - (0)\right]$

***Q*. 21**: $\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$

SOLUTION:

Divide uo and down by $\cos\theta$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta+\cos\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\frac{\sec\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{\cos\theta}}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta\ \sec\theta}{\tan\theta+1}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\tan\theta+1}\, d\theta$$

Here $f(x) = \tan\theta + 1 \quad \Longrightarrow f'(x) = \sec^2\theta$

$$= [\ln|\tan\theta + 1|]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\ln\left|\tan\frac{\pi}{4} + 1\right|\right) - (\ln|\tan 0 + 1|)\right]$$

$$= [(\ln|1+1|) - (\ln|0+1|)]$$

$$= \ln|2| - \ln|1| = \ln 2 - 0 = \ln 2$$

Q.22: $\int_{-1}^{5} |x-3|\, dx$

***SOLUTION*:**

$$\int_{-1}^{5} |x-3|\, dx$$

$$= \int_{-1}^{3} |x-3|\, dx + \int_{3}^{5} |x-3|\, dx$$

$$= \int_{-1}^{3} -(x-3)\, dx + \int_{3}^{5} (x-3)\, dx$$

$$= -\int_{-1}^{3} (x-3).1\, dx + \int_{3}^{5} (x-3).1\, dx$$

$$= -\left|\frac{(x-3)^2}{2}\right|_{-1}^{3} + \left|\frac{(x-3)^2}{2}\right|_{3}^{5}$$

$$= -\frac{1}{2}[((3-3)^2) - ((-1-3)^2)] + \frac{1}{2}[((5-3)^2) - ((3-3)^2)]$$

$$= -\frac{1}{2}[0-16] + \frac{1}{2}[4-0]$$

$$= -\frac{1}{2}[0-16] + \frac{1}{2}[4-0] = 8 + 2 = 10$$

Q.23: $\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}}\, dx$

$$\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}}\, dx = 3\int_{\frac{1}{8}}^{1} \left(x^{\frac{1}{3}}+2\right)^2 .\frac{1}{3}x^{-\frac{2}{3}}\, dx =$$

$$3\left|\frac{\left(x^{\frac{1}{3}}+2\right)^{2+1}}{2+1}\right|_{\frac{1}{8}}^{1} = \frac{3}{3}\left|\left(x^{\frac{1}{3}}+2\right)^3\right|_{\frac{1}{8}}^{1} = \left(\left((1)^{\frac{1}{3}}+2\right)^3\right) - \left(\left(\left(\frac{1}{8}\right)^{\frac{1}{3}}+2\right)^3\right)$$

$$= ((1+2)^3) - \left(\left((2^{-3})^{\frac{1}{3}}+2\right)^3\right) = (3)^3 - \left(\frac{1}{2}+2\right)^3 =$$

$$27 - \left(\frac{1+4}{2}\right)^3 = 27 - \frac{125}{8} = \frac{216-125}{8} = \frac{91}{8}$$

$\int_1^3 \frac{x^2-2}{x+1}\, dx$ *(Improper fraction)*

$$\begin{array}{r} x-1 \\ x+1\ \overline{)\ x^2-2} \\ \underline{\pm x^2 \pm x} \\ -x-2 \\ \underline{\mp x \mp 1} \\ -1 \end{array}$$

$$= \int_1^3 \left(Q + \frac{R}{D}\right) dx$$

$$= \int_1^3 \left(x - 1 - \frac{1}{1+x}\right) dx$$

$$= \left|\frac{x^2}{2} - x - \ln|x+1|\right|_1^3$$

$$= \left(\frac{3^2}{2} - 3 - \ln|3+1|\right) - \left(\frac{1}{2} - 1 - \ln|1+1|\right)$$

$$= \left(\frac{9}{2} - 3 - \ln 4\right) - \left(\frac{1}{2} - 1 - \ln 2\right)$$

$$= \frac{9}{2} - 3 - \ln 4 - \frac{1}{2} + 1 + \ln 2$$

$$= \frac{9}{2} - 2 - \frac{1}{2} - \ln 4 + \ln 2$$

$$= \frac{9-4-1}{2} - \ln 2^2 + \ln 2$$

$$= 2 - 2\ln 2 + \ln 2 = 2 - \ln 2$$

Q.25: $\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx$

***SOLUTION*:**

$\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx = \int_2^3 \frac{3x^2-2x+1}{x^3-x^2+x-1}\, dx = |\ln|x^3 - x^2 + x - 1||_2^3$

$$= (\ln|3^3 - 3^2 + 3 - 1|) - (\ln|2^3 - 2^2 + 2 - 1|)$$

$$= (\ln|27 - 9 + 3 - 1|) - (\ln|8 - 4 + 2 - 1|)$$

$$= \ln 20 - \ln 5 = \ln\frac{20}{5} = \ln 4$$

Q.26: $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx$

***SOLUTION*:**

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos x \cos x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec x \tan x - \sec^2 x)\, dx$$

$$= |\sec x - \tan x|_0^{\frac{\pi}{4}}$$

$$= \left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right) - (\sec 0 - \tan 0)$$

$$= (\sqrt{2} - 1) - (1 + 0)$$

$$= \sqrt{2} - 1 - 1 = \sqrt{2} - 2$$

Q.27: $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}\, dx$

***SOLUTION*:**

$$= \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{1-\sin^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x}\, dx = -\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx$$

$$= -\int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$= -\int_0^{\frac{\pi}{4}}\left(\frac{\sin x}{\cos x\cos x} - \frac{1}{\cos^2 x}\right) dx$

$= -\int_0^{\frac{\pi}{4}}(\sec x\tan x - \sec^2 x)\,dx$

$= -|\sec x - \tan x|_0^{\frac{\pi}{4}}$

$= -\left[\left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right) - (\sec 0 - \tan 0)\right]$

$= -[(\sqrt{2}-1) - (1+0)]$

$= -\sqrt{2}+1+1 = 2-\sqrt{2}$

Q.28: $\int_0^1 \frac{3x}{\sqrt{4-3x}}\,dx$

SOLUTION:

$\int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx = -\int_0^1 \frac{-3x}{\sqrt{4-3x}}\,dx = -\int_0^1\frac{4-3x-4}{\sqrt{4-3x}}\,dx = -\int_0^1\frac{4-3x}{\sqrt{4-3x}}\,dx - \int_0^1 \frac{-4}{\sqrt{4-3x}}\,dx = -\int_0^1\sqrt{4-3x}\,dx + 4\int_0^1\frac{1}{\sqrt{4-3x}}\,dx$

$= -\frac{1}{-3}\int_0^1(4-3x)^{\frac{1}{2}}(-3)dx + \frac{4}{-3}\int_0^1(4-3x)^{-\frac{1}{2}}(-3)dx$

$= \frac{1}{3}\left|\frac{(4-3x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_0^1 = \frac{1}{3}\left|\frac{(4-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^1 = \frac{1}{3}.\frac{2}{3}\left|(4-3x)^{\frac{3}{2}}\right|_0^1 - \frac{4}{3}.\frac{2}{1}\left|(4-3x)^{\frac{1}{2}}\right|_0^1$

$= \frac{2}{9}\left|(4-3x)^{\frac{3}{2}}\right|_0^1 - \frac{8}{3}\left|(4-3x)^{\frac{1}{2}}\right|_0^1 = \frac{2}{9}\left[\left((4-3(1))^{\frac{3}{2}}\right) - \left((4-3(0))^{\frac{3}{2}}\right)\right] - \frac{8}{3}\left[\left((4-3(1))^{\frac{1}{2}}\right) - \left((4-3(0))^{\frac{1}{2}}\right)\right]$

$= \frac{2}{9}\left[(1)^{\frac{3}{2}}-(4)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(1)^{\frac{1}{2}}-(4)^{\frac{1}{2}}\right] = \frac{2}{9}\left[1-(2^2)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(1)^{\frac{1}{2}}-(2^2)^{\frac{1}{2}}\right] = \frac{2}{9}[1-2^3]-\frac{8}{3}[1-2^1]$

$= \frac{2}{9}[1-8]-\frac{8}{3}[1-2] = \frac{2}{9}[-7]-\frac{8}{3}[-1] = \frac{-14}{9}+\frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9}$

Q.29: $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos x}{\sin x(2+\sin x)}\,dx$

SOLUTION:

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2}{\sin x(2+\sin x)}.\cos x\;dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{(2+\sin x)-\sin x}{\sin x(2+\sin x)}.\cos x\;dx$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[\frac{(2+\sin x)}{\sin x(2+\sin x)} - \frac{\sin x}{\sin x(2+\sin x)}\right]\cos x\;dx$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[\frac{1}{\sin x}-\frac{1}{2+\sin x}\right]\cos x\;dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[\frac{\cos x}{\sin x}-\frac{\cos x}{2+\sin x}\right]dx$

$= \frac{1}{2}\left|ln|\sin x| - ln|2+\sin x|\right|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(ln\left|\sin\frac{\pi}{2}\right| - ln\left|2+\sin\frac{\pi}{2}\right|\right) - \left(ln\left|\sin\frac{\pi}{6}\right| - ln\left|2+\sin\frac{\pi}{6}\right|\right)\right]$

$= \frac{1}{2}\left[(\ln(1) - ln|2+1|) - \left(ln\frac{1}{2} - ln\left|2+\frac{1}{2}\right|\right)\right]$

$= \frac{1}{2}\left[0-\ln 3-ln\frac{1}{2}+ln\frac{5}{2}\right] = \frac{1}{2}[-\ln 3-(ln1-ln2)+(\ln 5-ln2)]$

$= \frac{1}{2}[-\ln 3-ln1+ln2+\ln 5-ln2] = \frac{1}{2}[-\ln 3-0+\ln 5]$

$= \frac{1}{2}[\ln 5-\ln 3] = \frac{1}{2}\left[\ln\frac{5}{3}\right] = \frac{1}{2}\ln\frac{5}{3}$

Q.30: $\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx$

SOLUTION:

$\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx = \int_0^{\frac{\pi}{2}}\frac{1}{(1+\cos x)(2+\cos x)}\sin x\,dx$

$= \int_0^{\frac{\pi}{2}}\frac{(2+\cos x)-(1+\cos x)}{(1+\cos x)(2+\cos x)}\sin x\,dx$

$= \int_0^{\frac{\pi}{2}}\left[\frac{(2+\cos x)}{(1+\cos x)(2+\cos x)} - \frac{(1+\cos x)}{(1+\cos x)(2+\cos x)}\right]\sin x\,dx$

$= \int_0^{\frac{\pi}{2}}\left[\frac{1}{(1+\cos x)} - \frac{1}{(2+\cos x)}\right]\sin x\,dx$

$= \int_0^{\frac{\pi}{2}}\left[\frac{\sin x}{(1+\cos x)} - \frac{\sin x}{(2+\cos x)}\right]dx$

$= \frac{-1}{-1}\int_0^{\frac{\pi}{2}}\left[\frac{\sin x}{(1+\cos x)} - \frac{\sin x}{(2+\cos x)}\right]dx$

$= \frac{1}{-1}\int_0^{\frac{\pi}{2}}\left[\frac{-\sin x}{(1+\cos x)} - \frac{-\sin x}{(2+\cos x)}\right]dx$

$= -[\ln|1+\cos x| - \ln|2+\cos x|]_0^{\frac{\pi}{2}}$

$= -\left[\left(\ln\left|1+\cos\frac{\pi}{2}\right| - \ln\left|2+\cos\frac{\pi}{2}\right|\right) - (\ln|1+\cos 0| - \ln|2+\cos 0|)\right]$

$= -[(\ln|1+0| - \ln|2+0|) - (\ln|1+1| - \ln|2+1|)]$

$= -[\ln(1) - \ln(2) - \ln(2) + \ln(3)]$

$= -[0 - 2\ln(2) + \ln 3] = 2\ln(2) - \ln(3)$

$= \ln(2)^2 - \ln(3) = \ln 4 - \ln(3) = \ln\frac{4}{3}$

$$\because \left(\begin{array}{c} \boldsymbol{a\ln b = \ln b^a} \\ \boldsymbol{\because \ln a - \ln b} \\ \boldsymbol{= \ln\frac{a}{b}} \end{array}\right)$$

Application of definate integral: Area under the curve:

Case1. *if* $f(x) \geq 0\ \forall x \in [a,b]$ *then curve lies above* $x-axis.$

$$A=\int_a^b f(x)dx \ \ where\ a<b$$

A is area of region above $x-axis.$ *under the curve of function* $y=f(x)$ *from a to b*

Case II *if* $f(x) \leq 0\ \forall x \in [a,b]$ *then curve lies below*

$$x-axis. so\ A=-\int_a^b f(x)dx \ \ where\ a<b$$

A is area of region below $x-axis,$ *under* The curve of function $y=f(x)$ *from a to b*

Exercise 3.7

Q.1: Find the area between the x − axis and the curve $\boldsymbol{y=x^2+1}$ ***from*** $\boldsymbol{x=1}$ ***to*** $\boldsymbol{x=2}$

SOLUTION:

Given $y=x^2+1$ *As* $y=x^2+1>0$ *in* $[1,2]$, *therefore curve is above* $x-axis$

$$Required\ Area=\int_1^2 y\ dx=\int_1^2 (x^2+1)\ dx$$

$$=\left|\frac{x^3}{3}+x\right|_1^2=\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)$$

$$=\left(\frac{2^3}{3}+2\right)-\left(\frac{1^3}{3}+1\right)$$

$$=\left(\frac{8+6}{3}\right)-\left(\frac{1+3}{3}\right)=\frac{14}{3}-\frac{4}{3}$$

$$=\frac{14-4}{3}=\frac{10}{3}\ \ square\ unit$$

Q.2: Find the area above the x − axis and under the curve $\boldsymbol{y=5-x^2}$ ***from*** $\boldsymbol{x=-1}$ ***to*** $\boldsymbol{x=2}$

SOLUTION:

Given $y=5-x^2$ *As* $y=5-x^2>0$ *in* $[-1,2]$, *therefore curve is above* $x-axis$

$$Required\ Area=\int_{-1}^2 y\ dx=\int_{-1}^2 (5-x^2)\ dx$$

$$=\left|5x-\frac{x^3}{3}\right|_{-1}^2$$

$$=\left(5.2-\frac{2^3}{3}\right)-\left(5(-1)-\frac{(-1)^3}{3}\right)$$

$$=\left(10-\frac{8}{3}\right)-\left(-5-\frac{-1}{3}\right)$$

$$=\left(\frac{30-8}{3}\right)-\left(\frac{-15+1}{3}\right)=\frac{22}{3}-\frac{-14}{3}$$

$$=\frac{22+14}{3}=\frac{36}{3}=12\ \ sq.unit$$

Q.3: Find the area below the curve $\boldsymbol{y=3\sqrt{x}}$ ***and above the x − axis between*** $\boldsymbol{x=1}$ ***to*** $\boldsymbol{x=4}$

SOLUTION:

Given $y=3\sqrt{x}$ *As* $y=3\sqrt{x}>0$

in $[1,4]$, *therefore curve is above* $x-axis$

$$Required\ Area=\int_1^4 y\ dx=3\int_{-1}^2 \sqrt{x}\ \ dx$$

$$=3\int_1^4 (x)^{\frac{1}{2}}.1\ dx=3\left|\frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4$$

$$=3\left|\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^4=3.\frac{2}{3}\left|(x)^{\frac{3}{2}}\right|_1^4$$

$$=2\left((4)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right)=2\left((2^2)^{\frac{3}{2}}-1\right)$$

$$=2(2^3-1)=2(8-1)=2(7)$$

$$=14\ \ \ sq.unit$$

Q.4: Find the area bounded by cos function from $\boldsymbol{x=-\frac{\pi}{2}}$ ***to*** $\boldsymbol{x=\frac{\pi}{2}}$

SOLUTION:

Given $y=\cos x$ *As* $y=\cos x\geq 0$ *in* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, *therefore curve is above* $x-axis$

$$Required\ Area=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\ dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\ \ dx=|\sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)=1-(-1)$$

$$=1+1=2\ \ \ sq.unit$$

Q.5: Find the area betweeen the x − axis and the curve $\boldsymbol{y=4x-x^2}$.

SOLUTION:

Given $y=4x-x^2$

Put $y=0$
$$4x-x^2=0$$
$$x(4-x)=0$$
$$x=0\ \ or\ \ 4-x=0$$
$$x=4$$

As $y=4x-x^2\geq 0$ *in* $[0,4]$, *therefore curve is above* $x-axis$

$$Required\ Area=\int_0^4 y\ dx=\int_0^4 (4x-x^2)\ \ dx$$

$$=\left|4\frac{x^2}{2}-\frac{x^3}{3}\right|_0^4$$

$$=\left(2.4^2-\frac{4^3}{3}\right)-\left(0^2-\frac{0^3}{3}\right)$$

$$=\left(32-\frac{64}{3}\right)-(0-0)=\frac{96-64}{3}$$

$$=\frac{32}{3}\ sq.unit$$

Q.6: Determine the area bounded by the parabola $y = x^2 + 2x - 3$ and the $x-axis$.

SOLUTION:

$Given \quad y = x^2 + 2x - 3$

$As\ y = \ y = x^2 + 2x - 3 \leq 0\ in\ [-3,1], therefore\ curve\ is\ below\ x - axis$

$$Required\ Area = -\int_{-3}^{1} y\ dx = -\int_{-3}^{1} (x^2 + 2x - 3)\ dx$$

$$= -\left|\frac{x^3}{3} + 2\frac{x^2}{2} - 3x\right|_{-3}^{1}$$

$$= -\left|\frac{x^3}{3} + x^2 - 3x\right|_{-3}^{1}$$

$$= -\left[\left(\frac{1^3}{3} + 1^2 - 3.1\right) - \left(\frac{(-3)^3}{3} + (-3)^2 - 3(-3)\right)\right]$$

$$= -\left[\left(\frac{1}{3} + 1 - 3\right) - \left(\frac{-27}{3} + 9 + 9\right)\right]$$

$$= -\left[\frac{1}{3} + 1 - 3 + \frac{27}{3} - 9 - 9\right]$$

$$= -\left[\frac{1}{3} + \frac{27}{3} - 20\right]$$

$$= -\left[\frac{1+27-60}{3}\right]$$

$$= -\left[\frac{-32}{3}\right] = \frac{32}{3}\ sq.unit$$

$Put\ y = o\ \ x^2 + 2x - 3 = 0$

$x^2 + 3x - x - 3 = 0$

$x(x + 3) - 1(x + 3) = 0$

$(x + 3) - (x - 1) = 0$

$x + 3 = 0\ \ or\ x - 1 = 0$

$x = -3\ \ ,\ \ x = 0$

Q.7: Find the area bounded by the curve $y = x^3 + 1$, the $x - axis$ and line $x = 2$

SOLUTION:

$Given \quad y = x^3 + 1 \qquad Put\ y = 0 \qquad x^3 + 1 = 0$

$(x + 1)(x^2 - x + 1) = 0$

$x + 1 = 0 \quad or\ \ x^2 - x + 1 = 0\ \ (Solve\ itself)$

$x = -1$

$After\ solving\ this\ equation\ gives\ imaginary\ roots\ so\ neglect.$

$As\ y = x^3 + 1 \geq 0\ in\ [-1,2], therefore\ curve\ is\ above\ x - axis$

$$Required\ Area = \int_{-1}^{2} y\ dx = \int_{-1}^{2} (x^3 + 1)\ dx$$

$$= \left|\frac{x^4}{4} + x\right|_{-1}^{2}$$

$$= \left(\frac{2^4}{4} + 2\right) - \left(\frac{(-1)^4}{4} + (-1)\right)$$

$$= (4 + 2) - \left(\frac{1}{4} - 1\right) = (6) - \left(\frac{1 - 4}{4}\right)$$

$$= 6 - \frac{-3}{4} = 6 + \frac{3}{4} = \frac{24 + 3}{4}$$

$$= \frac{27}{4} \qquad square\ unit.$$

Q.8: Find the area bounded by the curve $y = x^3 - 4x$, and the $x - axis$.

SOLUTION:

$Put\ y = 0 \quad x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x(x - 2)(x + 2) = 0$

$x = 0\ \ or\ \ x - 2 = 0\ \ ,or\ \ x + 2 = 0$

$x = 0\ \ or\ \ x = 2\ \ ,or\ \ x = -2$

$Given \quad y = x^3 - 4x$

$As\ y = x^3 - 4x \geq 0\ in\ [-2,0], therefore\ the\ curve\ is\ above\ x - axis$

$As\ y = x^3 - 4x \leq 0\ in\ [0,2], therefore\ the\ curve\ is\ below\ x - axis$

$$Required\ Area = \int_{-2}^{0} y\ dx - \int_{0}^{2} y\ dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\ dx - \int_{0}^{2} (x^3 - 4x)\ dx$$

$$= \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{-2}^{0} - \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{0}^{2}$$

$$= \left[\left(\frac{0^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)\right]$$

$$- \left[\left(\frac{(2)^4}{4} - 2(2)^2\right) - \left(\frac{(0)^4}{4} - 2(0)^2\right)\right]$$

$$= [(0 - 0) - (4 - 8)]$$

$$- [(4 - 8) - (0 - 0)] = 0 + 4 + 4 + 0$$

$$= 8 \quad square\ unit.$$

Q.9: Find the area between the curve $y = x(x - 1)(x + 1)$, and the $x - axis$.

SOLUTION:

$Given \quad y = x(x - 1)(x + 1) = x^3 - x$

$As\ y = x^3 - x \geq 0\ in\ [-1,0], therefore\ the\ curve\ is\ above\ x - axis$

$As\ y = x^3 - x \leq 0\ in\ [0,1], therefore\ the\ curve\ is\ below\ x - axis$

$$Required\ Area = \int_{-1}^{0} y\ dx - \int_{0}^{1} y\ dx$$

$$= \int_{-1}^{0} (x^3 - x)\ dx - \int_{0}^{1} (x^3 - x)\ dx$$

$$= \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{-1}^{0} - \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{0}^{1}$$

$$= \left[\left(\frac{0^4}{4} - \frac{(0)^2}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)\right]$$

$$- \left[\left(\frac{(1)^4}{4} - \frac{(1)^2}{2}\right) - \left(\frac{(0)^4}{4} - \frac{(0)^2}{2}\right)\right]$$

$$= \left[(0 - 0) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$- \left[\left(\frac{1}{4} - \frac{1}{2}\right) - (0 - 0)\right]$$

$$= 0 - \frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 0$$

$$= \frac{-1 + 2 - 1 + 2}{4} = \frac{1}{2}$$

Q.10: Find the area above the $x - axis$, bounded by the curve $y^2 = 3 - x$ from $x = -1$ to $x = 2$

SOLUTION:

$Given \quad y^2 = 3 - x \implies y = \sqrt{3 - x} \qquad As\ y = \sqrt{3 - x} \geq 0\ in\ [-1,2], therefore\ curve\ is\ above\ x - axis$

$$Required\ Area = \int_{-1}^{2} y\ dx = \int_{-1}^{2} \sqrt{3-x}\ \ dx$$
$$= -\int_{-1}^{2} (3-x)^{\frac{1}{2}}(-1)\ dx$$
$$= -\left|\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{2} = -\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-1}^{2}$$
$$= -\frac{2}{3}\left[\left((3-2)^{\frac{3}{2}}\right)-\left((3-(-1))^{\frac{3}{2}}\right)\right]$$
$$= -\frac{2}{3}\left[1-\left((4)^{\frac{3}{2}}\right)\right] = -\frac{2}{3}[1-8]$$
$$= \frac{14}{3}\ \ sq.unit$$

Q.11: Find the area betweeen the x − axis and the curve $y = \cos\frac{1}{2}x$ from $x = -\pi$ to π.

SOLUTION:

Given $y = \cos\frac{1}{2}x$ *As* $y = \cos\frac{1}{2}x \geq 0$ *in* $[-\pi,\pi]$, *therefore curve is above* $x - axis$

$$Required\ Area = \int_{-\pi}^{\pi} y\ dx = \int_{-\pi}^{\pi} \cos\frac{1}{2}x\ dx$$
$$= \left|\frac{\sin\frac{1}{2}x}{\frac{1}{2}}\right|_{-\pi}^{\pi} = 2\left[\left(\sin\frac{1}{2}(\pi)\right)-\left(\sin\frac{1}{2}(-\pi)\right)\right]$$
$$= 2[1-(-1)] = 4$$

Q.12: Find the area betweeen the x − axis and the curve $y = \sin 2x$ from $x = 0$ to $\frac{\pi}{3}$.

SOLUTION:

Given $y = \sin 2x$ *As* $y = \sin 2x \geq 0$ *in* $\left[0,\frac{\pi}{3}\right]$, *therefore curve is above* $x - axis$

$$Required\ Area = \int_{0}^{\frac{\pi}{3}} y\ dx = \int_{0}^{\frac{\pi}{3}} \sin 2x\ dx$$
$$= \left|\frac{-\cos 2x}{2}\right|_{0}^{\frac{\pi}{3}} = -\frac{1}{2}\left[\left(\cos\frac{2\pi}{3}\right)-(\cos 2(0))\right]$$
$$= -\frac{1}{2}\left[-\frac{1}{2}-1\right] = \frac{3}{4}\ \ sq.unit$$

Q.13: Find the area betweeen the x − axis and the curve $y = \sqrt{2ax - x^2}$ when $a > 0$.

SOLUTION:

Given $y = \sin 2x$ *As* $y = \sin 2x \geq 0$ *in* $\left[0,\frac{\pi}{3}\right]$, *therefore curve is above* $x - axis$

$$Required\ Area = \int_{0}^{2a} y\ dx = \int_{0}^{2a} \sqrt{2ax - x^2}\ dx = \int_{0}^{2a} \sqrt{a^2 - a^2 + 2ax - x^2}\ dx$$
$$= \int_{0}^{2a} \sqrt{a^2 - (a^2 - 2ax + x^2)}\ dx$$
$$= \int_{0}^{2a} \sqrt{a^2 - (x-a)^2}\ dx$$

Using formula $\int \sqrt{a^2 - x^2}\, dx$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + c$$

$$= \left[\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + \left(\frac{x-a}{2}\right)\sqrt{a^2-(x-a)^2}\right]_0^{2a}$$
$$= \left(\frac{a^2}{2}\sin^{-1}\left(\frac{2a-a}{a}\right) + \left(\frac{2a-a}{2}\right)\sqrt{a^2-(2a-a)^2}\right) - \left(\frac{a^2}{2}\sin^{-1}\left(\frac{0-a}{a}\right) + \left(\frac{0-a}{2}\right)\sqrt{a^2-(0-a)^2}\right)$$
$$= \left(\frac{a^2}{2}\sin^{-1}(1) + \left(\frac{a}{2}\right)\sqrt{a^2-a^2}\right) - \left(\frac{a^2}{2}\sin^{-1}(-1) + \left(\frac{-a}{2}\right)\sqrt{a^2-a^2}\right) = \left(\frac{a^2}{2}.\frac{\pi}{2}+0\right) - \left(\frac{a^2}{2}\left(-\frac{\pi}{2}\right)-0\right) = \frac{a^2\pi}{4} + \frac{a^2\pi}{4}$$
$$= \frac{a^2\pi + a^2\pi}{4} = \frac{2a^2\pi}{4} = \frac{a^2\pi}{2}\ \ square\ unit.$$

- ***Differential equation:***
  An equation containing atleast one derivative of a dependent variable with respect to an independent variable is called differential equation. e.g.
  $y\frac{dy}{dx} + 2x = 0$ *and* $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0$
- ***Order of Differential equation:***
  The order of the differential equation is the order of the highest derivative in the equation.
  $y\frac{dy}{dx} + 2x = 0$ (*1st order differential equation*)
  $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0$ (*2nd order differential equation*)
- ***Degree of Differential equation:***
  The degree of a differential equation is the greatest power of the highest order derivative in the equation.
  $x\frac{d^4y}{dx^4} + \frac{dy}{dx} - x\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + 2x = 0$
  (*1st degree differential equation*)
  $x\left(\frac{d^4y}{dx^4}\right)^3 + \frac{dy}{dx} - x\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + 2x = 0$
  (*3rd degree differential equation*)
- ***General solution:***
  The solution of differential equation which contains arbitrary constants is called general solution.
- ***Particular solution:***
  The solution obtained from general solution by applying the initial conditions is called particular solution.
- ***Initial value conditions:***
  The arbitrary constants involving in the solution of a differential equation can be determine by the someven conditions. such conditions are called Initial
  TIT BIT: *General soltion of the differential equation of order n contains n arbitrary constants which can be determined by n initial values condtions.*

Exercise 3.8

Q.1: Check that each of the following equations written against the differential equation is its solution.

$\boldsymbol{i)\ x\frac{dy}{dx} = 1 + y}$

Prove that $\boldsymbol{y = cx - 1}$

SOLUTION:

$x\frac{dy}{dx} = 1 + y$

Separating the variables:

$x\,dy = (1 + y)dx$

$\frac{1}{1+y}dy = \frac{1}{x}\,dx$

Integrating both sides

$\int \frac{1}{1+y}dy = \int \frac{1}{x}\,dx$

$ln|1 + y| = ln|x| + ln|c|$

$ln|1 + y| = ln|cx|$

$1 + y = cx$

$y = cx - 1$

$\boldsymbol{ii)\ x^2(2y + 1)\frac{dy}{dx} - 1 = 0}$

prove that $\boldsymbol{y^2 + y = c - \frac{1}{x}}$

SOLUTION:

$x^2(2y + 1)\frac{dy}{dx} - 1 = 0$

Separating the variables:

$x^2(2y + 1)\frac{dy}{dx} = 1$

$(2y + 1)dy = \frac{1}{x^2}\,dx$

Integrating both sides

$\int(2y + 1)dy = \int x^{-2}\,dx$

$2\frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$

$y^2 + y = \frac{x^{-1}}{-1} + c$

$y^2 + y = -\frac{1}{x} + c$

$\boldsymbol{iii)\ y\frac{dy}{dx} - 1 = 0}$

prove that $\boldsymbol{y^2 + y = c - \frac{1}{x}}$

SOLUTION:

$x^2(2y + 1)\frac{dy}{dx} - 1 = 0$

Separating the variables:

$x^2(2y + 1)\frac{dy}{dx} = 1$

$(2y + 1)dy = \frac{1}{x^2}\,dx$

Integrating both sides

$\int(2y + 1)dy = \int x^{-2}\,dx$

$2\frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$

$y^2 + y = \frac{x^{-1}}{-1} + c$

$y^2 + y = -\frac{1}{x} + c$

$\boldsymbol{iv)\ \frac{1}{x}\frac{dy}{dx} - 2y = 0}$

Prove that $\boldsymbol{y = ce^{x^2}}$

SOLUTION:

$\frac{1}{x}\frac{dy}{dx} = 2y$

Separating the variables:

$\frac{1}{y}\,dy = 2x\,dx$

Integrating both sides

$\int \frac{1}{y}dy = \int 2x\,dx$

$ln|y| = 2\frac{x^2}{2} + c_1$

$ln|y| = x^2 + c_1$

$e^{ln|y|} = e^{x^2 + c_1}$

$y = e^{x^2} + e^{c_1}$

$y = e^{x^2} + c$

$\boldsymbol{v)\ \frac{dy}{dx} = \frac{y^2+1}{e^{-x}}}$

Prove $\boldsymbol{y = \tan(e^x + c)}$

SOLUTION:

$\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$

Separating the variables:

$\frac{1}{y^2+1}\,dy = \frac{1}{e^{-x}}dx$

$\frac{1}{1+y^2}\,dy = e^x dx$

Integrating both sides

$\int \frac{1}{1+y^2}dy = \int e^x\,dx$

$\tan^{-1} y = e^x + c$

$y = \tan(e^x + c)$

$\boldsymbol{Q.2:\ \frac{dy}{dx} = -y}$

SOLUTION:

$\frac{dy}{dx} = -y$

Separating the variables:

$\frac{1}{y}dy = -\,dx$

Integrating both sides

$\int \frac{1}{y}dy = -\int 1\,dx$

$\ln y = -x + c_1$

$e^{\ln y} = e^{-x + c_1}$

$y = e^{-x}\,e^{c_1}$

$y = c\,e^{-x}$

$\boldsymbol{Q.3:\ y\,dx + x\,dy = 0}$

SOLUTION:

$y\,dx + x\,dy = 0$

Separating the variables:

$x\,dy = -y\,dx$

$\frac{1}{y}\,dy = -\frac{1}{x}\,dx$

Integrating both sides

$\int \frac{1}{y}dy = -\int \frac{1}{x}\,dx$

$\ln y = -\ln x + \ln c$

$\ln y + \ln x = +\ln c$

$\ln(xy) = +\ln c$

$xy = c$

$\boldsymbol{Q.4:\ \frac{dy}{dx} = \frac{1-x}{y}}$

SOLUTION:

$\frac{dy}{dx} = \frac{1-x}{y}$

Separating the variables:

$y\,dy=(1-x)\,dx$

Integrating both sides

$\int y\,dy=\int(1-x)\,dx$

$\frac{y^2}{2}=x-\frac{x^2}{2}+c_1$

$y^2=2x-x^2+2c_1$

$y^2=x(2-x)+c$

Q.5: $\frac{dy}{dx}=\frac{y}{x^2}$

SOLUTION:

$\frac{dy}{dx}=\frac{y}{x^2}$

Separating the variables:

$\frac{1}{y}\,dy=\frac{1}{x^2}\,dx$

Integrating both sides

$\int\frac{1}{y}dy=\int x^{-2}\,dx$

$\ln y=\frac{x^{-2+1}}{-2+1}+c_1$

$\ln y=\frac{x^{-1}}{-1}+c_1$

$\ln y=-\frac{1}{x}+c_1$

$e^{\ln y}=e^{-\frac{1}{x}+c_1}$

$y=e^{-\frac{1}{x}}\ e^{c_1}$

$y=c\,e^{-\frac{1}{x}}$

Q.6: $\sin y\csc x\,\frac{dy}{dx}=1$

SOLUTION:

$\sin y\csc x\,\frac{dy}{dx}=1$

Separating the variables:

$\sin y\frac{1}{\sin x}\,dy=dx$

$\sin y\ \,dy=\sin x\,dx$

Integrating both sides

$\int\sin y\ \,dy=\int\sin x\ \,dx$

$-\cos y=-\cos x+c_1$

$-\cos y=-(\cos x-c_1)$

$\cos y=\cos x-c_1$

$\cos y=\cos x+c$

Q.7: $x\,dy+y(x-1)dx=0$

SOLUTION:

$x\,dy+y(x-1)dx=0$

Separating the variables:

$x\,dy=-y(x-1)dx$

$\frac{1}{y}\,dy=-\left(\frac{x-1}{x}\right)dx$

$\frac{1}{y}\,dy=-\left(1-\frac{1}{x}\right)\,dx$

Integrating both sides

$\int\frac{1}{y}dy=\int-1+\frac{1}{x}\,dx$

$\ln y=-x+\ln x+\ln c$

$\ln y=\ln(xc)-x$

$\ln y-\ln(xc)=-x$

$\ln\left(\frac{y}{xc}\right)=-x$

$e^{\ln\left(\frac{y}{xc}\right)}=e^{-x}$

$\frac{y}{xc}=e^{-x}$

$y=cx\,e^{-x}$

Q.8: $\frac{x^2+1}{y+1}=\frac{x}{y}\cdot\frac{dy}{dx}$

SOLUTION:

$\frac{x^2+1}{y+1}=\frac{x}{y}\cdot\frac{dy}{dx}$

Separating the variables:

$(x^2+1)y\,dx=x(y+1)dy$

$x(y+1)\,dy=(x^2+1)y\,dx$

$\frac{y+1}{y}\,dy=\frac{x^2+1}{x}\,dx$

$1+\frac{1}{y}\,dy=x+\frac{1}{x}\,dx$

Integrating both sides

$\int\left(1+\frac{1}{y}\right)dy=\int\left(x+\frac{1}{x}\right)\,dx$

$y+\ln y=\frac{x^2}{2}+\ln(x)+\ln c$

$y+\ln y=\frac{x^2}{2}+\ln(xc)$

$\ln y-\ln(xc)=\frac{x^2}{2}-y$

$\ln\left(\frac{y}{xc}\right)=\frac{x^2}{2}-y$

$e^{\ln\left(\frac{y}{xc}\right)}=e^{\frac{x^2}{2}-y}$

$\frac{y}{xc}=e^{\frac{x^2}{2}}\,e^{-y}$

$\frac{y}{e^{-y}}=xc\,e^{\frac{x^2}{2}}$

$y\,e^{y}=cx\,e^{\frac{x^2}{2}}$

Q.9: $\frac{1}{x}\frac{dy}{dx}=\frac{1}{2}\,(1+y^2)$

SOLUTION:

$\frac{1}{x}\frac{dy}{dx}=\frac{1}{2}\,(1+y^2)$

Separating the variables:

$\frac{1}{1+y^2}dy=\frac{1}{2}\,(x)dx$

Integrating both sides

$\int\frac{1}{1+y^2}\,dy=\frac{1}{2}\int\ x\,dx$

$\tan^{-1}y=\frac{1}{2}\,\frac{x^2}{2}+c$

$y=\tan\left(\frac{x^2}{4}+c\right)$

Q.10: $2x^2y\frac{dy}{dx}=x^2-1$

SOLUTION:

$2x^2y\frac{dy}{dx}=x^2-1$

Separating the variables:

$y\,dy=\frac{x^2-1}{2x^2}dx$

$y\,dy=\frac{1}{2}\left(\frac{x^2-1}{x^2}\right)dx$

Integrating both sides

$\int y\,dy=\frac{1}{2}\int\ \left(1-\frac{1}{x^2}\right)\,dx$

$\int y\,dy=\frac{1}{2}\int\ (1-x^{-2})\,dx$

$\frac{y^2}{2}=\frac{1}{2}\left(x-\frac{x^{-1}}{-1}\right)+c$

$y^2=\ x+\frac{1}{x}+c$

Q.11: $\frac{dy}{dx}+\frac{2xy}{2y+1}=x$

SOLUTION:

$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

Separating the variables:

$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$

$\frac{dy}{dx} = \frac{x(2y+1)-2xy}{2y+1}$

$\frac{dy}{dx} = \frac{2xy+x-2xy}{2y+1}$

$\frac{dy}{dx} = \frac{x}{2y+1}$

$(2y+1)dy = xdx$

Integrating both sides

$\int(2y+1)\,dy = \int x\,dx$

$2\frac{y^2}{2} + y = \frac{x^2}{2} + c$

$y(y+1) = \frac{x^2}{2} + c$

$Q.12$: $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2$

SOLUTION:

$\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2$

Separating the variables:

$x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$

$x^2(1-y)dy = -y^2(1+x)dx$

$\frac{1-y}{y^2}\,dy = -\frac{1+x}{x^2}dx$

$\frac{1}{y^2} - \frac{y}{y^2}\,dy = -\left(\frac{1}{x^2} + \frac{x}{x^2}\right)dx$

$\frac{1}{y^2} - \frac{1}{y}\,dy = -\left(\frac{1}{x^2} + \frac{1}{x}\right)dx$

$\left(y^{-2} - \frac{1}{y}\right)\,dy = -\left(x^{-2} + \frac{1}{x}\right)dx$

Integrating both sides

$\int\left(y^{-2} - \frac{1}{y}\right)\,dy = -\int\left(x^{-2} + \frac{1}{x}\right)\,dx$

$\frac{y^{-2+1}}{-2+1} - \ln y = -\left(\frac{x^{-2+1}}{-2+1} + \ln x\right) + c_1$

$\frac{y^{-1}}{-1} - \ln y = -\left(\frac{x^{-1}}{-1} + \ln x\right) + c_1$

$-\frac{1}{y} - \ln y = -\left(-\frac{1}{x} + \ln x\right) + c_1$

$\ln y + \frac{1}{y} = \left(-\frac{1}{x} + \ln x\right) - c_1$

$\ln y + \frac{1}{y} = \ln x - \frac{1}{x} + c$

$Q.13$: $\sec^2 x\,\tan y\,dx + \sec^2 y\,\tan x\,dy = 0$

SOLUTION:

$\sec^2 x\ \tan y\ dx + \sec^2 y\ \tan x\ dy = 0$

Separating the variables:

$\sec^2 y\ \tan x\ dy = \sec^2 x\ \tan y\ dx$

$\frac{\sec^2 y}{\tan y}\,dy = \frac{\sec^2 x}{\tan x}\ dx$

Integrating both sides

$\int\frac{\sec^2 y}{\tan y}\ dy = -\int\frac{\sec^2 x}{\tan x}\ dx$

$\ln(\tan y) = -\ln(\tan x) + \ln c$

$\ln(\tan y) + \ln(\tan x) = \ln c$

$\ln(\tan y\tan x) = \ln c$

$\tan y\tan x = c$

$Q.14$: $\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

SOLUTION:

$\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

Separating the variables:

$y - x\frac{dy}{dx} = 2y^2 + 2\frac{dy}{dx}$

$-x\frac{dy}{dx} - 2\frac{dy}{dx} = 2y^2 - y$

Multiplying both sides by -1

$x\frac{dy}{dx} + 2\frac{dy}{dx} = y - 2y^2$

$(x+2)\frac{dy}{dx} = y - 2y^2$

$(x+2)dy = y(1-2y)\ dx$

$\frac{1}{y(1-2y)}dy = \frac{1}{x+2}\ dx$

Integrating both sides

$\int\frac{(1-2y)+2y}{y(1-2y)}\ dy = \int\frac{1}{x+2}\ dx$

$\int\left[\frac{(1-2y)}{y(1-2y)} + \frac{2y}{y(1-2y)}\right]\ dy = \int\frac{1}{x+2}\ dx$

$\int\left[\frac{1}{y} + \frac{2}{(1-2y)}\right]\ dy = \int\frac{1}{x+2}\ dx$

$\int\frac{1}{y}\ dy - \int\frac{2}{2y-1}\ dy = \int\frac{1}{x+2}\ dx$

$\ln(y) + \ln(2y-1) = \ln(x+2) + \ln(c)$

$\ln\left(\frac{y}{2y-1}\right) = \ln c(x+2)$

$\frac{y}{2y-1} = c(x+2)$

$Q.15$: $1 + \cos x\tan y\frac{dy}{dx} = 0$

SOLUTION:

$1 + \cos x\tan y\frac{dy}{dx} = 0$

Separating the variables:

$\cos x\tan y\frac{dy}{dx} = -1$

$\tan y\ dy = -\frac{1}{\cos x}\ dx$

Integrating both sides

$\int\frac{-\sin y}{\cos y}\ dy = \int\sec x\ dx$

$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$

$\ln(\cos y) = \ln[c(\sec x + \tan x)]$

$\cos y = c(\sec x + \tan x)$

$Q.16$: $y - x\frac{dy}{dx} = 3\left(1 + x\frac{dy}{dx}\right)$

SOLUTION:

$y - x\frac{dy}{dx} = 3\left(1 + x\frac{dy}{dx}\right)$

Separating the variables:

$y - x\frac{dy}{dx} = 3 + 3x\frac{dy}{dx}$

$y - 3 = 3x\frac{dy}{dx} + x\frac{dy}{dx}$

$y - 3 = 4x\frac{dy}{dx}$

$4x\frac{dy}{dx} = y - 3$

$\frac{1}{y-3}\ dy = \frac{1}{4x}dx$

Integrating both sides

$\int\frac{1}{y-3}\ dy = \frac{1}{4}\int\frac{1}{x}\ dx$

$\ln(y-3) = \frac{1}{4}\ln x + \ln c$

$\ln(y-3) = \ln x^{\frac{1}{4}} + \ln c$

$\ln(y-3) = \ln\left(cx^{\frac{1}{4}}\right)$

$y - 3 = cx^{\frac{1}{4}}$

$y = 3 + cx^{\frac{1}{4}}$

***Q*.17**: $\sec x + \tan y \frac{dy}{dx} = 0$

SOLUTION:

$\sec x + \tan y \frac{dy}{dx} = 0$

Separating the variables:

$\tan y \frac{dy}{dx} = -\sec x$

$\tan y \ dy = -\sec x \ dx$

Integrating both sides

$\int \tan y \ dy = -\int \sec x \ dx$

$\int \frac{-\sin y}{\cos y} \ dy = \int \sec x \ dx$

$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$

$\ln(\cos y) = \ln[c(\sec x + \tan x)]$

$\cos y = c(\sec x + \tan x)$

***Q*.18**: $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

SOLUTION:

$(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$ *Separating the variables*:

$dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Integrating both sides

$\int 1 \ dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

$y = \ln(e^x + e^{-x}) + c$

Q*.19**: ***Find the general solution of the following equation $\frac{dy}{dx} - x = xy^2$***. Also find the Particular solution if*** $y = 1$ ***when*** $x = 0$

SOLUTION:

$\frac{dy}{dx} - x = xy^2$

Separating the variables:

$\frac{dy}{dx} = xy^2 + x$

$\frac{dy}{dx} = x(y^2 + 1)$

$\frac{1}{y^2+1} dy = x \ dx$

Integrating both sides

$\int \frac{1}{1+y^2} \ dy = \int x \ dx$

$\mathrm{Tan}^{-1} y = \frac{x^2}{2} + c$

(General solution) (1)

At $x = 0$, $y = 1$

$\tan^{-1}(1) = \frac{(0)^2}{2} + c$

$\frac{\pi}{4} = c$ *(Put in* 1*)*

$\tan^{-1} y = \frac{x^2}{2} + \frac{\pi}{4}$

(Particular solution)

Q*.20**: ***Solve the differential equation $\frac{dx}{dt} = 2x$ ***given that*** $x = 4$ ***when*** $t = 0$

SOLUTION:

$\frac{dx}{dt} = 2x$

Separating the variables:

$dx = 2x \ dt$

$\frac{1}{x} dx = 2 \ dt$

Integrating both sides

$\int \frac{1}{x} \ dy = 2\int 1 \ dx$

$\ln x = 2t + c_1$

$e^{\ln x} = e^{2t+c_1}$

$x = e^{2t} \ e^{c_1}$

$x = ce^{2t}$ *Where* $e^{c_1} = c$

(General solution) (1)

At $x = 4$, $t = 0$

$4 = ce^{2(0)}$

$4 = ce^0$

$4 = c$ *Put in* (1) $\because e^0 = 1$

$x = 4e^{2t}$

(Particular solution)

Q*.21**: ***Solve the differential equation $\frac{ds}{dt} + 2st = 0$***. Also find the Particular solution if*** $s = 4e$ ***when*** $t = 0$

SOLUTION:

$\frac{ds}{dt} + 2st = 0$

Separating the variables:

$\frac{ds}{dt} = -2st$

$\frac{1}{s} ds = -2t \ dt$

Integrating both sides

$\int \frac{1}{s} ds = -\int 2t \ dx$

$\ln s = -2\frac{t^2}{2} + c_1$

$\ln s = -t^2 + c_1$

$s = e^{-t^2 + c_1}$

$s = e^{-t^2} \ e^{c_1}$

$s = ce^{-t^2}$ *Where* $e^{c_1} = c$

(General solution) (1)

At $s = 4e$, $t = 0$

$4e = ce^{-(0)^2}$

$4e = ce^0$

$4e = c$ *Put in* (1) $\because e^0 = 1$

$s = 4e \,.\, e^{-t^2}$

$s = 4 \, e^{1-t^2}$

(Particular solution)

Q22. In a culture, bacteria increase number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution:

Let P be numbers of bacteria then

$$\frac{dP}{dt} \propto P$$

⇨ $\frac{dP}{dt} = kP$

⇨ $\frac{1}{P} \ dP = kdt$

take integral

⇨ $\int \frac{1}{P} dP = k \int dt$

⇨ $lnP = kt + lnc$

⇨ $lnp - lnc = kt$

⇨ $\ln \frac{p}{c} = kt$

⇨ $\frac{P}{c} = e^{kt}$

⇨ $p = ce^{kt} \rightarrow (i)$

put p=200 , t=0 (condition1)

$200 = ce^{k(0)} = ce^0$

⇨ $c = 200 \quad \because e^0 = 1$

So $(i) p = 200e^{kt} \rightarrow (ii)$

Put$p = 400$ *when* $t = 2$ $(conditionII)$

$so(i) \quad \Rightarrow \quad 400 = 200e^{kt}$

⇨ $2 = e^{kt} \Rightarrow ln2 = lne^{2k}$

⇨ $2k = ln2$

⇨ $k = \frac{1}{2} ln2$

So $(ii) \Rightarrow p = 200e^{\frac{1}{2}ln2}$

⇨ $p = 200e^{\frac{ln2}{2}(4)} \quad for\ t = 4$

⇨ $p = 200^{2ln2} = 200e^{ln2^2} = 200e^{ln4}$

⇨ $p = 200(4) \Rightarrow p = 800$

Which is required number of bacteria present four latter.

Q.23 a ball is thrown vertically upward with a velocity of 2450cm/sec neglecting air resistance, find

i. Velocity of ball at any time t

ii. Distance traveled in any time t

iii. Maximum height attained by the ball

Solution:

Let v *is velocity and* g *is acceleration, so*

$$i) \ \frac{dv}{dt} = -g \quad for\ upward$$

⇨ $dv = -gdt$

⇨ $\int dv = -g \int dt$

⇨ $v = -gt + c_1$

Put $v = 2450 , t = 0$ *so*

$2450 = -g(0) + c_1 \Rightarrow c_1 = 2450$

$v = -gt + 2450 \quad \because g = 9.8m/sec$

Thus $v = -980t + 2450 \quad \Rightarrow g = 980cm/sec$

ii) *let h be height so*

$$v = \frac{dh}{dt}$$

⇨ $\frac{dh}{dt} = v$

⇨ $\frac{dh}{dt} = -980 + 2450$

⇨ $dh = -980tdt + 2450dt$

⇨ $\int dh = -980 \int tdt + 2450 \int dt$

⇨ $h = -980\frac{t^2}{2} + 2450t + c_2$

put $h = 0 , t = 0$

$0 = -490(0)^2 + 2450(0) + c_2$

$\Rightarrow c_2 = 0$

so $h = -490t^2 + 2450$

(iii) *For max. hight,* $v = 0$

So $0 = -980t^2 + 2450$ *from* (i)

⇨ $980t = \frac{2450}{980}$

⇨ $t = \frac{5}{2}$

So $h = 2450\left(\frac{5}{2}\right) - 490\left(\frac{5}{2}\right)^2$

$= 6125 - 30625$

$\Rightarrow h = 3062.5$

So max. hight $= 3062.5cm$

⇨ max *hight* $= 30.6m \ (\div by\ 100)$

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