Nature of Roots (Page 165)

The roots of the quadratic equation

$$ax^2+bx+c=0$$

are

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(Where we take a,b & c as rational)

The nature of the roots of an equation depends on the value of the expression b^2-4ac called *discriminant.*

Case I: If $b^2-4ac = 0$

Then roots of the equation are $-\frac{b}{2a}$ and $-\frac{b}{2a}$. So the roots are real (rational) and repeated equal.

Case II: If $b^2-4ac < 0$

Then the roots are complex/imaginary and distinct/unequal.

Case III: If $b^2-4ac > 0$

Then the roots are real and distinct/unequal.

However, if b^2-4ac is a perfect square then $\sqrt{b^2-4ac}$ will be rational and so the roots are rational and unequal. And if b^2-4ac is not a perfect square then $\sqrt{b^2-4ac}$ will be irrational and so the roots are irrational and unequal.

Question # 1

Discuss the nature of the roots of the following equations:

(i) $4x^2+6x+1 = 0$

(ii) $x^2-5x+6 = 0$

(iv) $25x^2-30x+9 = 0$

Solution

(i)

$$4x^2+6x+1 = 0$$

Here $a=4$, $b=6$, $c=1$

Disc. $= b^2-4ac$

$= (6)^2-4(4)(1) = 36-16$

$= 20 > 0$

Discriminant is not perfect square therefore the roots are irrational (real) and unequal.

(ii) $x^2-5x+6 = 0$

$a=1$, $b=-5$, $x=6$

Disc. $= b^2-4ac$

$= (-5)^2-4(1)(6)$

$= 25-24 = 1 > 0$

Disc. is perfect square therefore roots are rational (real) and unequal.

(iii) *Do yourself as* (i)

(iv) $25x^2-30x+9 = 0$

$a=25$, $b=-30$, $c=9$

Disc. $= b^2-4ac$

$= (-30)^2-4(25)(9)$

$= 900-900 = 0$

$\therefore$ roots are rational (real) and equal.

Question # 2

Show that the roots of the following equations will be real:

(i) $x^2-2\left(m+\frac{1}{m}\right)x+3 = 0$

(ii) $(b-a)x^2+(c-a)x+(a-b) = 0$

Solution

(i)

$$x^2-2\left(m+\frac{1}{m}\right)x+3 = 0$$

Here $a=1$, $b=-2\left(m+\frac{1}{m}\right)$, $c=3$

Disc. $= b^2-4ac$

$= \left(-2\left(m+\frac{1}{m}\right)\right)^2-4(1)(3)$

$= 4\left(m^2+\frac{1}{m^2}+2\right)-12$

$= 4\left(m^2+\frac{1}{m^2}+2-3\right)$

$= 4\left(m^2+\frac{1}{m^2}-1\right)$

$= 4\left(m^2+\frac{1}{m^2}-2+1\right)$

$= 4\left(\left(m-\frac{1}{m}\right)^2+1\right) > 0$

Hence roots are real.

(ii)

$$(b-a)x^2+(c-a)x+(a-b) = 0$$

Here $A=b-c$, $B=c-a$, $C=a-b$

Disc. $= b^2-4ac$

$= (c-a)^2-4(b-c)(a-b)$

$= c^2+a^2-2ca-4\left(ab-b^2-ac+bc\right)$

$= c^2+a^2-2ac-4ab+4b^2+4ac-4bc$

$= \left(a^2+c^2+2ac\right)-4ab-4bc+4b^2$

$= (a+c)^2-4b(a+c)+(2b)^2$

$= (a+c-2b)^2 > 0$

Hence roots are real.

Question # 3

Show that the roots of the following equations will be rational:

(i) $(p+q)x^2-px-qb^2-4ac = 0$

Solution

(i) $(p+q)x^2-px-qb^2-4ac = 0$

Here $a=p+q$, $b=-p$, $c=-q$

Disc. $= b^2-4ac$

$= (-p)^2-4(p+q)(-q)$

$= p^2+4pq+4q^2$

$= (p+2q)^2$

$\therefore$ the roots are rational.

(ii) $px^2-(p-q)x-q = 0$

Do yourself

Question # 4

For what values of m will be roots of the following equations be equal?

(i) $(m+1)x^2+2(m+3)x+m+8 = 0$

Solution

(i) $(m+1)x^2+2(m+3)x+m+8 = 0$

$a=m+1$, $b=2(m+3)$, $c=m+8$

Disc. $= b^2-4ac$

$= \left(2(m+3)\right)^2-4(m+1)(m+8)$

$= 4\left(m^2+6m+9\right)-4\left(m^2+8m+m+8\right)$

$= 4\left(m^2+6m+9-m^2-8m-m-8\right)$

$= 4(-3m+1)$

For equal roots, we have

Disc. = 0

$\Rightarrow 4(-3m+1) = 0$

$\Rightarrow -3m+1 = 0$

$\Rightarrow 3m = 1 \Rightarrow \boxed{m = \frac{1}{3}}$

(ii) *Do yourself*

(iii) *Do yourself*

Question # 5

Show that the roots of $x^2+(mx+c)^2=a^2$ will be equal, if $c^2=a^2\left(1+m^2\right)$.

Solution

$x^2+(mx+c)^2 = a^2$

$\Rightarrow x^2+m^2x^2+2mcx+c^2-a^2 = 0$

$\Rightarrow x^2\left(1+m^2\right)+2mcx+c^2-a^2 = 0$

Here $A=1+m^2$, $B=2mc$, $C=c^2-a^2$

So Disc. $= B^2-4AC$

$=(2mc)^2-4\left(1+m^2\right)\left(c^2-a^2\right)$

$= 4m^2c^2-4\left(c^2-a^2+m^2c^2-m^2a^2\right)$

$= 4\left(m^2c^2-c^2+a^2-m^2c^2+m^2a^2\right)$

$= 4\left(-c^2+a^2+m^2a^2\right)$

For equal roots, we have

Disc. $= 0$

$-c^2+a^2+m^2a^2 = 0$

$\Rightarrow c^2 = a^2+m^2a^2$

$\Rightarrow c^2 = a^2\left(1+m^2\right)$

as required.

Question # 6

Show that the roots of $(mx+c)^2=4ax$ will be equal, if $c=\frac{a}{m}$; $m\neq 0$.

Solution

$(mx+c)^2 = 4ax$

$\Rightarrow m^2x^2+2mcx+c^2-4ax = 0$

$\Rightarrow m^2x^2+2(mc-2a)x+c^2 = 0$

$A=m^2$, $B=2(mc-2a)$, $C=c^2$

Disc. $= B^2-4AC$

$= \left[2(mc-2a)\right]^2-4m^2c^2$

$= 4\left(m^2c^2+4a^2-4amc-m^2c^2\right)$

$= 4\left(4a^2-4amc\right)$

For equal roots, we must have

Disc. $= 0$

$\Rightarrow 4\left(4a^2-4amc\right) = 0$

$\Rightarrow 16a(a-mc) = 0$

$\Rightarrow a-mc = 0 \quad \Rightarrow a = mc$

$\Rightarrow \frac{a}{m} = c$ or $\boxed{c = \frac{a}{m}}$

Question # 7

Prove that $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$ will have equal roots, if

$c^2 = a^2m^2 + b^2$; $a\neq 0, b\neq 0$.

Solution

$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow b^2x^2+a^2(mx+c)^2 = a^2b^2$

$\Rightarrow b^2x^2+a^2\left(m^2x^2+c^2+2mcx\right)-a^2b^2 = 0$

$\Rightarrow b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2-a^2b^2=0$

$$\Rightarrow \left(b^2+a^2m^2\right)x^2+2a^2mcx+a^2\left(c^2-b^2\right)=0$$

Here $A=b^2+a^2m^2$, $B=2a^2mc$, $C=a^2(c^2-b^2)$

Disc. $= B^2-4AC$

$$=(2a^2mc)^2-4(b^2+a^2m^2)\cdot a^2(c^2-b^2)$$

$$=4a^4m^2c^2-4a^2\left(c^2b^2-b^4+a^2c^2m^2-a^2b^2m^2\right)$$

$$=4a^2\left(a^2m^2c^2-c^2b^2+b^4-a^2c^2m^2+a^2b^2m^2\right)$$

$$=4a^2\left(-b^2c^2+b^4+a^2b^2m^2\right)$$

For equal roots we must have

Disc. $= 0$

$$\Rightarrow \quad 4a^2b^2\left(-c^2+b^2+a^2m^2\right) = 0$$

$$\Rightarrow \quad -c^2+b^2+a^2m^2 = 0 \qquad \because a\neq 0,\ b\neq 0$$

$$\Rightarrow \quad c^2 = a^2m^2+b^2$$

Question # 8

Show that the roots of the equation $\left(a^2-bc\right)x^2+2\left(b^2-ca\right)x+c^2-ab=0$ will be equal if either $a^3+b^3+c^3=3abc$ or $b=0$.

Solution

$$\left(a^2-ba\right)x^2+2\left(b^2-ac\right)x+c^2-ab = 0$$

$A=a^2-bc$, $B=2\left(b^2-ac\right)$, $C=c^2-ab$

Disc. $= B^2-4AC$

$$=\left[2\left(b^2-ac\right)\right]^2-4\left(a^2-bc\right)\left(c^2-ab\right)$$

$$=4\left(b^4+a^2c^2-2ab^2c\right) -4\left(a^2c^2-a^3b+bc^3-ab^2c\right)$$

$$=4\left(b^4+a^2c^2-2ab^2c -a^2c^2+a^3b+bc^3-ab^2c\right)$$

$$=4\left(a^3b+b^4+bc^3-3ab^2c\right)$$

$$=4b\left(a^3+b^3+c^3-3abc\right)$$

For equal roots, we must have

$$B^2-4AC = 0$$

$$\Rightarrow \quad 4b\left(a^3+b^3+c^3-3abc\right) = 0$$

$$\Rightarrow \quad 4b = 0 \quad \text{or} \quad a^3+b^3+c^3-3abc = 0$$

$$\Rightarrow \quad b = 0 \quad \text{or} \quad a^3+b^3+c^3 = 3abc$$

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