Exercise 4.4

Cube Roots of Unity.

$$x^3 = 1$$

$$x^3 - 1^3 = 0$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$(x-1)(x^2+x+1) = 0$$

$$x - 1 = 0 \quad , \qquad x^2 + x + 1 = 0$$

$$\boxed{x = 1}$$

real

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$x = \frac{-1 \pm \sqrt{3}\, i}{2} \quad \checkmark$$

$$\checkmark\ x = \frac{-1+\sqrt{3}\, i}{2} \quad , \qquad \checkmark\ x = \frac{-1-\sqrt{3}\, i}{2}$$

complex complex

So three cube roots of unity are

$$1, \ \frac{-1+\sqrt{3}\, i}{2}, \ \frac{-1-\sqrt{3}\, i}{2}$$

(complex roots of unity are conjugate of each other).

1- Each complex cube root of unity is square of other.

$$\checkmark \quad \left(\frac{-1+\sqrt{3}\, i}{2}\right)^2 = \frac{(-1)^2 + (\sqrt{3}\, i)^2 + 2(-1)(\sqrt{3}\, i)}{4} = \frac{1}{4}\left[1 + 3i^2 - 2\sqrt{3}\, i\right]$$

$$= \frac{1}{4}\left(1 - 3 - 2\sqrt{3}\, i\right) = \frac{1}{4}\left(-2 - 2\sqrt{3}\, i\right) = \frac{-1-\sqrt{3}\, i}{2} \quad \checkmark$$

and $$\left(\frac{-1-\sqrt{3}\, i}{2}\right)^2 = \frac{-1+\sqrt{3}\, i}{2} \quad . \qquad \checkmark$$

If we let $\omega = \frac{-1+\sqrt{3}\, i}{2}$, then $\omega^2 = \frac{-1-\sqrt{3}\, i}{2}$

2- $1 + \omega + \omega^2 = 0$

If $\omega = \frac{-1+\sqrt{3}\, i}{2}$, then

$$1 + \omega + \omega^2 = 1 + \frac{-1+\sqrt{3}\, i}{2} + \frac{-1-\sqrt{3}\, i}{2}$$

$$= \frac{2 - 1 + \sqrt{3}\, i - 1 - \sqrt{3}\, i}{2} = \frac{0}{2} = 0.$$

3- $1 \cdot \omega \cdot \omega^2 = 1$ $\quad (\omega^3 = 1)$.

$$1 \cdot \omega \cdot \omega^2 = 1 \cdot \left(\frac{-1+\sqrt{3}\, i}{2}\right)\left(\frac{-1-\sqrt{3}\, i}{2}\right) = \frac{(-1)^2 - (\sqrt{3}\, i)^2}{4} = \frac{1 - 3i^2}{4} = \frac{1+3}{4} = \frac{4}{4} = 1.$$

Fourth roots of unity

$$x^4 = 1.$$

$$x^4 - 1 = 0$$

$$(x^2)^2 - 1^2 = 0$$

$$(x^2 + 1)(x^2 - 1) = 0$$

$$x^2 + 1 = 0 \quad , \quad x^2 - 1 = 0$$

$$x^2 = -1 \quad , \quad (x+1)(x-1) = 0$$

$$x = \pm\sqrt{-1} \quad , \quad x + 1 = 0 \quad , \quad x - 1 = 0$$

$$x = \pm i \quad , \quad x = -1 \quad , \quad x = 1$$

$$x = i \quad , \quad x = -i$$

Four fourth roots of unity are $\underbrace{1, -1}_{\text{real}}, \underbrace{i, -i}_{\text{complex.}}$

Sum of four fourth roots of unity is zero.

$$1 + (-1) + i + (-i) = 1 - 1 + i - i = 0$$

Product of four fourth roots of unity is -1.

$$(1)(-1)(i)(-i) = (1)(i^2) = -1$$

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Suppose Math.

1- Find the three cube roots of $8, -8, 27, -27, 64$.

(i) We solve $x^3 = 8$

$x^3 - 8 = 0$

$x^3 - 2^3 = 0$

$(x-2)(x^2+2x+4) = 0$

$x-2=0$, $x^2+2x+4=0$

$x=2$ ✓

$$x = \frac{-2 \pm \sqrt{2^2-4(1)(4)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm \sqrt{4\times3}\, i}{2}$$

$$x = \frac{-2 \pm 2\sqrt{3}\, i}{2}$$

$$x = 2\left(\frac{-1 \pm \sqrt{3}\, i}{2}\right)$$

$x = 2\left(\frac{-1+\sqrt{3}\, i}{2}\right)$, $x = 2\left(\frac{-1-\sqrt{3}\, i}{2}\right)$

$x = 2\omega$, $x = 2\omega^2$

So three cube roots of 8 are $2, 2\omega, 2\omega^2$.

(ii) $x^3 = -8$

$x^3 + 8 = 0$

$x^3 + 2^3 = 0$ $\qquad a^3+b^3=(a+b)(a^2-ab+b^2)$

$(x+2)(x^2-2x+4) = 0$

$x+2=0$, $x^2-2x+4=0$

$x=-2$

$$x = \frac{2 \pm \sqrt{(-2)^2-4(1)(4)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{-12}}{2}$$

$$x = \frac{2 \pm \sqrt{4\times3}\, i}{2}$$

$$x = \frac{2 \pm 2\sqrt{3}\, i}{2}$$

$$x = -2\left(\frac{-1 \mp \sqrt{3}\, i}{2}\right)$$

$x = -2\left(\frac{-1-\sqrt{3}\, i}{2}\right)$, $x = -2\left(\frac{-1+\sqrt{3}\, i}{2}\right)$

$x = -2\omega^2$, $x = -2\omega$

So three cube roots of -8 are $-2, -2\omega, -2\omega^2$.

(iii) $x^3 = 27$

$x^3 - 27 = 0$

$x^3 - 3^3 = 0$

$(x-3)(x^2+3x+9) = 0$

$x-3=0$, $\quad x^2+3x+9=0$

$x=3$, $\quad x = \dfrac{-3 \pm \sqrt{3^2-4(1)(9)}}{2(1)}$

$x = \dfrac{-3 \pm \sqrt{-27}}{2}$

$x = \dfrac{-3 \pm \sqrt{9\times3}\,i}{2}$

$x = \dfrac{-3 \pm 3\sqrt{3}\,i}{2}$

$x = \left(\dfrac{-1 \pm \sqrt{3}\,i}{2}\right)$

$x = 3\omega, \quad x = 3\omega^2$

So cube roots of 27 are 3, 3ω and $3\omega^2$.

(iv) $x^3 = -27$

$x^3 + 27 = 0$

$x^3 + 3^3 = 0$

$(x+3)(x^2-3x+9) = 0$

$x+3=0$, $\quad x^2-3x+9=0$

$x=-3$, $\quad x = \dfrac{3 \pm \sqrt{(-3)^2-4(1)(9)}}{2(1)}$

$x = \dfrac{3 \pm \sqrt{-27}}{2}$

$x = \dfrac{3 \pm \sqrt{9\times3}\,i}{2}$

$x = \dfrac{3 \pm 3\sqrt{3}\,i}{2}$

$x = -3\left(\dfrac{-1 \pm \sqrt{3}\,i}{2}\right)$

$x = -3\omega, \quad x = -3\omega^2$

So cube roots of -27 are -3, -3ω and $-3\omega^2$.

(v) $x^3 = 64$

$x^3 - 64 = 0$

$x^3 - 4^3 = 0$

$(x-4)(x^2+4x+16) = 0$

$x-4=0$, $\quad x^2+4x+16=0$

$x=4$, $\quad x = \dfrac{-4 \pm \sqrt{4^2-4(1)(16)}}{2(1)} = \dfrac{-4 \pm \sqrt{-48}}{2}$

$x = \dfrac{-4 \pm \sqrt{16\times3}\,i}{2} = \dfrac{-4 \pm 4\sqrt{3}\,i}{2}$

$x = 4\left(\dfrac{-1 \pm \sqrt{3}\,i}{2}\right)$

$x = 4\omega, \quad 4\omega^2$

So cube roots of 64 are 4, 4ω and $4\omega^2$.

2. Evaluate

(i) $(1+\omega-\omega^2)^8$

$1+\omega+\omega^2 = 0$ ✓

$\omega^3 = 1$

$= (-\omega^2-\omega^2)^8$

$= (-2\omega^2)^8$

$2\times2\times2\times2\times2\times2\times2\times2$

$= (-2)^8(\omega^2)^8$

$= 2^8\ \omega^{16}$

$= 256\ \omega^{15}\cdot\omega$

$= 256\ (\omega^3)^5\cdot\omega$

$= 256\ (1)^5\cdot\omega$

$= 256\ \omega.$

(ii) $\omega^{28}+\omega^{29}+1$

$= \omega^{27}\cdot\omega + \omega^{27}\cdot\omega^2 + 1$

$= (\omega^3)^9\omega + (\omega^3)^9\omega^2 + 1$

$= (1)^9\omega + (1)^9\omega^2 + 1$

$= \omega + \omega^2 + 1$

$= 0.$

$1+\omega+\omega^2 = 0 \longrightarrow \omega^2 = -1-\omega$

$1+\omega = -\omega^2$

(iii) $(1+\omega-\omega^2)(1-\omega+\omega^2)$ ✓

$= (-\omega^2-\omega^2)(1-\omega-1-\omega)$

$= (-2\omega^2)(-2\omega)$

$= 4\omega^3$

$= 4(1)$

$= 4.$

(iv) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9 + \left(\frac{-1-\sqrt{-3}}{2}\right)^9$ ✓

$= \omega^9 + (\omega^2)^9$

$= \omega^9 + \omega^{18}$

$= (\omega^3)^3 + (\omega^3)^6$

$= (1)^3 + (1)^6$

$= 1 + 1$

$= 2.$

For videos, visit YouTube

Suppose Math.