This is a Title with Each Word Capitalized and Support α, β and Break of Line.

A thesis
submitted in partial fulfillment of the
requirement for the degree of
Master of Science in Mathematics

by

DDD Umer Din
CIIT/FA14-RMT-001/ATK

COMSATS INSTITUTE OF INFORMATION TECHNOLOGY
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COMSATS INSTITUTE OF INFORMATION TECHNOLOGY ATTOCK CAMPUS

FINAL APPROVAL

This thesis titled

This is a Title with Each Word Capitalized and Support α, β and Break of Line.

submitted to the Department of Mathematics

by

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Contents

Abstract

A function is convex if the line segment joining two points on the graph lies above the graph. These functions have important properties and applications in mathematics. Specially, they are very important in optimization and minimization problems. Also these functions are used in statistic and functional analysis. A positive function f is logarithmic convex if log f is convex. It would seem that log convex functions unremarkable because they are so simply related to convex functions. But they have some surprising properties.

we consider the difference due to the difference in Giaccardi's inequality, and prove the logarithmic convexity and Lyapunov type inequality of these functionals for different classes of functions.

In the first chapter, we organize some basic notions and results.

In the second chapter we consider the difference of Giaccardi's type inequality for convex functions and prove the logarithmic convexity and Lyapunov type inequality of this functional by considering the class of convex functions

In the third chapter we consider the difference of Giaccardi's type inequality for star-shaped functions and prove the logarithmic convexity and Lyapunov type inequality of this functional by considering the class of star-shaped functions

In the fourth chapter we generalize results for logarithmic convexity of Giaccardi's difference for classes of functions with the help of divided difference.

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Praise is to ***Almighty Allah***, *WHO* is Lord of the world, the Answerer of prayers and the Source of peace, whose blessing and exaltation flourished to the scared wealth of knowledge.

Special praises and regards for His Last Messenger, Holy Prophet ***Hazrat Muhammad (PBUH)***. Holy Prophet said that I AM the light, whoever follows ME, will never be in the darkness.

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Student Name

Notations

The notation and concepts used in this monograph are more or less specified. The reader is assumed to be familiar with the elements of Mathematical Analysis, as well as General Algebra, Matrix Theory and Topology, and since the standard notation and concepts were used, it was believed unnecessary to define all of them.

We give some of the Notation used in the Monograph.

$\mathbb{Z}$	the set of integer
$\mathbb{N}$	the set of positive integer
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers

Chapter 1

Introduction

(i) First item

(ii) Second item

(iii) Third item

(iv) Fourth item

$$\lim_{x\to\infty}\sqrt{b^2-4ac}\frac{n!}{r!\,(n-r)!}$$

$\lim_{x\to\infty}\sqrt{b^2-4ac}\frac{n!}{r!(n-r)!}\frac{n!}{r!(n-r)!}\frac{-b\pm\sqrt{b^2-4ac}}{2a}\sqrt{b^2-4ac}$

In this chapter we will give the notions and definitions of monotone function, convex function, some important inequalities and properties which will frequently use in the rest of chapters to proof our results.

1.1 Monotone function

The monotone function are such function which maintain the order of inequalities. It is often defined on the interval I, where I is also an order set (see [10, 24]).

Definition 1.1.1. A function $f : I \to \mathbb{R}$ is said to be nondecreasing (respectively nonincreasing) if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$ (respectively $f(x_1) \geq f(x_2)$) for x_1, $x_2 \in I$. We say that f is increasing (respectively decreasing) if $x_1 < x_2$ implies $f(x_1) < f(x_2)$ (respectively $f(x_1) > f(x_2)$) for all x_1, $x_2 \in I$.

Example 1.1.2. The function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^2$ is increasing on $[0, \infty)$, while $f : (-\infty, 0] \to \mathbb{R}$ given by $f(x) = x^2$ is decreasing on $(-\infty, 0]$

Definition 1.1.3. A function is said to be monotone on an interval I, if it is either increasing or decreasing.

The following criteria is often used to investigate the monotonicity of function (see [22, 23]).

Proposition 1.1.4. *Suppose that a function f is continuous on the closed interval $[a, b]$ and has a derivative at each point of the open interval (a, b).*

1. *If $f'(x)$ is positive for all x in (a, b), then f is increasing function on $[a, b]$.*
2. *If $f'(x)$ is negative for all x in (a, b), then f is decreasing function on $[a, b]$.*

1.2 Convex function

Convex geometry as a new field of mathematics takes it origin from the publication of the book by Minkowski [19]. This book influenced the formation of a new field in mathematics, viz., functional analysis [12]. Convex function are closely related to the theory of inequalities, and many important inequalities are the consequence of the application of convex function. For example, the important Arithmetic mean-Geometric mean inequality [8] or the general inequality between between means of r and s, such as Hölder's [21] and Minkowski's inequality, are all consequence of Jensen inequality [6] for convex function [19]. In [15] the following definition of convex function is given.

Definition 1.2.1. A function $f : I \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.2.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.
A function f is said to strictly convex on I if (1.2.1) is strict for $x \neq y$.

If f is convex function on I, then for $x_1 < x_2 < x_3$ the inequality

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3} \tag{1.2.2}$$

holds.

Lemma 1.2.2. *This is an example of lemma $x + 2y$ and this is good.*

Proposition 1.2.3. *This is an example of proposition* $x + 2y$ *and this is good.*

Theorem 1.2.4. *This is an example of thorem* $x + 2y$ *and* $2x - y$ *and this is good.*

Corollary 1.2.5. *This is an example of thorem* $x + 2y$ *and* $2x - y$ *and this is good.*

(1) This is first

(2) This is second

(3) This is third

Here is one more.

A- This is first

B- This is second

C- This is third

Chapter 2

This is a 2nd chapter

Here we can write.

2.1 Introduction

Let $I = [0, a) \subset \mathbb{R}$ be an interval, $(x_1, \ldots, x_n) \in I^n$ and $(p_1, \ldots, p_n)$ be a non-negative n-tuple such that

$$\sum_{i=1}^{n} p_i x_i \geq x_j \text{ for } j = 1, \ldots, n \text{ and } \sum_{i=1}^{n} p_i x_i \in I. \tag{2.1.1}$$

If $f : I \to \mathbb{R}$ be a convex function, then

$$f(\sum_{i=1}^{n} p_i x_i) \leq \sum_{i=1}^{n} p_i f(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) f(0). \tag{2.1.2}$$

The above inequality is known as Petrović inequality [20] (see also [19, p. 154]). There is a lot of literature available on Petrović inequality, for example, (see [19, 9]).

The generalization of Petrović inequality is given by F. Giaccardi [19]. It is given in the following theorem.

Lemma 2.1.1. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval , $(p_1, ..., p_n)$ be a non-negative n-tuple, $(x_1, ..., x_n)$ be n-tuple in I^n and $x_0 \in I$ such that $\tilde{x}_n := \sum_{k=1}^{n} p_k x_k \in I$ and*

$$(x_i - x_0)(\tilde{x}_n - x_i) \geq 0 \text{ for } i = 1, ..., n, \quad \tilde{x}_n \neq x_0. \tag{2.1.3}$$

If f is a convex function, then

$$\sum_{k=1}^{n} p_k f(x_k) \leq A f(\tilde{x}_n) + B f(x_0), \tag{2.1.4}$$

holds, where

$$A = \frac{\sum_{k=1}^{n} p_k \left(x_k - x_0\right)}{\tilde{x}_n - x_0}, B = \frac{\left(\sum_{k=1}^{n} p_k - 1\right) \tilde{x}_n}{\tilde{x}_n - x_0}. \tag{2.1.5}$$

2.1.1 This is a subsection

If f is a convex function, then

$$\sum_{k=1}^{n} p_k f\left(x_k\right) \leq A f\left(\tilde{x}_n\right) + B f\left(x_0\right), \tag{2.1.6}$$

holds, where

$$A = \frac{\sum_{k=1}^{n} p_k \left(x_k - x_0\right)}{\tilde{x}_n - x_0}, B = \frac{\left(\sum_{k=1}^{n} p_k - 1\right) \tilde{x}_n}{\tilde{x}_n - x_0}. \tag{2.1.7}$$

Notation 2.1.2. This is notation and I like it N

2.2 1st section

Gdgasdfg adsklfjasdlkf jlkasdjf lkja sdlfjlkasdj flkasd

2.2.1 Sub section

adfasdfsad fa fsdaf sdafasdfds fds

2.2.1.1 Sub sub section

adsfasdf adf asdf adsf df daf asdfs

Chapter 3

Properties of Giaccardi's difference for star-shaped function

3.1 Introduction

In this chapter we consider the Giaccardi's type inequality under conditions, which are weaker than the conditions given in the previous chapter. In [18], the following Giaccardi's type inequality is given.

Theorem 3.1.1. *Let $f : I \to \mathbb{R}$, where I is an interval. Let $(p_1, ..., p_n)$ be a positive n-tuple and $(x_1, ..., x_n)$ be a real n-tuple, $x_0, \tilde{x}_n = \sum_{k=1}^{n} p_k \in I$ such that $x_i \neq x_0$ $(i = 1, ..., n)$ and*

$$(x_i - x_0)(\tilde{x}_n - x_i) \geq 0 \text{ for } i = 1, ..., n, \quad \tilde{x}_n \neq x_0. \tag{3.1.1}$$

If $\frac{f(x)}{x-x_0}$ is increasing function for $x \in I \setminus \{x_0\}$, then

$$\sum_{k=1}^{n} p_k f(x_k) \leq A f(\tilde{x}_n) \tag{3.1.2}$$

holds, where

$$A = \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\tilde{x}_n - x_0}. \tag{3.1.3}$$

Proof. Let $x, y \in I$, with $x < y$. Then

$$\frac{f(x)}{x - x_0} \leq \frac{f(y)}{y - x_0}, \tag{3.1.4}$$

this gives

$$(y - x_0) f(x) \leq (x - x_0) f(y). \tag{3.1.5}$$

Take $x = x_k$, $y = \tilde{x}_n$, where $x_0 < x_k < \tilde{x}_n$, we have

$$(\tilde{x}_n - x_0) f(x_k) \leq (x_k - x_0) f(\tilde{x}_n).$$

This implies that

$$(\tilde{x}_n - x_0) \sum_{k=1}^{n} p_k f(x_k) \leq f(\tilde{x}_n) \sum_{k=1}^{n} p_k (x_k - x_0),$$

or

$$\sum_{k=1}^{n} p_k f(x_k) \leq \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\tilde{x}_n - x_0} f(\tilde{x}_n).$$

This is equivalent to (3.1.2).

Now take $x = \tilde{x}_n$, $y = x_k$, i.e $\tilde{x}_n < x_k < x_0$ in (3.1.5), we have

$$(x_k - x_0) f(\tilde{x}_n) \leq (\tilde{x}_n - x_0) f(x_k). \tag{3.1.6}$$

Since $\tilde{x}_n - x_0 < 0$, therefore above inequality takes the form

$$f(x_k) \leq \frac{(x_k - x_0)}{(\tilde{x}_n - x_0)} f(\tilde{x}_n). \tag{3.1.7}$$

This implies that

$$\sum_{k=1}^{n} p_k f(x_k) \leq \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\tilde{x}_n - x_0} f(\tilde{x}_n).$$

By using the value of A defined in (3.1.3), we have the result. □

3.2 Main results

In this section we consider a functional due to the difference for Giaccardi's type inequality (3.1.2). We consider different classes of star-shaped type functions to prove logarithmic convexity of the functional. Also we derive Lyapunov type inequality for that functional.

Let $f : I \to \mathbb{R}$, where I is an interval. Let $\mathbf{p} = (p_1, ..., p_n)$ be a positive n-tuple and $\mathbf{x} = (x_1, ..., x_n)$ be a real n-tuple.

Let us consider the following Giaccardi's difference.

$$\Psi(\mathbf{x}; \mathbf{p}; f) = A f(\tilde{x}_n) - \sum_{k=1}^{n} p_k f(x_k). \tag{3.2.1}$$

Remark 3.2.1. Let $f : I \to \mathbb{R}$, where I is an interval. Let $\mathbf{p} = (p_1, ..., p_n)$ be a positive n-tuple and $\mathbf{x} = (x_1, ..., x_n)$ be a real n-tuple, $x_0, \tilde{x}_n = \sum_{k=1}^{n} p_k \in I$ such that $x_i \neq x_0$ $(i = 1, ..., n)$ and (3.1.1) satisfied. If $\frac{f(x)}{x - x_0}$ is increasing function for $x \in I \setminus \{x_0\}$, then

$$\Psi(\mathbf{x}; \mathbf{p}; f) \geq 0. \tag{3.2.2}$$

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