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						$I = \sum m_i r_i^2$			Dr	Ĩ	Prepar
						. Amir Mah			Dr Ar		Prepar
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ir	Ma	hmoo Proof:	The	Prepa e angula	ared r mome	entum	Dr.	Amir a rigid	body, in	the for	n of a set	red	by: article	Dr. es. abo	out an	Ma	hmood ntaneous a	Prepar xis
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r	Ma	h≓nod	<sub>x</sub> i -	$+L_y \mathbf{j} + l$	$L_z \mathbf{k} = \sum_{i}$	m <sub>i</sub> [	$(x_i^2)$	$+ y_i^2 + 2$	$z_i^2)(\omega_x \mathbf{i} -$	+ ω <sub>y</sub> j +	$\omega_z \mathbf{k} - (\mathbf{x})$	$\alpha_i \omega_x$ -	$+ y_i \omega_y$	$+ z_i c$	$v_z)(x_i \mathbf{i})$	$+ y_i \mathbf{j}$	$+z_i\mathbf{k}$	Prepare
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	Comate Easy by: Dr. Amir Mahmoog	Prepared by: Di	A Moment of Iner	repare
ir Mahmoo	Prepared by $= \omega_x \sum_i m_i (y_i^2 + z_i^2) - \omega_y \sum_i m_i (y_i^2 + z_i^2) - \omega_y$	$\sum_{i} m_i x_i y_i - \omega_z \sum_{i} m_i x_i z_i$	r. Amir Mahmood	Prepar
ir Mahmoo	Prepared by: Dr. Amir Mahmoor $\Rightarrow L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xy}\omega_y$	Prepared by: D	Amir Mahmood	Prepar
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ir Mahmood	d Prepared by: Dr. Amir Mahmood	Prepared by: D	r. Anir Mahmood	Prepare
ir Mahmoo	d Preparec : Dr. $L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yy}\omega_y$	$y_z \omega_z \qquad  \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad$	5. Mir Mahmood	Prepar
ir Mahmoo	and $L_z = I_{xz}\omega_x + I_{yz}\omega_y + I$ Eqs. (4), (5) and (6) in matrix form, we get	$z_z \omega_z \longrightarrow (0)$	P. Anir Mahmood	Prepar
ir Mahmood	Eqs. $(4)$ , $(5)$ and $(6)$ in matrix form, we get	l Pr <del>opa</del> red by: Di	r. <mark>Am</mark> ir Mahmood	Prepar
ir Mahmood	Prepared by: Dr. $\operatorname{Am}\begin{pmatrix} L_x \\ L_y \end{pmatrix} = \begin{pmatrix} I_{xx} & I_x \\ I_{xy} & I_y \end{pmatrix}$	$V I_{xz} (\omega_x) d by: Di$	r. <del>Am</del> ir Mahmood	Prepare
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ir Mahmoo	ן Prepared by: Dr. <mark>A⇒m[נן צן נו]נש</mark> ססס	Hence proved.	Amir Mahmood	Prepar
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ir Mahmo	Rotational-Lin	ear Paralle	l <b>s</b> od	Prepar
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r Mahmoo	Linear Motion Rot	ational Motion	d Id	Prepare
r Mahmoc			d	Prepare
ir Mahmod	Position x 6		ular position	Prepare
r Mahmoc	Velocity V O		ular velocity	Prepare
r Mahmod	Acceleration a d			Prepare
r Mahmoo	Motion equations $x = v t$	$\theta = \omega t$ Mot	ion equations	Prepare
ir Mahmod	$v = v_0 + at$ a	$\omega = \omega_0 + \alpha t$		Prepar
ir Mahmod	$x = v_0 t + \frac{1}{2}at^2 \qquad \theta$	$=\omega_0 t + \frac{1}{2}\alpha t^2$	pd	Prepare
r Mahmoo		$2^{\alpha}$		Prepare
r Mahmoo	$v^2 = v_0^2 + 2ax \qquad a$	$\omega^2 = \omega_0^2 + 2\alpha\theta$	d)	Prepare
ir Mahmod	Mass (linear inertia) <i>M</i>	Mor	ment of inertia	Prepare
r Mahmoc	Newton's second law $F_{=}$ $ma$ 7	T= I 🛛 Nev	vton's second law	Prepare
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ir Mahmod	Kinetic energy $\frac{1}{2}mv^2$ $\frac{1}{2}$	Kine	etic energy od	Prepare
ir Mahmod	Power Fv 1	$\omega$ Pow	ver d	Prepare
r Mah <del>meed</del>	Prepared by: Dr. Amir Mahmood	Prepared by: Dr	Amir Mahmood	Prepare

Prepared by: Dr. Amir Mahood In Mahmood Dr. Amir Mal Page 3 of 29 repared by: Dr. Amir Mahmood Page 3 repared by: Dr. Amir Mahmood Page 3 Mechanics Made EasyPage 4 of 29Moment of Inertia<br/>Moment of InertiaProblem: Prove that  $T = \frac{1}{2}M\mathbf{v}^2 + \frac{1}{2}\omega$ . L, where all the notations used have their usual meanings.(or) prove that  $T = T_{tr} + T_{rot}$ where, $T_{tr} = \frac{1}{2}M\mathbf{v}^2 =$  total translational kinetic energy of the systemand $T_{rot} = \frac{1}{2}\omega$ . L = total rotational kinetic energy of the system

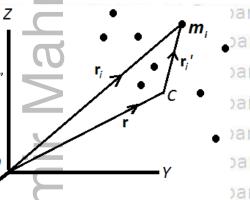
**Proof**: Consider a rigid body, in the form of a set of particles, which is in general state of motion (i.e, having both epartranslation and rotation) with respect to a fixed (inertial) frame of reference *Oxyz*.

r Mah Let, M = total mass of the body r. Amir M and Property of the body r.

 $\mathbf{r}_i$  = position vector of *i*-th particle of mass  $m_i$  with respect to origin "O"  $\mathbf{r}'_i$  = position vector of *i*-th particle of mass  $m_i$  with respect to centre of mass "C"

 $\mathbf{r}$  = position vector of centre of mass "*C*" with respect to origin "*O*"

 $\mathbf{v}_i$  = velocity of *i*-th particle of mass  $m_i$  with respect to origin "O"  $\mathbf{v}'_i$  = velocity of *i*-th particle of mass  $m_i$  with respect to centre of mass "C"



= velocity of centre of mass "C" with respect to origin "O' • = instantaneous angular velocity of body about instantaneous axis through centre of mass From figure, epared by: Dr. Amir Man  $\mathbf{r}_i = \mathbf{r} + \mathbf{r}'_i$  repared by: Dr. Amir Mahmo T Man Differentiating both sides with respect to time "t", we get P (D) and D (D) (Dir Mahmood Prepared by: Dr. Amir Mahmood r<sup>p</sup>repared by: Dr. Amir Mahm r Mahmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmo ir Mahmood Prepare  $\mathbf{v}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}'_i$ epar mood Prepare by: Dr. Am  $T_i = \frac{1}{2}m_i \mathbf{v}_i^2$ Kinetic energy of the *i*-th particle is Kinetic energy of the whole body is Prepared by: Dr. Am mood Prepared Amir Ma hmood  $= \sum T_i = \frac{1}{2} \sum m_i \mathbf{v}_i^2 = \frac{1}{2} \sum m_i \left( \mathbf{v}_i \cdot \mathbf{v}_i \right) = \frac{1}{2} \sum m_i \left\{ \left( \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i' \right) \cdot \left( \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i' \right) \right\}$  $\mathbf{v}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}'_i$ by: Dr. Amir  $(\boldsymbol{\omega} \times \mathbf{r}'_i) + (\boldsymbol{\omega} \times \mathbf{r}'_i) \cdot \mathbf{v} + (\boldsymbol{\omega} \times \mathbf{r}'_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}'_i)$  $\frac{1}{2} \sum m_i \{ \mathbf{v}^2 + 2\mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}'_i) + \boldsymbol{\omega} \cdot \mathbf{r}'_i \times (\boldsymbol{\omega} \times \mathbf{r}'_i) \} T \text{ red by: } Dr.$ nood Pr ebared by: Dr  $m_i \left| \mathbf{v}^2 + \right\rangle m_i \mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}'_i) + \frac{1}{2} \right\rangle m_i \{ \boldsymbol{\omega} \cdot \mathbf{r}'_i \times (\boldsymbol{\omega} \times \mathbf{r}'_i) \}$ 

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ir Mahmood	Prepared by:	$=\frac{1}{2}M\mathbf{v}^2+\mathbf{v}\cdot\left(\mathbf{u}\right)$	$\mathbf{v} \times \sum_{i} m_i \mathbf{r}'_i$	$+\frac{1}{2}\boldsymbol{\omega}\cdot\sum_{i}m_{i}\mathbf{r}_{i}^{\prime}$	× ( $\boldsymbol{\omega}$ × $\mathbf{r}_i^{\prime}$ )	Amir N	lahmood	Prepar
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r Mahmood	$\sum_i m_i \mathbf{r}'_i = 0$ , as $\mathbf{r}'_i$ i	Dr. Amir N	ector of <i>i</i> th par	Prepared ticle of mass m	by: Dr.	ect to cent	lahmood	Prepar
ir Mahmood	$\frac{1}{i}$ repared by:	Dr. Amir N	tahmood	Prepared	by: Dr.	Anir M	lahmood	Prepar
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ir Mahmood	Prepared by:	Dr <sup>2</sup> Amir <sup>2</sup>	14 mood	Prepared	by: Dr.	Amir N	lahmood	Prepar
	lar momentum <b>L</b> of t							
ir Mahmood	Prepared by:	Dr. Amir D	(ahmood	Prepared	by: Dr.	Amir N	lahmood	Prepar
ir Mahmood	$\mathbf{L} = \sum_{i} \mathbf{r}'_{i} \times (m_{i} \mathbf{v}'_{i}$							
ir Mahusing (2)	in (1), we get	Dr. Amir N	fahmood	Prepared	by: Dr	Amir N	lahmood	Prepar
r Mahmood	Prepared by:	Dr. Amir N	1ahmood	Prepared	by: Dr.	Amir N	lahmood	Prepar
ir Mahmood	Prepared by:	Dr. Amir 🛛	$T = -M\mathbf{v}^2 + \mathbf{v}^2 + \mathbf{v}$	2 <sup>-ω.L</sup> 2 <sup>-μ</sup> eppred	by: Dr.	Amir N	lahmood	Prepar
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r Mahmood	Find moment of in	Dr. Amir N	anmood	Prepared	by: Dr.	Amir M	lahmood	Prepare
r Mah direction	cosines are $(\lambda, \mu, \nu)$ .	Dr. Amir	shmood	given nne pa	by: Dr		lahmood	Prepare
r Mah Solution:	Consider a rigid bod	y, in the form o	f a set of parti	cles. And let us	take given	line as z-a	axis, as shown	Prepare
r Mah the figure	Prepared by:	Dr. Amir <mark>N</mark>	lahmood	Prepared	by:	$\sigma^{z}$	🕳 Given lir	ne/)are
r Mahnaepod	Prepared by:	Dr. Amir N	ehmood	Prepared	by: I	n l	d <sub>i</sub>	are
$r Mah_M = tota$	l mass of the body	Dr. Amir N	lahmood	Prepared	by:		• 7 "	bar
r Mahmood $\mathbf{r}_i = x_i \mathbf{i} + \mathbf{i}_i$	$y_i \mathbf{j} + z_i \mathbf{k} = \text{position}$	vector of <i>i</i> -th p	article of mass	Precared m; w.r.t. origin	1 <i>"O"</i>	ຕ/	<b>1</b> /	pare
r Mahmood	Prepare () endicular distance o	Dr. Amir IV	anmood	Prepared				bar
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ir Ivlanmood	e between position v	Dr. Amir 🎼	anmood	Prepared	by:			bar
r Mahmood	vector in the direction	n of given line <i>l</i>	lahmood	Prepared	by: Dr	Amir IV	lahmood	Prepare
r Mah Then, e	$= \lambda \mathbf{i} + \mu \mathbf{j} + \nu \mathbf{k}$ , where	e, ( $\lambda, \mu, \nu$ ) are di	rection cosine	s of the given li	ne l.: Dr.	Amir M	lahmood	Prepare
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Prepared	by: Dr. Amir Mahood	Dr. Amir N	Page 5 of 2	29 repared	by: Dr.	Amir N	Page 1ahmood	Prepar

ir Mahmood, Prepared by: Dr. Amir Mahmood, Prepared by: Dr. Amir Mahmood, Prepar Page 6 of 29 Mechanics Made Easy The required moment of inertia  $I_1$  is given by d bv: Dr. Amir Mahmood repared  $\sum m_i d_i^2 = \sum m_i (|\mathbf{r}_i| \sin \theta_i)^2 = \sum m_i (|\mathbf{e} \times \mathbf{r}_i|)$  $\sin \theta_i = \frac{\alpha_i}{|\mathbf{r}_i|}$  and  $|\mathbf{r}_i| \sin \theta_i = |\mathbf{e} \times \mathbf{r}_i|$ Amir enared by: Dr epared  $\mathbf{e} \times \mathbf{r}_i = \begin{vmatrix} \lambda & \mu & \nu \end{vmatrix} = (\mu z_i - \nu y_i)\mathbf{i} + (\nu x_i - \lambda z_i)\mathbf{j} + (\lambda y_i - \mu x_i)\mathbf{k}$  $Z_i$ Dr  $\lambda z_i)^2 + (\lambda y_i - \mu x_i)^2$  $\Rightarrow |\mathbf{e} \times \mathbf{r}_i| = (\mu z_i - \nu y_i)^2 + (\nu x_i - \nu y_i)^2 + (\nu x$ in (1), we get Prepared  $= \sum m_i [(\mu z_i)$  $(\nu y_i)^2 + (\nu x_i - \lambda z_i)^2 + (\lambda y_i)^2$  $- \mu x_i)^2$ ari r. Amir repared  $m_i[(\mu^2 z_i)]$  $2\mu\nu y_i z_i) + (\nu^2 x_i)$  $2\lambda v x_i z_i) + (\lambda^2 y_i^2)$  $-\sum m_i x_i y_i$  $= \lambda^2 \sum m_i (y_i^2 + z_i^2) + \mu^2 \sum m_i (x_i^2 + z_i^2) + \nu^2 \sum m_i (x_i^2 + z_i^2)$  $v_i^2$ ) +  $2\lambda\mu$  $-\sum m_i y_i z_i + 2\lambda v (-\sum m_i x_i z_i)$  $2\mu\nu$  $= \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda \mu I_{xy} + 2\mu \nu I_{\nu z} + 2\lambda \nu I_{xz}$ This is the required moment of inertia. Problem: Find the equation of "ellipsoid of inertia" or "momental ellipsoid" of a rigid body. **Solution**: As we know that moment of inertia of a rigid body about a given line *l* having direction cosines ( $\lambda$ ) with respect to a coordinate system Oxyz, whose origin "O" lies on the line l, is given by hmood Prepar $I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu\nu I_{yz} + 2\lambda\nu I_{xz} - - - -$ On the line l, choose a point P such that  $|\overline{OP}| = 1/\sqrt{I_l}$ . If coordinates of P are (x, y, z), then <sup>µ</sup>, recipied OP OP Eliminating  $\lambda$ ,  $\mu$  and  $\nu$  from (1) and (2), we get  $I_{l} = I_{l} (I_{xx}x^{2} + I_{yy}y^{2} + I_{zz}z^{2} + 2I_{xy}xy + 2I_{yz}yz$  $+2I_{xz}xz$ )  $I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{yz}yz + 2I_{yz}xz = 1$ Since,  $I_{xx}$ ,  $I_{yy}$  and  $I_{zz}$  are all positive, therefore, above equation represents an ellipsoid called "ellipsoid of inertia" or "momental ellipsoid" of the rigid body.mood Prepared by: Dr. Amir Mahmood Page 6 Mahmood Prepared by: Dr. Amir Mahood Page 6 of 29 repared by: Dr. Amir

r	N	lał	hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Pre Page 7 of 29	epar
r	M	ał	Mechanics Made Easy Moment of Inertia Note:	epar
r	N	lal	(i) The momental ellipsoid of a rigid body contains information about moments and product of inertia of that	epar
r	N	lał	body. Imood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Pre	epar
ŕ	N	la	( <i>ii</i> ) The centre of momental ellipsoid lies at the origin of the coordinate system.	epar
r	N	lał	( <i>ii</i> ) If <i>P</i> is any point on momental ellipsoid, then and a presented by: Dr. Shir Mahmood P	epar
r	N	la	hmood Prepared V: Dr. Amir $ \overline{OP}  = \frac{1}{\sqrt{I_l}} \Rightarrow d^l = \frac{1}{ \overline{OP} ^2}$ red by: Dr. Onir Mahmood Pr	epar
r	N	lal	showing that moment of inertia about line $\overrightarrow{OP}$ is equal to the reciprocal of square of distance of point P from	epar
r	N	lal	origin 0. hmood Prepared by: Dr. Amir Mahmood Propared by: Dr. Amir Mahmood Pr	epar
r	M	ał	Problem: State and prove perpendicular axis theorem for a set of particles.	epar
r	N	lal	hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Pre	epar
r	N	lał	<b>Statement</b> : The moment of inertia of a plane rigid body in the form of a set of particles about a given axis perpendicular to the plane of the body is equal to the sum of moments of inertia about two mutually	epar
ŕ	N	a	perpendicular axes lying in the plane of the body and meeting at a common point on the given axis.	epar
r	N	lał	<b>Proof</b> : We choose Cartesian coordinate system <i>Oxyz</i> such that <i>xy</i> -plane lies in the plane of the body, while <i>z</i> -axis	epar
r	N	la	lies perpendicular to it, which is assumed to the given axis. hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Pre	epar
r	N	lal	Let, $\mathbf{r}_i = x_i \mathbf{i} + y_i \mathbf{j}$ be the position vector of <i>i</i> -th particle of mass $m_i$ w.r.t. origin " $O$ ". Then moment of inertia of the badwebent e evision	epar
r	M	ał	the body about z-axis is of the Amir Manmood Prepared by: Dr. Amir	par
	M	ah	$I_{zz} = \sum m_i  \mathbf{r}_i ^2 = \sum m_i (x_i^2 + y_i^2) = \sum m_i x_i^2 + \sum m_i y_i^2 = I_{xx} + I_{yy}$	par
r	M	ał	$I_{zz} = \sum m_i  \mathbf{r}_i ^2 = \sum m_i (x_i^2 + y_i^2) = \sum m_i x_i^2 + \sum m_i y_i^2 = I_{xx} + I_{yy}$ moot Prepared by: Dr. Amir Mahmiood Prepared by: Dr. An	par
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r	M	ał	nmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Anz	par
r	M	ah	Problem: State and prove perpendicular axis theorem for a continuous mass distribution.	par
r	N	lał	Statement: The moment of inertia of a plane rigid body in the form of continuous mass distribution about a	epar
r	M	ał	given axis perpendicular to the plane of the body is equal to the sum of moments of inertia of same body about two mutually perpendicular axes lying in the plane of body and meeting at a common point on the given axis.	par
r	M	ał	nmood Prepare <del>d by</del> : Dr. Amir <del>Ma</del> hmood Prepared by: Dr <del>. Am</del> ir Mahmood Pre	par
r	M	ah	<b>Proof</b> : We choose Cartesian coordinate system <i>Oxyz</i> such that <i>xy</i> -plane lies in the plane of the body having mass <i>M</i> , while <i>z</i> -axis lies perpendicular to it, which is assumed to the given axis.	par
r	M	ał	Let, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of elementary particle of body of mass dm w.r.t. origin "0".	ar
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			Then moment of inertia of the body about z-axis is	ar
r	M	ał	nmood Prepared (); Dr. Amir Manmood Prepared by: Dr. (h)	ar
r	N	lał	$I_{zz} = \int_{M}  \mathbf{r} ^2 dm = \int_{M} (x^2 + y^2) dm = \int_{M} x^2 dm + \int_{M} y^2 dm = I_{xx} + I_{yy}$	Jar
		- 62	mood Preparedby: Dr $\Rightarrow$ $I_{zz} = I_{yy} + I_{yy}$ Hence proved. Dr $x$	iar
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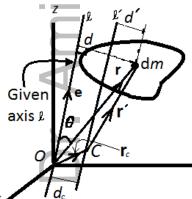
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ir Mahmood	Prepared by: Dr.	$A I_l = I_{l'} + M d_c^2 $	Hence proved.	by: Dr. Ami	r Mahmood	Prepar
ir Mahmood	Prepared by: Dr.	Amir <mark>Ma</mark> hmood	Prepared	by: Dr. Ami	r Mahmood	Prepar
Droblom	State and prove parall	al avis theorem for the	case of momo	nt of inartia for	a continuous m	Prenar

Problem: State and prove parallel axis theorem for the case of moment of inertia for a continuous mass epa distribution.

**Statement**: The moment of inertia of a rigid body in the form of a continuous mass distribution about a given axis is equal to the sum of moment of inertia of same body about a parallel axis (to the given axis) through the centre of mass of the body and the moment of inertia due to the total mass of the body placed at its centre of mass, about given axis.

Proof: Consider a rigid body, in the form of a continuous mass ir M distribution. Let l be the given and l' be an axis which is parallel to l and r Mai passing through centre of mass of the body. r Mał Let, M = total mass of the body

 $\mathbf{r}$  = position vector of *i*-th particle of mass  $m_i$  with respect to origin "O"  $\mathbf{r}'$  = position vector of *i*-th particle of mass  $m_i$  with respect to centre of mass "C"



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by: Dr. A mir Mahmood Prepare r Mah  $\mathbf{r}_c$  = position vector of centre of mass "C" with respect to origin "O" Prepar ir Mahmood Prepared  $\theta$  = angle between position vector  $\mathbf{r}_i$  and given line lir Mahmood, Prepared by: Dr. Amir Mahmood d = perpendicular distance of *i*-th particle of mass  $m_i$  from given axis *l* Prepare r Mahmood Prepare d' = perpendicular distance of *i*-th particle of mass  $m_i$  from parallel axis l'Prepare  $d_c$  = perpendicular distance of centre of mass C from given axis l = perpendicular distance between l and l' $\mathbf{e} = \mathbf{e}$  unit vector in the direction of given line llahmood Prepare r Mah From figure, epare lahmood Prepar  $\sin \theta = \frac{d}{|\mathbf{r}|} \Rightarrow d = |\mathbf{r}| \sin \theta$  $= |\mathbf{e} \times \mathbf{r}|$ r Mahmood Prepared b): Dr. Amir N Prepare Similarly,  $d' = |\mathbf{e} \times \mathbf{r}'|$  and  $d_c = |\mathbf{e} \times \mathbf{r}_c|$ ir Mahmood Preparec<del>yby</del>: Dr. Amir 🕅 Prepar Moment of inertia of the body about given axis *l* is given by mood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Ar Prepar repared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahood Page 9 of 29 ir Mahmood

ir Mahmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare Page 10 of 29 Mechanics Made Easv  $\int d^2 dm = \int (|\mathbf{e} \times \mathbf{r}|)^2 dm = \int (\mathbf{e} \times \mathbf{r}) \cdot (\mathbf{e} \times \mathbf{r}) dm$ Plen mood Mahmood (from figure) repared  $\int [\mathbf{e} \times (\mathbf{r}_c + \mathbf{r}')] \cdot [\mathbf{e} \times (\mathbf{r}_c + \mathbf{r}')] dm \qquad \because \mathbf{r} = \mathbf{r}_c + \mathbf{r}'$ Prepared by: Dr. Amir Mahm by: Dr.  $(\mathbf{e} \times \mathbf{r}_c + \mathbf{e} \times \mathbf{r}') \cdot (\mathbf{e} \times \mathbf{r}_c + \mathbf{e} \times \mathbf{r}') dm$ ed by:  $= \left[ \left[ (\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}_c) + 2(\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}') + (\mathbf{e} \times \mathbf{r}') \cdot (\mathbf{e} \times \mathbf{r}') \right] dm \right]$ Prepared by: Dr.  $= \int [(|\mathbf{e} \times \mathbf{r}_c|)^2 + 2(\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}') + (|\mathbf{e} \times \mathbf{r}'|)^2] \mathrm{d}m$  $= \left(\int_{M} \mathrm{d}m\right) (|\mathbf{e} \times \mathbf{r}_{c}|)^{2} + 2(\mathbf{e} \times \mathbf{r}_{c}) \cdot \int_{M} (\mathbf{e} \times \mathbf{r}') \mathrm{d}m + \int_{M} (|\mathbf{e} \times \mathbf{r}'|)^{2} \mathrm{d}m$  $= M d_c^{2} + 2(\mathbf{e} \times \mathbf{r}_c) \cdot \left( \mathbf{e} \times \left( \mathbf{r}' dm \right) + \int d'^{2} dm \right)$ eparee pared by: Dr. Amir where,  $M = \int dm = \text{total mass of the body}$ (ammoo of mass element dm Also, **0**, as **r**' is the position vector with respect to centre of mass ahmoom  $I_{l'} = \sum m_i d$ inertia of the body axis lood repare Amir Mahmood Pre  $\Rightarrow I_1 = I_1 + Md_c^2 d$ Hence proved. Prepare **Problem:** Prove in matrix notation that  $[\dot{\mathbf{L}}] = [\boldsymbol{\omega} \times \mathbf{L}] + [\mathbf{I}][\dot{\boldsymbol{\omega}}]$ , where, all the notations used have their usual meanings. Prebared by: Dr. Proof: As we know that the angular momentum of a system of particles is given by r Mahmood  $\sim \sum m_i \mathbf{r}_i \times \mathbf{v}_i$  $\mathbf{r}_i \times (m_i \mathbf{v}_i) =$ Differentiating both sides with respect to time "t", we get  $m_i \dot{\mathbf{r}}_i \times \mathbf{v}_i + \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{v}}_i = \sum_i m_i \mathbf{v}_i$  $\sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_i$ i  $= \sum m_i \mathbf{r}_i \times \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\omega} \times \mathbf{r}_i) \quad \forall i \in \mathbf{v}_i \times \mathbf{v}_i = \mathbf{0} \text{ and } \dot{\mathbf{v}}_i = \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\omega} \times \mathbf{r}_i)$  $m_i \mathbf{r}_i \times [(\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) + (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)] = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) + \sum_i m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)$ 

Prepared by: Dr. Amir Mahood r Mahmood Page 10 of 29 repared by: Dr. Amir Mahmood Page 10 r Mahmood Page 10 of 29 repared by: Dr. Amir Mahmood Page 10 ir Mahmood, Prepared by: Dr. Amir Mahmood, Prepared by: Dr. Amir Mahmood, Prepar Page 11 of 29 <u>Mechanics Made Easy</u> Writing in matrix form, we get Prepared by: hmood repared /lahmood Preparec[**L**] =  $\left[\sum_{i} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i)\right] + \left[\sum_{i} m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)\right]$ epar mood We also know that, par  $[\mathbf{L}] = [\mathbf{I}][\boldsymbol{\omega}]$ ahmood Prepared  $\mathbf{r}_i \times (m_i \mathbf{v}_i) = [\mathbf{I}][\boldsymbol{\omega}]$  $\therefore \mathbf{L} = \sum \mathbf{r}_i \times (m_i \mathbf{v}_i)$ = [Ι][ω]  $m_i \mathbf{r}_i \times \mathbf{v}_i$ i geared  $m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = [\mathbf{I}][\boldsymbol{\omega}]$  $\omega \times \mathbf{r}_i$  $\left[\sum_{i}^{n}\right]$ Dr. Amir repared by  $\dot{\omega}$  on both sides, we get mir Mahmood Prepared by: Dr. Amir  $m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) = [\mathbf{I}][\dot{\boldsymbol{\omega}}]$ Prepared Now consider, pared nood Prepared b renared by: Dr.  $m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \sum m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) = \sum m_i \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = \sum m_i \mathbf{r}_i \times [(\boldsymbol{\omega} \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{r}_i]$  $= \sum_{i} m_{i} [(\boldsymbol{\omega} \cdot \mathbf{r}_{i})(\mathbf{r}_{i} \times \boldsymbol{\omega}) - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_{i} \times \mathbf{r}_{i})] = \sum_{i} m_{i} (\boldsymbol{\omega} \cdot \mathbf{r}_{i})(\mathbf{r}_{i} \times \boldsymbol{\omega}) - \cdots$  $\rightarrow$  (3)  $\mathbf{r}_i \times \mathbf{r}_i$ arien rein Further consider that Prepared by: Dr. Amir Mahmood  $\omega \times (\mathbf{r}_i \times \mathbf{v}_i) = \omega \times [\mathbf{r}_i \times (\omega \times \mathbf{r}_i)] = \omega \times [(\mathbf{r}_i \cdot \mathbf{r}_i)\omega - (\mathbf{r}_i \cdot \omega)\mathbf{r}_i] = (\mathbf{r}_i \cdot \mathbf{r}_i)(\omega \times \omega) - (\mathbf{r}_i \cdot \omega)(\omega \times \mathbf{r}_i)$  $= -(\mathbf{r}_i \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{r}_i) = (\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) \cdot$ → (4)  $\therefore \omega \times \omega = 0$ ahmood Using (4) in (3), we get epared by: Dr. Amir  $\times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \sum m_i \, \boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{v}_i) = \boldsymbol{\omega} \times \sum \mathbf{r}_i \times (m_i \mathbf{v}_i) = \boldsymbol{\omega} \times \mathbf{L}$  $\therefore \mathbf{L} = \sum \mathbf{r}_i$  $m_i \mathbf{r}_i$ Writing in matrix form, we get py: Dr Predared  $m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i)$  $= [\omega \times L]$ red and (5) in (1), we get ed by: Using (2)  $[\omega \times L]$ [I][w] Hence proved.  $\pm$ of rank 2. Problem: Show that inertia matrix [I] is a Cartesian Prepared by: Dr. Amir Mahood Page 11 of 29 Page 11 Page 11

ir Mahmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepar Page 12 of 29 Mechanics Made Easy Proof: As we know that the angular momentum of a system of particles is given by Mahmood Prepared by: Dr  $\mathbf{L} = \sum \mathbf{r}_{\alpha} \times (m_{\alpha} \mathbf{v}_{\alpha}) = \sum m_{\alpha} (\mathbf{r}_{\alpha} \times \mathbf{v}_{\alpha}) = \sum m_{\alpha} (\mathbf{r}_{\alpha} \times (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}))$ Amir Mahmood Prepared by: Prepared  $\mathbf{L} = \sum m_{\alpha} [(\mathbf{r}_{\alpha} \cdot \mathbf{r}_{\alpha})\boldsymbol{\omega} - (\mathbf{r}_{\alpha} \cdot \boldsymbol{\omega})\mathbf{r}_{\alpha}] = \sum m_{\alpha} [\mathbf{r}_{\alpha}^{2}\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_{\alpha})\mathbf{r}_{\alpha}]$ ared by: Dr  $\boldsymbol{\omega} = (\omega_1, \, \omega_2, \, \omega_3)$ hmood  $= (L_1, L_2, L_3),$ and  $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ ahmood Prepared pared  $+ \omega_2 x_{\alpha,2} + \omega_3 x_{\alpha,3} = \sum \omega_j x_{\alpha,j}$ <u>j=1</u> ebared by be written as P/ spared Amn  $\sum_{j=1}^{\infty} \omega_j x_{j,\alpha} \left( x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3} \right) \right)$  $(L_1, L_2, L_3) = \sum m$  $m_{\alpha} \left[ \mathbf{r}_{\alpha}^{2}(\omega_{1}, \omega_{2}, \omega_{3}) - \right]$ ared by: Dr. Amir Mahmood apared  $\sum \omega_j x_{\alpha,j} x_{\alpha,i}$  , or  $m_{\alpha} \mathbf{r}_{\alpha}^2 \omega_i -$ = 1.2.3Dr. Anir Mann god  $\sum \omega_j x_{\alpha,j} x_{\alpha,i}$  $m_{\alpha} |\mathbf{r}_{\alpha}^2 \sum \omega_j \delta_{ij} -$  $: \omega_i = \sum \omega_i \delta_{ij}$ Vfahmood Prepa  $= \sum m_{\alpha} \sum [\mathbf{r}_{\alpha}^{2} \delta_{ij} - x_{\alpha,j} x_{\alpha,i}] \omega_{j} = \sum \omega_{j} \sum m_{\alpha} [\mathbf{r}_{\alpha}^{2} \delta_{ij} - x_{\alpha,i} x_{\alpha,j}]$ A3  $=\sum m_{\alpha} \sum [\mathbf{r}_{\alpha}^2 \delta_{ij}]$  $-x_{\alpha,j}x_{\alpha,i}]\omega_{j} = \sum \omega_{j} \sum m_{\alpha} [\mathbf{r}_{\alpha}^{2}\delta_{ij} - x_{\alpha,i}x_{\alpha,j}] = \sum \omega_{j} I_{ij}$ ared by: Dr hmood Prepared by: Dr. Amir  $I_{ij} = \sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] = ij'$ th component of inertia tensor where, mood Prepare d by: Dr. Amir Mahmood Prepared by: D ir Mahmood Since, both the angular velocity  $\boldsymbol{\omega} = (\omega_i)$  and the angular momentum  $\mathbf{L} = (L_i)$  are known to be vectors (i.e., Cartesian tensors of rank 1), it follows from equation (2) and quotient theorem that the inertia tensor  $[\mathbf{1}] = (I_{ij})$  is a Cartesian tensor of rank 2. Problem: Express angular momentum in tensor notation. **Solution:** As we know that the angular momentum of a system of particles is given by mood  $\mathbf{r}_{\alpha} \times (m_{\alpha} \mathbf{v}_{\alpha}) = \sum m_{\alpha} (\mathbf{r}_{\alpha} \times \mathbf{v}_{\alpha}) = \sum m_{\alpha} (\mathbf{r}_{\alpha} \times (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}))$ Prepared by: mir Mahmood  $-(\mathbf{r}_{\alpha}\cdot\boldsymbol{\omega})\mathbf{r}_{\alpha}] = \sum m_{\alpha}[\mathbf{r}_{\alpha}^{2}\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r}_{\alpha})\mathbf{r}_{\alpha}]$  $\mathbf{L} = \sum m_{\alpha} [(\mathbf{r}_{\alpha} \cdot \mathbf{r}_{\alpha}) \boldsymbol{\omega}]$ Prepared by: Dr. Amir Mahmood Prepared by: Dr nood  $\mathbf{L} = (L_1, L_2, L_3), \qquad \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and  $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ Let, Prepared by: Dr. Amir Mahood

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ir Mahmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Page 13 of 29 **Mechanics Made Easy** 3  $\omega_3 x_{\alpha,3} = \sum \omega_j x_{\alpha,j}$ Then,  $\Pr = \boldsymbol{\omega} \cdot \mathbf{r}_{\alpha} = \omega_1 x_{\alpha,1}$ ahmood Prepared by: So, (1) can be written as  $\rangle \omega_j x_{j,\alpha}$  $(L_1, L_2, L_3) =$  $m_{\alpha} | \mathbf{r}_{\alpha}^2(\omega_1, \omega_2, \omega_3)$  $(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ amir j=19ared 3  $m_{\alpha} | \mathbf{r}_{\alpha}^2 \omega_i \omega_j x_{\alpha,j} x_{\alpha,i}$ 30 3  $m_{\alpha}$   $\mathbf{r}_{\alpha}^{2}$  $\sum \omega_j x_{\alpha,j} x_{\alpha,i}$  $\omega_j \delta_{ij} \omega_i =$  $\sum \omega_j \delta_{ij}$  $\sum \omega_j \sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij}]$  $m_{\alpha} \sum [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i}] \omega_j =$  $x_{\alpha,i}x_{\alpha,j} =$  $\sum \omega_j I_{ij}$ mood Prepared  $[x_{\alpha,i}x_{\alpha,j}] = ij$ th component of inertia tensor  $\sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij}]$ Equation (2) is required tensor form of angular momentum. Problem: Express rotational kinetic energy in tensor notation. repared by: Dr. Solution: As we know that the rotational kinetic energy of a system is given  $\frac{1}{2}\omega \cdot \mathbf{L}$ Prepared nmood  $\boldsymbol{\omega}=(\omega_1,\,\omega_2,\,\omega_3),$  $\mathbf{L} = (L_1, L_2, L_3)$ Let, Mahmood Prepared  $\frac{1}{2}(\omega_1L_1+\omega_2L_2)$  $\omega_3 L_3) =$  $\omega_i L_i$ 3  $= \sum \omega_j \sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij}]$  $\omega_j \rangle m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij}]$  $\omega_i$  $-x_{\alpha,i}x_{\alpha,i}$  $\overline{j=1}$ 3  $\omega_i$  $\sum \omega_i / \gamma$  $m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij}]$  $\sim \omega_i \omega_j I_{ij}$  $x_{\alpha,i}x_{\alpha,j}$  $\sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}]$ *ij*th component of inertia tensor where, Pre Prepared Equation (1) is required tensor form of rotational kinetic energy. Problem: Express parallel axis theorem in tensor notation. Solution: Consider a rigid body, in the form of a set of particles. Let, C be the centre of mass of the body. We consider two parallel coordinate systems Oxyz and Cx'y'z', as shown in the figure. Prepared by: Dr. Amir Mahood

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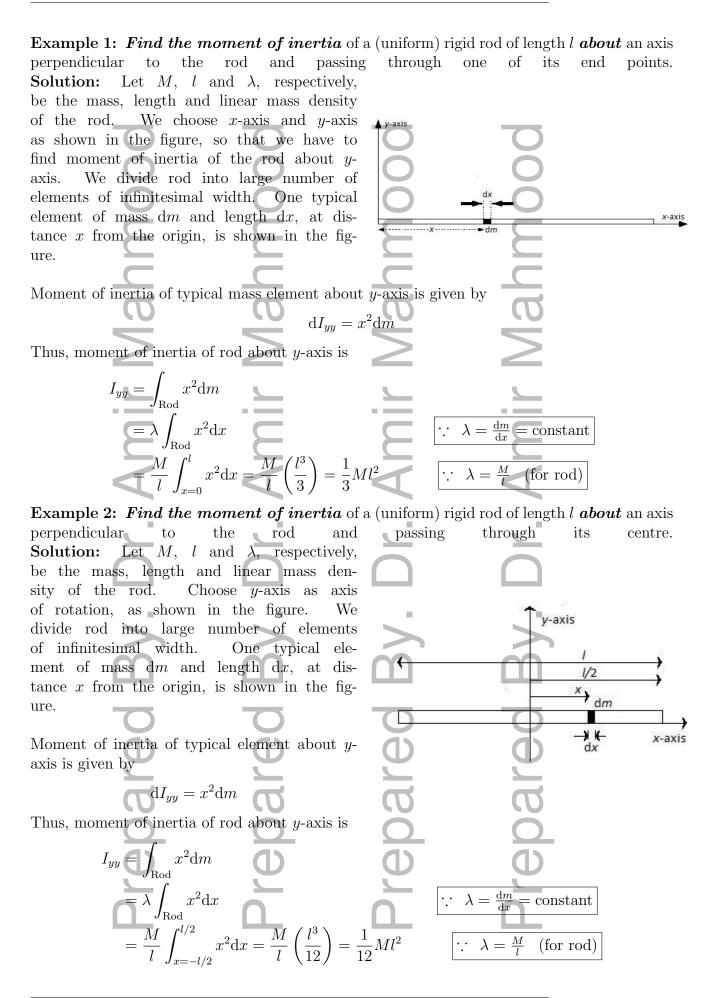
ir Mahmood, Prepared by: Dr. Amir Mahmood, Prepared by: Dr. Amir Mahmood, Prepar Page 14 of 29 Mechanics Made Easy Let, M = total mass of the body mood Prepared by: Dr. Amir **r**<sub> $\alpha$ </sub> = position vector of  $\alpha$ -th particle of mass  $m_{\alpha}$  with respect to origin "0" hmood Prepared by: Dr. Amir Mahmood Prepared by: Di  $\mathbf{r}'_{\alpha}$  = position vector of  $\alpha$ -th particle of mass  $m_{\alpha}$  with respect to centre of mass "C" mood Prep  $\mathbf{r}_c$  = position vector of centre of mass "C" with respect to origin "O" Amir From figure,  $\mathbf{r}_{\alpha} = \mathbf{r}_{c} + \mathbf{r}_{\alpha}' - - - - - - - - \rightarrow (1)$ repared Let,  $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}), \quad \mathbf{r}_{c} = (x_{c,1}, x_{c,2}, x_{c,3}) \text{ and } \mathbf{r}_{\alpha}' = (x_{\alpha,1}', x_{\alpha,2}', x_{\alpha,3}')$ Amir Manmood Prepared b Equation (1) becomes  $(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$ Prepared by: Dr. Amir  $\Rightarrow x_{\alpha,i} = x_{c,i} + x'_{\alpha,i},$ i = 1, 2, 3→ (2) repared Amir Vla As we know that  $I_{ij} = \sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}]$ Vlahmood Prepared ared Prepared by: Dr. An  $=\sum_{\alpha}m_{\alpha}[((\mathbf{r}_{c}+\mathbf{r}_{\alpha}')\cdot(\mathbf{r}_{c}+\mathbf{r}_{\alpha}'))\delta_{ij}-(x_{c,i}+x_{\alpha,i}')(x_{c,j}+x_{\alpha,j}')]$ (by using (1) and (2))  $-x_{c,i}x_{c,j}-x_{c,i}x_{\alpha,j}'$  $\sum m_{\alpha} [(\mathbf{r}_{c} \cdot \mathbf{r}_{c} + 2 \mathbf{r}_{c} \cdot \mathbf{r}_{\alpha}' + \mathbf{r}_{\alpha}' \cdot \mathbf{r}_{\alpha}') \delta_{ii}]$  $-x_{c,j}x'_{\alpha,i}-x'_{\alpha,i}x'_{\alpha,j}$ pared ed by:  $= \sum m_{\alpha} \left[ (\mathbf{r}_{c}^{2} + 2 \, \mathbf{r}_{c} \cdot \mathbf{r}_{\alpha}' + \mathbf{r}_{\alpha}'^{2}) \delta_{ij} \right]$  $x_{c,i} x_{\alpha,j}$  $-x_{c,j}x'_{\alpha,i}-x'_{\alpha,i}x'_{\alpha,j}]$  $-x_{c,i}x_{c,j}$  $m_{\alpha} \big[ \mathbf{r}_{\alpha}^{\prime 2} \delta_{ij} - x_{\alpha,i}^{\prime} x_{\alpha,j}^{\prime} \big] + 2 \, \mathbf{r}_{c} \cdot$  $m_{\alpha} \mathbf{r}_{c}^{\prime 2} \delta_{ij}$  $\delta_{ij}$  $\left(\sum_{\alpha}m_{\alpha}\mathbf{r}_{\alpha}'\right)$ Ŧ  $m_{\alpha}$  $\langle \alpha$ repared α  $\sum m_{\alpha} x'_{\alpha,j} x_{c,i} \left(\sum m_{\alpha} x'_{\alpha,i}\right) x_{c,j}$  $\rightarrow$  (3)  $\sum m_{\alpha} [\mathbf{r}_{c}^{2} \delta_{ij} - x'_{\alpha,i} x'_{\alpha,j}] = I'_{ij} = ij$ th component of inertia tensor with respect to Cx'y'z' system Preparet = 0, as  $\mathbf{r}'_{\alpha}$  is the position vector of  $\alpha$ -th particle of mass  $m_{\alpha}$  with respect to centre of mass Also,  $m_{\alpha}\mathbf{r}'_{\alpha}$ Dr. Amir Ma Prepared by: Dr. Amir Mahmo repared  $m_{\alpha}\mathbf{r}_{\alpha}' = \mathbf{0}$  $\Rightarrow$   $n_{\alpha}x'_{\alpha,i} = 0, i = 1, 2, 3$ >  $m_{\alpha}(x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3}) = (0, 0, 0)$ lanmood  $m_{\alpha} = M =$  total mass of the body And, repared by: So equation (3) becomes  $I_{ij} = I'_{ij} + M\mathbf{r}_c^{\prime 2}\delta_{ij} - Mx_{c,i}x_{c,j}$ nood Prepared by:

Prepared by: Dr. Amir Mahood r Mahmood Prepared by: Dr. Amir Mal Page 14 of 29 repared by: Dr. Amir Mahmood Prep ir Mahmood, Prepared by: Dr. Amir Mahmood, Prepared by: Dr. Amir Mahmood, Prepar Page 15 of 29 Mechanics Made Easy This is required tensor form of parallel axis theorem. Prepared by: Dr. Amir Mahmood Prepared by Mahmood Problem: State and prove parallel axis theorem for the case of products of inertia for a set of particles. mood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood **Statement**: Consider a rigid body, in the form of a set of particles. Let, C be the centre of mass of the body. If Oxyz and Cx'y'z' be two parallel coordinate systems as shown in the figure, then we have mood  $= I_{ij}' - M x_{c,i} x_{c,j},$  $i, j \in \{1, 2, 3\}$ r. Amir Mahmood repared by: Dr. mood Prepare  $I_{ii}$  = product of inertia with respect to *Oxyz*-system imood Prepared = product of inertia with respect to Cx'y'z'-system  $(x_{c,1}, x_{c,2}, x_{c,3}) =$  position vector of centre of mass "C" with respect to origin "O" M =total mass of the body Proof: Consider a rigid body, in the form of a set of particles. Amir Mahmoo ood Prepare Preparec Let,  $\mathbf{r}_{\alpha} = \text{position vector of } \alpha$ -th particle of mass  $m_{\alpha}$  with respect to origin "O"  $\mathbf{r}'_{\alpha}$  = position vector of  $\alpha$ -th particle of mass  $m_{\alpha}$  with respect to centre of mass "C"  $\mathbf{r}_c$  = position vector of centre of mass "*C*" with respect to origin "*O*" From figure, Let,  $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}), \quad \mathbf{r}_{c} = (x_{c,1}, x_{c,2}, x_{c,3}) \text{ and } \mathbf{r}_{\alpha}' = (x_{\alpha,1}', x_{\alpha,2}', x_{\alpha,3}')$ So, equation (1) becomes  $(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$  $A_{i} = 1, 2, 3$ repared by: Dr $x_{\alpha,i} = x_{c,i} + x'_{\alpha,i},$ ahmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Am Now consider for  $i \neq j$ ,  $I_{ij} = -\sum m_{\alpha} x_{\alpha,i} x_{\alpha,j} = -\sum m_{\alpha} (x_{c,i} + x'_{\alpha,i}) (x_{c,j} + x'_{\alpha,j})$  $x_{c,i}x_{c,j} - \left(\sum m_{\alpha}x'_{\alpha,j}\right)x_{c,i} - \frac{1}{2}$  $\left(\sum m_{\alpha} x'_{\alpha,i}\right) x_{c,j}$  $m_{\alpha} | x_{c,i} x_{c,j} -\sum m_{\alpha} x'_{\alpha,i} x'_{\alpha,j}$ epared by: Dr. Amir Mahmood ahmood  $m_{\alpha} = M =$  total mass of the body, where, arec α by: Dr. Amir N hmood Prepared by: Dr. Am pared  $m_{\alpha} x'_{\alpha,i} x'_{\alpha,j} = I'_{ij}$  = product of inertia with respect to Cx'y'z'-system Also, ared by: Dr. An hmood Pren  $m_{\alpha}\mathbf{r}_{\alpha}'=\mathbf{0}$  $m_{\alpha}(x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3}) = (0, 0, 0)$  $m_{\alpha}x'_{\alpha,i}=0,$ i = av: Dr. aparec a Amir nmooc So equation (3) gives (Apy: Dr. Amir Mahmood Propared by: Dr. Amir  $I_{ii} = I'_{ii}$  $-Mx_{c,i}x_{c,j}$  $i, j \in \{1, 2, 3\}$ Hence proved. ≠ j, Prepared by: Dr. Amir Mahood Page 15

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irN	Ial	Mechanic Problem:	<u>s Made Easy</u> State and pr	ove paralle	l axis the	orem for the	case of pr	oducts o	f inertia	Moment of In for a continuous r	ertia nass
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ir f	Vla	$\mathbf{r}_c = \text{positive}$	tion vector of	centre of ma	ass " <i>C</i> " wi	th respect to o	origin "O"	ed by:	Dr. Ar	nir Mahmoo	d Prepar
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z-axis

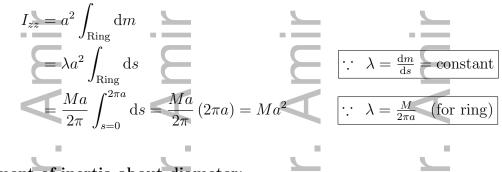
## **Example 3:** *Find the moment of inertia* of a (uniform) circular ring of radius *a about*

(i) an axis passing through its centre and perpendicular to its plane,

(*ii*) its diameter. Solution: (*i*) Moment of inertia about central axis: Let M, a and  $\lambda$ , respectively, be the mass, radius and linear mass density of the ring. Choose coordinate axes as shown in the figure. We divide ring into large number of elements of infinitesimal width. One typical element of mass dm and length ds is shown in the figure.

Moment of inertia of typical element about z-axis is given by

Thus, moment of inertia of ring about z-axis is



 $\mathrm{d}I_{zz} = a^2 \mathrm{d}m$ 

(*ii*) Moment of inertia about diameter: By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx},$$

**Example 4:** *Find the moment of inertia* of a (uniform) circular disc of Mass M and radius a *about* 

 $\Rightarrow I_{xx} = \frac{1}{2}Ma^2$ 

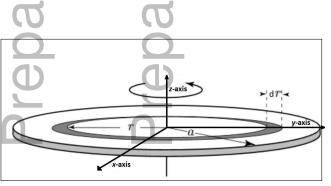
(i) an axis passing through its centre and perpendicular to its plane, (ii) its diameter.

Solution: (i) Moment of inertia about central axis: Let M, a and  $\sigma$ , respectively, be the mass, radius and surface (areal) mass density of the disc. Choose axis of rotation as z-axis, as shown in figure.

We divide disc into large number of concentric circular rings of infinitesimal width. One typical elementary ring of mass dm, radius r, width dr and area dA is shown in the figure.

Moment of inertia of typical elementary ring about z-axis is given by

$$\mathrm{d}I_{zz} = r^2 \mathrm{d}m$$



 $\therefore I_{xx} = I_{yy}$  (by symmetry)

Thus, moment of inertia of disc about z-axis is

$$I_{zz} = \int_{\text{Disc}} r^2 dm$$

$$= 2\pi\sigma \int_{\text{Disc}} r^3 dr$$

$$= \frac{2M}{a^2} \int_{r=0}^{a} r^3 dr = \frac{2M}{a^2} \left(\frac{a^4}{4}\right) = \frac{1}{2}Ma^2 \qquad \boxed{\because \sigma = \frac{dm}{dA} = \frac{dm}{(2\pi r)dr} = \text{constant}}$$

$$(ii) \text{ Moment of inertia about diameter:} \qquad (1)$$

By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx},$$
  

$$\Rightarrow I_{xx} = \frac{1}{4}Ma^{2}$$

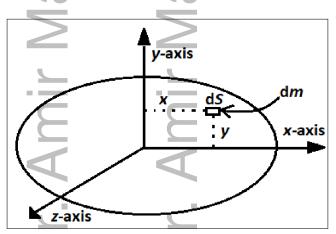
**Example 5:** *Find the moment of inertia* of a (uniform) elliptical plate with semi-major axis and semi minor axis *a* and *b*, respectively *about* 

- (i) major axis,
- (*ii*) minor axis,

(iii) an axis passing through centre of plate and perpendicular to its plane.

Solution: Consider an elliptical plate in xyplane whose boundary curve is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$



(2)

Let M and  $\sigma$ , respectively, be the mass and surface (areal) mass density of the elliptical plate. To find moment of inertia about major axis (x-axis), we proceed as follows. We divide plate into large number of elementary rectangular pieces of infinitesimal area with sides parallel to x and y axis. One typical area element having mass dm, area dS, length dx and width dy is shown in the figure at point (x, y).

Moment of inertia of typical area element about x-axis is given by

Thus, moment of inertia of elliptical plate about x-axis is  

$$I_{xx} = \int_{\text{Elliptical plate}} y^2 dm$$

$$= \sigma \int_{\text{Elliptical plate}} y^2 dx dy$$

$$= \frac{M}{\pi ab} \int_{\text{Elliptical plate}} y^2 dx dy$$

$$= \frac{M}{\pi ab} \int_{x=-a}^a \left( \int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 dy \right) dx$$

$$= \frac{M}{\pi ab} \left( \frac{2b^3}{3a^3} \right) \int_{x=-a}^a (a^2 - x^2)^{3/2} dx$$

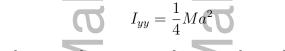
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$$I_{xx} = \frac{4Mb^2}{3\pi a^4} \int_{x=0}^a (a^2 - x^2)^{3/2} \mathrm{d}x$$
  $\therefore$  integrand is even

Put  $x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta$ ,  $x = 0 \Rightarrow \theta = 0$ ,  $x = a \Rightarrow \theta = \pi/2$ 

$$I_{xx} = \frac{4Mb^2}{3\pi a^4} \int_{x=0}^{\pi/2} a^4 \cos^4\theta d\theta$$
Using Wallis cosine formula, we get
$$I_{xx} = \frac{4Mb^2}{3\pi} \left(\frac{3}{4}\right) \left(\frac{1}{2}\right) \left(\frac{\pi}{2}\right) = \frac{1}{4}Mb^2$$
(3)
Similarly, moment of inertia about minor axis is

S



By perpendicular axis theorem, the moment of inertia about the axis passing through centre of the elliptical plate and perpendicular to its plane, is

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{4}M(a^2 + b^2)$$
(4)

Corollary: The moment of inertia of a (uniform) circular disc of radius a about (i) its diameter and (ii) an axis passing through its centre and perpendicular to its plane can be obtained by putting b = a in (3) and (4), to give (respectively)

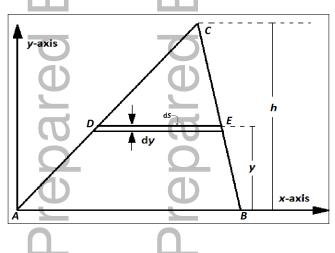
$$I_{xx} = \frac{1}{4}Ma^2 \tag{5}$$

and

Note that, the results obtained in (5) and (6) are in accordance (as they should be) with the results, obtained in (2) and (1), respectively.

Example 6: Find the moment of inertia of a (uniform) triangular lamina (i.e., two dimensional triangular plate) of mass *M* about one of its sides.

**Solution:** Let M and  $\sigma$ , respectively, be the mass and surface (areal) mass density of the triangular lamina in xy-plane. Choose x-axis and y-axis as shown in figure. We divide lamina into large number of strips of infinitesimal width parallel to the base AB of lamina. One typical elementary strip DE of mass dm, width dy and area dS is shown in the figure.



Moment of inertia of typical elementary strip about side AB (x-axis) is given by

$$\mathrm{d}I_{xx} = y^2 \mathrm{d}m$$

Thus, moment of inertia of triangular lamina about x-axis is

$$I_{xx} = \int_{\text{Triangular lamina}} y^2 dm$$

$$= \sigma \int_{\text{Triangular lamina}} y^2 |\text{DE}| dy$$

$$= \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \frac{|\text{DE}|}{|\text{AB}|} dy$$

$$\therefore \sigma = \frac{dm}{dS} = \frac{dm}{|\text{DE}|dy} = \text{constant}$$

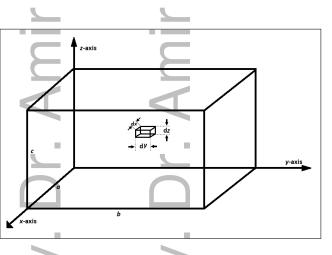
$$\therefore \sigma = \frac{dm}{dS} = \frac{dm}{|\text{DE}|dy} = \text{constant}$$
From equivalent triangles ABC and DEC, we have are equivalent triangles, therefore
$$\frac{|\text{DE}|}{|\text{AB}|} = \frac{\text{height of DEC}}{\text{height of ABC}} = \frac{h - y}{h}$$

$$I_{xx} = \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \left(\frac{h - y}{h}\right) dy$$

$$= \frac{2M}{h^2} \int_{y=0}^{h} y^2 (h - y) dy = \frac{2M}{h^2} \left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{1}{6}Mh^2$$

**Example 7:** Calculate the inertia matrix of a (uniform solid) rectangular box (rectangular parallelopiped or cuboid) of mass M at one of its corners, by taking coordinate axes along its edges.

**Solution:** Let M and  $\rho$ , respectively, be the mass and volume mass density of the rectangular box. Let the lengths of adjacent edges be a, b and c. Choose coordinate axis along the edges of box, as shown in figure. We divide lamina into large number of elementary rectangular boxes of infinitesimal volume. One typical elementary volume element



of mass dm, volume dV and dimensions dx, dy and dz, is shown in the figure.

Moment of inertia of typical elementary volume element about x-axis is given by

Thus, moment of inertia of triangular lamina about x-axis is  

$$I_{xx} = \int_{\text{Rectangular box}} (y^2 + z^2) dm$$

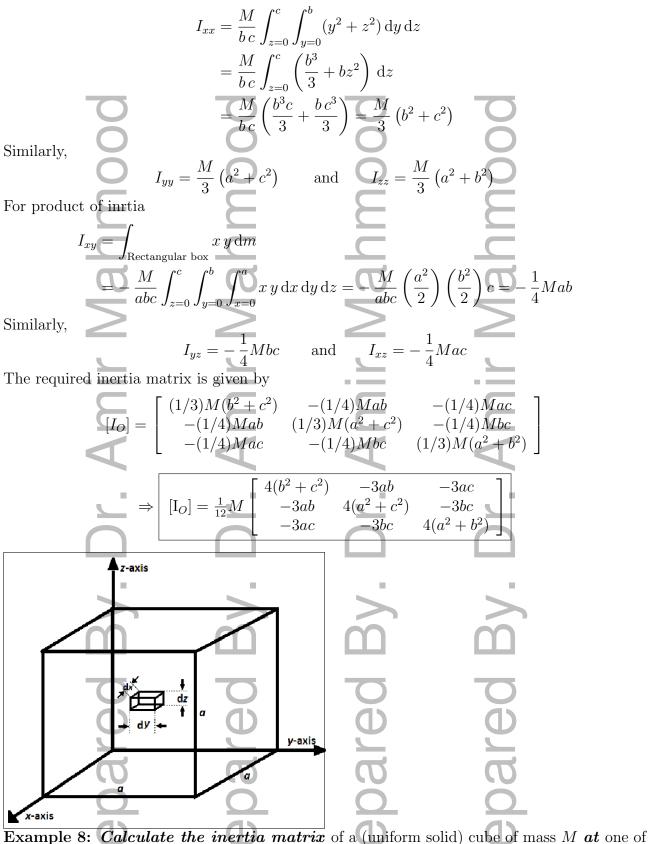
$$= \rho \int_{\text{Rectangular box}} (y^2 + z^2) dx dy dz$$

$$= \frac{M}{a \, b \, c} \int_{\text{Rectangular box}} (y^2 + z^2) dx dy dz$$

$$= \frac{M}{a \, b \, c} \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (y^2 + z^2) dx dy dz$$

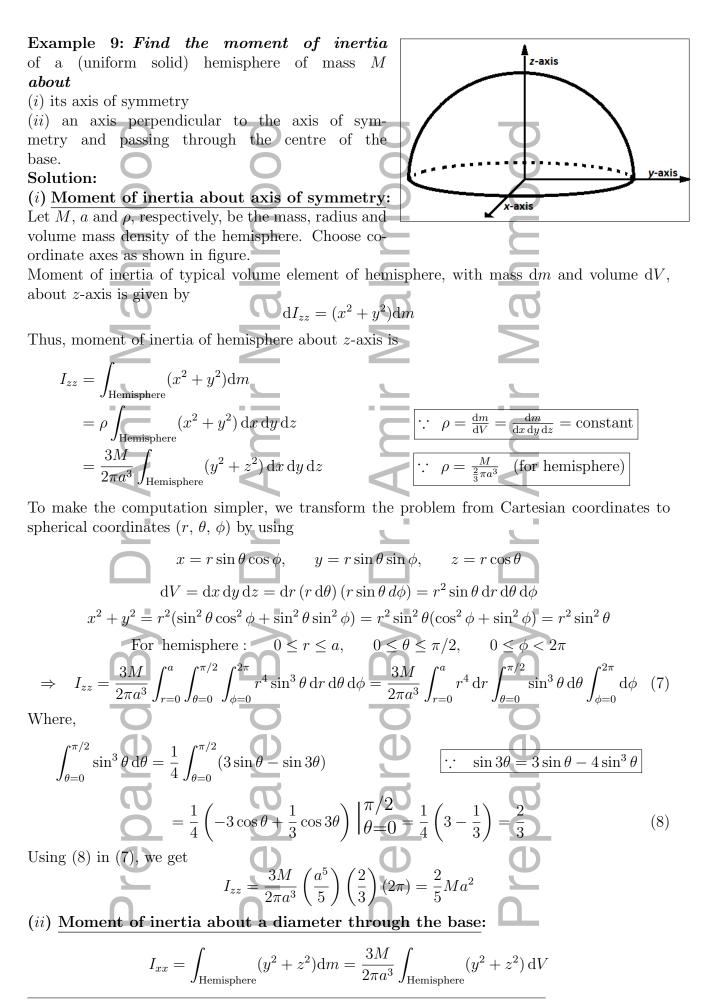
$$= \frac{M}{a \, b \, c} \left[ \int_{z=0}^{a} dx \right] \left[ \int_{z=0}^{c} \int_{y=0}^{b} (y^2 + z^2) dy dz \right]$$

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**Example 8:** Calculate the inertia matrix of a (uniform solid) cube of mass M at one of its corners, by taking coordinate axes along its edges. Solution: Repeat example 7 for a = b = c and get

$$[I_O] = \frac{1}{12} M a^2 \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$



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Transforming problem in spherical coordinates  $(r, \theta, \phi)$ , we get

$$I_{xx} = \frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 (\sin^3\theta \sin^2\phi + \cos^2\theta \sin\theta) dr d\theta d\phi$$
  
=  $\frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4 dr \left( \int_{\theta=0}^{\pi/2} \sin^3\theta d\theta \int_{\phi=0}^{2\pi} \sin^2\phi d\phi + \int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi \right)$ (9)

Where,

$$\int_{\phi=0}^{2\pi} \sin^2 \phi \, \mathrm{d}\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 - \cos 2\phi) \, \mathrm{d}\phi = \frac{1}{2} \left( \phi - \frac{1}{2} \sin 2\phi \right) \Big|_{\phi=0}^{2\pi} = \frac{1}{2} (2\pi) = \pi \tag{10}$$

and

$$\int_{\theta=0}^{\pi/2} \cos^2 \theta \sin \theta \, \mathrm{d}\theta = -\frac{1}{3} \cos^3 \theta \Big|_{\theta=0}^{\pi/2} = \frac{1}{3} \tag{11}$$

Using (8), (10) and (11), (9) gives

$$I_{xx} = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5}\right) \left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5}\right) \left(\frac{4\pi}{3}\right) = \frac{2}{5}Ma^2$$

Example 10: *Find three products of inertia* of a (uniform) solid hemisphere of mass *M* with respect to coordinate axes as in figure of example 9. Solution:

$$I_{xy} = -\int_{\text{Hemisphere}} x y \, dm = \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} x y \, dV$$
$$= -\frac{3M}{2\pi a^3} \int_{r=0}^a \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^2 \theta \sin \phi \cos \phi \, dr \, d\theta \, d\phi$$
$$= -\frac{3M}{2\pi a^3} \int_{r=0}^a r^4 \, dr \int_{\theta=0}^{\pi/2} \sin^2 \theta \, d\theta \int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, d\phi$$

But

$$\int_{\phi=0}^{2\pi} \sin\phi \cos\phi \,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi \Big|_{\phi=0}^{2\pi} = 0 \quad \Longrightarrow \quad I_{xy} = 0$$

Now,

But  

$$\int_{\phi=0}^{2\pi} \cos \phi \, d\phi = \sin \phi \Big|_{\phi=0}^{2\pi} = 0 \implies I_{xz} = 0 = I_{yz},$$

$$x \, z \, dm = \frac{3M}{2\pi a^3} \int_{Hemisphere}^{\pi/2} x \, y \, dV$$

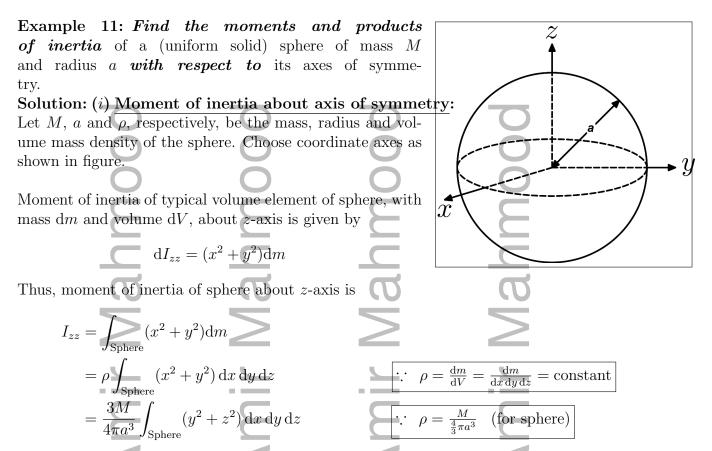
$$= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\phi=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin \theta \cos \theta \, d\theta \int_{\phi=0}^{2\pi} \cos \phi \, d\phi$$

$$= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4 \, dr \int_{\theta=0}^{\pi/2} \sin \theta \cos \theta \, d\theta \int_{\phi=0}^{2\pi} \cos \phi \, d\phi$$

$$\therefore I_{xz} = I_{yz} \text{ (by symmetry)}$$

Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0$$



To make the computation simpler, we transform the problem from Cartesian coordinates to spherical coordinates  $(r, \theta, \phi)$  by using

$$x = r \sin \theta \cos \phi, \qquad y = r \sin \theta \sin \phi, \qquad z = r \cos \theta$$
$$dV = dx \, dy \, dz = dr \, (r \, d\theta) \, (r \sin \theta \, d\phi) = r^2 \sin \theta \, dr \, d\theta \, d\phi$$
$$x^2 + y^2 = r^2 (\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi) = r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) = r^2 \sin^2 \theta$$
For sphere :  $0 \le r \le a, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$ 
$$\Rightarrow I_{zz} = \frac{3M}{2\pi a^3} \int_{r=0}^a \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 \sin^3 \theta \, dr \, d\theta \, d\phi = \frac{3M}{2\pi a^3} \int_{r=0}^a r^4 \, dr \int_{\theta=0}^{\pi} \sin^3 \theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$
ere,

Where,

$$\int_{\theta=0}^{\pi} \sin^{3}\theta \, d\theta = \frac{1}{4} \int_{\theta=0}^{\pi} (3\sin\theta - \sin 3\theta)$$

$$\boxed{\vdots \sin 3\theta - 4\sin^{3}\theta}$$

$$\boxed{\vdots \sin 4\theta - 4\sin^{3}\theta}$$

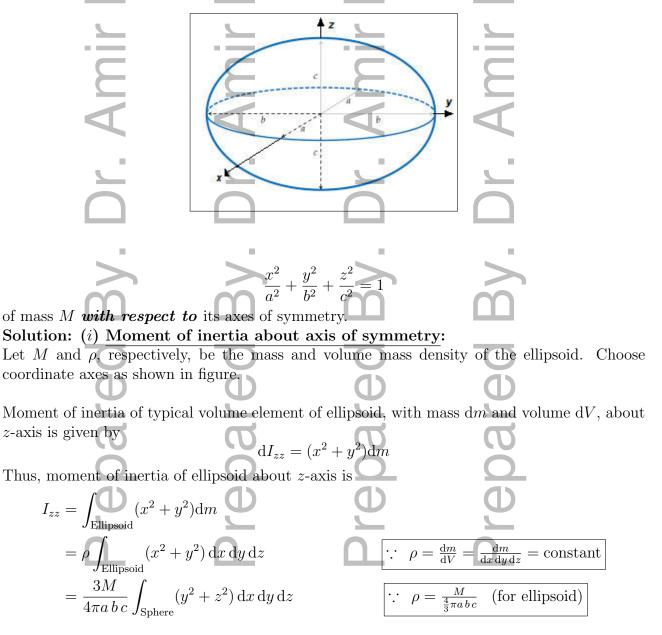
$$\boxed{i \sin 4\theta - 4\sin^{3}\theta}$$

$$\boxed{i$$

## (ii) Products of inertia with respect to axes of symmetry:

$$I_{xy} = -\int_{\text{Sphere}} x y \, dm = \frac{3M}{4\pi a^3} \int_{\text{Sphere}} x y \, dV$$
$$= -\frac{3M}{4\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 \sin^2 \theta \sin \phi \cos \phi \, dr \, d\theta \, d\phi$$
$$= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4 \, dr \int_{\theta=0}^{\pi} \sin^2 \theta \, d\theta \int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, d\phi$$
But
$$\int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, d\phi = \frac{1}{2} \sin^2 \phi \Big|_{\phi=0}^{2\pi} = 0 \implies I_{xy} = 0$$
Similarly,
$$I_{yz} = I_{xz} = 0$$
$$\because I_{xy} = I_{yz} = I_{xz} \text{ (by symmetry)}$$

Example 12: Find the moments and products of inertia of a (uniform) solid ellipsoid



$$I_{zz} = \int_{\text{Ellipsoid}} (x^2 + y^2) \mathrm{d}m \tag{12}$$

Let us substitute

$$x/a = x', \quad y/a = y', \quad z/a = z'$$
  

$$\Rightarrow dx/a = dx', \quad dy/a = dy', \quad dz/a = dz', \quad dx \, dy \, dz = a \, b \, c \, dx' \, dy' \, dz'$$

Under the above transformation, the given ellipsoid is transformed into the unit sphere

1

1

$$S: x'^{2} + y'^{2} + z'^{2} = 1.$$

$$\Rightarrow I_{zz} = \frac{3M}{4\pi a \, b \, c} \int_{S} (a^{2} x'^{2} + b^{2} y'^{2}) (a \, b \, c \, dx' \, dy' \, dz')$$

$$= \frac{3M}{4\pi} \int_{S} (a^{2} x'^{2} + b^{2} y'^{2}) dx' \, dy' \, dz'$$

$$\therefore \int_{S} x'^{2} dx' \, dy' \, dz' = \int_{S} y'^{2} dx' \, dy' \, dz' \qquad \text{(by symmetry)}$$

$$\Rightarrow I_{zz} = \frac{3M(a^{2} + b^{2})}{4\pi} \int_{S} x'^{2} dx' \, dy' \, dz'$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x',y', z') to spherical coordinates  $(r, \theta, \phi)$  by using

$$x' = r \sin \theta \cos \phi, \qquad y' = r \sin \theta \sin \phi, \qquad z' = r \cos \theta$$
$$dV = dx' dy' dz' = dr (r d\theta) (r \sin \theta d\phi) = r^2 \sin \theta dr d\theta d\phi$$
e,
$$0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

For unit sphere,

<

$$\Rightarrow I_{zz} = \frac{3M(a^2 + b^2)}{4\pi} \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 \sin^3 \theta \cos^2 \phi \, dr \, d\theta \, d\phi$$
$$= \frac{3M(a^2 + b^2)}{4\pi} \int_{r=0}^{1} r^4 \, dr \int_{\theta=0}^{\pi} \sin^3 \theta \, d\theta \int_{\phi=0}^{2\pi} \cos^2 \phi \, d\phi$$

Where, 
$$\int_{\theta=0}^{\pi} \sin^3 \theta \, \mathrm{d}\theta = \frac{1}{4} \int_{\theta=0}^{\pi} (3\sin\theta - \sin 3\theta) \qquad \because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$= \frac{1}{4} \left( -3\cos\theta + \frac{1}{3}\cos 3\theta \right) \left[ \frac{\pi}{\theta=0} = \frac{1}{4} \left[ \left( 3 - \frac{1}{3} \right) - \left( -3 + \frac{1}{3} \right) \right] = \frac{4}{3}$$
and 
$$\int_{\theta=0}^{2\pi} \cos^2 \theta \, \mathrm{d}\phi = \frac{1}{4} \int_{\theta=0}^{2\pi} (1+\cos 2\phi) \, \mathrm{d}\phi = \frac{1}{4} \left[ (4+\cos 2\phi) + \frac{1}{2}\cos^2 \theta + \frac{1}{3}\cos^2 \theta \right] = \frac{4}{3}$$

д

$$\int_{\phi=0}^{2\pi} \cos^2 \phi \, \mathrm{d}\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1+\cos 2\phi) \, \mathrm{d}\phi = \frac{1}{2} \left( \phi + \frac{1}{2} \sin 2\phi \right) \left| \begin{array}{c} 2\pi & \phi \\ \phi = 0 = \frac{1}{2} (2\pi) = \pi \\ \phi = 0 = \frac{1}{2} (2\pi) = \pi \\ \Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi} \left( \frac{1}{5} \right) \left( \frac{4}{3} \right) (\pi) = \frac{1}{5} M(a^2+b^2)$$

Similarly,

$$I_{xx} = \frac{1}{5}M(b^2 + c^2)$$
 and  $I_{yy} = \frac{1}{5}M(a^2 + c^2)$ 

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## (*ii*) Products of inertia with respect to axes of symmetry:

$$i_{xy} = -\int_{\text{Ellipsoid}} x y \, dm = -\frac{3M}{4\pi a b c} \int_{\text{Ellipsoid}} x y \, dV$$

$$= -\frac{3M}{4\pi a b c} \int_{S} (a b x' y') (a b c dx' dy' dz')$$

$$= -\frac{3a b M}{4\pi} \int_{S} x' y' dx' dy' dz'$$

$$= -\frac{3a b M}{4\pi} \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 \sin^2 \theta \sin \phi \cos \phi \, dr \, d\theta \, d\phi$$

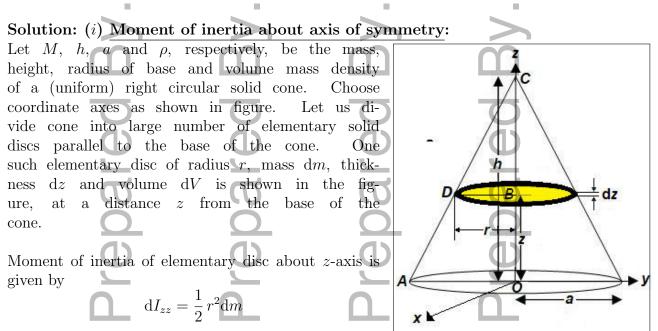
$$I_{xy} = -\frac{3 a b M}{4\pi} \int_{r=0}^{1} r^4 \, dr \int_{\theta=0}^{\pi} \sin^2 \theta \, d\theta \int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, d\phi$$
But
$$\int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, d\phi = \frac{1}{2} \sin^2 \phi \Big|_{\phi=0}^{2\pi} = 0 \implies I_{xy} = 0$$
Similarly, it is not difficult to show that
$$I_{yz} = I_{xz} = 0$$

Bu

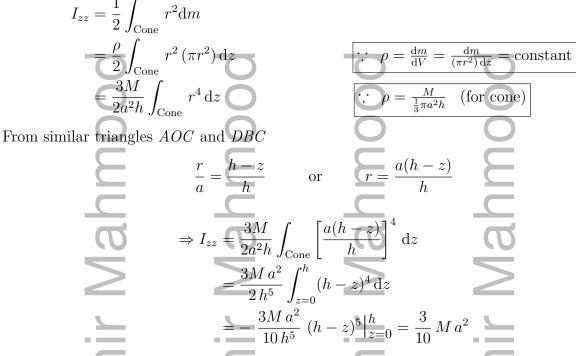
Corollary: The moment and product of inertia of a (uniform) solid sphere of mass Mand radius a with respect to its axis of symmetry can be obtained by putting a = b = c in results of above example 12. The obtained results are in accordance (as they should be) with the results of example 11.

**Example 13:** Find the moment of inertia of a (uniform) right circular solid cone about (i) its axis of symmetry and

(ii) any diameter of the base.



Thus, moment of inertia of cone about z-axis is



(ii) Moment of inertia about diameter of the base: In this case, the moment of inertia of the elementary disc of mass dm about a diameter, along DB, is given by

$$\mathrm{d}I_o = \frac{1}{4} r^2 \mathrm{d}m$$

We note that the diameter passes through the center (which is also the centroid) of the elementary disc. Hence, by parallel axis theorem, the moment of inertia of the elementary disc about a parallel axis along AO (through the centre of the base of cone) is given by

$$dI_{yy} = dI_o + (dm) z^2$$
  
=  $\frac{1}{4} r^2 dm + (dm) z^2 = \left(\frac{1}{4} r^2 + z^2\right) dm$   
=  $\frac{3M}{a^2 h} \left(\frac{1}{4} r^4 + r^2 z^2\right) dz$ ,  $\therefore dm = \rho dV = \frac{3M}{a^2 h} (r^2 dz)$ 

Therefore, the moment of inertia of whole cone about diameter of the base is given by

$$I_{yy} = \frac{3M}{a^2h} \int_{z=0}^{h} \left[ \frac{1}{4} \left( \frac{a(h-z)}{h} \right)^4 + \left( \frac{a(h-z)}{h} \right)^2 z^2 \right] dz$$

$$= \frac{3M}{a^2h} \int_{z=0}^{h} \left[ \frac{a^4}{4h^4} (h-z)^4 + \frac{a^2}{h^2} (h^2 z^2 - 2h z^3 + z^4) \right] dz$$

$$= \frac{3M}{a^2h} \left[ -\frac{a^4}{20 h^4} (h-z)^5 + \frac{a^2}{h^2} \left( h^2 \frac{z^3}{3} - h \frac{z^4}{2} + \frac{z^5}{5} \right) \right] \left| \begin{array}{c} h \\ z=0 \end{array} \right|$$

$$= \frac{3M}{a^2h} \left[ \frac{a^4 h}{20} + \frac{a^2}{h^2} \left( \frac{h^5}{3} - \frac{h^5}{2} + \frac{h^5}{5} \right) \right] = \frac{3M}{a^2h} \left[ \frac{a^4 h}{20} + a^2 h^3 \left( \frac{10 - 15 + 6}{30} \right) \right]$$

$$= \frac{3M}{a^2h} \left[ \frac{a^4 h}{20} + \frac{a^2 h^3}{30} \right] = \frac{3M}{a^2h} \left[ \frac{3a^4 h + 2a^2 h^3}{60} \right] = \frac{1}{20} M (3a^2 + 2h^2)$$

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