Moment of Inertia

Definition: Moment of inertia of a particle of mass m about a line (called axis of rotation) is defined as

$$I = m\,r^2$$

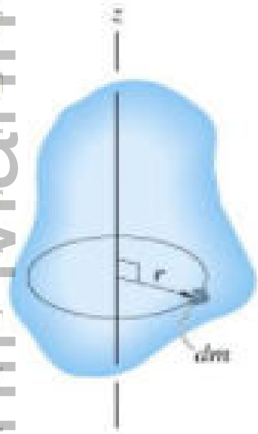
Where, r is the perpendicular distance of particle from line.

Definition: Moment of inertia of a system of a number of particles with masses m_i, about a line (called axis of rotation) is defined as

$$I = \sum_i m_i r_i^2$$

Where, r_i is the perpendicular distance of i-th particle from line.

Definition: Moment of inertia of a continuous distribution of mass, such as the solid rigid body (shown in the figure), having mass M and constant density ρ, about a line is defined as



$$I = \int_M r^2\,\mathrm{d}m = \rho \int_M r^2\,\mathrm{d}V$$

Where, r is the perpendicular distance of point mass element dm of the body and dV is its elementary volume.

Moments of inertia with respect to Cartesian coordinate axes are defined in the following table:

Moment of inertia	Moment of inertia of a particle with respect to 3-dimensioal Cartesian coordinate system	Moment of inertia of a set of particles with respect to 3-dimensioal Cartesian coordinate system	Moment of inertia of a continuous rigid body with respect to 3-dimensioal Cartesian coordinate system
About x-axis $I_{xx} = I_{11}$	$m(y^2+z^2)$	$\sum_i m_i(y_i^2+z_i^2)$	$\int_M (y^2+z^2)\mathrm{d}m$
About y-axis $I_{yy} = I_{22}$	$m(x^2+z^2)$	$\sum_i m_i(x_i^2+z_i^2)$	$\int_M (x^2+z^2)\mathrm{d}m$
About z-axis $I_{zz} = I_{33}$	$m(x^2+y^2)$	$\sum_i m_i(x_i^2+y_i^2)$	$I_{zz} = I_{33} = \int_M (x^2+y^2)\mathrm{d}m$

Products of inertia with respect to Cartesian coordinate axes are defined in the following table:

Product of inertia	Product of inertia of a particle with respect to 3-dimensioal Cartesian coordinate system	Product of inertia of a set of particles with respect to 3-dimensioal Cartesian coordinate system	Product of inertia of a continuous rigid body with respect to 3-dimensioal Cartesian coordinate system
$I_{xy} = I_{yx} = I_{12} = I_{21}$	$-mxy$	$-\sum_i m_i\,x_i y_i$	$-\int_M xy\,\mathrm{d}m$

$I_{yz} = I_{zy} = I_{23} = I_{32}$	$-myz$	$-\sum_i m_i y_i z_i$	$-\int_M yz\,dm$
$I_{xz} = I_{zx} = I_{13} = I_{31}$	$-mxz$	$-\sum_i m_i x_i z_i$	$-\int_M xz\,dm$

Definition: Radius of gyration k of a rigid body of mass M with respect to a line l is defined as

$$k = \sqrt{\frac{I}{M}}$$

where, I is the moment of inertia of the body with respect to l.

Problem: Prove in matrix notation that $[\mathbf{L}] = [\,\mathbf{I}\,][\boldsymbol{\omega}]$, where, all the notations used have their usual meanings.

Proof: The angular momentum of a rigid body, in the form of a set of particles, about an instantaneous axis through a fixed point, is given by

$$\mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i) = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i) = \sum_i m_i\big(\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)\big) \qquad \because \mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

$$= \sum_i m_i(\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)) = \sum_i m_i[(\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i]$$

Let, $\mathbf{L} = L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}$, $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ and $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$

$$\Rightarrow \quad \mathbf{r}_i \cdot \mathbf{r}_i = x_i^2 + y_i^2 + z_i^2 \quad \text{and} \quad \mathbf{r}_i \cdot \boldsymbol{\omega} = x_i\boldsymbol{\omega}_x + y_i\omega_y + z_i\omega_z$$

$$\Rightarrow \quad L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k} = \sum_i m_i\big[(x_i^2 + y_i^2 + z_i^2)(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) - (x_i\omega_x + y_i\omega_y + z_i\omega_z)(x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k})\big]$$

Comparing corresponding components on both sides of above vector equation, we get

$$L_x = \sum_i m_i\big[(x_i^2 + y_i^2 + z_i^2)\omega_x - (x_i\omega_x + y_i\omega_y + z_i\omega_z)x_i\big] \longrightarrow (1)$$

$$L_y = \sum_i m_i\big[(x_i^2 + y_i^2 + z_i^2)\omega_y - (x_i\omega_x + y_i\omega_y + z_i\omega_z)y_i\big] \longrightarrow (2)$$

$$L_z = \sum_i m_i\big[(x_i^2 + y_i^2 + z_i^2)\omega_z - (x_i\omega_x + y_i\omega_y + z_i\omega_z)z_i\big] \longrightarrow (3)$$

From Eq. (1), we get

$$L_x = \sum_i m_i\big[(x_i^2 + y_i^2 + z_i^2)\omega_x - (x_i\omega_x + y_i\omega_y + z_i\omega_z)x_i\big]$$

$$= \sum_i m_i\big[x_i^2\omega_x + (y_i^2 + z_i^2)\omega_x - x_i^2\omega_x - x_iy_i\omega_y - x_iz_i\omega_z\big]$$

$$= \omega_x \sum_i m_i\left(y_i^2 + z_i^2\right) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i$$

$$\Rightarrow L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \quad ------\rightarrow \quad (4)$$

Similarly, from (2) and (3), we get

$$L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \quad ------\rightarrow \quad (5)$$

and $$L_z = I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z \quad ------\rightarrow \quad (6)$$

Writing Eqs. (4), (5) and (6) in matrix form, we get

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$\Rightarrow$ **[L] = [I][ω]** **Hence proved.**

For your information:

Rotational-Linear Parallels

	Linear Motion	Rotational Motion	
Position	x	θ	Angular position
Velocity	v	ω	Angular velocity
Acceleration	a	α	Angular acceleration
Motion equations	$x = \bar{v}t$	$\theta = \bar{\omega}t$	Motion equations
	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$	
	$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	
	$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$	
Mass (linear inertia)	m	I	Moment of inertia
Newton's second law	$F = ma$	$\tau = I\alpha$	Newton's second law
Momentum	$p = mv$	$L = I\omega$	Angular momentum
Work	Fd	$\tau\theta$	Work
Kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$	Kinetic energy
Power	Fv	$\tau\omega$	Power

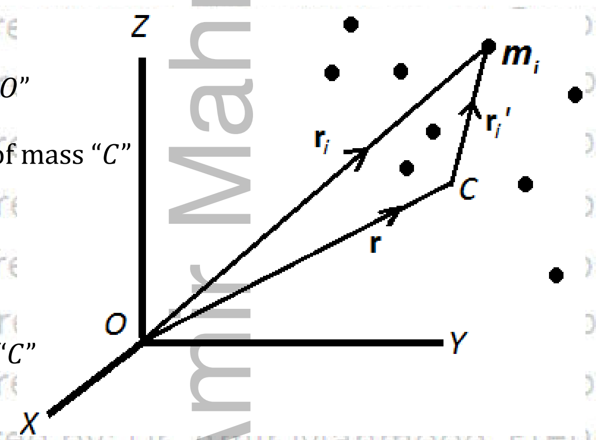
Problem: Prove that $T = \frac{1}{2}M\mathbf{v}^2 + \frac{1}{2}\boldsymbol{\omega}.\mathbf{L}$, where all the notations used have their usual meanings.

(or) prove that $T = T_{tr} + T_{rot}$

where, $T_{tr} = \frac{1}{2}M\mathbf{v}^2 =$ total translational kinetic energy of the system

and $T_{rot} = \frac{1}{2}\boldsymbol{\omega}.\mathbf{L} =$ total rotational kinetic energy of the system

Proof: Consider a rigid body, in the form of a set of particles, which is in general state of motion (i.e, having both translation and rotation) with respect to a fixed (inertial) frame of reference $Oxyz$.



Let, $M =$ total mass of the body

$\mathbf{r}_i =$ position vector of i-th particle of mass m_i with respect to origin "O"

$\mathbf{r}_i' =$ position vector of i-th particle of mass m_i with respect to centre of mass "C"

$\mathbf{r} =$ position vector of centre of mass "C" with respect to origin "O"

$\mathbf{v}_i =$ velocity of i-th particle of mass m_i with respect to origin "O"

$\mathbf{v}_i' =$ velocity of i-th particle of mass m_i with respect to centre of mass "C"

$\mathbf{v} =$ velocity of centre of mass "C" with respect to origin "O"

$\boldsymbol{\omega} =$ instantaneous angular velocity of body about instantaneous axis through centre of mass "C"

From figure, $\mathbf{r}_i = \mathbf{r} + \mathbf{r}_i'$

Differentiating both sides with respect to time "t", we get

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}} + \dot{\mathbf{r}}_i'$$

$$\mathbf{v}_i = \mathbf{v} + \mathbf{v}_i'$$

$$\mathbf{v}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i' \qquad \because \quad \mathbf{v}_i' = \boldsymbol{\omega} \times \mathbf{r}_i'$$

Kinetic energy of the i-th particle is $T_i = \frac{1}{2}m_i\mathbf{v}_i{}^2$

Kinetic energy of the whole body is

$$T = \sum_i T_i = \frac{1}{2}\sum_i m_i\mathbf{v}_i{}^2 = \frac{1}{2}\sum_i m_i\,(\mathbf{v}_i \cdot \mathbf{v}_i) = \frac{1}{2}\sum_i m_i\,\{(\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i') \cdot (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i')\} \quad \because \quad \mathbf{v}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i'$$

$$= \frac{1}{2}\sum_i m_i\,\{\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}_i') + (\boldsymbol{\omega} \times \mathbf{r}_i') \cdot \mathbf{v} + (\boldsymbol{\omega} \times \mathbf{r}_i') \cdot (\boldsymbol{\omega} \times \mathbf{r}_i')\}$$

$$= \frac{1}{2}\sum_i m_i\,\{\mathbf{v}^2 + 2\mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}_i') + \boldsymbol{\omega} \cdot \mathbf{r}_i' \times (\boldsymbol{\omega} \times \mathbf{r}_i')\}T$$

$$= \frac{1}{2}\left(\sum_i m_i\right)\mathbf{v}^2 + \sum_i m_i\mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}_i') + \frac{1}{2}\sum_i m_i\{\boldsymbol{\omega} \cdot \mathbf{r}_i' \times (\boldsymbol{\omega} \times \mathbf{r}_i')\}$$

$$T = \frac{1}{2}M\mathbf{v}^2 + \mathbf{v}\cdot\left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i'\right) + \frac{1}{2}\boldsymbol{\omega}\cdot\sum_i m_i \mathbf{r}_i' \times (\boldsymbol{\omega} \times \mathbf{r}_i')$$

where, $M = \sum_i m_i$ = total mass of the body

Also, $\sum_i m_i \mathbf{r}_i' = \mathbf{0}$, as $\mathbf{r}_i'$ is the position vector of ith particle of mass m_i with respect to centre of mass "C"

$$\Rightarrow \quad T = \frac{1}{2}M\mathbf{v}^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\sum_i m_i \mathbf{r}_i' \times (\boldsymbol{\omega} \times \mathbf{r}_i') \quad ---------(1)$$

But, angular momentum $\mathbf{L}$ of the body with respect to centre of mass "C" is given by

$$\mathbf{L} = \sum_i \mathbf{r}_i' \times (m_i \mathbf{v}_i') = \sum_i \mathbf{r}_i' \times \{m_i(\boldsymbol{\omega} \times \mathbf{r}_i')\} = \sum_i m_i \mathbf{r}_i' \times (\boldsymbol{\omega} \times \mathbf{r}_i') \quad ---------(2)$$

Using (2) in (1), we get

$$\boxed{T = \frac{1}{2}M\mathbf{v}^2 + \frac{1}{2}\boldsymbol{\omega}.\mathbf{L}}$$

$$\boxed{T = T_{tr} + T_{rot}}$$

where, $T_{tr} = \frac{1}{2}M\mathbf{v}^2$ = total translational kinetic energy of the system

and $T_{rot} = \frac{1}{2}\boldsymbol{\omega}.\mathbf{L}$ = total rotational kinetic energy of the system

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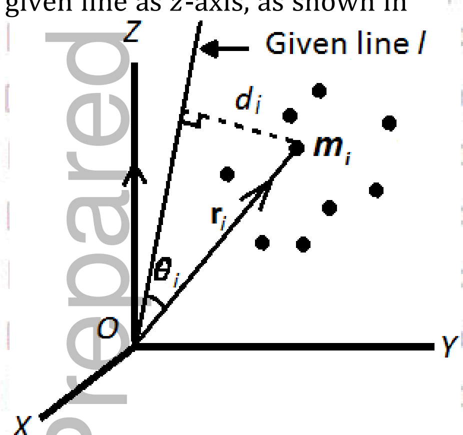
Problem: Find moment of inertia of a rigid body about a given line passing through the origin and having direction cosines are (λ, μ, ν).

Solution: Consider a rigid body, in the form of a set of particles. And let us take given line as z-axis, as shown in the figure.



Let,

M = total mass of the body

$\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$ = position vector of i-th particle of mass m_i w.r.t. origin "O"

d_i = perpendicular distance of i-th particle of mass m_i from given line l

θ_i = angle between position vector $\mathbf{r}_i$ and given line l

$\mathbf{e}$ = unit vector in the direction of given line l

Then, $\mathbf{e} = \lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k}$, where, (λ, μ, ν) are direction cosines of the given line l.

The required moment of inertia I_l is given by

$$I_l = \sum_i m_i d_i^2 = \sum_i m_i(|\mathbf{r}_i|\sin\theta_i)^2 = \sum_i m_i(|\mathbf{e}\times\mathbf{r}_i|)^2 - - - - - (1) \quad \because \sin\theta_i = \frac{d_i}{|\mathbf{r}_i|} \text{ and } |\mathbf{r}_i|\sin\theta_i = |\mathbf{e}\times\mathbf{r}_i|$$

Now, $$\mathbf{e}\times\mathbf{r}_i = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda & \mu & \nu \\ x_i & y_i & z_i \end{vmatrix} = (\mu z_i - \nu y_i)\mathbf{i} + (\nu x_i - \lambda z_i)\mathbf{j} + (\lambda y_i - \mu x_i)\mathbf{k}$$

$$\Rightarrow |\mathbf{e}\times\mathbf{r}_i| = (\mu z_i - \nu y_i)^2 + (\nu x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2 - - - - - - - (2)$$

Using (2) in (1), we get

$$I_l = \sum_i m_i[(\mu z_i - \nu y_i)^2 + (\nu x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2]$$

$$= \sum_i m_i[(\mu^2 {z_i}^2 + \nu^2 {y_i}^2 - 2\mu\nu y_i z_i) + (\nu^2 {x_i}^2 + \lambda^2 {z_i}^2 - 2\lambda\nu x_i z_i) + (\lambda^2 {y_i}^2 + \mu^2 {x_i}^2 - 2\lambda\mu x_i y_i)]$$

$$= \lambda^2 \sum_i m_i({y_i}^2 + {z_i}^2) + \mu^2 \sum_i m_i({x_i}^2 + {z_i}^2) + \nu^2 \sum_i m_i({x_i}^2 + {y_i}^2) + 2\lambda\mu\left(-\sum_i m_i x_i y_i\right)$$

$$+ 2\mu\nu\left(-\sum_i m_i y_i z_i\right) + 2\lambda\nu\left(-\sum_i m_i x_i z_i\right)$$

$$\boxed{I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu\nu I_{yz} + 2\lambda\nu I_{xz}}$$

This is the required moment of inertia.

Problem: Find the equation of "**ellipsoid of inertia**" or "**momental ellipsoid**" of a rigid body.

Solution: As we know that moment of inertia of a rigid body about a given line l having direction cosines (λ, μ, ν) with respect to a coordinate system $Oxyz$, whose origin "O" lies on the line l, is given by

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu\nu I_{yz} + 2\lambda\nu I_{xz} - - - - - - (1)$$

On the line l, choose a point P such that $|\overrightarrow{OP}| = 1/\sqrt{I_l}$. If coordinates of P are (x, y, z), then

$$\frac{x}{|\overrightarrow{OP}|} = \lambda, \qquad \frac{y}{|\overrightarrow{OP}|} = \mu, \qquad \frac{z}{|\overrightarrow{OP}|} = \nu$$

$$\Rightarrow \quad \lambda = x\sqrt{I_l}, \qquad \mu = y\sqrt{I_l}, \qquad \nu = z\sqrt{I_l} - - - - - - (2)$$

Eliminating λ, μ and ν from (1) and (2), we get

$$I_l = I_l(I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{yz}yz + 2I_{xz}xz)$$

$$\boxed{I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{yz}yz + 2I_{xz}xz = 1}$$

Since, I_{xx}, I_{yy} and I_{zz} are all positive, therefore, above equation represents an ellipsoid called "ellipsoid of inertia" or "momental ellipsoid" of the rigid body.

Note:

(i) The momental ellipsoid of a rigid body contains information about moments and product of inertia of that body.

(ii) The centre of momental ellipsoid lies at the origin of the coordinate system.

(ii) If P is any point on momental ellipsoid, then

$$|\overrightarrow{OP}| = \frac{1}{\sqrt{I_l}} \Rightarrow I_l = \frac{1}{|\overrightarrow{OP}|^2},$$

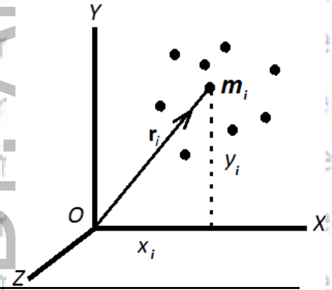
showing that moment of inertia about line $\overleftrightarrow{OP}$ is equal to the reciprocal of square of distance of point P from origin O.

Problem: State and prove perpendicular axis theorem for a set of particles.

Statement: The moment of inertia of a plane rigid body in the form of a set of particles about a given axis perpendicular to the plane of the body is equal to the sum of moments of inertia about two mutually perpendicular axes lying in the plane of the body and meeting at a common point on the given axis.

Proof: We choose Cartesian coordinate system $Oxyz$ such that xy-plane lies in the plane of the body, while z-axis lies perpendicular to it, which is assumed to the given axis.

Let, $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j}$ be the position vector of i-th particle of mass m_i w.r.t. origin "O". Then moment of inertia of the body about z-axis is



$$I_{zz} = \sum_i m_i|\mathbf{r}_i|^2 = \sum_i m_i({x_i}^2 + {y_i}^2) = \sum_i m_i{x_i}^2 + \sum_i m_i{y_i}^2 = I_{xx} + I_{yy}$$

$$\Rightarrow \quad I_{zz} = I_{xx} + I_{yy} \quad \textbf{Hence proved.}$$

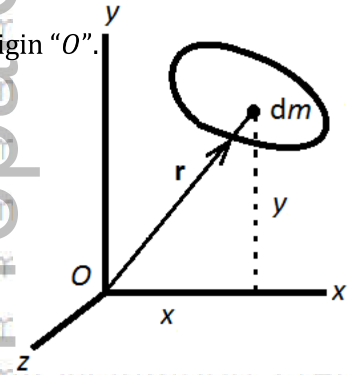
Problem: State and prove perpendicular axis theorem for a continuous mass distribution.

Statement: The moment of inertia of a plane rigid body in the form of continuous mass distribution about a given axis perpendicular to the plane of the body is equal to the sum of moments of inertia of same body about two mutually perpendicular axes lying in the plane of body and meeting at a common point on the given axis.

Proof: We choose Cartesian coordinate system $Oxyz$ such that xy-plane lies in the plane of the body having mass M, while z-axis lies perpendicular to it, which is assumed to the given axis.

Let, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of elementary particle of body of mass $\mathrm{d}m$ w.r.t. origin "O".

Then moment of inertia of the body about z-axis is



$$I_{zz} = \int_M |\mathbf{r}|^2 \mathrm{d}m = \int_M (x^2 + y^2)\mathrm{d}m = \int_M x^2 \mathrm{d}m + \int_M y^2 \mathrm{d}m = I_{xx} + I_{yy}$$

$$\Rightarrow \quad I_{zz} = I_{xx} + I_{yy} \quad \textbf{Hence proved.}$$

Problem: State and prove parallel axis theorem for the case of moment of inertia for a set of particles.

Statement: The moment of inertia of a rigid body in the form of a set of particles about a given axis is equal to the sum of moment of inertia of same body about a parallel axis (to the given axis) through the centre of mass of the body and the moment of inertia due to the total mass of the body placed at its centre of mass, about given axis.

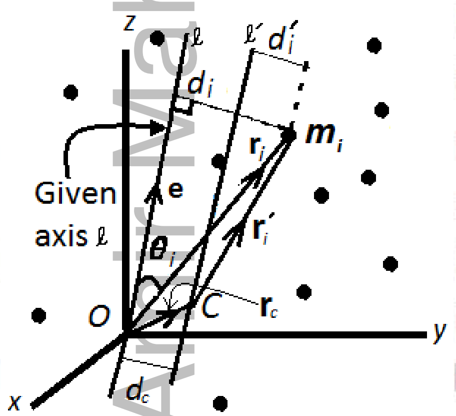
Proof: Consider a rigid body, in the form of a set of particles. Let l be the given and l' be an axis which is parallel to l and passing through centre of mass of the body.

Let, $M =$ total mass of the body

$\mathbf{r}_i =$ position vector of i-th particle of mass m_i with respect to origin "O"

$\mathbf{r}_i' =$ position vector of i-th particle of mass m_i with respect to centre of mass "C"

$\mathbf{r}_c =$ position vector of centre of mass "C" with respect to origin "O"



$\theta_i =$ angle between position vector $\mathbf{r}_i$ and given line l

$d_i =$ perpendicular distance of i-th particle of mass m_i from given axis l

$d_i' =$ perpendicular distance of i-th particle of mass m_i from parallel axis l'

$d_c =$ perpendicular distance of centre of mass C from given axis l

$\quad =$ perpendicular distance between l and l'

$\mathbf{e} =$ unit vector in the direction of given line l

From figure, $\sin\theta_i = \frac{d_i}{|\mathbf{r}_i|} \Rightarrow d_i = |\mathbf{r}_i|\sin\theta_i = |\mathbf{e}\times\mathbf{r}_i|$

Similarly, $d_i' = |\mathbf{e}\times\mathbf{r}_i'|$ and $d_c = |\mathbf{e}\times\mathbf{r}_c|$

Moment of inertia of the body about given axis l is given by

$$I_l = \sum_i m_i d_i^2 = \sum_i m_i(|\mathbf{e}\times\mathbf{r}_i|)^2 = \sum_i m_i(\mathbf{e}\times\mathbf{r}_i)\cdot(\mathbf{e}\times\mathbf{r}_i)$$

$$= \sum_i m_i[\mathbf{e}\times(\mathbf{r}_c+\mathbf{r}_i')]\cdot[\mathbf{e}\times(\mathbf{r}_c+\mathbf{r}_i')] \qquad \because \mathbf{r}_i = \mathbf{r}_c + \mathbf{r}_i' \text{ (from figure)}$$

$$= \sum_i m_i(\mathbf{e}\times\mathbf{r}_c + \mathbf{e}\times\mathbf{r}_i')\cdot(\mathbf{e}\times\mathbf{r}_c + \mathbf{e}\times\mathbf{r}_i')$$

$$= \sum_i m_i[(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}_c) + 2(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}_i') + (\mathbf{e}\times\mathbf{r}_i')\cdot(\mathbf{e}\times\mathbf{r}_i')]$$

$$= \sum_i m_i\left[(|\mathbf{e}\times\mathbf{r}_c|)^2 + 2(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}_i') + (|\mathbf{e}\times\mathbf{r}_i'|)^2\right]$$

$$= \left(\sum_i m_i\right)(|\mathbf{e}\times\mathbf{r}_c|)^2 + 2(\mathbf{e}\times\mathbf{r}_c)\cdot\sum_i m_i\,(\mathbf{e}\times\mathbf{r}_i') + \sum_i m_i(|\mathbf{e}\times\mathbf{r}_i'|)^2$$

$$= Md_c^{\,2} + 2(\mathbf{e}\times\mathbf{r}_c)\cdot\left(\mathbf{e}\times\sum_i m_i\,\mathbf{r}_i'\right) + \sum_i m_i d_i'^{\,2}$$

where, $M = \sum_i m_i$ = total mass of the body

Also, $\sum_i m_i \mathbf{r}_i' = \mathbf{0}$, as $\mathbf{r}_i'$ is the position vector of ith particle of mass m_i with respect to centre of mass "C" and

$I_{l'} = \sum_i m_i d_i'^2$ = moment of inertia of the body axis l'

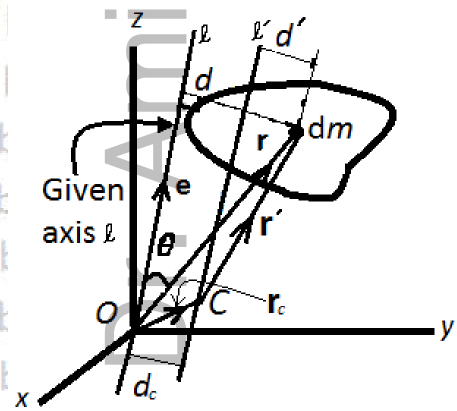
Therefore,

$$I_l = I_{l'} + Md_c^2 \quad \textbf{Hence proved.}$$

Problem: State and prove parallel axis theorem for the case of moment of inertia for a continuous mass distribution.

Statement: The moment of inertia of a rigid body in the form of a continuous mass distribution about a given axis is equal to the sum of moment of inertia of same body about a parallel axis (to the given axis) through the centre of mass of the body and the moment of inertia due to the total mass of the body placed at its centre of mass, about given axis.

Proof: Consider a rigid body, in the form of a continuous mass distribution. Let l be the given and l' be an axis which is parallel to l and passing through centre of mass of the body.



Let, M = total mass of the body

$\mathbf{r}$ = position vector of i-th particle of mass m_i with respect to origin "O"

$\mathbf{r}'$ = position vector of i-th particle of mass m_i with respect to centre of mass "C"

$\mathbf{r}_c$= position vector of centre of mass "C" with respect to origin "O"

θ = angle between position vector $\mathbf{r}_i$ and given line l

d = perpendicular distance of i-th particle of mass m_i from given axis l

d' = perpendicular distance of i-th particle of mass m_i from parallel axis l'

d_c = perpendicular distance of centre of mass C from given axis l = perpendicular distance between l and l'

$\mathbf{e}$ = unit vector in the direction of given line l

From figure, $\sin\theta = \frac{d}{|\mathbf{r}|} \Rightarrow d = |\mathbf{r}|\sin\theta = |\mathbf{e} \times \mathbf{r}|$

Similarly, $d' = |\mathbf{e} \times \mathbf{r}'|$ and $d_c = |\mathbf{e} \times \mathbf{r}_c|$

Moment of inertia of the body about given axis l is given by

$$I_l = \int_M d^2 \mathrm{d}m = \int_M (|\mathbf{e} \times \mathbf{r}|)^2 \mathrm{d}m = \int_M (\mathbf{e} \times \mathbf{r}) \cdot (\mathbf{e} \times \mathbf{r}) \mathrm{d}m$$

$$= \int_M [\mathbf{e} \times (\mathbf{r}_c + \mathbf{r}')] \cdot [\mathbf{e} \times (\mathbf{r}_c + \mathbf{r}')] \mathrm{d}m \qquad \because \mathbf{r} = \mathbf{r}_c + \mathbf{r}' \quad \text{(from figure)}$$

$$I_l = \int_M (\mathbf{e} \times \mathbf{r}_c + \mathbf{e} \times \mathbf{r}') \cdot (\mathbf{e} \times \mathbf{r}_c + \mathbf{e} \times \mathbf{r}') \mathrm{d}m$$

$$= \int_M [(\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}_c) + 2(\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}') + (\mathbf{e} \times \mathbf{r}') \cdot (\mathbf{e} \times \mathbf{r}')] \mathrm{d}m$$

$$= \int_M [(|\mathbf{e} \times \mathbf{r}_c|)^2 + 2(\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}') + (|\mathbf{e} \times \mathbf{r}'|)^2] \mathrm{d}m$$

$$= \left(\int_M \mathrm{d}m \right) (|\mathbf{e} \times \mathbf{r}_c|)^2 + 2(\mathbf{e} \times \mathbf{r}_c) \cdot \int_M (\mathbf{e} \times \mathbf{r}') \mathrm{d}m + \int_M (|\mathbf{e} \times \mathbf{r}'|)^2 \mathrm{d}m$$

$$= M {d_c}^2 + 2(\mathbf{e} \times \mathbf{r}_c) \cdot \left(\mathbf{e} \times \int_M \mathbf{r}' \mathrm{d}m \right) + \int_M d'^2 \mathrm{d}m$$

$$\text{where,} \quad M = \int_M \mathrm{d}m = \text{total mass of the body}$$

Also, $\int_M \mathbf{r}' \mathrm{d}m = \mathbf{0}$, as $\mathbf{r}'$ is the position vector of mass element $\mathrm{d}m$ with respect to centre of mass "C" and

$I_{l'} = \sum_i m_i {d'}^2$ = moment of inertia of the body axis l'

$$\Rightarrow \quad I_l = I_{l'} + M {d_c}^2 \qquad \textbf{Hence proved.}$$

Problem: Prove in matrix notation that $[\dot{\mathbf{L}}] = [\boldsymbol{\omega} \times \mathbf{L}] + [\,\mathbf{I}\,][\dot{\boldsymbol{\omega}}]$, where, all the notations used have their usual meanings.

Proof: As we know that the angular momentum of a system of particles is given by

$$\mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i) = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i$$

Differentiating both sides with respect to time "t", we get

$$\dot{\mathbf{L}} = \sum_i m_i \dot{\mathbf{r}}_i \times \mathbf{v}_i + \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{v}}_i = \sum_i m_i \mathbf{v}_i \times \mathbf{v}_i + \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{v}}_i$$

$$= \sum_i m_i \mathbf{r}_i \times \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\omega} \times \mathbf{r}_i) \qquad \because \quad \mathbf{v}_i \times \mathbf{v}_i = \mathbf{0} \ \text{ and } \ \dot{\mathbf{v}}_i = \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\omega} \times \mathbf{r}_i)$$

$$= \sum_i m_i \mathbf{r}_i \times [(\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) + (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)] = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) + \sum_i m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)$$

Writing in matrix form, we get

$$[\dot{\mathbf{L}}] = \left[\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i)\right] + \left[\sum_i m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)\right] -----\rightarrow (1)$$

We also know that, $[\mathbf{L}] = [\,\mathbf{I}\,][\boldsymbol{\omega}]$

$$\left[\sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i)\right] = [\,\mathbf{I}\,][\boldsymbol{\omega}] \qquad \because \mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i)$$

$$\left[\sum_i m_i \mathbf{r}_i \times \mathbf{v}_i\right] = [\,\mathbf{I}\,][\boldsymbol{\omega}]$$

$$\left[\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)\right] = [\,\mathbf{I}\,][\boldsymbol{\omega}] \qquad \because \mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

Replace $\boldsymbol{\omega}$ by $\dot{\boldsymbol{\omega}}$ on both sides, we get

$$\left[\sum_i m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)\right] = [\,\mathbf{I}\,][\dot{\boldsymbol{\omega}}] -----\rightarrow (2)$$

Now consider,

$$\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) = \sum_i m_i \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = \sum_i m_i \mathbf{r}_i \times [(\boldsymbol{\omega} \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{r}_i]$$

$$= \sum_i m_i [(\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_i \times \mathbf{r}_i)] = \sum_i m_i (\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) ---\rightarrow (3) \qquad \because \mathbf{r}_i \times \mathbf{r}_i = \mathbf{0}$$

Further consider that

$$\boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{v}_i) = \boldsymbol{\omega} \times [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = \boldsymbol{\omega} \times [(\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i] = (\mathbf{r}_i \cdot \mathbf{r}_i)(\boldsymbol{\omega} \times \boldsymbol{\omega}) - (\mathbf{r}_i \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{r}_i)$$

$$= -(\mathbf{r}_i \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{r}_i) = (\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) --------\rightarrow (4) \qquad \because \boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$$

Using (4) in (3), we get

$$\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \sum_i m_i\, \boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{v}_i) = \boldsymbol{\omega} \times \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i) = \boldsymbol{\omega} \times \mathbf{L} \qquad \because \mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i)$$

Writing in matrix form, we get

$$\left[\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i)\right] = [\boldsymbol{\omega} \times \mathbf{L}] -------\rightarrow (5)$$

Using (2) and (5) in (1), we get

$$\boxed{[\dot{\mathbf{L}}] = [\boldsymbol{\omega} \times \mathbf{L}] + [\,\mathbf{I}\,][\dot{\boldsymbol{\omega}}] \qquad \textbf{Hence proved.}}$$

Problem: Show that inertia matrix $[\,\mathbf{I}\,]$ is a Cartesian tensor of rank 2.

Proof: As we know that the angular momentum of a system of particles is given by

$$\mathbf{L} = \sum_{\alpha} \mathbf{r}_\alpha \times (m_\alpha \mathbf{v}_\alpha) = \sum_{\alpha} m_\alpha (\mathbf{r}_\alpha \times \mathbf{v}_\alpha) = \sum_{\alpha} m_\alpha \big(\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)\big) \qquad \because \mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

$$\mathbf{L} = \sum_{\alpha} m_\alpha [(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha)\boldsymbol{\omega} - (\mathbf{r}_\alpha \cdot \boldsymbol{\omega})\mathbf{r}_\alpha] = \sum_{\alpha} m_\alpha [\mathbf{r}_\alpha^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)\mathbf{r}_\alpha] ---------\rightarrow (1)$$

Let, $\mathbf{L} = (L_1,\, L_2,\, L_3)$, $\boldsymbol{\omega} = (\omega_1,\, \omega_2,\, \omega_3)$ and $\mathbf{r}_\alpha = (x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3})$

Then, $\boldsymbol{\omega} \cdot \mathbf{r}_\alpha = \omega_1 x_{\alpha,1} + \omega_2 x_{\alpha,2} + \omega_3 x_{\alpha,3} = \sum_{j=1}^{3} \omega_j x_{\alpha,j}$

So, (1) can be written as

$$(L_1,\, L_2,\, L_3) = \sum_{\alpha} m_\alpha \left[\mathbf{r}_\alpha^2 (\omega_1,\, \omega_2,\, \omega_3) - \left(\sum_{j=1}^{3} \omega_j x_{j,\alpha} \right) (x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3}) \right]$$

$$\Rightarrow L_i = \sum_{\alpha} m_\alpha \left[\mathbf{r}_\alpha^2 \omega_i - \left(\sum_{j=1}^{3} \omega_j x_{\alpha,j} \right) x_{\alpha,i} \right], \qquad i = 1, 2, 3$$

$$= \sum_{\alpha} m_\alpha \left[\mathbf{r}_\alpha^2 \sum_{j=1}^{3} \omega_j \delta_{ij} - \left(\sum_{j=1}^{3} \omega_j x_{\alpha,j} \right) x_{\alpha,i} \right] \qquad \because \omega_i = \sum_{j=1}^{3} \omega_j \delta_{ij}$$

$$= \sum_{\alpha} m_\alpha \sum_{j=1}^{3} [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i}] \omega_j = \sum_{j=1}^{3} \omega_j \sum_{\alpha} m_\alpha [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}]$$

$$= \sum_{\alpha} m_\alpha \sum_{j=1}^{3} [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i}] \omega_j = \sum_{j=1}^{3} \omega_j \sum_{\alpha} m_\alpha [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] = \sum_{j=1}^{3} \omega_j I_{ij} -------\rightarrow (2)$$

where, $I_{ij} = \sum_{\alpha} m_\alpha [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] = ij'$th component of inertia tensor

Since, both the angular velocity $\boldsymbol{\omega} = (\omega_j)$ and the angular momentum $\mathbf{L} = (L_i)$ are known to be vectors (i.e., Cartesian tensors of rank 1), it follows from equation (2) and quotient theorem that the inertia tensor $[\,\mathbf{I}\,] = (I_{ij})$ is a Cartesian tensor of rank 2.

Problem: Express angular momentum in tensor notation.

Solution: As we know that the angular momentum of a system of particles is given by

$$\mathbf{L} = \sum_{\alpha} \mathbf{r}_\alpha \times (m_\alpha \mathbf{v}_\alpha) = \sum_{\alpha} m_\alpha (\mathbf{r}_\alpha \times \mathbf{v}_\alpha) = \sum_{\alpha} m_\alpha \big(\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)\big) \qquad \because \mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

$$\mathbf{L} = \sum_{\alpha} m_\alpha [(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha)\boldsymbol{\omega} - (\mathbf{r}_\alpha \cdot \boldsymbol{\omega})\mathbf{r}_\alpha] = \sum_{\alpha} m_\alpha [\mathbf{r}_\alpha^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)\mathbf{r}_\alpha] ---------\rightarrow (1)$$

Let, $\mathbf{L} = (L_1,\, L_2,\, L_3)$, $\boldsymbol{\omega} = (\omega_1,\, \omega_2,\, \omega_3)$ and $\mathbf{r}_\alpha = (x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3})$

Then, $\boldsymbol{\omega}\cdot\mathbf{r}_\alpha = \omega_1 x_{\alpha,1}+\omega_2 x_{\alpha,2}+\omega_3 x_{\alpha,3} = \sum_{j=1}^{3}\omega_j x_{\alpha,j}$

So, (1) can be written as

$$(L_1,\, L_2,\, L_3) = \sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2(\omega_1,\,\omega_2,\,\omega_3) - \left(\sum_{j=1}^{3}\omega_j x_{j,\alpha}\right)(x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3})\right]$$

$$\Rightarrow\ L_i = \sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\omega_i - \left(\sum_{j=1}^{3}\omega_j x_{\alpha,j}\right)x_{\alpha,i}\right],\qquad i=1,2,3$$

$$= \sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\sum_{j=1}^{3}\omega_j\delta_{ij} - \left(\sum_{j=1}^{3}\omega_j x_{\alpha,j}\right)x_{\alpha,i}\right]\qquad \because\ \omega_i = \sum_{j=1}^{3}\omega_j\delta_{ij}$$

$$= \sum_\alpha m_\alpha\sum_{j=1}^{3}\left[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,j}x_{\alpha,i}\right]\omega_j = \sum_{j=1}^{3}\omega_j\sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}\right] = \sum_{j=1}^{3}\omega_j I_{ij} ------\rightarrow (2)$$

where, $I_{ij} = \sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}\right] = ij$th component of inertia tensor

Equation (2) is required tensor form of angular momentum.

Problem: Express rotational kinetic energy in tensor notation.

Solution: As we know that the rotational kinetic energy of a system is given by

$$= \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$$

Let, $\boldsymbol{\omega} = (\omega_1,\,\omega_2,\,\omega_3)$, $\mathbf{L} = (L_1,\, L_2,\, L_3)$

$$\Rightarrow\ T_{rot} = \frac{1}{2}(\omega_1 L_1 + \omega_2 L_2 + \omega_3 L_3) = \frac{1}{2}\sum_{i=1}^{3}\omega_i L_i$$

$$= \frac{1}{2}\sum_{i=1}^{3}\omega_i\left(\sum_{j=1}^{3}\omega_j\sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}\right]\right)\qquad \because\ L_i = \sum_{j=1}^{3}\omega_j\sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{3}\omega_i\sum_{j=1}^{3}\omega_j\left(\sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}\right]\right) = \frac{1}{2}\sum_{i,j=1}^{3}\omega_i\omega_j I_{ij} ------\rightarrow (1)$$

where, $I_{ij} = \sum_\alpha m_\alpha\left[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}\right] = ij$th component of inertia tensor

Equation (1) is required tensor form of rotational kinetic energy.

Problem: Express parallel axis theorem in tensor notation.

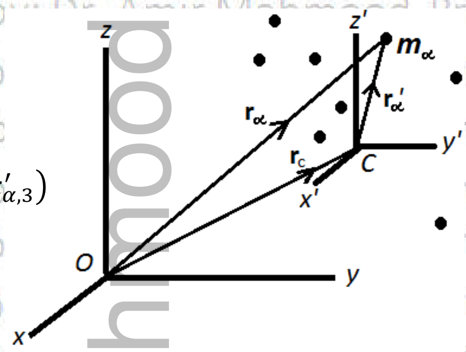
Solution: Consider a rigid body, in the form of a set of particles. Let, C be the centre of mass of the body. We consider two parallel coordinate systems $Oxyz$ and $Cx'y'z'$, as shown in the figure.

Let, $M =$ total mass of the body

$\mathbf{r}_\alpha =$ position vector of α-th particle of mass m_α with respect to origin "O"

$\mathbf{r}'_\alpha =$ position vector of α-th particle of mass m_α with respect to centre of mass "C"

$\mathbf{r}_c =$ position vector of centre of mass "C" with respect to origin "O"



From figure, $\mathbf{r}_\alpha = \mathbf{r}_c + \mathbf{r}'_\alpha \longrightarrow (1)$

Let, $\mathbf{r}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$, $\mathbf{r}_c = (x_{c,1}, x_{c,2}, x_{c,3})$ and $\mathbf{r}'_\alpha = (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$

Equation (1) becomes

$$(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$$

$$\Rightarrow \quad x_{\alpha,i} = x_{c,i} + x'_{\alpha,i}, \qquad i = 1,2,3 \longrightarrow (2)$$

As we know that

$$I_{ij} = \sum_\alpha m_\alpha \left[\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}\right]$$

$$= \sum_\alpha m_\alpha \left[\left((\mathbf{r}_c + \mathbf{r}'_\alpha)\cdot(\mathbf{r}_c + \mathbf{r}'_\alpha)\right)\delta_{ij} - (x_{c,i} + x'_{\alpha,i})(x_{c,j} + x'_{\alpha,j})\right] \qquad \text{(by using (1) and (2))}$$

$$= \sum_\alpha m_\alpha \left[(\mathbf{r}_c \cdot \mathbf{r}_c + 2\,\mathbf{r}_c \cdot \mathbf{r}'_\alpha + \mathbf{r}'_\alpha \cdot \mathbf{r}'_\alpha)\delta_{ij} - x_{c,i}x_{c,j} - x_{c,i}x'_{\alpha,j} - x_{c,j}x'_{\alpha,i} - x'_{\alpha,i}x'_{\alpha,j}\right]$$

$$= \sum_\alpha m_\alpha \left[(\mathbf{r}_c^2 + 2\,\mathbf{r}_c \cdot \mathbf{r}'_\alpha + \mathbf{r}'^2_\alpha)\delta_{ij} - x_{c,i}x_{c,j} - x_{c,i}x'_{\alpha,j} - x_{c,j}x'_{\alpha,i} - x'_{\alpha,i}x'_{\alpha,j}\right]$$

$$= \sum_\alpha m_\alpha \left[\mathbf{r}'^2_\alpha \delta_{ij} - x'_{\alpha,i}x'_{\alpha,j}\right] + 2\,\mathbf{r}_c \cdot \left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha\right)\delta_{ij} + \left(\sum_\alpha m_\alpha\right)\mathbf{r}'^2_c \delta_{ij} - \left(\sum_\alpha m_\alpha\right) x_{c,i}x_{c,j}$$

$$- \left(\sum_\alpha m_\alpha x'_{\alpha,j}\right) x_{c,i} - \left(\sum_\alpha m_\alpha x'_{\alpha,i}\right) x_{c,j} \longrightarrow (3)$$

Now, $\sum_\alpha m_\alpha \left[\mathbf{r}_c^2 \delta_{ij} - x'_{\alpha,i}x'_{\alpha,j}\right] = I'_{ij} = ij$th component of inertia tensor with respect to $Cx'y'z'$ system

Also, $\sum_\alpha m_\alpha \mathbf{r}'_\alpha = \mathbf{0}$, as $\mathbf{r}'_\alpha$ is the position vector of α-th particle of mass m_α with respect to centre of mass "C"

$$\sum_\alpha m_\alpha \mathbf{r}'_\alpha = \mathbf{0} \;\Rightarrow\; \sum_\alpha m_\alpha (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3}) = (0,0,0) \;\Rightarrow\; \sum_\alpha m_\alpha x'_{\alpha,i} = 0, \quad i = 1,2,3$$

And, $\sum_\alpha m_\alpha = M =$ total mass of the body

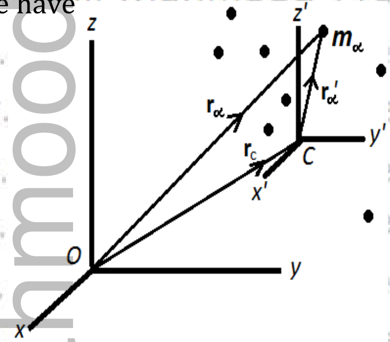
So equation (3) becomes

$$\boxed{I_{ij} = I'_{ij} + M\mathbf{r}'^2_c \delta_{ij} - M x_{c,i} x_{c,j}}$$

This is required tensor form of parallel axis theorem.

Problem: State and prove parallel axis theorem for the case of products of inertia for a set of particles.

Statement: Consider a rigid body, in the form of a set of particles. Let, C be the centre of mass of the body. If $Oxyz$ and $Cx'y'z'$ be two parallel coordinate systems as shown in the figure, then we have



$$I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}, \quad i \neq j, \quad i,j \in \{1,2,3\}$$

I_{ij} = product of inertia with respect to $Oxyz$-system

I'_{ij} = product of inertia with respect to $Cx'y'z'$-system

$(x_{c,1}, x_{c,2}, x_{c,3})$ = position vector of centre of mass "C" with respect to origin "O"

M = total mass of the body

Proof: Consider a rigid body, in the form of a set of particles.

Let, $\mathbf{r}_\alpha$ = position vector of α-th particle of mass m_α with respect to origin "O"

$\mathbf{r}'_\alpha$ = position vector of α-th particle of mass m_α with respect to centre of mass "C"

$\mathbf{r}_c$ = position vector of centre of mass "C" with respect to origin "O"

From figure, $\quad \mathbf{r}_\alpha = \mathbf{r}_c + \mathbf{r}'_\alpha$ ——————→ (1)

Let, $\mathbf{r}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$, $\mathbf{r}_c = (x_{c,1}, x_{c,2}, x_{c,3})$ and $\mathbf{r}'_\alpha = (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$

So, equation (1) becomes $\quad (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$

$$\Rightarrow \quad x_{\alpha,i} = x_{c,i} + x'_{\alpha,i}, \qquad i = 1,2,3 \quad ——————→ (2)$$

Now consider for $i \neq j$, $\quad I_{ij} = -\sum_\alpha m_\alpha x_{\alpha,i}x_{\alpha,j} = -\sum_\alpha m_\alpha (x_{c,i} + x'_{\alpha,i})(x_{c,j} + x'_{\alpha,j})$

$$= -\left(\sum_\alpha m_\alpha\right)x_{c,i}x_{c,j} - \left(\sum_\alpha m_\alpha x'_{\alpha,j}\right)x_{c,i} - \left(\sum_\alpha m_\alpha x'_{\alpha,i}\right)x_{c,j} - \sum_i m_\alpha x'_{\alpha,i}x'_{\alpha,j} ————→ (3)$$

where, $\quad \sum_\alpha m_\alpha = M$ = total mass of the body,

Also, $\quad -\sum_i m_\alpha x'_{\alpha,i}x'_{\alpha,j} = I'_{ij}$ = product of inertia with respect to $Cx'y'z'$-system

$$\sum_\alpha m_\alpha \mathbf{r}'_\alpha = \mathbf{0} \;\Rightarrow\; \sum_\alpha m_\alpha (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3}) = (0,0,0) \;\Rightarrow\; \sum_\alpha m_\alpha x'_{\alpha,i} = 0, \quad i = 1,2,3$$

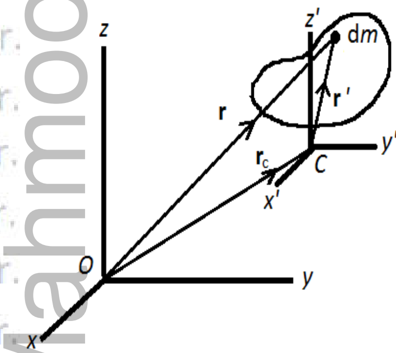
So equation (3) gives

$$\boxed{I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}, \quad i \neq j, \quad i,j \in \{1,2,3\}} \quad \textbf{Hence proved.}$$

Problem: State and prove parallel axis theorem for the case of products of inertia for a continuous mass distribution.

Statement: Consider a rigid body, in the form of a continuous mass distribution. Let, C be the centre of mass of the body. If $Oxyz$ and $Cx'y'z'$ be two parallel coordinate systems as shown in the figure, then we have

$$I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}, \quad i \neq j, \quad i,j \in \{1,2,3\}$$



I_{ij} = product of inertia with respect to $Oxyz$-system

I'_{ij} = product of inertia with respect to $Cx'y'z'$-system

$(x_{c,1}, x_{c,2}, x_{c,3})$ = position vector of centre of mass "C" with respect to origin "O"

M = total mass of the body

Proof: Consider a rigid body, in the form of a set of particles.

$\mathbf{r}$ = position vector of elementary mass $\mathrm{d}m$ with respect to origin "O"

$\mathbf{r}'$ = position vector of elementary mass $\mathrm{d}m$ with respect to centre of mass "C"

$\mathbf{r}_c$ = position vector of centre of mass "C" with respect to origin "O"

From figure, $\quad \mathbf{r} = \mathbf{r}_c + \mathbf{r}' \longrightarrow (1)$

Let, $\mathbf{r} = (x_1, x_2, x_3)$, $\mathbf{r}_c = (x_{c,1}, x_{c,2}, x_{c,3})$ and $\mathbf{r}' = (x'_1, x'_2, x'_3)$

So, equation (1) becomes $\quad (x_1, x_2, x_3) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_1, x'_2, x'_3)$

$$\Rightarrow \quad x_i = x_{c,i} + x'_i, \quad i = 1,2,3 \longrightarrow (2)$$

Now consider for $i \neq j$, $\quad I_{ij} = -\int_M x_i x_j \mathrm{d}m = -\int_M (x_{c,i} + x'_i)(x_{c,j} + x'_j)\mathrm{d}m$

$$= -\left(\int_M \mathrm{d}m\right)x_{c,i}x_{c,j} - \left(\int_M x'_j \mathrm{d}m\right)x_{c,i} - \left(\int_M x'_i \mathrm{d}m\right)x_{c,j} - \int_M x'_i x'_j \mathrm{d}m$$

where, $\quad \int_M \mathrm{d}m = M$ = total mass of the body,

Also, $\quad -\int_M x'_i x'_j \mathrm{d}m = I'_{ij}$ = product of inertia with respect to $Cx'y'z'$-system

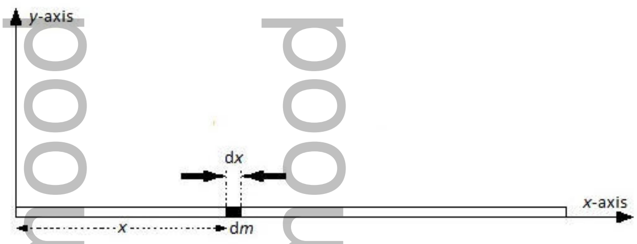
And $\int_M \mathbf{r}'\mathrm{d}m = \mathbf{0} \Rightarrow \int_M (x'_1, x'_2, x'_3)\mathrm{d}m = \left(\int_M x'_1 \mathrm{d}m, \int_M x'_2 \mathrm{d}m, \int_M x'_2 \mathrm{d}m\right) = (0,0,0) \Rightarrow \int_M x'_i \mathrm{d}m = 0, \quad i = 1,2,3$

So equation (3) gives

$$\boxed{I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}, \quad i \neq j, \quad i,j \in \{1,2,3\} \qquad \textbf{Hence proved.}}$$

Example 1: ***Find the moment of inertia*** of a (uniform) rigid rod of length l ***about*** an axis perpendicular to the rod and passing through one of its end points.

Solution: Let M, l and λ, respectively, be the mass, length and linear mass density of the rod. We choose x-axis and y-axis as shown in the figure, so that we have to find moment of inertia of the rod about y-axis. We divide rod into large number of elements of infinitesimal width. One typical element of mass dm and length dx, at distance x from the origin, is shown in the figure.



Moment of inertia of typical mass element about y-axis is given by

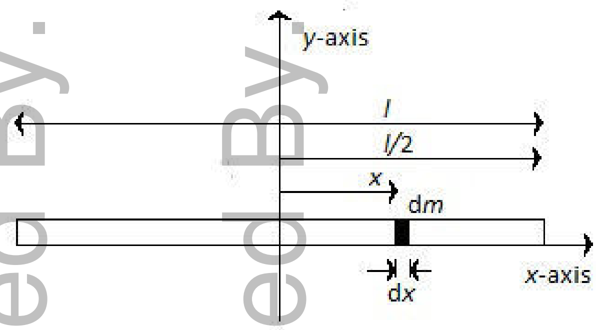
$$\mathrm{d}I_{yy} = x^2 \mathrm{d}m$$

Thus, moment of inertia of rod about y-axis is

$$\begin{aligned} I_{yy} &= \int_{\text{Rod}} x^2 \mathrm{d}m \\ &= \lambda \int_{\text{Rod}} x^2 \mathrm{d}x & \because\ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}x} = \text{constant} \\ &= \frac{M}{l}\int_{x=0}^{l} x^2 \mathrm{d}x = \frac{M}{l}\left(\frac{l^3}{3}\right) = \frac{1}{3}Ml^2 & \because\ \lambda = \tfrac{M}{l}\ \ \text{(for rod)} \end{aligned}$$

Example 2: ***Find the moment of inertia*** of a (uniform) rigid rod of length l ***about*** an axis perpendicular to the rod and passing through its centre.

Solution: Let M, l and λ, respectively, be the mass, length and linear mass density of the rod. Choose y-axis as axis of rotation, as shown in the figure. We divide rod into large number of elements of infinitesimal width. One typical element of mass dm and length dx, at distance x from the origin, is shown in the figure.



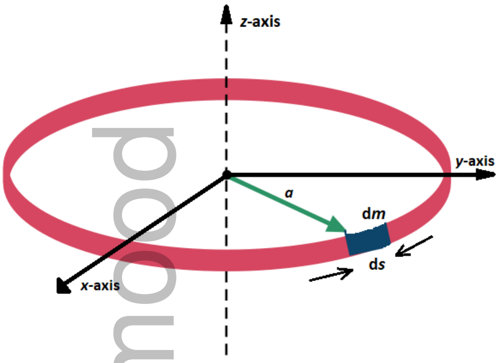
Moment of inertia of typical element about y-axis is given by

$$\mathrm{d}I_{yy} = x^2 \mathrm{d}m$$

Thus, moment of inertia of rod about y-axis is

$$\begin{aligned} I_{yy} &= \int_{\text{Rod}} x^2 \mathrm{d}m \\ &= \lambda \int_{\text{Rod}} x^2 \mathrm{d}x & \because\ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}x} = \text{constant} \\ &= \frac{M}{l}\int_{x=-l/2}^{l/2} x^2 \mathrm{d}x = \frac{M}{l}\left(\frac{l^3}{12}\right) = \frac{1}{12}Ml^2 & \because\ \lambda = \tfrac{M}{l}\ \ \text{(for rod)} \end{aligned}$$

Example 3: ***Find the moment of inertia*** of a (uniform) circular ring of radius a ***about***
(i) an axis passing through its centre and perpendicular to its plane,
(ii) its diameter.



Solution: (i) **Moment of inertia about central axis:**
Let M, a and λ, respectively, be the mass, radius and linear mass density of the ring. Choose coordinate axes as shown in the figure. We divide ring into large number of elements of infinitesimal width. One typical element of mass dm and length ds is shown in the figure.

Moment of inertia of typical element about z-axis is given by

$$\mathrm{d}I_{zz} = a^2\mathrm{d}m$$

Thus, moment of inertia of ring about z-axis is

$$I_{zz} = a^2\int_{\text{Ring}}\mathrm{d}m$$

$$= \lambda a^2\int_{\text{Ring}}\mathrm{d}s \qquad \because\ \lambda = \frac{\mathrm{d}m}{\mathrm{d}s} = \text{constant}$$

$$= \frac{Ma}{2\pi}\int_{s=0}^{2\pi a}\mathrm{d}s = \frac{Ma}{2\pi}(2\pi a) = Ma^2 \qquad \because\ \lambda = \frac{M}{2\pi a}\ \text{(for ring)}$$

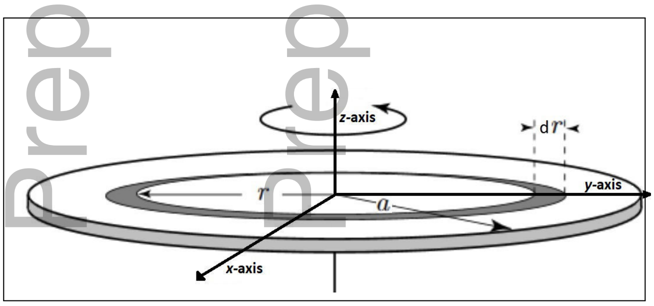
(ii) **Moment of inertia about diameter:**
By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx}, \qquad \because\ I_{xx} = I_{yy}\ \text{(by symmetry)}$$

$$\Rightarrow I_{xx} = \frac{1}{2}Ma^2$$

Example 4: ***Find the moment of inertia*** of a (uniform) circular disc of Mass M and radius a ***about***
(i) an axis passing through its centre and perpendicular to its plane,
(ii) its diameter.
Solution: (i) **Moment of inertia about central axis:** Let M, a and σ, respectively, be the mass, radius and surface (areal) mass density of the disc. Choose axis of rotation as z-axis, as shown in figure.

We divide disc into large number of concentric circular rings of infinitesimal width. One typical elementary ring of mass dm, radius r, width dr and area dA is shown in the figure.



Moment of inertia of typical elementary ring about z-axis is given by

$$\mathrm{d}I_{zz} = r^2\mathrm{d}m$$

Thus, moment of inertia of disc about z-axis is

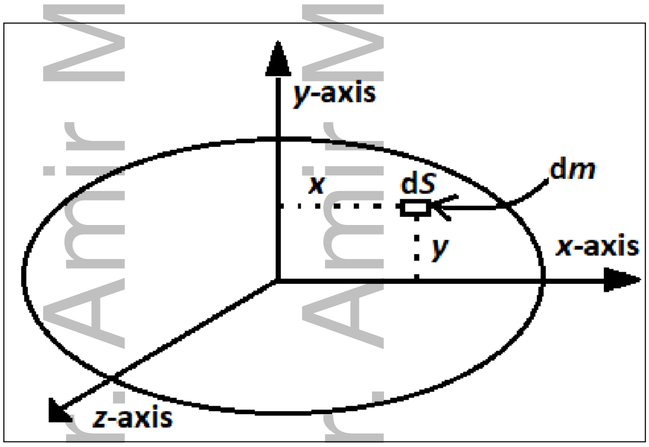
$$I_{zz} = \int_{\text{Disc}} r^2 \mathrm{d}m$$
$$= 2\pi\sigma \int_{\text{Disc}} r^3 \mathrm{d}r \qquad \boxed{\because\ \sigma = \frac{\mathrm{d}m}{\mathrm{d}A} = \frac{\mathrm{d}m}{(2\pi r)\mathrm{d}r} = \text{constant}}$$
$$= \frac{2M}{a^2} \int_{r=0}^{a} r^3 \mathrm{d}r = \frac{2M}{a^2}\left(\frac{a^4}{4}\right) = \frac{1}{2}Ma^2 \qquad \boxed{\because\ \sigma = \frac{M}{\pi a^2}\ \ \text{(for disc)}} \qquad (1)$$

(*ii*) <u>Moment of inertia about diameter</u>:
By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx}, \qquad \boxed{\because\ I_{xx} = I_{yy}\ \text{(by symmetry)}}$$

$$\Rightarrow I_{xx} = \frac{1}{4}Ma^2 \qquad (2)$$

Example 5: ***Find the moment of inertia*** of a (uniform) elliptical plate with semi-major axis and semi minor axis a and b, respectively ***about***
(*i*) major axis,
(*ii*) minor axis,
(*iii*) an axis passing through centre of plate and perpendicular to its plane.



Solution: Consider an elliptical plate in xy-plane whose boundary curve is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$

Let M and σ, respectively, be the mass and surface (areal) mass density of the elliptical plate. To find moment of inertia about major axis (x-axis), we proceed as follows. We divide plate into large number of elementary rectangular pieces of infinitesimal area with sides parallel to x and y axis. One typical area element having mass dm, area dS, length dx and width dy is shown in the figure at point (x, y).

Moment of inertia of typical area element about x-axis is given by

$$\mathrm{d}I_{xx} = y^2 \mathrm{d}m$$

Thus, moment of inertia of elliptical plate about x-axis is

$$I_{xx} = \int_{\text{Elliptical plate}} y^2 \mathrm{d}m$$
$$= \sigma \int_{\text{Elliptical plate}} y^2\, \mathrm{d}x\, \mathrm{d}y \qquad \boxed{\because\ \sigma = \frac{\mathrm{d}m}{\mathrm{d}S} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y} = \text{constant}}$$
$$= \frac{M}{\pi ab} \int_{\text{Elliptical plate}} y^2 \mathrm{d}x\, \mathrm{d}y \qquad \boxed{\because\ \sigma = \frac{M}{\pi a\, b}\ \ \text{(for elliptical plate)}}$$
$$= \frac{M}{\pi ab} \int_{x=-a}^{a} \left(\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 \mathrm{d}y \right) \mathrm{d}x$$
$$= \frac{M}{\pi ab}\left(\frac{2b^3}{3a^3}\right) \int_{x=-a}^{a} (a^2 - x^2)^{3/2} \mathrm{d}x$$

$$I_{xx} = \frac{4Mb^2}{3\pi a^4}\int_{x=0}^{a}(a^2 - x^2)^{3/2}\mathrm{d}x \qquad \boxed{\because \text{ integrand is even}}$$

Put $x = a\sin\theta \Rightarrow \mathrm{d}x = a\cos\theta\mathrm{d}\theta,\ x = 0 \Rightarrow \theta = 0,\ x = a \Rightarrow \theta = \pi/2$

$$I_{xx} = \frac{4Mb^2}{3\pi a^4}\int_{x=0}^{\pi/2} a^4\cos^4\theta\mathrm{d}\theta$$

Using Wallis cosine formula, we get

$$I_{xx} = \frac{4Mb^2}{3\pi}\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{1}{4}Mb^2 \tag{3}$$

Similarly, moment of inertia about minor axis is

$$I_{yy} = \frac{1}{4}Ma^2$$

By perpendicular axis theorem, the moment of inertia about the axis passing through centre of the elliptical plate and perpendicular to its plane, is

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{4}M(a^2 + b^2) \tag{4}$$

Corollary: ***The moment of inertia*** of a (uniform) circular disc of radius a ***about*** (i) its diameter and (ii) an axis passing through its centre and perpendicular to its plane can be obtained by putting $b = a$ in (3) and (4), to give (respectively)

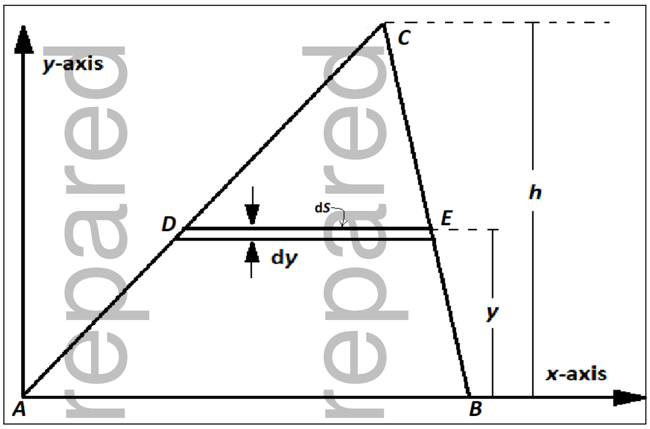
$$I_{xx} = \frac{1}{4}Ma^2 \tag{5}$$

and

$$I_{zz} = \frac{1}{2}Ma^2 \tag{6}$$

Note that, the results obtained in (5) and (6) are in accordance (as they should be) with the results, obtained in (2) and (1), respectively.

Example 6: ***Find the moment of inertia*** of a (uniform) triangular lamina (i.e., two dimensional triangular plate) of mass M ***about*** one of its sides.



Solution: Let M and σ, respectively, be the mass and surface (areal) mass density of the triangular lamina in xy-plane. Choose x-axis and y-axis as shown in figure. We divide lamina into large number of strips of infinitesimal width parallel to the base AB of lamina. One typical elementary strip DE of mass dm, width dy and area dS is shown in the figure.

Moment of inertia of typical elementary strip about side AB (x-axis) is given by

$$\mathrm{d}I_{xx} = y^2\mathrm{d}m$$

Thus, moment of inertia of triangular lamina about x-axis is

$$I_{xx} = \int_{\text{Triangular lamina}} y^2 \mathrm{d}m$$

$$= \sigma \int_{\text{Triangular lamina}} y^2 |\mathrm{DE}| \mathrm{d}y \qquad \boxed{\because \ \sigma = \frac{\mathrm{d}m}{\mathrm{d}S} = \frac{\mathrm{d}m}{|\mathrm{DE}|\mathrm{d}y} = \text{constant}}$$

$$= \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \frac{|\mathrm{DE}|}{|\mathrm{AB}|} \mathrm{d}y \qquad \boxed{\because \ \sigma = \frac{M}{\frac{1}{2}|\mathrm{AB}|h} \quad \text{(for tiangular lamina)}}$$

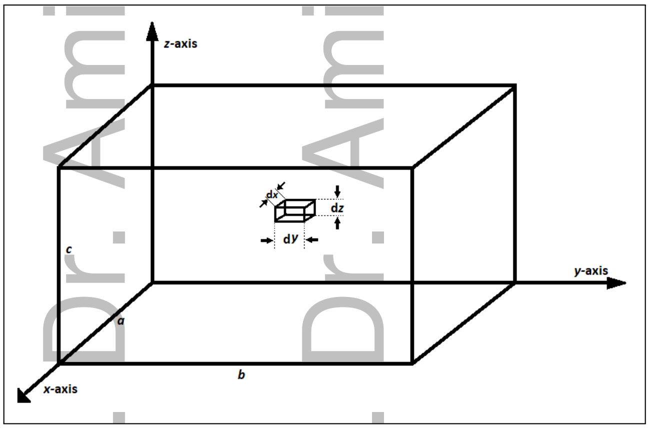
From equivalent triangles ABC and DEC, we have are equivalent triangles, therefore

$$\frac{|\mathrm{DE}|}{|\mathrm{AB}|} = \frac{\text{height of DEC}}{\text{height of ABC}} = \frac{h-y}{h}$$

$$\Rightarrow I_{xx} = \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \left(\frac{h-y}{h} \right) dy$$

$$= \frac{2M}{h^2} \int_{y=0}^{h} y^2 (h-y) dy = \frac{2M}{h^2} \left(\frac{h^4}{3} - \frac{h^4}{4} \right) = \frac{1}{6} M h^2$$

Example 7: *Calculate the inertia matrix* of a (uniform solid) rectangular box (rectangular parallelopiped or cuboid) of mass M ***at*** one of its corners, by taking coordinate axes along its edges.



Solution: Let M and ρ, respectively, be the mass and volume mass density of the rectangular box. Let the lengths of adjacent edges be a, b and c. Choose coordinate axis along the edges of box, as shown in figure. We divide lamina into large number of elementary rectangular boxes of infinitesimal volume. One typical elementary volume element of mass $\mathrm{d}m$, volume $\mathrm{d}V$ and dimensions $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, is shown in the figure.

Moment of inertia of typical elementary volume element about x-axis is given by

$$\mathrm{d}I_{xx} = (y^2 + z^2)\mathrm{d}m$$

Thus, moment of inertia of triangular lamina about x-axis is

$$I_{xx} = \int_{\text{Rectangular box}} (y^2 + z^2)\mathrm{d}m$$

$$= \rho \int_{\text{Rectangular box}} (y^2 + z^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \boxed{\because \ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z} = \text{constant}}$$

$$= \frac{M}{a\,b\,c} \int_{\text{Rectangular box}} (y^2 + z^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad \boxed{\because \ \rho = \frac{M}{a\,b\,c} \quad \text{(for rectangular box)}}$$

$$= \frac{M}{a\,b\,c} \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (y^2 + z^2) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$= \frac{M}{a\,b\,c} \left[\int_{x=0}^{a} \mathrm{d}x \right] \left[\int_{z=0}^{c} \int_{y=0}^{b} (y^2 + z^2)\, \mathrm{d}y\, \mathrm{d}z \right]$$

$$I_{xx} = \frac{M}{b\,c}\int_{z=0}^{c}\int_{y=0}^{b}(y^2+z^2)\,\mathrm{d}y\,\mathrm{d}z$$
$$= \frac{M}{b\,c}\int_{z=0}^{c}\left(\frac{b^3}{3}+bz^2\right)\mathrm{d}z$$
$$= \frac{M}{b\,c}\left(\frac{b^3c}{3}+\frac{b\,c^3}{3}\right) = \frac{M}{3}\left(b^2+c^2\right)$$

Similarly,

$$I_{yy} = \frac{M}{3}\left(a^2+c^2\right) \qquad \text{and} \qquad I_{zz} = \frac{M}{3}\left(a^2+b^2\right)$$

For product of inrtia

$$I_{xy} = \int_{\text{Rectangular box}} x\,y\,\mathrm{d}m$$
$$= -\frac{M}{abc}\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} x\,y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\frac{M}{abc}\left(\frac{a^2}{2}\right)\left(\frac{b^2}{2}\right)c = -\frac{1}{4}Mab$$

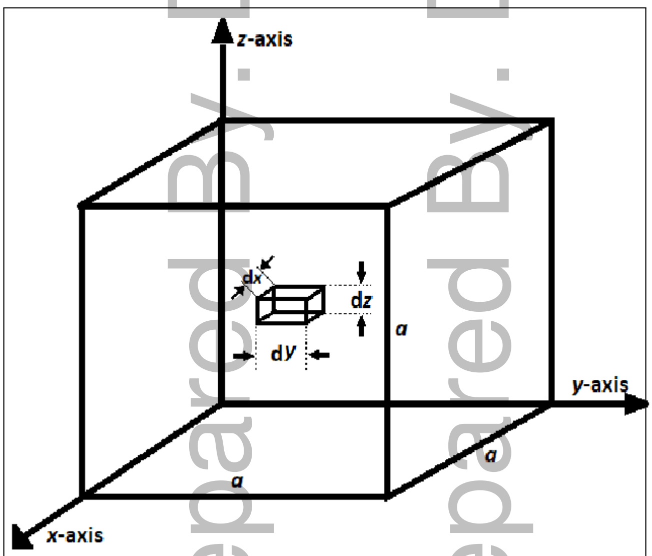
Similarly,

$$I_{yz} = -\frac{1}{4}Mbc \qquad \text{and} \qquad I_{xz} = -\frac{1}{4}Mac$$

The required inertia matrix is given by

$$[I_O] = \begin{bmatrix} (1/3)M(b^2+c^2) & -(1/4)Mab & -(1/4)Mac \\ -(1/4)Mab & (1/3)M(a^2+c^2) & -(1/4)Mbc \\ -(1/4)Mac & -(1/4)Mbc & (1/3)M(a^2+b^2) \end{bmatrix}$$

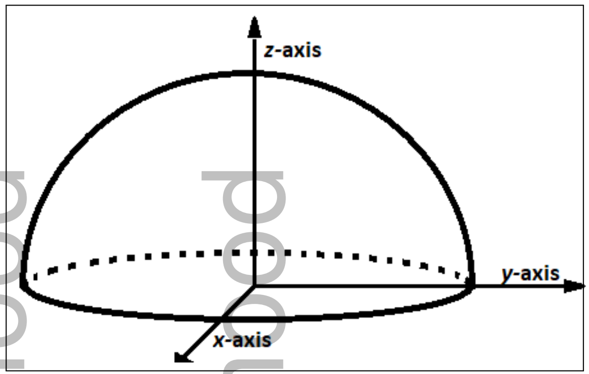
$$\Rightarrow \boxed{[\mathrm{I}_O] = \tfrac{1}{12}M\begin{bmatrix} 4(b^2+c^2) & -3ab & -3ac \\ -3ab & 4(a^2+c^2) & -3bc \\ -3ac & -3bc & 4(a^2+b^2) \end{bmatrix}}$$



Example 8: ***Calculate the inertia matrix*** of a (uniform solid) cube of mass M ***at*** one of its corners, by taking coordinate axes along its edges.
Solution: Repeat example 7 for $a = b = c$ and get

$$[I_O] = \frac{1}{12}Ma^2\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

Example 9: ***Find the moment of inertia*** of a (uniform solid) hemisphere of mass M ***about***
(i) its axis of symmetry
(ii) an axis perpendicular to the axis of symmetry and passing through the centre of the base.



Solution:

(i) **Moment of inertia about axis of symmetry:**

Let M, a and ρ, respectively, be the mass, radius and volume mass density of the hemisphere. Choose coordinate axes as shown in figure.
Moment of inertia of typical volume element of hemisphere, with mass dm and volume dV, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of hemisphere about z-axis is

$$I_{zz} = \int_{\text{Hemisphere}} (x^2 + y^2)\mathrm{d}m$$

$$= \rho \int_{\text{Hemisphere}} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \boxed{\because \ \ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}}$$

$$= \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} (y^2 + z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \boxed{\because \ \ \rho = \frac{M}{\frac{2}{3}\pi a^3} \ \ \text{(for hemisphere)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates to spherical coordinates (r, θ, ϕ) by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\sin^2\theta$$

$$\text{For hemisphere :} \quad 0 \le r \le a, \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4\,\mathrm{d}r \int_{\theta=0}^{\pi/2} \sin^3\theta\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi \quad (7)$$

Where,

$$\int_{\theta=0}^{\pi/2} \sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2} (3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Bigg|_{\theta=0}^{\pi/2} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3} \quad (8)$$

Using (8) in (7), we get

$$I_{zz} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

(ii) **Moment of inertia about a diameter through the base:**

$$I_{xx} = \int_{\text{Hemisphere}} (y^2 + z^2)\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} (y^2 + z^2)\,\mathrm{d}V$$

Transforming problem in spherical coordinates (r, θ, ϕ), we get

$$I_{xx} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4(\sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\left(\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi + \int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\right) \quad (9)$$

Where,

$$\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1-\cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi \quad (10)$$

and

$$\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta = -\frac{1}{3}\cos^3\theta\Big|_{\theta=0}^{\pi/2} = \frac{1}{3} \quad (11)$$

Using (8), (10) and (11), (9) gives

$$I_{xx} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4\pi}{3}\right) = \frac{2}{5}Ma^2$$

Example 10: ***Find three products of inertia*** of a (uniform) solid hemisphere of mass M ***with respect to*** coordinate axes as in figure of example 9.
Solution:

$$I_{xy} = -\int_{\text{Hemisphere}} x\,y\,\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,y\,\mathrm{d}V$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin^2\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Now,

$$I_{xz} = -\int_{\text{Hemisphere}} x\,z\,\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,y\,\mathrm{d}V$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin\theta\cos\theta\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

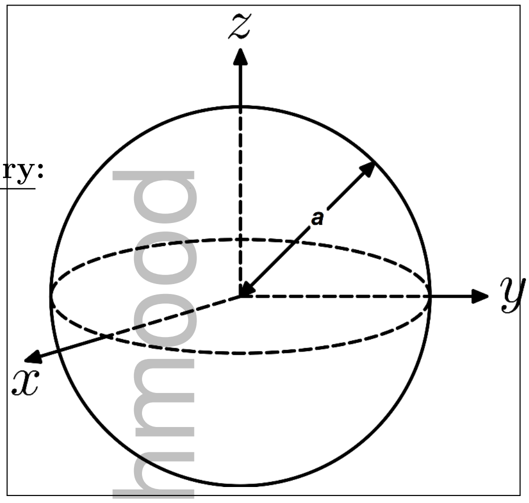
$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = \sin\phi\Big|_{\phi=0}^{2\pi} = 0 \implies I_{xz} = 0 = I_{yz}, \qquad \boxed{\because\ I_{xz} = I_{yz}\ \text{(by symmetry)}}$$

Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0$$



Example 11: ***Find the moments and products of inertia*** of a (uniform solid) sphere of mass M and radius a ***with respect to*** its axes of symmetry.

Solution: (i) Moment of inertia about axis of symmetry:
Let M, a and ρ, respectively, be the mass, radius and volume mass density of the sphere. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of sphere, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of sphere about z-axis is

$$\begin{aligned} I_{zz} &= \int_{\text{Sphere}} (x^2 + y^2)\mathrm{d}m \\ &= \rho \int_{\text{Sphere}} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \because\ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant} \\ &= \frac{3M}{4\pi a^3}\int_{\text{Sphere}} (y^2 + z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \because\ \rho = \frac{M}{\frac{4}{3}\pi a^3} \quad \text{(for sphere)} \end{aligned}$$

To make the computation simpler, we transform the problem from Cartesian coordinates to spherical coordinates $(r,\,\theta,\,\phi)$ by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\sin^2\theta$$

$$\text{For sphere}: \quad 0 \le r \le a, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi$$

Where,

$$\begin{aligned} \int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta &= \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) && \because\ \sin 3\theta = 3\sin\theta - 4\sin^3\theta \\ &= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{4}{3} \end{aligned}$$

Thus,

$$I_{zz} = \frac{3M}{4\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

Similarly,

$$I_{xx} = I_{yy} = \frac{2}{5}Ma^2 \qquad \because\ I_{xx} = I_{yy} = I_{zz} \ \text{(by symmetry)}$$

(*ii*) **Products of inertia with respect to axes of symmetry:**

$$\begin{aligned} I_{xy} &= -\int_{\text{Sphere}} x\,y\,\mathrm{d}m = \frac{3M}{4\pi a^3}\int_{\text{Sphere}} x\,y\,\mathrm{d}V \\ &= -\frac{3M}{4\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^2\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi \end{aligned}$$

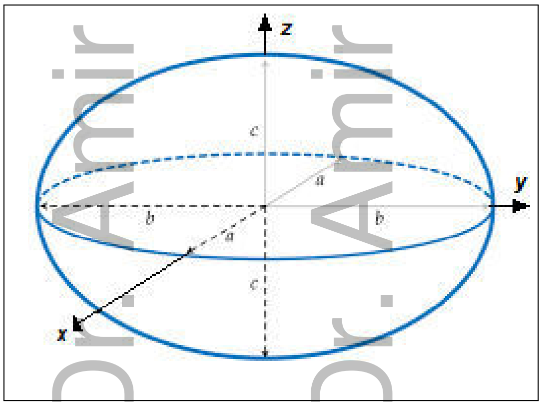
But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Rightarrow \qquad I_{xy} = 0$$

Similarly,

$$I_{yz} = I_{xz} = 0 \qquad \boxed{\because\ I_{xy} = I_{yz} = I_{xz}\ \text{(by symmetry)}}$$

Example 12: ***Find the moments and products of inertia*** of a (uniform) solid ellipsoid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

of mass M ***with respect to*** its axes of symmetry.

Solution: (*i*) **Moment of inertia about axis of symmetry:**

Let M and ρ, respectively, be the mass and volume mass density of the ellipsoid. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of ellipsoid, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of ellipsoid about z-axis is

$$\begin{aligned} I_{zz} &= \int_{\text{Ellipsoid}} (x^2 + y^2)\mathrm{d}m \\ &= \rho\int_{\text{Ellipsoid}} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because\ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}} \\ &= \frac{3M}{4\pi a\,b\,c}\int_{\text{Sphere}} (y^2 + z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \boxed{\because\ \rho = \frac{M}{\frac{4}{3}\pi a\,b\,c}\quad \text{(for ellipsoid)}} \end{aligned}$$

$$I_{zz} = \int_{\text{Ellipsoid}} (x^2 + y^2)\mathrm{d}m \tag{12}$$

Let us substitute

$$x/a = x', \quad y/a = y', \quad z/a = z'$$

$$\Rightarrow \quad \mathrm{d}x/a = \mathrm{d}x', \quad \mathrm{d}y/a = \mathrm{d}y', \quad \mathrm{d}z/a = \mathrm{d}z', \qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

Under the above transformation, the given ellipsoid is transformed into the unit sphere

$$S : x'^2 + y'^2 + z'^2 = 1.$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{4\pi a\,b\,c}\int_S (a^2x'^2 + b^2y'^2)(\,a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z')$$
$$= \frac{3M}{4\pi}\int_S (a^2x'^2 + b^2y'^2)\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

$$\because \quad \int_S x'^2\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \int_S y'^2\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' \qquad \text{(by symmetry)}$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\int_S x'^2\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x', y', z') to spherical coordinates (r, θ, ϕ) by using

$$x' = r\sin\theta\cos\phi, \qquad y' = r\sin\theta\sin\phi, \qquad z' = r\cos\theta$$

$$\mathrm{d}V = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

For unit sphere,

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\cos^2\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos^2\phi\,\mathrm{d}\phi$$

Where, $$\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) \qquad \because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Bigg|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{4}{3}$$

and

$$\int_{\phi=0}^{2\pi}\cos^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1+\cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right)\Bigg|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi$$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\left(\frac{1}{5}\right)\left(\frac{4}{3}\right)(\pi) = \frac{1}{5}M(a^2+b^2)$$

Similarly,

$$I_{xx} = \frac{1}{5}M(b^2+c^2) \qquad \text{and} \qquad I_{yy} = \frac{1}{5}M(a^2+c^2)$$

(ii) **Products of inertia with respect to axes of symmetry:**

$$\Rightarrow \quad I_{xy} = -\int_{\text{Ellipsoid}} x\,y\,\mathrm{d}m = -\frac{3M}{4\pi a\,b\,c}\int_{\text{Ellipsoid}} x\,y\,\mathrm{d}V$$
$$= -\frac{3M}{4\pi a\,b\,c}\int_{\text{S}}(a\,b\,x'\,y')(a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z')$$
$$= -\frac{3a\,b\,M}{4\pi}\int_{\text{S}} x'\,y'\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$
$$= -\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$I_{xy} = -\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^2\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Similarly, it is not difficult to show that

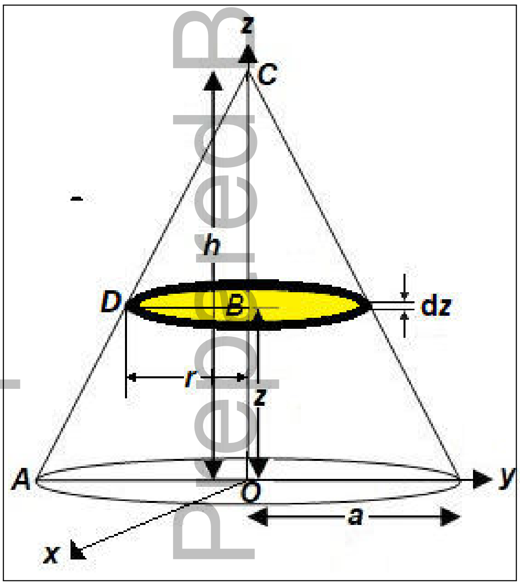
$$I_{yz} = I_{xz} = 0$$

Corollary: ***The moment and product of inertia*** of a (uniform) solid sphere of mass M and radius a ***with respect to*** its axis of symmetry can be obtained by putting $a = b = c$ in results of above example 12. The obtained results are in accordance (as they should be) with the results of example 11.

Example 13: ***Find the moment of inertia*** of a (uniform) right circular solid cone ***about***
(i) its axis of symmetry and
(ii) any diameter of the base.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M, h, a and ρ, respectively, be the mass, height, radius of base and volume mass density of a (uniform) right circular solid cone. Choose coordinate axes as shown in figure. Let us divide cone into large number of elementary solid discs parallel to the base of the cone. One such elementary disc of radius r, mass $\mathrm{d}m$, thickness $\mathrm{d}z$ and volume $\mathrm{d}V$ is shown in the figure, at a distance z from the base of the cone.



Moment of inertia of elementary disc about z-axis is given by

$$\mathrm{d}I_{zz} = \frac{1}{2}\,r^2\mathrm{d}m$$

Thus, moment of inertia of cone about z-axis is

$$\begin{aligned} I_{zz} &= \frac{1}{2}\int_{\text{Cone}} r^2 \mathrm{d}m \\ &= \frac{\rho}{2}\int_{\text{Cone}} r^2\,(\pi r^2)\,\mathrm{d}z & \boxed{\because\ \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \tfrac{\mathrm{d}m}{(\pi r^2)\,\mathrm{d}z} = \text{constant}} \\ &= \frac{3M}{2a^2 h}\int_{\text{Cone}} r^4\,\mathrm{d}z & \boxed{\because\ \rho = \tfrac{M}{\frac{1}{3}\pi a^2 h}\quad \text{(for cone)}} \end{aligned}$$

From similar triangles AOC and DBC

$$\frac{r}{a} = \frac{h-z}{h} \qquad \text{or} \qquad r = \frac{a(h-z)}{h}$$

$$\begin{aligned} \Rightarrow I_{zz} &= \frac{3M}{2a^2 h}\int_{\text{Cone}} \left[\frac{a(h-z)}{h}\right]^4 \mathrm{d}z \\ &= \frac{3M\,a^2}{2\,h^5}\int_{z=0}^{h} (h-z)^4\,\mathrm{d}z \\ &= -\frac{3M\,a^2}{10\,h^5}\,(h-z)^5\Big|_{z=0}^{h} = \frac{3}{10}\,M\,a^2 \end{aligned}$$

(*ii*) **<u>Moment of inertia about diameter of the base</u>:**
In this case, the moment of inertia of the elementary disc of mass dm about a diameter, along DB, is given by

$$\mathrm{d}I_o = \frac{1}{4}\,r^2 \mathrm{d}m$$

We note that the diameter passes through the center (which is also the centroid) of the elementary disc. Hence, by parallel axis theorem, the moment of inertia of the elementary disc about a parallel axis along AO (through the centre of the base of cone) is given by

$$\begin{aligned} \mathrm{d}I_{yy} &= \mathrm{d}I_o + (\mathrm{d}m)\,z^2 \\ &= \frac{1}{4}\,r^2\mathrm{d}m + (\mathrm{d}m)\,z^2 = \left(\frac{1}{4}\,r^2 + z^2\right)\mathrm{d}m \\ &= \frac{3M}{a^2 h}\left(\frac{1}{4}\,r^4 + r^2 z^2\right)\mathrm{d}z, & \boxed{\because \mathrm{d}m = \rho\,\mathrm{d}V = \tfrac{3M}{a^2 h}(r^2\,\mathrm{d}z)} \end{aligned}$$

Therefore, the moment of inertia of whole cone about diameter of the base is given by

$$\begin{aligned} I_{yy} &= \frac{3M}{a^2 h}\int_{z=0}^{h}\left[\frac{1}{4}\left(\frac{a(h-z)}{h}\right)^4 + \left(\frac{a(h-z)}{h}\right)^2 z^2\right]\mathrm{d}z & \boxed{\because r = \tfrac{a(h-z)}{h}} \\ &= \frac{3M}{a^2 h}\int_{z=0}^{h}\left[\frac{a^4}{4\,h^4}(h-z)^4 + \frac{a^2}{h^2}(h^2 z^2 - 2h\,z^3 + z^4)\right]\mathrm{d}z \\ &= \frac{3M}{a^2 h}\left[-\frac{a^4}{20\,h^4}(h-z)^5 + \frac{a^2}{h^2}\left(h^2\frac{z^3}{3} - h\frac{z^4}{2} + \frac{z^5}{5}\right)\right]\Big|_{z=0}^{h} \\ &= \frac{3M}{a^2 h}\left[\frac{a^4\,h}{20} + \frac{a^2}{h^2}\left(\frac{h^5}{3} - \frac{h^5}{2} + \frac{h^5}{5}\right)\right] = \frac{3M}{a^2 h}\left[\frac{a^4\,h}{20} + a^2\,h^3\left(\frac{10-15+6}{30}\right)\right] \\ &= \frac{3M}{a^2 h}\left[\frac{a^4\,h}{20} + \frac{a^2\,h^3}{30}\right] = \frac{3M}{a^2 h}\left[\frac{3a^4\,h + 2a^2\,h^3}{60}\right] = \frac{1}{20}M(3a^2 + 2h^2) \end{aligned}$$