Projectile Motion.

Before Starting this chapter, some basic concepts of parabola.

$$(x-h)^2 = -4a(y-k).$$

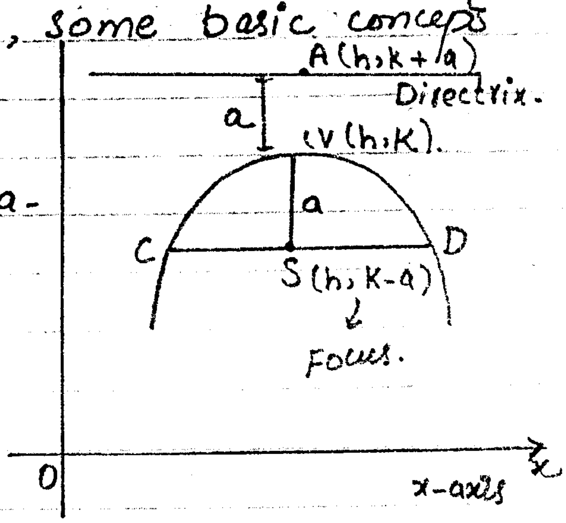
(i) Vertex $V(h,k)$ of Parabola.

(ii) Length of Latus Rectum $|CD| = |4a|$.

(iii) Distance b/w Focus and Vertex i-e $|VS| = a = |VA|$

(iv) coordinates of focus. $S(h, k-a)$.

(V) hieght of directrix line is $k+a$.



Note: If we take vertex at origin i-e $h=0, k=0$ then $S(0,-a)$ $V(0,0)$ Directrix $=a$ & eq is $x^2 = -4ay$.

Introduction:-

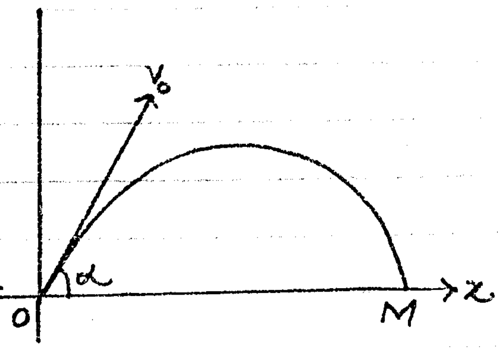
Vertical Plane:-

When we consider the xy-Plane as the vertical plane and take the x-axis horizontally and y-axis vertically upward.

Projectile:- is the particle projected in air

OR

projectile is the particle droped from a moving plane.



Angle of Projection:- is the angle α which a projectile (particle) makes with horizontal line ox.

Velocity of Projection is the velocity V_0 with which a particle is projected.

<u>Point of projection</u> is the point 'o' from where a paticle is projected.
<u>Trajectory</u> is the path along which projectile (particle) moves.
<u>Range</u> (|OM|) is the distance from the point of projection to the point where projectile hits.
<u>Time of flight</u> is the time taken by the particle for it's Range from 'o' to 'M'.
<u>Note:-</u>
In this chapter we shall assume that
(i) there is no air resistance and particle does not go too far from the surface of earth, therefore Value of 'g' will be constant.
(ii) The accelration due to gravity has constant magnitude and it acts vertically downward.

Q:- Find the equation of the projectile with speed V_o and angle of projection 'α'- Also show that it's path is Parabolic-

OR.

Discuss the motion of projectile in air with velocity V_o and angle of projection 'α'. Also show that it's path is Parabolic-

Solution:-

Suppose a particle of mass 'm' is projected from a point 'O' in xy-Plane, with initial velocity 'V_o' making an angle 'α' with x-axis- Then the horizontal and vertical components of 'V_o' are $V_o\cos\alpha$ & $V_o\sin\alpha$ respectively then,

$$V_o = V_o\cos\alpha\,\hat{i} + V_o\sin\alpha\,\hat{j} \quad \text{—— (1)}$$

Now position of a moving particle at any time 't' is given by

$$\vec{r} = x\hat{i} + y\hat{j} \quad \text{—— (2)}$$

Since the air resistance has been neglected, so the only force acting on the particle is gravitational force which is acting vertically downward.

Therefore its eq of motion is

$$F = -mg\hat{j}$$
$$m\ddot{r} = -mg\hat{j}$$
$$\ddot{r} = -g\hat{j}$$

But the principle of independent of forces gravity effects in vertical motion & has no effect on horizontal motion-

Integrating

$$\dot{r} = -gt\hat{j} + A \quad \text{—— (3)}$$

Initially when $t=0$, velocity $= V_o = \dot{r}$

$$\dot{r} = -gt\hat{j} + V_o \qquad \because V_o = 0 + A \text{ from (3)}$$

Integrating

$\vec{r} = -\frac{1}{2}gt^2\hat{j} + V_0t + B$ ——(4) initially when $t=0, \vec{r}=0$

$\Rightarrow B=0$

$\Rightarrow \vec{r} = -\frac{1}{2}gt^2\hat{j} + V_0t$

$\because \vec{r} = x\hat{i} + y\hat{j}$

$x\hat{i} + y\hat{j} = -\frac{1}{2}gt^2\hat{j} + (V_0\sin\alpha\,\hat{j} + V_0\cos\alpha\,\hat{i})t$ using ①

$x\hat{i} + y\hat{j} = (V_0\cos\alpha t)\,\hat{i} + (V_0\sin\alpha - \frac{1}{2}gt^2)\hat{j}$

By comparing components of $\hat{i}$ & $\hat{j}$

$\therefore x = (V_0\cos\alpha)t$ —— ⑤

& $y = (V_0\sin\alpha)t - \frac{1}{2}gt^2$ —— ⑥

eq ⑤ & ⑥ gives the coord of pt 'P' at any time 't'. Also can be regarded as parametric eqs of the curve with 't' as parameter.

To get the equations of trajectory, eliminate 't' from ⑤ & ⑥

$y = (V_0\sin\alpha)t - \frac{1}{2}gt^2$

$\because t = \frac{x}{V_0\cos\alpha}$ from ⑤

$$y = (V_0\sin\alpha)\left(\frac{x}{V_0\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{V_0\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha} \quad \text{—— ⑦}$$

This is equation of Trajectory of projectile.

Nature of trajectory.

We have from ⑦

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}$$

$$\frac{gx^2}{2V_0^2\cos^2\alpha} = x\tan\alpha - y$$

$$x^2 = x\tan\alpha\left(\frac{2V_0^2\cos^2\alpha}{g}\right) - y\left(\frac{2V_0^2\cos^2\alpha}{g}\right)$$

$\sin 2\alpha = 1 \Rightarrow 2\alpha = \sin^{-1}(1) = \pi/2 \Rightarrow \alpha = \pi/4$

Hence Max. Range is $R_{max} = \dfrac{V_0^2}{g}(1) = \dfrac{V_0^2}{g}$

TIME OF FLIGHT:-

"It is the time 'T' taken by the projectile in moving from the point of projection 'O' to the horizontal plane 'A'. For point 'A' put $y=0$ since the hieght of a particle at any time 't' is

$y = (V_0 \sin\alpha)t - \frac{1}{2}gt^2$

For time of flight put $y=0$ & $t=T$

$0 = (V_0 \sin\alpha)T - \frac{1}{2}gT^2$

$0 = (V_0 \sin\alpha - \frac{1}{2}gT)T$

$\therefore$ either $T = 0$ (when particle is at pt. 'O') or

$0 = V_0 \sin\alpha - \frac{1}{2}gT$

$\Rightarrow \frac{1}{2}gT = V_0 \sin\alpha$

$T = \dfrac{2V_0 \sin\alpha}{g}$

MAXIMUM HEIGHT OF PROJECTILE:-

ordinate of vertex i-e 'k' is the hieght i-e point from x-axis.

$\therefore$ Max. hieght $= \dfrac{V_0^2 \sin^2\alpha}{2g}$

IInd Method to find Range:-

We know

$x = (V_0 \cos\alpha)t$ —— (1)

$y = (V_0 \sin\alpha)t - \frac{1}{2}gt^2$ —— (2)

First find time of flight by putting $y=0$ in (ii) we get

$t = \dfrac{2V_0 \sin\alpha}{g}$ put in (1)

$x = (V_0 \cos\alpha)\left(\dfrac{2V_0 \sin\alpha}{g}\right) \Rightarrow x = \dfrac{V_0^2 \sin 2\alpha}{g}$

$\Rightarrow x = \dfrac{V_0^2 \sin 2\alpha}{g}$

Velocity OF Projectile:-

At any time 't' let P be the position of the particle such that

$$\overline{OP} = \bar{r} = x\hat{i} + y\hat{j}$$

If V is the velocity of projectile at 'P' Then by principle of energy-

"Decrease in K.E" = "Work done against gravity"

$$\frac{1}{2}mV_0^2 - \frac{1}{2}mV^2 = mgy$$

$$\frac{1}{2}m(V_0^2 - V^2) = mgy$$

$$\Rightarrow V_0^2 - V^2 = 2gy$$

$$V^2 = V_0^2 - 2gy$$

$$= 2g\left[\frac{V_0^2}{2g} - y\right]$$

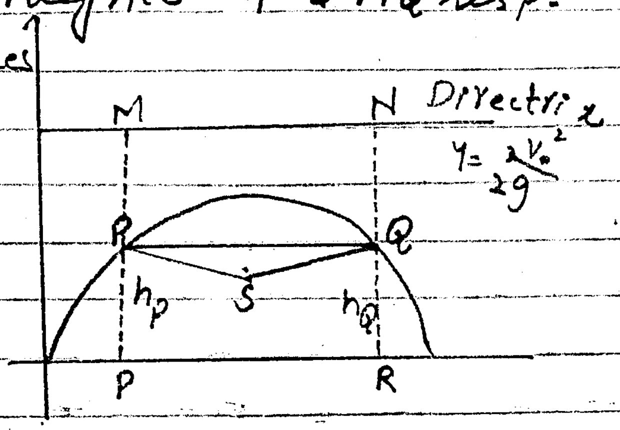
$$= 2g\,(\text{hieght of directrix} - \text{ordinate of } P)$$

$$V^2 = 2g\,(\text{hieght of directrix above } P)$$

Hence K.E & magnitude of velocity at any point 'P' of the path of projectile is the same as would be acquired in falling freely under gravity 'P' from the point on the direction $y = V_0^2/2g$ which is vertically above P.

To Find the Least speed with which a particle must be projected so That it passes through two points P & Q at hieghts h_P & h_Q resp.

Let PM & QN be the distances of P & Q from directrix $y = V_0^2/2g$. then Speed V_P of Projectile at Point 'P' is given by



$$x^2 - \frac{2V_0^2\cos\alpha\sin\alpha}{g}.x = -\left(\frac{2V_0^2\cos^2\alpha}{g}\right)y.$$

Completing square.

$$x^2 - \frac{2V_0^2\cos\alpha\sin\alpha}{g}.x + \frac{V_0^4\cos^2\alpha\sin^2\alpha}{g^2} - \frac{V_0^4\cos^2\alpha\sin^2\alpha}{g^2} = -\left(\frac{2V_0^2\cos^2\alpha}{g}\right)y$$

$$\left(x - \frac{V_0^2\cos\alpha\sin\alpha}{g}\right)^2 = \frac{V_0^4\cos^2\alpha\sin^2\alpha}{g} - \left(\frac{2V_0^2\cos^2\alpha}{g}\right)y.$$

$$\left(x - \frac{V_0^2\cos\alpha\sin\alpha}{g}\right)^2 = \frac{-2V_0^2\cos^2\alpha}{g}\left(y - \frac{V_0^2\sin^2\alpha}{2g}\right) \quad \text{—— Ⓐ}$$

Compare Ⓐ with general Equation of parabola.

$$(x-h)^2 = -4a(y-k).$$

to get vertex $V(h,k)$ as $V\left(\frac{V_0^2\cos\alpha\sin\alpha}{g}, \frac{V_0^2\sin^2\alpha}{2g}\right)$

and Latus rectum i-e $4a = \frac{2V_0^2\cos^2\alpha}{g}$

K = hieght of vertex $= \frac{V_0^2\sin^2\alpha}{2g}$

Distance b/w focus & Directrix i-e $a = \frac{2V_0^2\cos^2\alpha}{4g}$

$$a = \frac{V_0^2\cos^2\alpha}{2g}.$$

Equation of Directrix i-e $Y = k + a$

or hieght of directrix $= \frac{V_0^2\sin^2\alpha}{2g} + \frac{V_0^2\cos^2\alpha}{2g}$

$$Y = \frac{V_0^2}{2g}(1) = \frac{V_0^2}{2g}$$

Focus $= (h, k-a) = \left(\frac{V_0^2\cos\alpha\sin\alpha}{g}, \frac{V_0^2\sin^2\alpha}{2g} - \frac{V_0^2\cos^2\alpha}{2g}\right)$

$$= \left(\frac{V_0^2 \sin 2\alpha}{2g}, -\frac{V_0^2 \cos 2\alpha}{2g} \right)$$

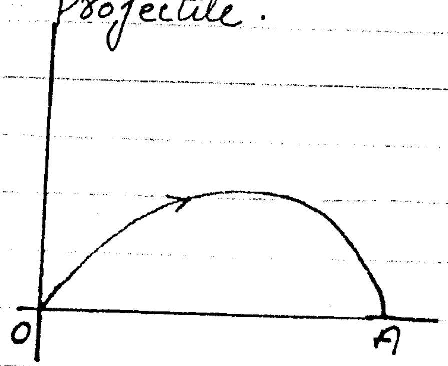
Note:- Eq ⑤ determines the distance by the particle in time 't' along x-axis with uniform speed $(V_0 \cos\alpha)$ and Eq ⑥ gives the distance travelled by particle vertically upward in time 't' with initial velocity $(V_0 \sin\alpha)$ under retardation 'g'.

HORIZONTAL RAGE OF PROJECTILE:-

"It is defined as the distance b/w the point of projection and the point 'A' where the projectile meets the horizontal plane."

Taking OA along x-axis - For pt 'A' $y = 0$ ∵ x-axis.

$\therefore \; y = x\tan\alpha - \dfrac{gx^2}{2V_0^2\cos^2\alpha}$ Eq. of trajectory of projectile.



$\because \; y = 0 \Rightarrow 0 = x\tan\alpha - \dfrac{gx^2}{2V_0^2\cos^2\alpha}$

$0 = x\left(\tan\alpha - \dfrac{gx}{2V_0^2\cos^2\alpha} \right)$

either $x = 0$ or $\tan\alpha - \dfrac{gx}{2V_0^2\cos^2\alpha} = 0$

$\tan\alpha = \dfrac{gx}{2V_0^2\cos^2\alpha}$ x corresponds to pt of projection.

$\dfrac{\sin\alpha}{\cos\alpha} \cdot \dfrac{2V_0^2\cos^2\alpha}{g} = x$

$\dfrac{V_0^2 \sin 2\alpha}{g} = x \quad \therefore \; O(0,0)$ & $A\left(\dfrac{V_0^2 \sin 2\alpha}{g}, 0 \right)$

∴ The horizontal Range $= \overline{OA} = \dfrac{V_0^2 \sin 2\alpha}{g}$

Max. Range will be when $\sin 2\alpha$ is max i-e

$V_p^2 = V_0^2 - 2g(h_p)$

$V_p^2 = 2g\left[\frac{V_0^2}{2g} - h_p\right]$

$V_p^2 = 2g\,|PM|$

comparing we get

$\Rightarrow \frac{V_0^2}{2g} - h_p = |PM|$

$\frac{V_0^2}{2g} = |PM| + h_p$ —— (i)

Similarly

$\frac{V_0^2}{2g} = |QN| + h_Q$ —— (ii)

Adding (i) & (ii)

$\frac{V_0^2}{2g} + \frac{V_0^2}{2g} = h_p + |PM| + h_Q + |QN|$

$\frac{V_0^2}{g} = h_p + h_Q + |PS| + |QS|$ $\because$ using Focus directix propety of Parabola. Where S is Focus.

$|SP| = e\,|PM|$

$|SQ| = e\,|QN|$

For Parabola $e = 1$.

$V_0^2 = g[h_p + h_Q + |SP| + |QS|]$

Now V_0^2 is least when $|SP| + |SQ|$ is least.

Which is least when 'S' lies on PQ i-e when $|SP| + |SQ| = |PQ|$

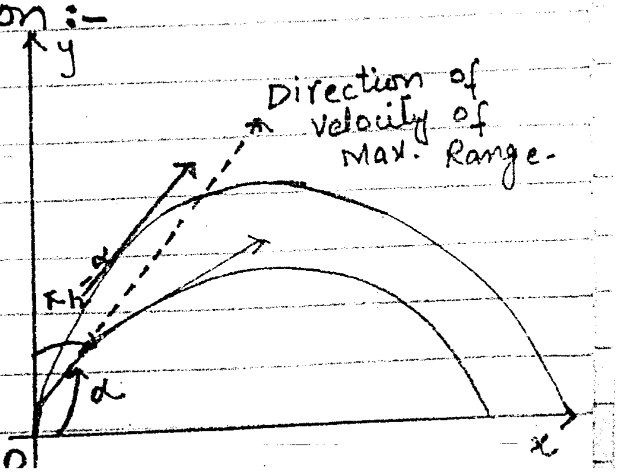
or $|PM| + |QN| = |PQ|$

Hence

$V_{0(min)} = \sqrt{g(h_p + h_Q + PQ)}$

Direction of Projection:-

When a particle is projected from origin with speed 'V_0' making an angle 'α' with its horizontal range, there are two possible directions of projection which are equally inclined to the



direction of maximum range.

Range, $R = \frac{V_0^2 \sin 2\alpha}{g} = \frac{2V_0^2 \sin\alpha\cos\alpha}{g}$

$= \frac{2V_0^2 \cos(\pi/2-\alpha) \sin(\pi/2-\alpha)}{g}$.

Hence for given range R & V_0 there are two angles of projection

(i) α with x-axis

(ii) $\pi/2-\alpha$ with y-axis.

When R is max for given Velocity of projection V_0, there is only one angle of projection '$\pi/4$'. The directions α & $\pi/2-\alpha$ are equally inclined to x-axis and y-axis resp. Hence direction of max. range bisects the angle between them.

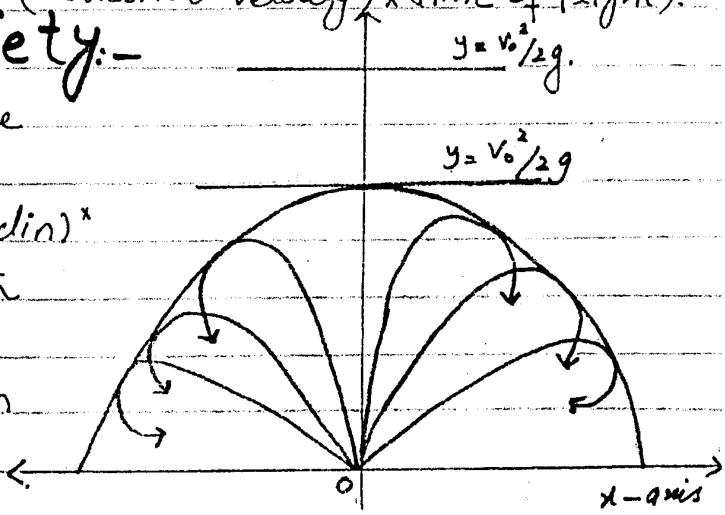
Note:-

(i) We can find the range of a projectile projected from 'O' for any velocity and angle of projection because $R = \frac{2V_0^2 \sin\alpha\cos\alpha}{g} = (V_0\cos\alpha) \times \left(\frac{2V_0\sin\alpha}{g}\right)$

(Horizontal velocity) x (Time of flight).

Parabola OF Safety:-

"parabola of Safety is the boundry curve in the vertical plane which (inclin)ˣ includes all possible path of projectile projected with same speed 'V_0' in different directions".



Eq of Parabola OF Safety:-

The eq of the trajectory of a projectile projected from 'o' with velocity V_0 and angle of projection α, in a vertical plane is

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2V_0^2}\sec^2\alpha$$

$$= x\tan\alpha - \frac{gx^2}{2V_0^2}(1+\tan^2\alpha)$$

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2} - \frac{gx^2}{2V_0^2}\tan^2\alpha$$

For a given value of V_0, eq ① depens on 'α' i-e for different trajectories so x & y are fns of parameter 'α'.

$$\frac{gx^2}{2V_0^2}\tan^2\alpha - x\tan\alpha + \left(y + \frac{gx^2}{2V_0^2}\right) = 0 \quad ①$$

① represents a family of projectile trajectories with the same initial speed V_0, 'α' being parameter ① is quadratic in $\tan\alpha$, it's Envelope is $B^2-4AC=0$

Envelope is a curve touching all the trajectories of a given family of curves.

$$\Rightarrow x^2 - 4\left(\frac{gx^2}{2V_0^2}\right)\left(y + \frac{gx^2}{2V_0^2}\right) = 0$$

$$\Rightarrow x^2 - 2\frac{gx^2}{V_0^2}y - \frac{g^2x^4}{V_0^4} = 0$$

$$\Rightarrow x^2\left[1 - \frac{2g}{V_0^2}y - \frac{g^2x^2}{V_0^4}\right] = 0$$

$\because x \neq 0$ Then $1 - \frac{2g}{V_0^2}y - \frac{g^2x^2}{V_0^4} = 0$

$$1 - \frac{2gy}{V_0^2} = \frac{g^2x^2}{V_0^4} \Rightarrow \frac{V_0^4}{g^2} - \frac{V_0^2\,2y}{g} = x^2$$

$$\Rightarrow (x-0)^2 = -\frac{2V_0^2}{g}\left(y - \frac{V_0^2}{2g}\right) \quad ②$$

is called Eq. of parabola of safety.

From eq. of parabola of safety.

$$(x-0)^2 = -\frac{2V_0^2}{g}\left(y-\frac{V_0^2}{2g}\right) \quad \text{--- (2)}$$

compare with $(x-h)^2 = -4a(y-k)$ we get.

Vertex. $V(h,k) = V\left(0, \frac{V_0^2}{2g}\right)$

Latus rectum i-e $4a = 2V_0^2/g$

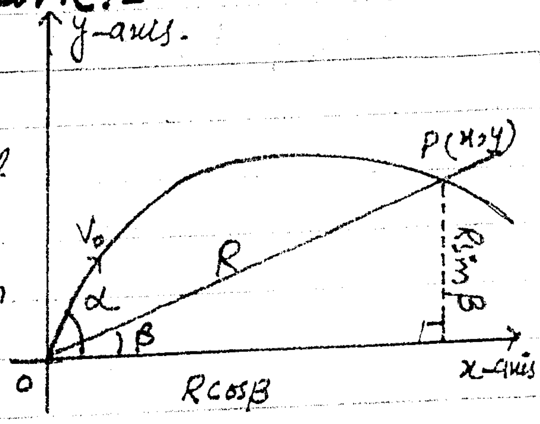
$$a = \frac{1}{4}\left(\frac{2V_0^2}{g}\right) = \frac{V_0^2}{2g}.$$

Focus $(h, k-a) = \left(0, \frac{V_0^2}{2g} - \frac{V_0^2}{2g}\right) = (0,0)$ i,e focus of parabola of safety is origin.

Eq of directrix i,e $y = k+a = \frac{V_0^2}{2g} + \frac{V_0^2}{2g} = \frac{V_0^2}{g}$.

Range on Inclined Plane:-

Let a particle is projected in vertical plane with speed V_0, making an angle α with horizontal. Let it hit the inclined plane at $P(x,y)$. inclined plane is making an angle 'β' ($\beta < \alpha$) with horizontal. We have to find range i,e



$|OP| = R$

$\frac{x}{R} = \cos\beta \Rightarrow x = R\cos\beta$

$\frac{y}{R} = \sin\beta \Rightarrow y = R\sin\beta$

$\therefore$ point $P(x,y)$ is given by $P(R\cos\beta, R\sin\beta)$

Point 'P' should satisfy eq of trajectory, because it lies on the path of projectile.

$$\therefore \quad y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha} \quad \text{--- (1) Eq. of trajectory.}$$

$$R\sin\beta = (R\cos\beta)\tan\alpha - \frac{g(R\cos\beta)^2}{2V_0^2\cos^2\alpha}$$ $\because$ Putting $x = R\cos\beta$, $y = R\sin\beta$

$$\cancel{R}\sin\beta = \cancel{R}\left[\frac{\cos\beta\sin\alpha}{\cos\alpha} - \frac{gR\cos^2\beta}{2V_0^2\cos^2\alpha}\right]$$

$$\sin\beta = \frac{\cos\beta\sin\alpha}{\cos\alpha} - \frac{gR\cos^2\beta}{2V_0^2\cos^2\alpha}$$

$$\frac{gR\cos^2\beta}{2V_0^2\cos^2\alpha} = \frac{\cos\beta\sin\alpha}{\cos\alpha} - \sin\beta$$

$$R = \frac{2V_0^2\cos^2\alpha}{g\cos^2\beta}\left(\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha}\right)$$

$$R = \frac{2V_0^2\cos\alpha}{g\cos^2\beta}\left(\sin(\alpha-\beta)\right)$$

$$R = \frac{V_0^2}{g\cos^2\beta}\left(2\cos\alpha\sin(\alpha-\beta)\right)$$

$\because 2\cos a\sin b = \sin(a+b) - \sin(a-b)$

$$R = \frac{V_0^2}{g\cos^2\beta}\left(\sin(\alpha+\alpha-\beta) - \sin(\alpha-\alpha+\beta)\right)$$

$$R = \frac{V_0^2}{g\cos^2\beta}\left(\sin(2\alpha-\beta) - \sin\beta\right) \quad \text{--- (3)}$$

is required range of inclined plane.

For Max. Range:-

put $\sin(2\alpha-\beta) = 1$

$\because$ Since β is fixed angle so for a given value V_0, R will depend on 'α' only.

$$\Rightarrow R_{max} = \frac{V_0^2}{R^2\cos^2\beta}\left(1-\sin\beta\right)$$

$$= \frac{V_0^2}{g}\,\frac{(1-\sin\beta)}{(1+\sin\beta)(1-\sin\beta)}$$

$\because \cos^2\beta = 1-\sin^2\beta = (1+\sin\beta)(1-\sin\beta)$

$$= \frac{V_0^2}{g}\left(\frac{1}{1+\sin\beta}\right) \quad \text{--- (4)}$$

$$R_{max} = \frac{V_0^2}{g(1+\sin\beta)}$$

For time of Flight:-

$$x = V_0\cos\alpha\cdot t$$

$$R\cos\beta = V_0\cos\alpha\cdot t \qquad \because x = R\cos\beta$$

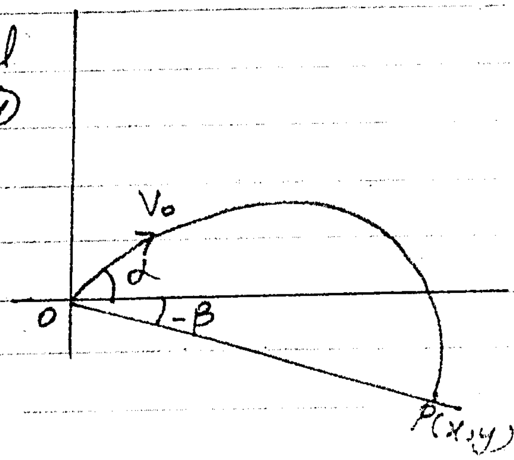
$$\frac{R\cos\beta}{V_0\cos\alpha} = t$$

$$\left[\left(\frac{V_0^2}{g\cos^2\beta}\right)2\cos\alpha\sin(\alpha-\beta)\right]\frac{\cos\beta}{V_0\cos\alpha} = t \qquad \because \text{using } (2)$$

$$\frac{2V_0\sin(\alpha-\beta)}{g\cos\beta} = t \quad \text{—— } (5)$$

Note:-

If the inclination of inclined Plane be '$-\beta$' with horizontal then put $\beta = -\beta$ in eq (2), (3), (4) & (5) to get Range on Inclined Plane down, Max Range & Time of flight.

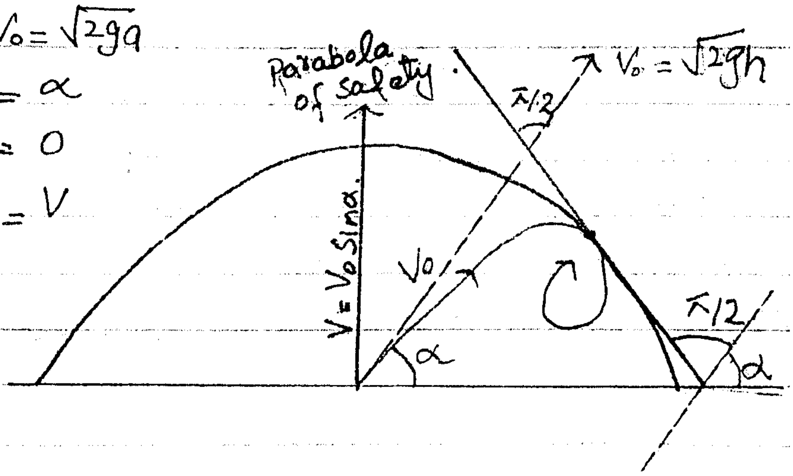


Exercise

Q No: 1.

A particle is projected at time $t=0$ in a fixed vertical plane from given point 'o' with speed $\sqrt{2ga}$ of which the vertical component is V. Show that at time $t = 2a/V$ the particle is on a fixed parabola (parabola of safety), that its path touches parabola and that its direction of motion is then perpendicular to its direction of projection.

Solution:-



Velocity of projection $V_0 = \sqrt{2ga}$

Angle of projection $= \alpha$

Point of Projection $= O$

Vertical component of velocity $= V$

$$V_0 \sin\alpha = V$$

$$\sqrt{2ga}\,\sin\alpha = V$$

given $\quad t = \dfrac{2a}{V}$

$$t = \frac{2a}{\sqrt{2ga}\,\sin\alpha}$$

Position of projectile at any time 't' is given by $P(x_1, y_1)$ where.

$$x_1 = (V_0\cos\alpha)t = \sqrt{2ga}\cos\alpha \cdot \frac{2a}{\sqrt{2ga}\,\sin\alpha} = 2a\cot\alpha \quad \text{--- ①}$$

and

$$y_1 = V_0\sin\alpha\, t - \frac{1}{2}gt^2$$

$$= \sqrt{2ga}\,\sin\alpha \cdot \frac{2a}{\sqrt{2ga}\,\sin\alpha} - \frac{1}{2}g\left[\frac{4a^2}{2ga\sin^2\alpha}\right]$$

$$y_1 = 2a - a\,\mathrm{cosec}^2\alpha$$

$$y_1 = 2a - a(1+\cot^2\alpha)$$

$$y_1 = 2a - a - a\cot^2\alpha$$

$$y_1 = a - a\cot^2\alpha \quad \text{--- (2)}$$

Now we check whether $P(2a\cot\alpha,\ a-a\cot^2\alpha)$ lies on parabola of safety or not.

$\therefore$ Eq of Parabola of safety is

$$x^2 = -\frac{2V_0^2}{g}\left[y - \frac{V_0^2}{2g}\right]$$

$$x^2 = -\frac{2(2ga)}{g}\left[y - \frac{2ga}{2g}\right] \qquad \because V = \sqrt{2ga}$$

$$x^2 = -4a(y-a) \quad \text{--- (3)}$$

$$(2a\cot\alpha)^2 = -4a(\cancel{a} - a\cot^2\alpha - \cancel{a}) \qquad \text{at } P(x_1, y_1)$$

$$4a^2\cot^2\alpha = 4a^2\cot^2\alpha$$

Hence point $P(x_1, y_1)$ lies on parabola of safety i.e touches parabola of safety.

Now we find direction of motion at time $t = 2a/V$

If 'θ' be the inclination of the tangent at 'P' then

$$\tan\theta = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{V_0\sin\alpha - gt}{V_0\cos\alpha}$$

$$= \frac{V_0\sin\alpha - g(2a/V)}{V_0\cos\alpha}$$

$$= \frac{VV_0\sin\alpha - 2ag}{VV_0\cos\alpha}$$

$$= \frac{V_0^2\sin^2\alpha - V_0^2}{V_0^2\sin\alpha\cos\alpha} \qquad \because V = V_0\sin\alpha,\ V_0 = \sqrt{2ga}$$

$$= \frac{\cancel{V_0^2}(\sin^2\alpha - 1)}{\cancel{V_0^2}\sin\alpha\cos\alpha}$$

$$= -\frac{\cos^2\alpha}{\sin\alpha\cos\alpha} \qquad \because \cos^2\alpha + \sin^2\alpha = 1$$

$\tan\theta = -\cot\alpha$

$\tan\theta = \tan(\pi/2 + \alpha)$

$\Rightarrow \theta = \frac{\pi}{2} + \alpha$

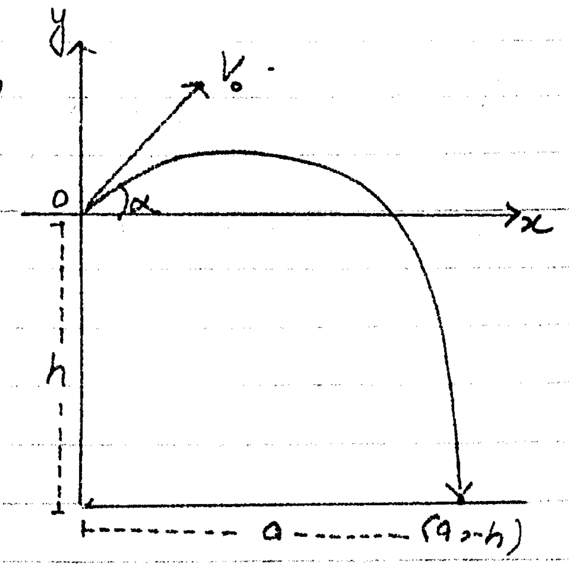
which shows that direction of motion is ⊥ar to direction of projection 'α'.

Hence proved !!

Q No: 2.

prove that the speed required to project a particle from a hieght 'h' to fall a horizontal distance 'a' from the point of projection is at least $\sqrt{g(\sqrt{a^2+h^2} - h)}$.

Solution:-



Point of projection of Particle } = 0

Angle of projection of Particle } = α

Speed of Projection of Particle } = V_0

Point of projection 'o' is at a hieght } = h

Let the particle fall at a horizontal distance 'a' from 'o'.

Therefore (a,-h) lies on the Path of projectile. Hence it satisfies the trajectory.

$\therefore y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_0^2}$ (Eq. of trajectory.

$-h = a\tan\alpha - \frac{ga^2(1+\tan^2\alpha)}{2V_0^2}$ put $x = a$, $y = -h$

$-2V_0^2h = 2V_0^2 a\tan\alpha - ga^2 - ga^2\tan^2\alpha$

$(ga^2)\tan^2\alpha - (2V_0^2 a)\tan\alpha + (ga^2 - 2V_0^2 h) = 0$ —— ①

It is Quadratic in $\tan\alpha$. For actual motion 'α' must be real, and so $\tan\alpha$ must be real i-e roots are real of eq ①

Roots of ① are real when Discriminent

$B^2 - 4AC \geq 0$

$\therefore \quad (-2V_0^2 a)^2 - 4ga^2(ga^2 - 2V_0^2 h) \geq 0$

$\Rightarrow \quad 4a^2V_0^4 - 4a^4g^2 + 8a^2gV_0^2h \geq 0$

$\Rightarrow \quad 4a^2(V_0^4 - a^2g^2 + 2V_0^2gh) \geq 0$

$\Rightarrow \quad V_0^4 - a^2g^2 + 2V_0^2gh \geq 0$

$V_0^4 + 2V_0^2gh \geq a^2g^2$

completing square

$V_0^4 + 2V_0^2gh + h^2g^2 \geq a^2g^2 + h^2g^2$

$\Rightarrow \quad (V_0^2 + hg)^2 \geq g^2(a^2+h^2)$

$V_0^2 + hg \geq \sqrt{g^2(a^2+h^2)}$

$V_0^2 \geq g\sqrt{a^2+h^2} - hg$

$V_0^2 \geq g(\sqrt{a^2+h^2} - h)$

$\Rightarrow \quad V_0 \geq \sqrt{g((a^2+h^2) - h)}$

Hence least Velocity is $V_0 = \sqrt{g((a^2+h^2)-h)}$

Hence proved!!

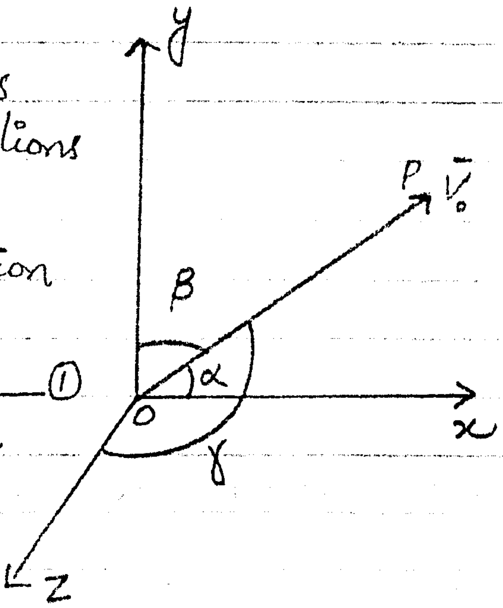
Q NO: 3-

Any Number of Particles are projected from the same point at the same Instant in various directions with speed V_0 - Prove that at any subsequent time 't' They will all be on a sphere of radius

$V_0 t$ and determine the motion of the centre of this sphere.

Solution:-

Let 'o' be the point of projection. Since the particles are projected in all directions (i.e in space), so we suppose velocity of projection $\bar{V}_0$ as



$$V_0 = V_0\cos\alpha\,\hat{i} + V_0\cos\beta\,\hat{j} + V_0\cos\gamma\,\hat{k} \quad \text{---} (1)$$

Let at any time 't' the projected particle be at point $P(x,y,z)$ then

$$\overline{OP} = \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{---} (2)$$

Let the force of gravity act along -ve direction of y-axis then the equation of motion of particle is

$$m\ddot{r} = -mg\,\hat{j}$$

$$\ddot{r} = -g\,\hat{j}$$

Integrating

$$\dot{r} = -gt\hat{j} + a$$

at $t=0$, $V = V_0$ $\therefore$ $V_0 = -0 + a \Rightarrow V_0 = a$

$$\therefore \quad \dot{r} = -gt\hat{j} + V_0$$

Integrating

$$\bar{r} = \frac{-gt^2}{2}\hat{j} + V_0 t + b$$

at $t=0$, $r=0$, $\therefore b = 0$

$$\therefore \quad \bar{r} = \frac{-gt^2}{2}\hat{j} + V_0 t$$

$$x\hat{i} + y\hat{j} + z\hat{k} = \frac{-gt^2}{2}\hat{j} + (V_0\cos\alpha\,\hat{i} + V_0\cos\beta\,\hat{j} + V_0\cos\gamma\,\hat{k})t.$$

On comparing

$\Rightarrow x = (V_0 \cos\alpha)t$ —— ①′

$\Rightarrow y = (V_0 \cos\beta)t - \dfrac{gt^2}{2}$ —— ②′

$\Rightarrow z = (V_0 \cos\gamma)t$ —— ③′

From ②

$$y + \frac{gt^2}{2} = (V_0 \cos\beta)t \quad \text{—— ④′}$$

Squaring and adding ①′, ③′ & ④′

$$x^2 + \left(y + \frac{gt^2}{2}\right)^2 + z^2 = (V_0 \cos\alpha)^2 t^2 + (V_0 \cos\beta)^2 t^2 + (V_0 \cos\gamma)^2 t^2$$

$$(x-0)^2 + \left(y - \left(-\frac{gt^2}{2}\right)\right)^2 + (z-0)^2 = (V_0^2 t^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$(x-0)^2 + \left(y - \left(-\frac{gt^2}{2}\right)\right)^2 + (z-0)^2 = V_0^2 t^2$$

$\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

which is eq of sphere having radius '$V_0 t$' and centre at $(0, -gt^2/2, 0)$ clearly the centre lies on $-ve$ part of y-axis, and as time 't' varies centre moves in $-ve$ direction of y-axis.

Hence proved !!

Q NO:- 4:-

A Shell bursts on a contact with the ground and pieces from all directions with all speed upto 80feet/sec- prove that a man is 100 feet away is in danger for $5\sqrt{2}$ seconds-

Solution:-

$V_0 = 80 \, ft/sec$

$g = 32 \, ft/sec^2$

$$R_{max} = \frac{V_0^2}{g}$$

$$= \frac{(80)^2}{32}$$

$$= 200 ft$$

* we shall consider posibility of a man being hit with a piece having speed 80ft/sec. Man is at 100ft/sec away from shell we find max. Range to know whether the shell piece reach 100ft or not then we find time of flight of piece.

Time of flight

$$T = \frac{2V_0 \sin\alpha}{g}$$

$$T = \frac{2 \times 80 \sin 45}{32} \quad \because \text{Rang Max. so } \alpha = 45^\circ$$

$$T = 5 \cdot \left(\frac{1}{\sqrt{2}}\right) \text{ sec}$$

Ans

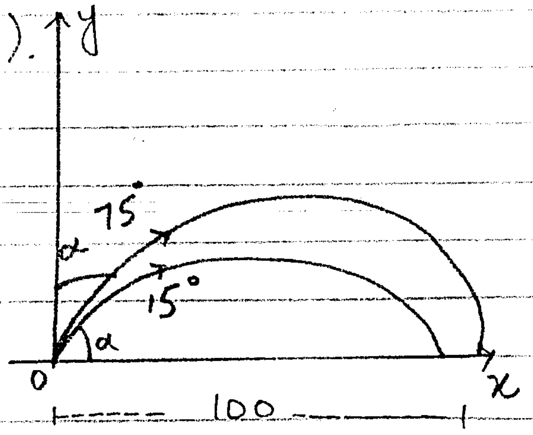
2ND Method:- (Logical).

$V_0 = 80 \, ft/sec$

$R = 100 ft$

Horizontal Range:

$$R = \frac{V_0^2 \sin 2\alpha}{g}$$



$$100 = \frac{80 \cdot 80 \sin 2\alpha}{32.}$$

$$\frac{1}{2} = \sin 2\alpha \Rightarrow 2\alpha = \sin^{-1}(1/2).$$
$$= 30^\circ, 150^\circ$$

$$\Rightarrow \alpha = 15^\circ, 75^\circ$$

Hence at $\alpha = 15^\circ$ & 75° are two directions for pieces to hit a man standing at 100 ft away.

Two direction of Projections
$\alpha, \pi/2 - \alpha$
$15, 90-15$
$75, 75$

Now let t_1 & t_2 be times of flights of these pieces. Then

$$t_1 = \frac{2V_0 \sin\alpha}{g} = \frac{2 \cdot 80 \sin 15}{32} = 5 \sin 15^\circ.$$

$$t_2 = \frac{2V_0 \sin\alpha}{g} = \frac{2 \cdot 80 \sin 75^\circ}{32} = 5 \sin 75^\circ$$

so time of danger $= t_2 - t_1$

$$= 5(\sin 75^\circ - \sin 15^\circ)$$

$$= 5\left(2 \sin\left(\frac{75-15}{2}\right) \cos\left(\frac{75+15}{2}\right)\right)$$

$$= 10 \sin 30^\circ \cos 45^\circ$$

$$= 10 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{5}{\sqrt{2}} \text{ sec}$$

Hence Proved !!

Q NO:5-

An aeroplane is flying with constant speed V_0 and at constant hieght h. Show that, if a gun is fired point blank at the aeroplane after it has passed directly over the gun when its angle of elevation as seen from the gun is α, the shell will hit the aeroplane provided that

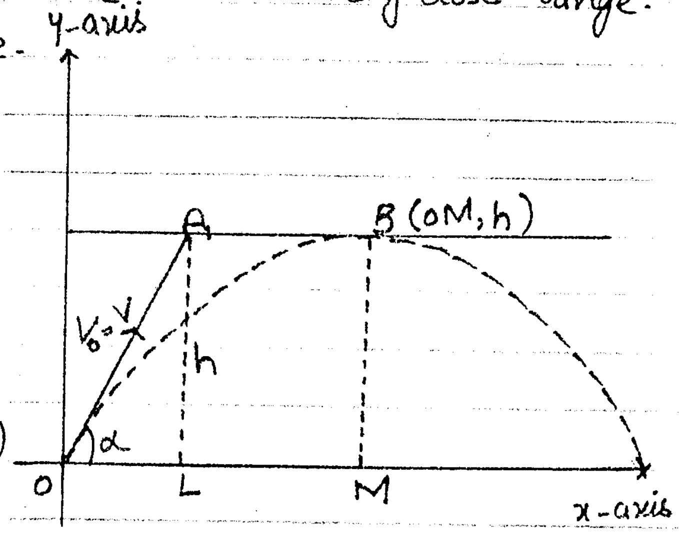
$$2(V\cos\alpha - V_0)\tan^2\alpha = gh$$

where V is the intial speed of the shot, the path being assumed parabolic-

solution:-

Let at t=0, the gun is at origin 'O' and 'A' is the position of aeroplane when the gun is fired. Let after time 't' the shell hit the aeroplane at point 'B', since the shot describes parabolic path.

{ Point blank = fired at very close range.



Coodinates of Point B(OM, h)

$= B(OL + LM, h)$

$\because LM = |AB| = V_0 t$ $\quad \because V_0$ = Speed of plane. $\quad \because S = vt$

$\because \dfrac{h}{OL} = \tan\alpha \Rightarrow \dfrac{h}{\tan\alpha} = OL \Rightarrow OL = h\cot\alpha.$

$= B(h\cot\alpha + V_0 t, h)$

This point B, lies on path of parabola so it should satisfy equations.

$y = V_0\sin\alpha t - \frac{1}{2}gt^2.$

$h = V_0 \sin\alpha \cdot t - \frac{1}{2} g t^2$ —— ①

position of moving particle at any time t

$x = V_0 \cos\alpha \, t$

$y = V_0 \sin\alpha \, t - \frac{1}{2} g t^2$

First Find 't'.

$\because V_0 = V$

$x = \cos\alpha \cdot t \; V_0$

$h\cot\alpha + V_0 t = V\cos\alpha \cdot t$

$\Rightarrow t = \dfrac{h\cot\alpha}{V\cos\alpha - V_0}$

Put in ①

$$h = V\sin\alpha \cdot \left(\frac{h\cot\alpha}{V\cos\alpha - V_0}\right) - \frac{g}{2}\left(\frac{h\cot\alpha}{V\cos\alpha - V_0}\right)^2$$

$$2h(V\cos\alpha - V_0)^2 = 2V\sin\alpha\left[h\cot\alpha(V\cos\alpha - V_0)\right] - g h^2\cot^2\alpha$$

$$2\cancel{h}(V\cos\alpha - V_0)^2 = 2\cancel{h}\left[\left[V\sin\alpha \cdot \frac{\cos\alpha}{\sin\alpha}(V\cos\alpha - V_0)\right] - g h\cot^2\alpha\right]$$

$$2(V\cos\alpha - V_0)^2 = 2V\cos\alpha(V\cos\alpha - V_0) - g h\cot^2\alpha.$$

$$gh\cot^2\alpha = 2V\cos\alpha(V\cos - V_0) - 2(V\cos\alpha - V_0)^2$$

$$gh\cot^2\alpha = 2(V\cos\alpha - V_0)\left[\cancel{V\cos\alpha} - \cancel{V\cos\alpha} + V_0\right]$$

$$gh\cot^2\alpha = 2(V\cos\alpha - V_0)V_0$$

$$gh = \frac{2(V\cos\alpha - V_0)V_0}{\cot^2\alpha}$$

$$gh = 2(V\cos\alpha - V_0)V_0\tan^2\alpha$$

Hence proved !!

Q NO: 6 -

Find a range of rifle bullet when α is the elevation of projection and V_0 the speed - show that if the rifle is fired with the same speed and elevation from a car travelling with speed V towards the target the range will be increased by $2V_0V\sin\alpha/g$ -

Solution:-

We Know Range = (Horizontal Velocity)(Time of flight).

$$R = (V_0 \cos\alpha)\left(\frac{2V_0 \sin\alpha}{g}\right)$$

$$R = \frac{V_0^2 \sin 2\alpha}{g}.$$

when the rifle is fired with the same elevation 'α' and speed V_0 from car moving with velocity V towards the target, then horizontal velocity is increased by V and becomes $V_0\cos\alpha + V$ then

$$R' = (V_0 \cos\alpha + V)\left(\frac{2V_0 \sin\alpha}{g}\right)$$

Increase in Range is $R' - R$.

$$R' - R = (V_0\cos\alpha + V)\cdot\frac{2V_0\sin\alpha}{g} - \frac{V_0^2 \sin 2\alpha}{g.}$$

$$= \frac{2V_0^2 \sin\alpha\cos\alpha + 2VV_0\sin\alpha}{g} - \frac{V_0^2 \sin 2\alpha}{g}$$

$$= \frac{V_0^2 \cancel{\sin 2\alpha} + 2VV_0 \sin\alpha - V_0^2 \cancel{\sin 2\alpha}}{g}$$

$$= \frac{2VV_0 \sin\alpha}{g}$$

Hence proved !!

Q: 7- The range of a rifle bullet is 1200 yards where α is the elevation of projection. Show that if the rifle is fired from the same elevation from a car travelling at

10 miles/hour towards the target. The range will be increased by $220\sqrt{\tan\alpha}$ feet.

Solution:-

1 mile = 1760 yards
1 yard = 3 feet
1 feet = 12 inches
1 inch = 2.54 cm

$$R = 1200 \text{ yards}$$
$$= 1200 \times 3 \text{ feet}$$
$$R = 3600 \text{ feet} \quad \text{---(1)}$$
$$V = 10 \text{ miles/hour}$$
$$= \frac{10 \times 1760 \times 3}{60 \times 60}$$
$$V = \frac{44}{3} \text{ ft/sec} \quad \text{---(ii)}$$
$$g = 32 \text{ ft/sec}^2$$
$$R = \frac{V_0^2 \sin 2\alpha}{g}$$
$$3600 = \frac{V_0^2 \cdot 2\sin\alpha\cos\alpha}{32}$$
$$\frac{57600}{\sin\alpha\cos\alpha} = V_0^2$$
$$\Rightarrow V_0 = \frac{240}{\sqrt{\sin\alpha\cos\alpha}} \quad \text{---(iii)}$$

$$\text{Increase in Range} = \frac{2VV_0 \sin\alpha}{g}$$
$$= \frac{2 \times 44}{3} \times \left(\frac{240}{\sqrt{\sin\alpha\cos\alpha}}\right) \cdot \frac{\sin\alpha}{32}$$
$$= 220\sqrt{\frac{\sin\alpha}{\cos\alpha}}$$
$$= 220\sqrt{\tan\alpha}$$

Hence proved!!

Q No: 8-

Determine the maximum possible range for a projectile fired from a cannon having muzzle Velocity V_0, and proved that hieght reached in this case is $\frac{V_0^2}{4g}$ -

Solution:-

$$\text{Range} = (\text{Horizontal Velocity})(\text{Time of flight}).$$
$$= (V_0\cos\alpha)\left(\frac{2V_0\sin\alpha}{g}\right)$$
$$R = \frac{V_0^2\sin 2\alpha}{g}.$$

Range is maximum when $\sin 2\alpha = 1$ $\because \sin 2\alpha$ is max.

$\because \sin 2\alpha = 1 \quad \Rightarrow 2\alpha = \sin^{-1}(1) = 90$

$\Rightarrow \quad \alpha = \frac{90}{2} = 45^\circ$

$\therefore R_{max} = \frac{V_0^2}{g} \cdot 1 = \frac{V_0^2}{g}.$

As the vertex of Parabola is highiest point

$\therefore$ Max. hieght = ordinate of Vertex i-e K.

$$= \frac{V_0^2\sin^2\alpha}{2g} = \frac{V_0^2\sin^2(\pi/4)}{2g}.$$
$$= \frac{V_0^2(1/2)}{2g}$$
$$= \frac{V_0^2}{4g}$$

Hence proved !!

Q NO: 9-

(a) What is the maximum range possible for a projectile fired from a cannon having muzzle velocity 1 mile/sec and (b) what is the hieght reached in this case-

Solution:-

(a) $V_0 = 1 \text{ mile/sec}$

$= 1 \times 1760 \times 3 \text{ ft/sec.}$

$= 5280 \text{ ft/sec.}$

$$R_{max} = \frac{V_0^2}{g} = \frac{5280 \times 5280}{32} = 871200 \text{ ft}$$

$$R_{max} = \frac{871200}{1760 \times 3} = 165 \text{ miles-}$$

$$Hieght_{max} = \frac{V_0^2}{4g}$$

$$= \frac{5280 \times 5280}{4 \times 32}$$

$$= 217800 \text{ ft}$$

$$Hieght_{max} = \frac{217800}{1760 \times 3} = 41.25 \text{ mils-}$$

Ans

Q: 10-

A cannon has its maximum Range R, Prove that (a) the hieght Reaced is $\frac{1}{4}R$ and (b) the time of flight is $\sqrt{\frac{2R}{g}}$.

Solution:-

(a) = $R_{max} = \frac{V_0^2}{g}$ — ①

$\because$ Hieght max = ordinate of vertex.

$$= \frac{V_0^2 \sin^2\alpha}{2g} = \frac{V_0^2 \sin^2(\pi/4)}{2g} \quad \because \alpha = \pi/4$$

$$= \frac{V_0^2 \cdot (1/2)}{g} = \frac{V_0^2}{4g}$$

$$\text{Hieght}_{Max} = \frac{1}{4}(R_{max}) \quad \text{using (i).}$$

(b) Time of flight $= \dfrac{2V_0 \sin\alpha}{g}$ —— (ii)

$\because$ $gR = V_0^2$ from (i)

$\sqrt{gR} = V_0$ —— (iii)

from (i) (ii) & (iii)

$$T = \frac{2\sqrt{gR}\,\sin(\pi/4)}{g}$$

$$= 2\cdot\frac{\sqrt{R}}{\sqrt{g}}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{R}}{\sqrt{g}}$$

$$T = \sqrt{\frac{2R}{g}}$$ Hence Proved !!

Q 11:-

A projectile having horizontal range R, reaches a maximum hieght H. Prove that it must have been launched with (a) an initial speed equal to

$$\sqrt{\frac{g(R^2+16H^2)}{8H}}$$ and (b) at angle with the horizontal is given by

$$\sin^{-1}\left(\frac{4H}{\sqrt{R^2+16H^2}}\right).$$

Solution:-

$$R = \frac{2V_0^2 \sin\alpha\cos\alpha}{g} \quad \text{---} \quad (i) \quad \text{Horizontal Range.}$$

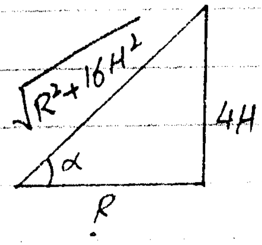
$$H = \frac{V_0^2 \sin^2\alpha}{2g} \quad \text{---} \quad (ii) \quad \text{Hiegth = ordinate of vertex.}$$

$$\frac{R}{H} = \frac{2V_0^2 \sin\alpha\cos\alpha}{g} \cdot \left(\frac{2g}{V_0^2 \sin^2\alpha}\right)$$

$$\frac{R}{H} = 4\cot\alpha.$$

$$\frac{H}{R} = \frac{1}{4}\tan\alpha$$

$$\frac{4H}{R} = \tan\alpha$$



$$\therefore \sin\alpha = \frac{4H}{\sqrt{R^2+16H^2}} \quad \text{---} \quad (iii)$$

$$\alpha = \sin^{-1}\left(\frac{4H}{\sqrt{R^2+16H^2}}\right) \quad \text{Proved.}$$

Also from (ii) $V_0^2 = \dfrac{2gH}{\sin^2\alpha}$

$$= \frac{2gH}{\frac{16H^2}{R^2+16H^2}}$$

$$V_0^2 = \frac{g(R^2+16H^2)}{8H}$$

$$V_0 = \sqrt{\frac{g(R^2+16H^2)}{8H}}$$

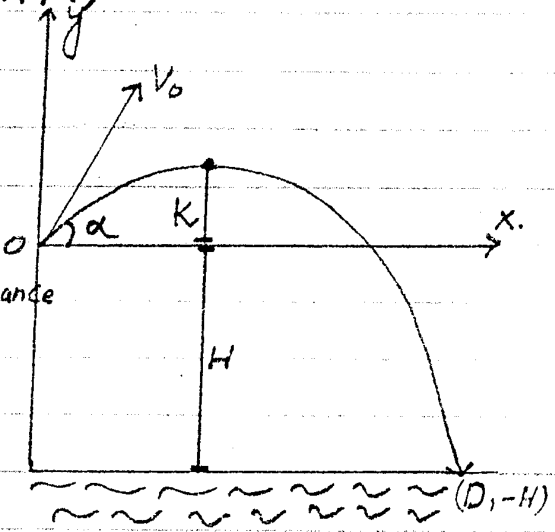
Proved !!

Q NO: 12-

A projectile is launched at angle α from a cliff of Hieght H above sea level. If it falls into the sea at a distance D from the base of the cliff, prove that the maximum hieght above sea level is

$$H + \frac{D^2 \tan^2\alpha}{4(H - D\tan\alpha)}.$$

Solution:-



Let V_0 be the velocity of projection, projected from cliff 'O' at a hieght 'H' above the Seal level.
Let it fall into the sea at a distance D base of cliff. Therefore the pt $(D, -H)$ lies on path of projectile, hence satisfies it,

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_0^2}$$

$$-H = D\tan\alpha - gD^2\sec^2\alpha / 2V_0^2$$

$$\frac{gD^2\sec^2\alpha}{2V_0^2} = D\tan\alpha + H$$

$$\frac{gD^2\sec^2\alpha}{2[D\tan\alpha + H]} = V_0^2$$

Max. hieght above sea level is $H + K$.

$$= H + \frac{V_0^2\sin^2\alpha}{2g} \qquad \left[K \text{ is ordinate of vertex } = V_0^2 \sin^2\alpha/2g \right]$$

$$= H + \frac{gD^2\sec^2\alpha}{2[D\tan\alpha + H]} \cdot \frac{\sin^2\alpha}{2g}$$

$$= H + \frac{D^2\tan^2\alpha}{4[D\tan\alpha + H]}$$

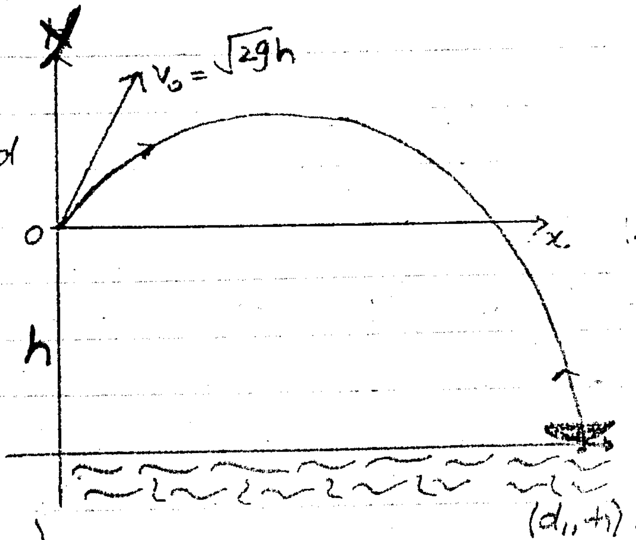
Hence proved!!

Q NO: 13

A fort and a ship are both armed with guns which give their projectiles a muzzle velocity $\sqrt{2gh}$ and the guns in the fort are at a hieght h above the guns in the ship. If d_1 and d_2 are the greatest horizontal ranges at which the fort and ship, respectively, can engage, prove that

$$\frac{d_1}{d_2} = \sqrt{\frac{K+h}{K-h}}.$$

Proof:-



The point $(d_1, -h)$ lies on path of projectile fired from guns in fort at a hieght 'h' above sea level.

$$\therefore\ Y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_0^2}$$

$$-h = d_1\tan\alpha - \frac{gd_1^2\sec^2\alpha}{2V_0^2}$$

$$-h = d_1\tan\alpha - \frac{gd_1^2(1+\tan^2\alpha)}{2(2gK)}$$

$$-4Kh = 4Kd_1\tan\alpha - d_1^2 - d_1^2\tan^2\alpha$$

$$d_1^2\tan^2\alpha - 4Kd_1\tan\alpha + d_1^2 - 4Kh = 0$$

This eq is Quadratic in $\tan\alpha$. For actual motion 'α' must be real, and so $\tan\alpha$ must be real, so roots are real i.e $B^2 - 4AC \geq 0$

$$\therefore\ 16K^2d_1^2 - 4d_1^2(d_1^2 - 4Kh) \geq 0$$

$$4K^2 - d_1^2 - 4Kh \geq 0$$

$$4K(K+h) \geq d_1^2$$

$$\sqrt{4K(K+h)} \geq d_1$$

Similarly for the gun in ship:

Put $h = -h$.

$\sqrt{4k(k-h)} \geq d_2$

divide d_1 by d_2

$$\frac{d_1}{d_2} = \frac{\sqrt{4k(k+h)}}{\sqrt{4k(k-h)}}$$

$$\frac{d_1}{d_2} = \frac{\sqrt{k+h}}{\sqrt{k-h}}$$

$$\frac{d_1}{d_2} = \sqrt{\frac{k+h}{k-h}}$$

Hence proved !!

Q NO:14.

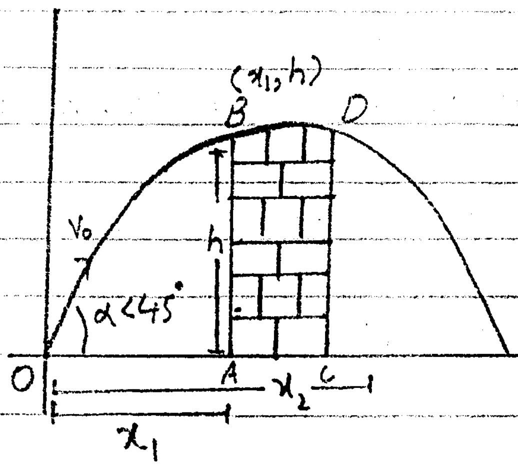
From a gun placed on a horizontal plane, which can fire a shell with speed $\sqrt{2gH}$, it is required to throw a shell over a wall of hieght h, and the elevation of the gun cannot exceed $\alpha < 45°$. Show that this will be possible only when $h < H\sin^2\alpha$, and that if this condition be satisfied, the gun must be fired from within a strip of the plane whose breadth is $4\cos\alpha\sqrt{H(H\sin^2\alpha - h)}$.

Proof :-

Let the origin 'o' be the position of gun and speed V_0 of shell is $V_0 = \sqrt{2gH}$

Let hieght of wall $= AB = h$

Let distance of wall from origin 'O' $= x_1$



Let the shell passes over the wall, so that point $B(x_1, h)$ lies on path of shell.

$$Y = x_1 \tan\alpha - \frac{1}{2} \frac{g x_1^2 \sec^2\alpha}{V_0^2}$$

$$h = x_1 \tan\alpha - \frac{g x_1^2 \sec^2\alpha}{2.2gH}$$

$$h = x_1 \tan\alpha - \frac{x_1^2}{4H\cos^2\alpha}$$

$$4H\cos^2\alpha \cdot h = 4H\cos^2\alpha \cdot x\tan\alpha - x^2$$

$$= 4H\cos\alpha\sin\alpha \cdot x - x^2$$

$$x^2 - (4H\cos\alpha\sin\alpha)x + 4Hh\cos^2\alpha = 0 \quad \text{—— (1)}$$

This eq is quadratic in x. It has real roots if

$$B^2 - 4AC > 0$$

$$16H^2\cos^2\alpha\sin^2\alpha - 4(1)(4)(Hh\cos^2\alpha) > 0$$

$$16H\cos^2\alpha\,(H\sin^2\alpha - h) > 0$$

$$H\sin^2\alpha - h > 0$$

$$H\sin^2\alpha > h$$

Now from (1)

$$x^2 - (4H\cos\alpha\sin\alpha)x + 4Hh\cos^2\alpha = 0$$

$$x = \frac{4H\cos\alpha\sin\alpha \pm \sqrt{16H^2\cos^2\alpha\sin^2\alpha - 16Hh\cos^2\alpha}}{2}$$

$$x = \frac{4H\cos\alpha\sin\alpha \pm 4\cos\alpha\sqrt{H^2\sin^2\alpha - Hh}}{2}$$

$$x = 2H\cos\alpha\sin\alpha \underset{x_1}{\overset{x_2}{\pm}} 2\cos\alpha\sqrt{H(H\sin^2\alpha - h)}$$

Breadth $= OC - OA$

$$x_2 - x_1 = 2H\cos\alpha\sin\alpha + 2\cos\alpha\sqrt{H(H\sin^2\alpha - h)} - \left(2H\cos\alpha\sin\alpha - 2\cos\alpha\sqrt{H(H\sin^2\alpha - h)}\right)$$

$$= 4\cos\alpha\sqrt{H(H\sin^2\alpha - h)}$$

Hence proved !!

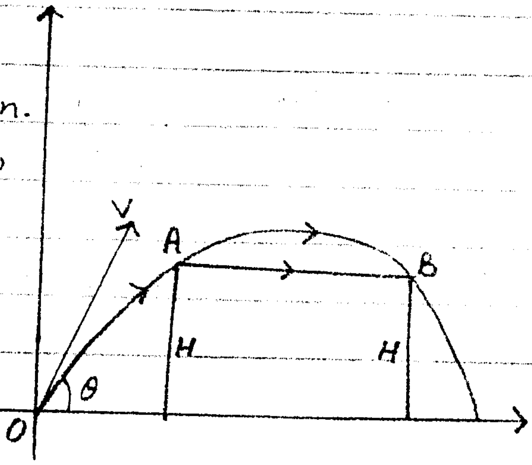
Q NO: 15-

A Shell is fired with speed V at an elevation θ hits an airship at hieght H- which is moving horizontally away from the gun with speed V_0. Show that, if

$$(2V\cos\theta - V_0)(V^2\sin^2\theta - 2gH)^{1/2} = V_0 V \sin\theta$$

the shell might also have hit the airship if the latter had remained stationary in the position it occupied when the gun was actually fired.

Proof :-



Let 'O' be the position of gun. Let 'A' be the position of airship when gun is fired Hieght of airship at A is 'H'.
when the shell is fired from the gun at 'O' the airship moves at constant hieght 'H' away from A. The shell has the only posibility to hit the airship at the position B at hieght 'H' on the trajectory of shell.
First we find time taken by shell when it attains hieght 'H' i.e to reach 'A'

$x = V\cos\theta \cdot t$ —— (i) (Position of moving particle at

$Y = V\sin\theta \cdot t - \frac{1}{2}gt^2$ —— (ii) any point is (x, y))-

$H = V\sin\theta \cdot t - \frac{1}{2}gt^2$

$2H = 2V\sin\theta \cdot t - gt^2$

$gt^2 - 2V\sin\theta \cdot t + 2H = 0$ quad in 't'

→ (1)

$$t = \frac{2V\sin\theta \pm \sqrt{4V^2\sin^2\theta - 8gH}}{2g}$$

$$t = \frac{V\sin\theta \pm \sqrt{V^2\sin^2\theta - 2gH}}{g}$$

let t_1 = time taken by shell to reach A from 'O'.

t_2 = time taken by shell to reach B from 'O'.

$$t_1 = \frac{V\sin\theta - \sqrt{V^2\sin^2\theta - 2gH}}{g} \quad \text{---(II)} \quad \because t_1 < t_2$$

$$t_2 = \frac{V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}}{g} \quad \text{---(III)}$$

The time taken from A to B $= t_2 - t_1$

$\therefore$ Horizontal distance $|AB| = V\cos\theta(t_2 - t_1)$.

$$|AB| = \frac{V\cos\theta \cdot 2 \cdot \sqrt{V^2\sin^2\theta - 2gH}}{g} \quad \text{---(IV)}$$

Now if the time taken by airship from A to B is equal to time taken by shell from O to B. then shell be hit airship at B.

$$\therefore \quad |AB| = V_0 t_2$$

from (III)

$$|AB| = V_0\left[\frac{V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}}{g}\right]$$

using (IV)

$$\frac{V\cos\theta\, 2\sqrt{V^2\sin^2\theta - 2gH}}{g} = V_0\left(\frac{V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}}{g}\right)$$

$$\sqrt{V^2\sin^2\theta - 2gH}\,(2V\cos\theta) - V_0\sqrt{V^2\sin^2\theta - 2gH} = V_0 V\sin\theta$$

$$\sqrt{V^2\sin^2\theta - 2gH}\,(2V\cos\theta - V_0) = V_0 V\sin\theta$$

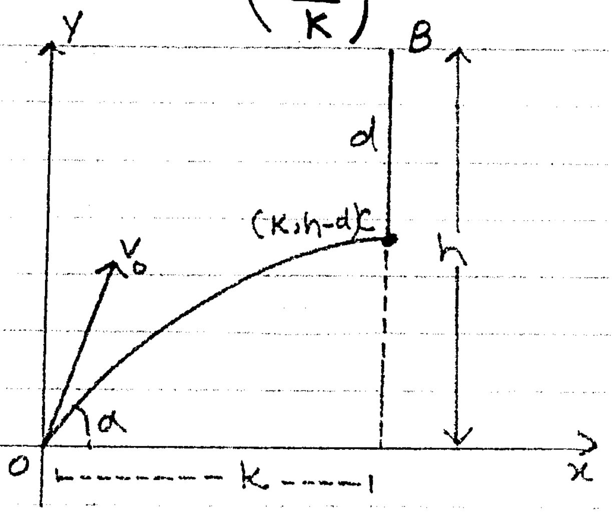
Hence proved!!

Q No: 16 -

A ball is dropped from the top of a tower of hieght h. At the same moment, another ball is thrown from a point of ground at a distance K from the foot of tower so as to strike the first ball at a depth d. Show that the initial speed and the direction of projection of the second ball are respectively

$$\sqrt{\frac{g(h^2-k^2)}{2d}} \quad \text{and} \quad \tan^{-1}\left(\frac{h}{k}\right).$$

Proof:-



let at $t=0$ the ball is dropped from point 'B' at a hieght 'h' and let at the same time ball is thrown from point 'o' with speed 'V_0' making angle 'α'. Distance between 'o' & tower is 'K'

First we find time taken by ball from 'B' to 'C'

using $S = ut + \frac{1}{2}gt^2$ (Newton's Law).

$d = 0.t + \frac{1}{2}gt^2$ initial velocity $u=0$

$$\sqrt{\frac{2d}{g}} = t$$

In the same time the second ball moves from 'o' to 'c'

∴ Pt 'C' (K, h-d) Lies on path of projectile.

∴ Parametric eq's of path of projectile-

$x = V_0 \cos\alpha . t$ $\qquad Y = V_0 \sin\alpha t - \frac{1}{2} g t^2$

$K = V_0 \cos\alpha \sqrt{\frac{2d}{g}}$ $\qquad h - d = V_0 \sin\alpha \sqrt{\frac{2d}{g}} - \frac{1}{2} g \left(\sqrt{\frac{2d}{g}}\right)^2$

$$h = V_0 \sin\alpha \sqrt{\frac{2d}{g}} - \frac{1}{2} g . \frac{2d}{g} + d$$

$$h = V_0 \sin\alpha \sqrt{\frac{2d}{g}}$$

Squaring & adding

$$K^2 + h^2 = V_0^2 \cos^2\alpha \left(\frac{2d}{g}\right) + V_0^2 \sin^2\alpha . \left(\frac{2d}{g}\right)$$

$$K^2 + h^2 = V_0^2 (\cos^2\alpha + \sin^2\alpha)\, 2d/g$$

$$\frac{g(K^2 + h^2)}{2d} = V_0^2$$

$$\sqrt{\frac{g(K^2 + h^2)}{2d}} = V_0$$

Now divide h by K.

$$\frac{h}{K} = \frac{V_0 \sin\alpha \sqrt{2d/g}}{V_0 \cos\alpha \sqrt{2d/g}}$$

$$\frac{h}{K} = \tan\alpha$$

$$\tan^{-1}\left(\frac{h}{K}\right) = \alpha$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{h}{k}\right)$$

Hence proved!!

Q NO: 17.

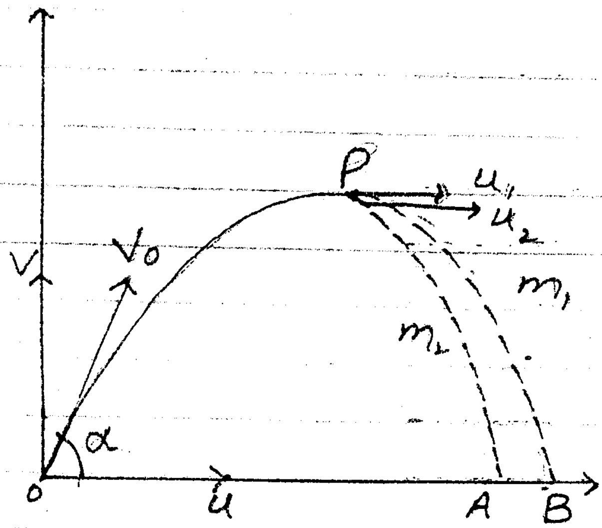
A shell of mass $(m_1 - m_2)$ is fired with a velocity whose horizontal and vertical components u, v and at the highiest point in its path the shell explodes into two fragments m_1, m_2. The explosion produces an additional Kinetic Energy E, and the fragments sepsate in a horizontal direction - show that they strike the ground at a distance apart which is equal to

$$\frac{V}{g}\left\{2E\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right\}^{1/2}$$

Proof:-

Suppose that shell of mass (m_1+m_2) is fired from origin with velocity 'V_0' at an angle of measure 'α'

$$\left.\begin{aligned} V_0\cos\alpha &= u \\ V_0\sin\alpha &= V \end{aligned}\right\} \text{ given.}$$



At the highiest pt 'P' the Shell explodes into two fragments 'm_1' 'm_2'. At hieghiest point let u_1, u_2 be the horizontal velocities of 'm_1' & 'm_2' respectively.

$(m_1+m_2)u = m_1u_1 + m_2u_2$ { ∵ At the hieghiest pt. vel. of Particle has only horizontal component.

Squaring

$(m_1+m_2)^2u^2 = m_1^2u_1^2 + m_2^2u_2^2 + 2m_1m_2u_1u_2$ ——(1)

Now explosion produces additional energy E.

$$\therefore \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1+m_2)u^2 = E$$

∵ Inc. in K.E $= E$

$m_1u_1^2 + m_2u_2^2 - (m_1+m_2)u^2 = 2E$ ——(2)

$(m_1+m_2).2E = (m_1+m_2)m_1u_1^2 + (m_1+m_2)m_2u_2^2 - (m_1+m_2)^2u^2$

∵ multiplying by (m_1+m_2)

$= (m_1^2u_1^2 + m_1m_2u_1^2 + m_1m_2u_2^2 + m_2^2u_2^2 -$

$[m_1^2u_1^2 + m_2^2u_2^2 + 2m_1m_2u_1u_2]$

using (1)

$$2(m_1+m_2)E = m_1^2u_1^2 + m_1m_2u_1^2 + m_1m_2u_2^2 + m_2^2u_2^2 - m_1^2u_1^2 - m_2^2u_2^2 - 2m_1m_2u_1u_2$$

$$= m_1m_2(u_1^2+u_2^2-2u_1u_2)$$

$$= m_1m_2(u_1-u_2)^2$$

$$\frac{2E(m_1+m_2)}{m_1m_2} = (u_1-u_2)^2$$

$$2E\left(\frac{m_1}{m_1m_2}+\frac{m_2}{m_1m_2}\right) = (u_1-u_2)^2$$

$$2E\left(\frac{1}{m_2}+\frac{1}{m_1}\right) = (u_1-u_2)^2$$

$$\sqrt{2E\left(\frac{1}{m_1}+\frac{1}{m_2}\right)} = u_1-u_2$$

It is relative velocity of one fragment w.r.t other at hieghiest point.

∴ Time of flight from hieghiest point P to ground.

$$T = \frac{1}{2}\left(\frac{2V_0 \sin\alpha}{g}\right)$$

$$= \frac{V_0 \sin\alpha}{g} = \frac{V}{g}$$

$\because S = Vt$ $\qquad |AB| = $ (Relative time)(Relative velocity).

$$= \frac{V}{g}\sqrt{2E\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}$$

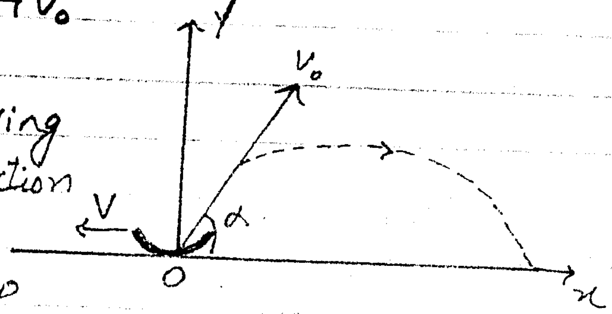
Proved !!

Example 1-

A battle Ship is Steaming ahead with V, and a gun is mounted on the battleship So as to point Straight backwards and is set an angle of elevation α. If V_0 is the speed of projection relative to gun. Show that the range is $\frac{2V_0 \sin\alpha}{g}(V_0\cos\alpha - V)$. Prove that the angle g of elevation for Max. Range is

$$\cos^{-1}\left(\frac{V+\sqrt{V^2+8V_0^2}}{4V_0}\right).$$

Proof:-



let the battleship is moving towards origin from direction of x-axis with speed 'V'
let at $t=0$ the battleship is at origin & the gun is fired with speed V_0 & angle α.

$\therefore$ Horizontal velocity of bullet relative to gun $= V_0\cos\alpha$.

" " " " to Ship $= V_0\cos\alpha - V$

Time of flight $= \frac{2V_0 \sin\alpha}{g}$

Range $=$ (horizontal velocity)(time)

$= (V_0\cos\alpha - V)\cdot\frac{2V_0\sin\alpha}{g}$.

Now max. Range

$$\frac{dR}{d\alpha} = 0, \quad \frac{d^2R}{d^2\alpha^2} < 0$$

So. $\frac{dR}{d\alpha} = \frac{2V_0}{g}\left[(V_0\cos\alpha - V)\cos\alpha + (-V_0\sin\alpha)\sin\alpha\right]$

$= \frac{2V_0}{g}\left(V_0\cos^2\alpha - V\cos\alpha - V_0\sin^2\alpha\right)$

$$= 2V_0/g\left[V_0\cos^2\alpha - V\cos\alpha - V_0(1-\cos^2\alpha)\right]$$

$$= \frac{2V_0}{g}\left\{V_0\cos^2\alpha - V\cos\alpha - V_0 + V_0\cos^2\alpha\right]$$

$$\frac{dR}{d\alpha} = \frac{2V_0}{g}\left\{2V_0\cos^2\alpha - V\cos\alpha - V_0\right\} = 0$$

$$\Rightarrow \quad 2V_0\cos^2\alpha - V\cos\alpha - V_0 = 0$$

$$\Rightarrow \quad \cos\alpha = \frac{V \pm \sqrt{V^2+8V_0^2}}{4V_0}$$

$$\frac{d^2R}{d\alpha^2} = \frac{2V_0}{g}\left\{-4V_0\cos\alpha\sin\alpha + \sin\alpha\right]$$

$$= \frac{2V_0\sin\alpha}{g}\left(-V_0\cdot 4\cos\alpha + V\right)$$

Put $\cos\alpha = \dfrac{V+\sqrt{V^2+8V_0^2}}{4V_0}$ in $\dfrac{d^2R}{d\alpha^2}$

$$\therefore \frac{d^2R}{d\alpha^2} = \frac{2V_0\sin\alpha}{g}\left(-V_0\cdot 4\cdot\left(\frac{V \pm \sqrt{V^2+8V_0^2}}{4V_0}\right) + V\right)$$

$\Rightarrow \dfrac{d^2R}{d\alpha^2} < 0$ Hence Range is max. at

$$\alpha = \cos^{-1}\left(\frac{V+\sqrt{V^2+8V_0^2}}{4V_0}\right)$$

Hence proved !!

Example 2-

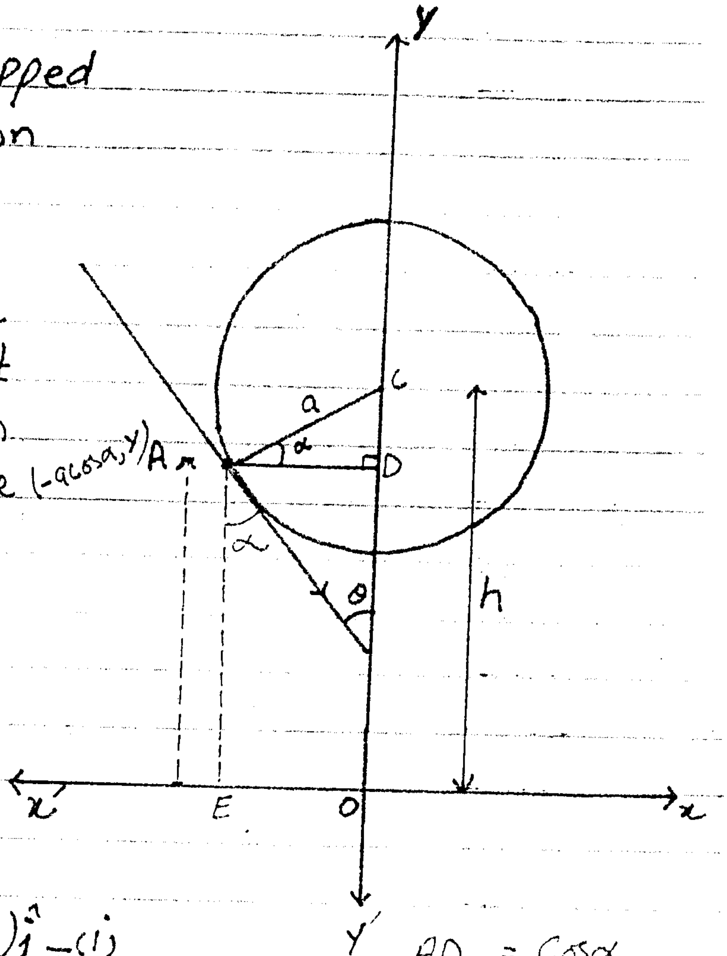
An aeroplane is flying with uniform speed V_0 in an arc of a verticle circle of radius 'a' whose centre is at a hieght 'h' vertically above a point 'o' of the ground. If a bomb is dropped from the aeroplane when at a hieght 'y' and strikes the ground at 'o', show that 'y' satisfies

$$KY^2 + Y(a^2 - 2hK) + K(h^2 - a^2) = 0$$

where $K = h + \frac{ga^2}{2V_0^2}$.

Solution:-

Let the bomb is dropped from the Point A on the circular path of plane. Let 'α' is the angle which aeroplane makes with vertical. It is also the angle which bomb makes with the vertical.



coordinates of A are $A(-a\cos\alpha, Y)$

Position of Pt A is $(-a\cos\alpha)\hat{i} + Y\hat{j}$

$\frac{AD}{AC} = \cos\alpha$, $AD = a\cos\alpha$.

The initial velocity of projectile is

$V_0 = (V_0\sin\alpha)\hat{i} - (V_0\cos\alpha)\hat{j}$ — (i)

$\because$ (along -ve Y i.e Y', so $-\hat{j}$)

$m\ddot{r} = -mg\hat{j}$

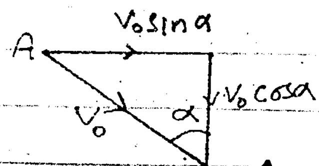
$\ddot{r} = -g\hat{j}$ where $\vec{r} = x\hat{i} + y\hat{j}$.

$\ddot{x}\hat{i} + \ddot{y}\hat{j} = -g\hat{j}$

$\ddot{x} = 0$ and $\ddot{y} = -g$

$\dot{x} = a$ and $\dot{y} = -gt + c$

$x = at + b$ & $y = -\frac{gt^2}{2} + ct + d$ — (iii)



At point A

initially, $t = 0$, $x = -a\cos\alpha$, $y = Y$

$V_0 = \dot{r} = \dot{x}\hat{i} + \dot{y}\hat{j}$

from (i) $V_0 = V_0\sin\alpha\,\hat{i} - V_0\cos\alpha\,\hat{j}$

$\therefore \dot{x} = V_0\sin\alpha$

$\dot{y} = -V_0\cos\alpha$.

$\therefore\ -a\cos\alpha = b + 0$ & $Y = 0 + 0 + d$.

$-a\cos\alpha = b$ —(iv) $\quad Y = d$ —(v)

Differentiate (ii) & (iii)

$\dot{x} = a$ & $\dot{y} = -gt + c$.

$V_0\sin\alpha = a$ —(vi) $\quad \dot{y} = 0 + c.$ at $t = 0$

$V_0\cos\alpha = c$ —(vii)

Then the trajectory is given by.

from (ii) $\quad x = (V_0\sin\alpha)t - a\cos\alpha$ —(viii) using (iv) & (vi)

from (iii) $\quad y = -\dfrac{gt^2}{2} - V_0\cos\alpha . t + Y$ —(ix) using (v) & (vii)

Since the projectile strikes the ground at 'O' (0,0).

$\therefore$ from (viii) $\quad 0 = (V_0\sin\alpha)t - a\cos\alpha \quad \because x = 0.$

$$t = \frac{a\cos\alpha}{V_0\sin\alpha} \quad \text{—(x)}$$

from (ix) $\quad 0 = -\dfrac{1}{2}gt^2 - V_0\cos\alpha\, t + Y \quad \because y = 0$

$$0 = -\frac{1}{2}g\left[\frac{a^2\cos^2\alpha}{V_0^2\sin^2\alpha}\right] - V_0\cos\alpha\left(\frac{a\cos\alpha}{V_0\sin\alpha}\right) + Y$$

$$0 = -ga^2\cos^2\alpha - 2a\cos^2\alpha\, V_0^2\sin\alpha + 2YV_0^2\sin^2\alpha.$$

$$2YV_0^2\sin^2\alpha = ga^2\cos^2\alpha + 2a\cos^2\alpha\, V_0^2\sin\alpha.$$

$$= ga^2(1-\sin^2\alpha) + 2a(1-\sin^2\alpha)V_0^2\sin\alpha.$$

$$2YV_0^2\sin^2\alpha = ga^2 - ga^2\sin^2\alpha + 2aV_0^2\sin\alpha - 2aV_0^2\sin^3\alpha.$$

$$2YV_0^2\sin^2\alpha + ga^2\sin^2\alpha + 2aV_0^2\sin^3\alpha = ga^2 + 2aV_0^2\sin\alpha.$$

$$(2YV_0^2 + ga^2 + 2aV_0^2\sin\alpha)\sin^2\alpha = ga^2 + 2V_0^2 a\sin\alpha.$$

From diagram.

$$\sin\alpha = \frac{CD}{AC} = \frac{OC - OD}{AC} = \frac{h - Y}{a}.$$

$$\left[2YV_0^2 + ga^2 + 2aV_0^2\left(\frac{h-Y}{a}\right)\right]\left(\frac{h-Y}{a}\right)^2 = ga^2 + 2aV_0^2\left(\frac{h-Y}{a}\right)$$

$$\left[2YV_0^2 + ga^2 + 2V_0^2h - 2V_0^2Y\right](h-Y)^2 = a^2\left[ga^2 + 2V_0^2(h-Y)\right]$$

$$(2V_0^2 h + ga^2)(h-Y)^2 = a^2(a^2 g + 2V_0^2 h - 2V_0^2 Y)$$

$$\cancel{2V_0^2}\left(h + \frac{ga^2}{2V_0^2}\right)(h-Y)^2 = \cancel{2V_0^2}\, a^2\left(\frac{ga^2}{2V_0^2} + h - Y\right)$$

$$K(h-Y)^2 = a^2(K-Y)$$

$$\because K = h + \frac{ga^2}{2V_0^2} \quad \because \text{given.}$$

$$K(h^2 + Y^2 - 2hY) - a^2(K-Y) = 0$$

$$Kh^2 + KY^2 - 2KhY - a^2 K + a^2 Y = 0$$

$$KY^2 + Y(a^2 - 2hK) + K(h^2 - a^2) = 0$$

Proved !!