Chapter: VIII

RECTILINEAR MOTION

# 81 RECTILINEAR MOTION:

The motion of a body along a straight line is called its rectilinear motion.

If the motion is rectilinear, so there is no distinction between vector equation & scalar equation,

\[ v = \frac{dx}{dt} \quad (i) \]

Can be expressed as:

\[ v = \frac{dx}{dt} \quad (ii) \]

And

\[ a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad (iii) \]

Can be denoted as:

\[ a = \frac{du}{dt} = \frac{d^2x}{dx \cdot dt} = \mu \frac{dv}{dx} \]

If \( v \) is considered as function of \( x \) so

\[ a = \frac{dv}{dx} \cdot \frac{dx}{dt} = \mu \cdot \frac{dv}{dx} \]

# MOTION WITH CONSTANT ACCELERATION:

Let a particle moving with constant acceleration along a straight line. Let at time \( t=0 \) the particle will be at point \( O \) and after some time \( t \) its velocity become \( v \) so acceleration \( a = \frac{du}{dt} \) \( (i) \)

\[ \Rightarrow \frac{dv}{dt} = a \cdot dt \quad (separating \ text \ variables) \]

Integrating with respect to \( t \):

\[ \int dv = \int a \cdot dt \quad (i.e.) \]

\[ v = at + c_1 \quad (ii) \]

Where \( c_1 \) is a constant of integration.

Applying condition when \( t=0 \) its velocity \( v=0 \)

So \( u = a \cdot 0 + c_1 \Rightarrow u = c_1 \)

So equation \( (ii) \) become:

\[ v = at + u \quad (iii) \]
We can write Equation (iii)
\[ v = \frac{dx}{dt} = u + at \]

Separating the variable \( \quad \int dx = (u + at) dt \)
Integrating above
\[ \int dx = \int (u + at) dt = \int ud\tau + at\,d\tau \]
\[ x = ut + \frac{at^2}{2} + C_2 \quad \ldots \ldots (iv) \]

where \( C_2 \) is another constant of integration.
\[ x = 0 \text{ when } t = 0 \]
\[ \therefore \quad 0 = u \cdot 0 + \frac{a \cdot 0^2}{2} + C_2 \Rightarrow C_2 = 0 \]
Thus Equation (iv) takes the form
\[ x = ut + \frac{1}{2} at^2 \quad \ldots \ldots (v) \]

\[ a = \frac{dv}{dt} = v \frac{dv}{dx} \quad \ldots \ldots (vi) \]

Separating the variable \( \quad vdv = axdx \)
Integrating above \( \quad \int vdv = \int axdx \)
\[ \Rightarrow \quad \frac{v^2}{2} = ax + C_3 \quad \ldots \ldots (vii) \]
where \( C_3 \) is a constant of integration
\[ v = u \text{ at } x = 0 \]
\[ \therefore \quad \frac{u^2}{2} = 0 + C_3 \Rightarrow C_3 = \frac{u^2}{2} \]
Thus Equation (vii) becomes
\[ \frac{v^2}{2} = ax + \frac{u^2}{2} \]
\[ v^2 = 2ax + u^2 \]
\[ \left| v^2 - u^2 \right| = 2ax \quad \ldots \ldots (viii) \]

Now if the particle starts from rest \( \quad v = 0 \)
\[ u = 0 \]
\[ v = u + at = 0 + at \]
\[ \Rightarrow \quad v = at \quad \ldots \ldots (ix) \]
\[ x = ut + \frac{1}{2}at^2 = 0t + \frac{1}{2}at^2 \]
\[ \Rightarrow x = \frac{1}{2}at^2 \quad (x) \]

\[ v^2 - u^2 = 2ax \]
\[ v^2 - 0 = 2ax \]
\[ \Rightarrow v^2 = 2ax \quad (xi) \]

If the particle moves with retardation \( a' \),
we replace \( a \) by \(-a'\) so
\[ v = u - a't \quad \text{so} \quad (xii) \]
\[ v' = ut' - \frac{1}{2}a't' \quad (xiii) \]
\[ v^2 = u^2 - 2a'x \quad (xiv) \]

The distance covered in \( nth \) second is given by
\[ x_n - x_{n-1} = u + \frac{1}{2} (2n-1)a \quad (i) \]

**Vertical Motion Under Gravity:**

1) Downward Motion:
   - If the bodies fall freely from rest, so
     \[ v = gt \quad (i) \]
     \[ x = \frac{1}{2}gt^2 \quad (i) \]
     \[ v^2 = 2gx \quad (iii) \]
   - Upward Motion:
     - If the body projected vertically upward with initial velocity \( u \) so body moves up with retardation and equation of up motion takes form
     \[ v = u - gt \quad (i) \]
     \[ x = ut - \frac{1}{2}gt^2 \quad (i) \]
     \[ v^2 - u^2 = -2gx \quad (iii) \]
     \[ \Rightarrow v^2 = u^2 - 2gx \quad (iv) \]

Math #8.3 Motion with Variable Acceleration:

Case (i) : Time-Dependent Acceleration
\[ a = f(t) \quad (i) \]
\[ \Rightarrow \frac{du}{dt} = a = f(t) \]
\[ \Rightarrow \text{Separating the variable}\ & \text{then integrating:} \]
\[ \int du = \int f(t) \, dt \]
\[ v = \int f(t)\,dt + C_1 \quad (ii) \]

where \( C_1 \) is a constant of integration.

\[ \frac{dx}{dt} = f(t) + C \]

\[ \int dx = \int [f(t) + C]\,dt \quad \text{i.e.,} \]

\[ x = \int f(t)\,dt + C_1 + C_2 \quad (iii) \]

where \( C_2 \) is another constant of integration.

The values of \( C_1 \) & \( C_2 \) can be determined by applying initial conditions of motion.

Case (ii): Distance-Dependent Acceleration

\[ a = f(x) \quad (i) \]

\[ \Rightarrow \quad v \cdot \frac{dv}{dx} = a = f(x) \quad \text{i.e.,} \]

\[ \int v \cdot dv = \int f(x)\,dx \]

\[ \Rightarrow \quad \frac{v^2}{2} = \int f(x)\,dx + C_1 \quad (ii) \]

where \( C_1 \) is a constant of integration.

\[ \Rightarrow \quad v^2 = 2\psi(x) + 2C_1 \quad \text{i.e.,} \]

\[ \frac{dx}{dt} = v = \pm \sqrt{2\psi(x) - 2C_1} \]

\[ \int dt = \pm \int \frac{1}{\sqrt{2\psi(x) - 2C_1}}\,dx \]

\[ \Rightarrow \quad t = \pm \int \frac{dx}{\sqrt{2\psi(x) + 2C_1}} + C_2 \quad (iii) \]

where \( C_2 \) is another constant of integration.

Case (iii): Velocity-Dependent Acceleration

\[ a = f(v) \quad (i) \]

\[ \Rightarrow \quad \frac{dv}{dt} = a = f(v) \quad \text{i.e.,} \]

\[ v \cdot \frac{dv}{dx} = a = f(x) \]

Seperating the variable \( x \) and then integrating.
\[
\int t \, dt = \int \frac{dv}{f(v)} \quad \text{i.e.,}
\]
\[
t = \int \frac{dv}{f(v)} + c_1 \quad \text{i.e.,}
\]
where \( c_1 \) is the constant of integration.

\[& \quad \int dx = \int \frac{vdv}{f(v)} \quad \text{i.e.,}
\]
\[
x = \int \frac{vdv}{f(v)} + c_2
\]
where \( c_2 \) is another constant of integration.

**Graphical Solution of Rectilinear Problems:**

Area \( \int_{x_1}^{x_2} y \, dx \)

\[
v = \frac{dx}{dt}
\]

\[
\alpha = \frac{dv}{dt}
\]

\[
\therefore \quad v = \frac{dx}{dt} \Rightarrow dx = v \, dt
\]

Integrating above

\[
\int_{t_1}^{t_2} dx = \int_{t_1}^{t_2} v \, dt
\]

\[
[x]_{t_1}^{t_2} = \int_{t_1}^{t_2} v \, dt \quad \text{i.e.,}
\]

\[
t_2 - t_1 = \int_{t_1}^{t_2} v \, dt
\]

The slope of velocity-time curve of a particle
moving in a straight line gives its acceleration and area under curve denotes distance and area under curve denotes distance covered by particle during some interval.

# 8.6 SIMPLE HARMONIC MOTION :-

The particle is said to move with S.H.M if it moves in a straight line with an acceleration which is proportional to its distance from fixed pt. and is always directed toward fixed pt.

\[ X^\prime \quad O \quad X \]

Let \( ox \) be a straight line along which particle is moving and fixed pt. \( O \) on line can be taken as origin. Consider \( P \) the position of particle at any time \( t \), where \( OP = x \). So the acceleration at \( P \) is proportional to \( x \) and hence become \( 2kx \) in magnitude. It is known that acceleration is directed toward \( O \) and is in opposite direction in which \( x \) increases, so the equation of motion takes the form,

\[
\frac{dx}{dt^2} = -kx \quad (i)
\]

or \[
\frac{dv}{dx} = -kx \quad (ii)
\]

Separating the variable and integrating \( (ii) \)

\[
\int dx = -k \int x \, dx \quad \Rightarrow \quad \frac{v^2}{2} = -\frac{x^2}{2} + c_1 \quad (iii)
\]

where \( c_1 \) is a constant of integration.

The particle is moving away from \( O \) and acceleration is towards \( O \), i.e., opposite direction.

So its velocity becomes zero at some pt. \( A \) (say) where \( OA = a \), i.e., \( v = 0 \) at \( x = a \), put values in \( (iii) \)

So \( 0 = -k \frac{a^2}{2} + c_1 \quad \Rightarrow \quad c_1 = -\frac{ka^2}{2} \)
The Eq (iii) become:

\[ v^2 = \frac{-\lambda}{2} \cdot \frac{a^2 - x^2}{x} + \frac{\lambda a}{2} \]

\[ v^2 = \lambda \left( a^2 - x^2 \right) \frac{2}{x} \]

\[ v = \pm \sqrt[4]{\lambda \left( a^2 - x^2 \right)} \]

It gives velocity at any displacement \( x \).

If particle is moving towards right and as \( t \) increases, \( x \) also increases so \( \frac{dx}{dt} \) is true.

and we get

\[ v = \frac{dx}{dt} = \sqrt[4]{\lambda \left( a^2 - x^2 \right)} \]  \( \text{(v)} \)

Separating the variables and then integrating

\[ \int \frac{dx}{\sqrt[4]{\lambda \left( a^2 - x^2 \right)}} = \int \sqrt[4]{\lambda} \cdot 1 \cdot dt \]

\[ \Rightarrow \sin^{-1} \frac{x}{a} = \sqrt[4]{\lambda} \cdot t + c_2 \]  \( \text{(vi)} \)

where \( c_2 \) is another constant of integration.

Applying condition i.e. \( t = 0 \) if time is measured from instant when particle is at \( A \) where \( x = a \)

\[ \sin^{-1} \left( \frac{x}{a} \right) = \sqrt[4]{\lambda} \cdot 0 + c_2 = c_2 \]

\[ c_2 = \sin^{-1} \frac{a}{a} = \frac{\pi}{2} \]

Put value of \( c_2 \) in \( \text{vi} \), so

\[ \sin^{-1} \frac{x}{a} = \sqrt[4]{\lambda} \cdot t + \frac{\pi}{2} \]

\[ \Rightarrow \frac{x}{a} = \sin \left( \sqrt[4]{\lambda} \cdot t + \frac{\pi}{2} \right) = \cos \sqrt[4]{\lambda} \cdot t \]

If \( t \) is measured from fixed pt. \( 0 \)

\[ x = 0 \text{ at } t = 0 \]

\[ \sin^{-1} \frac{x}{a} = \sqrt[4]{\lambda} \cdot 0 + c_2 = c_2 \]

\[ c_2 = \sin^{-1} 0 = 0 \]

Putting \( c_2 = 0 \) in \( \text{vi} \) we have \n
\[ \sin^{-1} \frac{x}{a} = \sqrt[4]{\lambda} \cdot t + 0 = \sqrt[4]{\lambda} \cdot t \]

\[ \Rightarrow \frac{x}{a} = \sin \sqrt[4]{\lambda} \cdot t \]

\[ x = a \sin \sqrt[4]{\lambda} \cdot t \]  \( \text{(vii)} \)

The Equation (vii) & (viii) give displacement of particle from fixed pt. \( 0 \) according as time is measured from end pt. or fixed pt. \( 0 \).
8-1 NATURE OF S.H.M:

If the particle is at pt. A i.e. \( t = 0 \), so S.H.M is given by

\[ x = a \cos \sqrt{\frac{a}{k}} t \] (i)

Differentiate \( w.r.t. t \),

\[ \frac{dx}{dt} = a(- \sin \sqrt{\frac{a}{k}} t) \sqrt{\frac{a}{k}} t \]

\[ \Rightarrow \frac{dx}{dt} = v = -a \sqrt{\frac{a}{k}} \sin \sqrt{\frac{a}{k}} t \] (ii)

Now, the distance of particle at any time \( t \) is given by

\[ x = a \cos \sqrt{\frac{a}{k}} t \]

\[ = a \cos (\sqrt{\frac{a}{k}} t + 2\pi) \quad \Rightarrow \cos \theta = \cos (2\pi + \theta) \]

\[ = a \cos (\sqrt{\frac{a}{k}} (t + 4\pi)) \]

\[ \text{or} \quad x = a \cos \sqrt{\frac{a}{k}} (t + 2\pi) \]

\[ = a \cos \sqrt{\frac{a}{k}} (t + 4\pi) \]

It shows that after time \( t + 2\frac{2\pi}{\sqrt{\frac{a}{k}}} \), \( t + 4\frac{2\pi}{\sqrt{\frac{a}{k}}} \), \ldots is same as at time \( t \). It means particle occupied some position after every \( 2\frac{2\pi}{\sqrt{\frac{a}{k}}} \) sec. now

\[ \frac{dx}{dt} = -a \sqrt{\frac{a}{k}} \sin (t + 2\pi) \]

\[ = -a \sqrt{\frac{a}{k}} \sin (\sqrt{\frac{a}{k}} t + 4\pi) \] (iii)

\[ \text{or} \quad \frac{dx}{dt} = -a \sqrt{\frac{a}{k}} \sin \sqrt{\frac{a}{k}} (t + \frac{4\pi}{\sqrt{\frac{a}{k}}} \) \]

\[ \Rightarrow \text{at time } t + 2\frac{2\pi}{\sqrt{\frac{a}{k}}} \]

\[ \Rightarrow \text{time period of oscillation (or motion) and denoted by } T \]

Thus we find that if at some time \( t \), the particle is at some pt. moving with velocity \( V \) in some direction, so after \( 2\frac{2\pi}{\sqrt{\frac{a}{k}}} \) units of time, it is again at \( t \) same pt. moving with same velocity \( V \) in same direction. Therefore, the motion is such that is repeats itself after \( 2\frac{2\pi}{\sqrt{\frac{a}{k}}} \) unit of time and time is known as time period of oscillation (or motion) and denoted by \( T \).

Hence the particle oscillates once about pt. A in...
in the term

\[ T = \frac{2\pi}{\lambda} \]  

The particle moves with \( x = a \) & \( x = -a \) so displacement of particle on either side of fixed pt O is called amplitude.

The number of vibrations completed by particle in a unit of time is called frequency denoted by \( \nu \)

Thus if \( \nu \) is frequency so

\[ \nu \cdot T = 1 \]  

\[ \therefore \nu = \frac{1}{T} = \frac{\sqrt{3}}{2\pi} \]  

# 8.8 GEOMETRICAL REPRESENTATION:

The motion of particle with uniform speed along a circle along a circle has relation with S.H.M. as motion is repeated every time the path has been directed completely.

Let a particle \( Q \) be moving along circle of radius \( a \) with uniform speed \( v \) so its angular velocity is \( w = \frac{v}{a} \) The particle moves around circle once in \( \frac{2\pi}{w} \) units of time, whereas \( OP \) is projection of \( Q \) on \( x \)-axis passing through centre \( O \) of circle.

The particle \( Q \) repeats its motion after every \( \frac{2\pi}{w} \) units of time and motion of \( P \) is also periodic having period \( \frac{2\pi}{w} \). The acceleration of \( P \) is same as that of \( Q \) \( \parallel \) to \( x \)-axis. The particle has acceleration \( a_{x_0} \) along \( \overrightarrow{OP} \) so it can be expressed as \( w^2 \overrightarrow{a_0} \).

\[ w^2 \overrightarrow{a} = w^2 (\overrightarrow{OP} + \overrightarrow{PO}) \]  

\[ \therefore w^2 \overrightarrow{a} = 0 \]
So acceleration of P is \(\omega^2 \vec{r}_0\), i.e. acceleration of P is proportionate to its distance from fixed pt (i.e. centre of circle) and is directed toward centre.

Hence P executes S.H.M. whose time period is \(\frac{2\pi}{\omega}\). 

Let \(\vec{r}_0 P = x_0 \hat{a} = \theta\) & \((x,y)\) be the cartesian co-ordinates of pt \(P\), so \(\frac{OP}{OA} = \cos \theta \Rightarrow \frac{x}{a} = \cos \theta\) \(\Rightarrow \theta\).

\[x = a \cos \theta \quad \text{and} \quad y = a \sin \theta\]

The pt \(P\) lies on \(x\)-axis so its co-ordinate are \((x,0)\) or \((a \cos \theta,0)\)

Then velocity and acceleration of P along \(x\)-axis are given by

\[
\dot{x} = \frac{dx}{dt} \quad \omega (-\sin \theta) \quad \dot{\omega} = -a \omega^2 \sin \theta.
\]

\[a \cos \theta \quad \frac{d^2x}{dt^2} = \frac{d}{dt} \left( \frac{dx}{dt} \right) \frac{d\theta}{dt} = \frac{d^2x}{d\theta^2} \frac{d\theta}{dt} \]

\[\Rightarrow \ddot{x} = \frac{d^2x}{dt^2} = -a \omega^2 \cos \theta \quad \frac{d\theta}{dt} = -a \omega^2 \cos \theta \Rightarrow -\omega^2 (a \cos \theta).
\]

\[\dddot{x} = -\omega^4 x \quad \text{(iii)}
\]

The last equation implies that P executes S.H.M. about \(O\) with time period \(\frac{2\pi}{\omega}\).

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