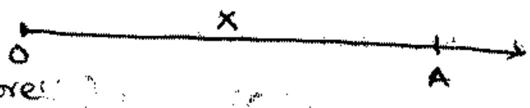
Monday

Chapter : VIII

RECTILINEAR MOTION

81 RECTILINEAR MOTION:

The motion of a body along a straight line is called its rectilinear motion.



If the motion is rectilinear so there is NO distinction between vector Equation & scalar equation.

i.e $\underline{v} = \frac{dx}{dt}\underline{i}$ ——— (i)

Can be expressed as:

$$v = \frac{dx}{dt} \text{ ——— (ii)}$$

and $\underline{a} = \frac{dv}{dt}\underline{i} = \frac{d^2x}{dt^2}\underline{i}$ ——— (iii)

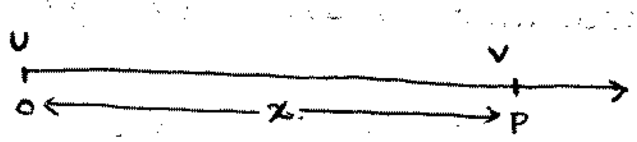
Can be denoted as. $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ ——— (iv)

If v is consider as function of x so

$$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\cdot\frac{dv}{dx}$$

MOTION WITH CONSTANT ACCELERATION:

Let a particle moving with constant acceleration along a st. line.



Let at time $t=0$ the particle will be at pt. O and after some time it velocity become v so

accelration $a = \frac{dv}{dt}$ ——— (i)

$$\Rightarrow dv = a\,dt \quad \text{(separating the variable)}$$

Integrating w.r.t t.

$$\int dv = \int a\,dt \text{ — (i.e.}$$

$$v = at + c_1 \text{ ——— (ii)}$$

where c_1 is a constant of integration

Applying condition when $t=0$ it velocity $v=u$

so $u = a\cdot 0 + c_1 \Rightarrow u = c_1$

So Equation (ii) become.

$$\boxed{v = at + u} \text{ ——— (iii)}$$

We can write Equation (iii)

$$v = \frac{dx}{dt} = u + at$$

Seperating the variable $dx = (u + at)\,dt$

Integrating above

$$\int dx = \int (u + at)\,dt = \int u\,dt + a\int t\,dt$$

$$\Rightarrow \quad x = u \cdot t + \frac{at^2}{2} + c_2 \quad \text{......(iv)}$$

where c_2 is another constant of integration.

$x = 0$ when $t = 0$

$$\therefore \quad 0 = u \cdot 0 + \frac{a}{2} \cdot 0^2 + c_2 \Rightarrow c_2 = 0$$

Thus equation (iv) takes form.

$$\boxed{x = ut + \frac{1}{2}at^2} \quad \text{——— (v)}$$

$$\because \quad a = \frac{dv}{dt} = v \cdot \frac{dv}{dx} \quad \text{——— (vi)}$$

$\Rightarrow$ Separating the variable.

$$v\,dv = a\,dx$$

Integrating above $\Rightarrow \int v\,dv = \int a\,dx$

$$\Rightarrow \quad \frac{v^2}{2} = ax + c_3 \quad \text{——— (vii)}$$

where c_3 is a constant of integration

$\because \quad v = u$ at $x = 0$

$$\therefore \quad \frac{u^2}{2} = a \cdot 0 + c_3 = c_3$$

Thus Equation (vii) become.

$$\frac{v^2}{2} = ax + \frac{u^2}{2}$$

$$\Rightarrow \quad v^2 = 2ax + u^2$$

$$\boxed{v^2 - u^2 = 2ax} \quad \text{——— (viii)}$$

Now if the particle starts from rest

So $\quad u = 0$

$$v = u + at = 0 + at$$

$$\Rightarrow \quad v = at \quad \text{——— (ix)}$$

$\because \quad x = ut + \frac{1}{2}at^2 = 0\cdot t + \frac{1}{2}at^2$

$\Rightarrow \quad x = \frac{1}{2}at^2$ —— (x)

& $\because \quad v^2 - u^2 = 2ax$

$v^2 - 0 = 2as$

$\Rightarrow \quad v^2 = 2ax$ —— (xi)

If the particle moves with retardation a' we replace a by $-a'$ so

$v = u - a't$ So —— (xii)

$v = ut - \frac{1}{2}a't'$ —— (xiii)

$v^2 = u^2 - 2a'x$ —— (xiv)

The distance covered in nth second is given by

$x_n - x_{n-1} = u + \frac{1}{2}(2n-1)a$ —— (i)

$\because x_n = un + \frac{1}{2}an^2$

$x_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2$

$\Rightarrow$

$x_n - x_{n-1} = \cancel{un} - \frac{1}{2}an^2$

$- \cancel{un} + u - \frac{1}{2}an^2 - 2an + a$

$= u + \frac{1}{2}a(\cancel{n^2} - \cancel{n^2} + 2n - 1)$

$= u + \frac{1}{2}a(2n-1)$

VERTICAL MOTION UNDER GRAVITY:

i) Down ward motion :-

If the bodies fall freely from rest ?

so $\quad v = gt$ —— (i)

$x = \frac{1}{2}gt^2$ —— (ii)

$v^2 = 2gx$ —— (iii)

(ii) Upward Motion:

If the body projected vertically upward with initial velocity u so body moves ~~un~~ with retardation and equation of ~~un~~ motion take form

$v = u - gt$ —— (i)

$x = ut - \frac{1}{2}gt^2$ —— (ii)

$\left.\begin{array}{l} v^2 - u^2 = -2gx \\ \Rightarrow v^2 = u^2 - 2gx \end{array}\right\}$ —— (iii)

~~Mat~~ # 8.3 MOTION WITH VARIABLE ACCELERATION:

Case (i) Time - Dependent Acceleration

$a = f(t)$ —— (i)

$\therefore \quad \frac{dv}{dt} = a = f(t)$

$\Rightarrow$ Seperating the variable & then integrating.

$\int dv = \int f(t)\,dt$

$\Rightarrow \quad v = \int f(t)\,dt + c_1$ —— (ii)

where c_1 is a constt. of integration.

$\Rightarrow$ put $\int f(t)\,dt = \phi(t)$

$\therefore \quad v = \phi(t) + c_1$ —— i.e.

$\frac{dx}{dt} = \phi(t) + c_1$

Separating the variable & integrating.

$\int dx = \int [\phi(t) + c_1]\,dt$ — i.e.

$x = \int \phi(t)\,dt + c_1 t + c_2$ —— (iii)

where c_2 is another constt. of integration

The values of c_1 & c_2 can be determine by applying initial Conditions of motion.

Case (ii): Distance - Dependent acceleration.

$a = f(x)$ —— (i)

$\Rightarrow \quad v \cdot \frac{dv}{dx} = a = f(x)$ —— i.e.

Seperating the variable and integrating

$\int v \cdot dv = \int f(x)\,dx$

$\Rightarrow \quad \frac{v^2}{2} = \int f(x)\,dx + c_1$ —— (ii)

where c_1 is a constant of integration.

put $\int f(x)\,dx = \psi(x)$

$\therefore \quad \frac{v^2}{2} = \psi(x) + c_1$

$\Rightarrow \quad v^2 = 2\psi(x) + 2c_1$ —— i.e.

$\frac{dx}{dt} = v = \pm\sqrt{2\psi(x) - 2c_1}$

Seperating the variables & then integrating

$\int dt = \pm \int \frac{1}{\sqrt{2\psi(x) - 2c_1}}\,dx$

$\Rightarrow \quad t = \pm \int \frac{dx}{\sqrt{2\psi(x) + 2c_1}} + c_2$ —— (iii)

where c_2 is another constant of integration.

Case (iii) Velocity - Dependent acceleration:

$a = f(v)$ —— (i)

$\Rightarrow \quad \frac{dv}{dt} = a = f(x)$ ——

& $\quad v \cdot \frac{dv}{dx} = a = f(x)$

Separating the variable & then integrating.

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$$\int 1 \cdot dt = \int \frac{dv}{f(v)} \quad \text{— i.e.}$$

$$t = \int \frac{dv}{f(v)} + c_1 \quad \text{— (ii)}$$

where c_1 is the constant of integration

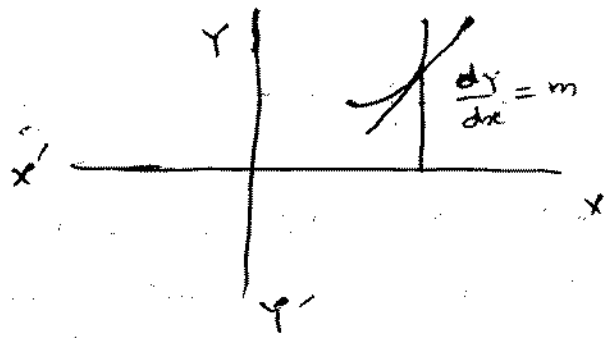
& $$\int dx = \int \frac{v\,dv}{f(v)} \quad \text{— i.e.}$$

$$x = \int \frac{v\,dv}{f(v)} + c_2$$

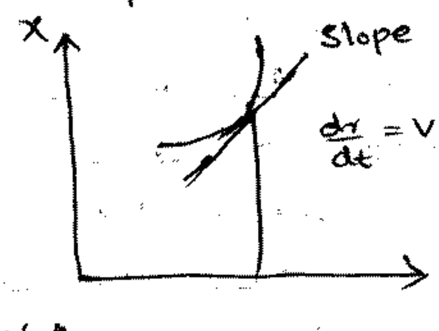
where c_2 is another constant of integration

GRAPHICAL SOLUTION OF RECTLINEAR PROBLEMS:

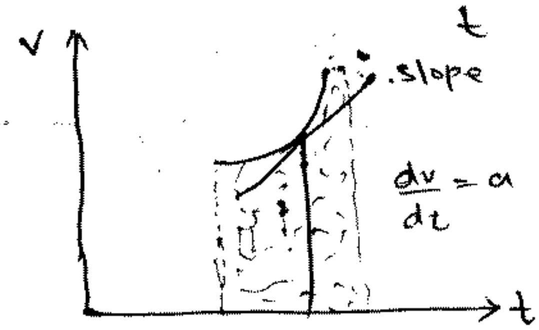
$$\text{Area} = \int_{x_1}^{x_2} Y\,dx$$



$$v = \frac{dx}{dt}$$



$$a = \frac{dv}{dt}$$



$$\because \quad v = \frac{dx}{dt} \Rightarrow dx = v\,dt$$

Integrating above

$$\int_{t_1}^{t_2} dx = \int_{t_1}^{t_2} v \cdot dt$$

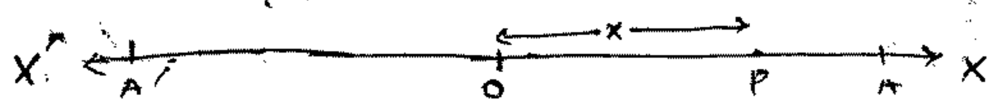
$$|x|_{t_1}^{t_2} = \int_{t_1}^{t_2} v\,dt \quad \text{— i.e.}$$

$$t_2 - t_1 = \int_{t_1}^{t} v\,dt$$

The slope of velocity-time curve of a particle moving in a st. line gives its acceleration and area under curve denotes distance and area under curve denotes distance covered by particle during some interval.

8.6 SIMPLE HARMONIC MOTION :-

The particle is said to move with S.H.M if it moves in a st. line with an acceleration which is proportional to its distance from fixed pt. and is always directed toward fixed pt. on line.



Let xox' be a st. line along which particle is moving and fixed pt. O on line can be taken as origin. Consider P the position of particle at any time t, where as $OP = x$. so the acceleration at P is proportional to x and hence become λx in magnitude. It is known that acceleration is directed toward O and is in opposite direction in which x increases, so the equation of motion taken form.

$$\frac{d^2x}{dt^2} = -\lambda x \quad \text{——} \quad (i)$$

$$\text{or } \; v\frac{dv}{dx} = -\mu x \quad \text{——} \quad (ii)$$

or $-\lambda x$

Separating the variable and integrating ii)

$$\int v\,dx = -\lambda \int x\,dx \quad \text{——} \quad \Rightarrow \frac{v^2}{2} = -\frac{x^2}{2} + c_1 \quad \text{——} (iii)$$

where c_1 is a constant of integration.

∵ The particle is moving away from O and acceleration is towards O i.e. opposite direction.

So its velocity becomes zero at some pt A (say) where $OA = a$ i.e. $v = 0$

at $x = a$ put values in (iii)

So $0 = -\lambda \frac{a^2}{2} + c_1 \Rightarrow c_1 = \frac{\lambda a^2}{2}$

$\Rightarrow$ The Eq (iii) become.

$$\frac{v^2}{2} = -\lambda \frac{x^2}{2} + \frac{\lambda a^2}{2}$$

$$\Rightarrow \quad v^2 = \lambda(a^2 - x^2) \text{ ——— i.e.}$$

$$v = \pm\sqrt{\lambda(a^2-x^2)}$$

It gives velocity at any displacement x.
$\because$ particle is moving towards right and as t increases, x also increases so $\frac{dx}{dt}$ is +ive and we get

$$v = \frac{dx}{dt} = \sqrt{\lambda}\sqrt{a^2-x^2} \text{ —— (v)}$$

Separating the verrables and then integrating

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\lambda}\int 1 \cdot dt$$

$$\Rightarrow \quad \sin^{-1}\frac{x}{a} = \sqrt{\lambda}\, t + c_2 \text{ —— (vi)}$$

where c_2 is another constant of integration.

Applying condition i.e $t=0$ If time is measured from distant when particle is at A where $x=a$

$$\sin^{-1}\left(\frac{a}{a}\right) = \sqrt{\lambda}\cdot 0 + c_2 = c_2$$

$$c_2 = \sin^{-1} 1 = \pi/2$$

Put value of c_2 in vi) So

$$\sin^{-1}\frac{x}{a} = \sqrt{\lambda}\, t + \pi/2$$

$$\frac{x}{a} = \sin\left(\sqrt{\lambda}\, t + \pi/2\right) = \cos\sqrt{\lambda}\, t$$

$$\Rightarrow \quad x = a\cos\sqrt{\lambda}\, t \text{ ——— (vii)}$$

If t is measured from fixed pt. O.

$$x = 0 \quad \text{at} \quad t = 0$$

$$\sin^{-1}\frac{0}{a} = \sqrt{\lambda}\cdot 0 + c_2 = c_2$$

$$c_2 = \sin^{-1} 0 = 0$$

Putting $c_2 = 0$ in vi) we have

$$\sin^{-1}\frac{x}{a} = \sqrt{\lambda}\, t + 0 = \sqrt{\lambda}\, t$$

$$\Rightarrow \quad \frac{x}{a} = \sin\sqrt{\lambda}\, t \text{ — i.e.}$$

$$x = a\sin\sqrt{\lambda}\, t \text{ ——— (viii)}$$

The Equation vii) & viii) give displacement of particle from fixed pt. O according as time is measured from end pt. or fixed pt. O.

8.7 NATURE OF S.H.M :-

If the particle is at pt. A i.e. $x = a$ & $t = 0$ so S.H.M is given by

$$x = a\cos\sqrt{\lambda} \quad \text{—— (i)}$$

Differentiate w.r.t. t

$$\frac{dx}{dt} = a(-\sin\sqrt{\lambda}t)\sqrt{\lambda}.t$$

$$\Rightarrow \quad \frac{dx}{dt} = v = -a\sqrt{\lambda}\sin\sqrt{\lambda}t \quad \text{—— (ii)}$$

Now the distance of particle at any time t is given by

$$x = a\cos\sqrt{\lambda}t$$
$$= a\cos(\lambda t + 2\pi) \qquad \because \cos\theta = \cos(2\pi \pm \theta)$$
$$= a\cos(\sqrt{\lambda}t + 4\pi)$$

or $x = a\cos\sqrt{\lambda}\left(t + \frac{2\pi}{\lambda}\right)$

$$= a\cos\sqrt{\lambda}\left(\frac{\sqrt{\lambda}}{t} + \frac{4\pi}{\sqrt{\lambda}}\right)$$

It showing that after time $t + \frac{2\pi}{\sqrt{\lambda}}$, $t + \frac{4\pi}{\sqrt{\lambda}}$, ---- is same as at time t. It means particle occupied same position after every $\frac{2\pi}{\sqrt{\lambda}}$ sec. now

$$\frac{dx}{dt} = -a\sqrt{\lambda}\sin\sqrt{\lambda}t = -a\sqrt{\lambda}\sin(\lambda t + 2\pi)$$
$$= -a\sqrt{\lambda}\sin(\sqrt{\lambda}t + 4\pi) \cdots$$

or $\frac{dx}{dt} = -a\sqrt{\lambda}\sin\sqrt{\lambda}\left(t + \frac{2\pi}{\lambda}\right) = -a\sqrt{\lambda}\sin\sqrt{\lambda}\left(t + 4\pi/\sqrt{\lambda}\right) = \cdots$

after time $t + 2\pi/\sqrt{\lambda}$, $t + 4\pi/\sqrt{\lambda}$ ---- is same as at time t.

Thus we find that if at some time t, the particle is at some pt. moving with velocity V in some direction, so after $\frac{2\pi}{\sqrt{\lambda}}$ units of time, it is again at same pt moving with same velocity V in same direction $\therefore$ Therefore the motion is such that it repeats itself after $\frac{2\pi}{\sqrt{\lambda}}$ unit of time and time is know as time periode of oscilation (or motion) and denoted by T.

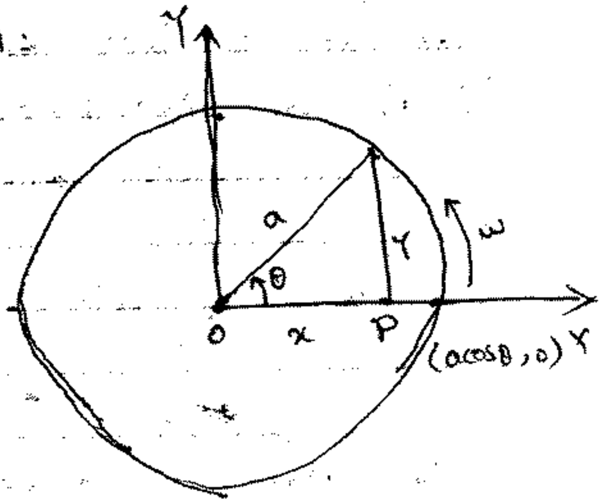
Hence the particle oscillates once about pt O in

in the time
$$T = \frac{2\pi}{\sqrt{\lambda}} \quad \text{---------- (i)}$$
The particle moves with $x = a$ & $x = -a$ so displacement of particle on either side of fixed pt O is called amplitude.

The number of vibration completed by particle in a unit of time is called frequency denoted by ν

Thus if ν is frequency so
$$\nu \cdot T = 1 \quad \text{------ (ii)}$$
$$\Rightarrow \quad \nu = \frac{1}{T} = \frac{\sqrt{\lambda}}{2\pi} \quad \text{------ (iii)}$$

8.8 GEOMETRICAL REPRESENTATION:



$\because \; v = r\omega$

$\Rightarrow \; \omega = \frac{v}{r}$

& $s = vt = r\omega t = 2\pi r$

$r\omega t = 2\pi r \Rightarrow \omega t = 2\pi$

$t = \frac{2\pi}{\omega}$ or $\omega = \frac{2\pi}{t}$

The motion of particle with uniform speed along a circle along a circle has relation with S.H.M. as motion is repeated every time the path has been directed completely.

Let a particle Q be moving along circle of radius a with uniform speed v so its angular velocity is $w = \frac{v}{a}$. The particle moves around circle once in $\frac{2\pi}{\omega}$ units of time, where as P is projection of Q on x-axis passing through centre O of circle.

The particle Q repeats its motion after every $\frac{2\pi}{\omega}$ units of time and motion of P is also periodic having period $\frac{2\pi}{\omega}$. The acceleration of P is same as that of Q ∥ to x-axis. The particle has acceleration $a\omega^2$ along $\overrightarrow{QO}$ so it can be expressed as $\omega^2 \overrightarrow{QO}$.

$$\therefore \omega^2 \overrightarrow{QO} = \omega^2 (\overrightarrow{QP} + PO) \quad \text{------ (i)}$$

So acceleration of P is $\omega^2 \vec{PO}$. i.e Acceleration of P is proportionate to its distance from Fixed pt (i.e centre of circle) and is directed toward centre.

Hence P executes S.H.M whose time period is $\frac{2\pi}{\omega}$ & centre of motion is O.

Let $\vec{Q\hat{O}P} = x\hat{O}Q = \theta$ & (x,y) be the cartesian co-ordinates of pt Q so

$$\frac{OP}{OQ} = \cos\theta \Rightarrow \frac{x}{a} = \cos\theta \text{ — i.e.}$$

$$x = a\cos\theta \quad \& \quad \frac{PQ}{OQ} = \sin\theta$$

$$\frac{y}{a} = \sin\theta \quad \text{i.e} \quad y = a\sin\theta.$$

The pt P lies on x-axis so its co-ordinate are $(x,0)$ or $(a\cos\theta, 0)$

Then velocity and acceleration of P along x-axis are given by

$$\dot{x} = \frac{dx}{dt}\ a(-\sin\theta)\,\omega = -a\omega\sin\theta.$$

$$\because \frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} = \frac{dx}{d\theta}\cdot\omega$$

$$\Rightarrow \ddot{x} = \frac{d^2x}{dt^2} = -a\omega\cos\theta\,\frac{d\theta}{dt}$$

$$= -a\omega^2\cos\theta = -\omega^2(a\cos\theta).$$

$$\ddot{x} = -\omega^2 x \text{ —— (iii)}$$

The last Equation implies that P executes S.H.M about O with time period $\frac{2\pi}{\omega}$,