CHAPTER No. 8

Rectilinear motion

The motion of a body along a straight line is called rectilinear motion.

Motion with constant acceleration

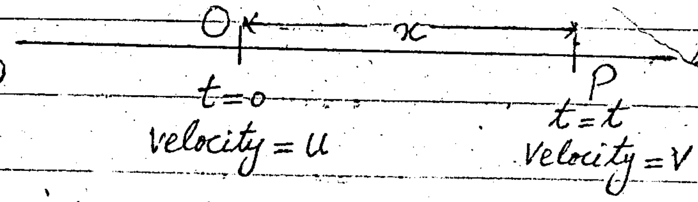
Question

Drive the expression for velocity and distance covered by a particle in time 't' when it is moving with constant acceleration.

Solution

Considering a particle which is moving with constant acceleration along a straight line.

Let at $t=0$ the particle is at point O and is moving with velocity u. After a time 't' the particle is at point 'P' at a distance 'x' from 'O' and attains a velocity v.



we know that

$$\frac{dx}{dt} = v$$

diff. w.r.t. 't'

$$\frac{d^2x}{dt^2} = \frac{dv}{dt}$$

$$\frac{d^2x}{dt^2} = a \qquad \because \frac{dv}{dt} = a \text{ (acceleration)}$$

Integrating w.r.t. 't'

$$\int \frac{d^2x}{dt^2}\, dt = a\int dt$$

$$\frac{dx}{dt} = at + A$$

$$V = at + A \quad \left(\because \frac{dx}{dt} = V\right) \quad \text{——— (1)}$$

Initially when $t = 0$, $V = u$.

So $\quad u = a(0) + A$

$\implies A = u$

From (1) we have

$$V = at + u$$

or $\quad V = u + at \quad$ is the velocity of the particle at any time 't'.

Again $\quad V = u + at$

then $\quad \frac{dx}{dt} = u + at$

$$dx = (u + at)\,dt$$

Integrating $\quad \int dx = \int (u + at)\,dt$

$$\int dx = u\int dt + a\int t\,dt$$

$$x = ut + a\frac{t^2}{2} + B \quad \text{——— (2)}$$

Initially when $t = 0$, $x = 0$

So, $\quad 0 = u(0) + a(0) + B$

$\implies B = 0$

So from (2) we have

$$x = ut + a\frac{t^2}{2} + 0$$

or $\quad x = ut + \frac{1}{2}at^2 \quad$ is the distance covered by the particle in time 't'.

Question

Drive the expression $v^2 - u^2 = 2ax$ where 'u' and 'v' are velocities of a particle at $t = 0$ and $t = t$ resp. 'a' is the acceleration and 'x' is the distance covered in time 't'.

<u>Solution</u>

we know that,

$$v = u + at \quad \text{———} (1)$$

$$x = ut + \frac{1}{2}at^2 \quad \text{———} (2)$$

Now from (1) $\quad t = \dfrac{v-u}{a}$

Using the value of t in (2)

$$x = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

$$x = \frac{u(v-u)}{a} + \frac{1}{2}a\frac{(v-u)^2}{a^2}$$

$\times 2a$

$$2ax = \cancel{2uv} - 2u^2 + v^2 + u^2 - \cancel{2uv}$$

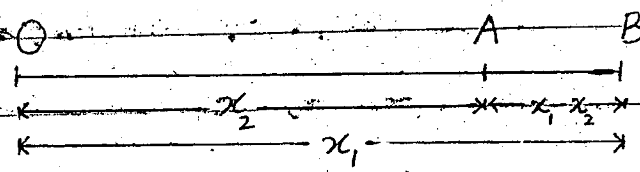
$$2ax = v^2 - u^2 \qquad \text{as required}$$

<u>Example P-151</u>

Find the distance travelled by the particle moving in a straight line with uniform acceleration, in the nth unit of time.

<u>Solution</u>

Considering a particle which is moving in a straight line with uniform acceleration



Let x_1 and x_2 be the distances travelled by the particle in the first n and $n-1$ units of time respectively.

Then by using the formula

$$x = ut + \frac{1}{2}at^2$$

we have $\quad x_1 = un + \frac{1}{2}an^2 \quad \text{———} (1)$

$$x_2 = u(n-1) + \frac{1}{2}a(n-1)^2 \quad \text{———} (2)$$

Hence the distance travelled in nth unit of time is given by

$$x_1 - x_2 = \left(un + \frac{1}{2}an^2\right) - \left(u(n-1) + \frac{1}{2}a(n-1)^2\right)$$

$$= \cancel{un} + \frac{1}{2}an^2 - \cancel{un} + u - \frac{1}{2}a(n^2+1-2n)$$

$$= u + \frac{1}{2}a(\cancel{n^2} - \cancel{n^2} - 1 + 2n)$$

$$= u + \frac{1}{2}a(2n-1)$$

Motion with variable acceleration

(i) Time dependent acceleration

If acceleration of a particle depends upon time, then such an acceleration is called time dependent acceleration.

Since $\frac{d^2x}{dt^2} = a(t)$

Integrating both the sides

$$\frac{dx}{dt} = \int a(t)\,dt + A$$

or $\frac{dx}{dt} = b(t) + A$ where $b(t) = \int a(t)\,dt$

On integrating again we get

$$x = \int b(t)\,dt + At + B$$

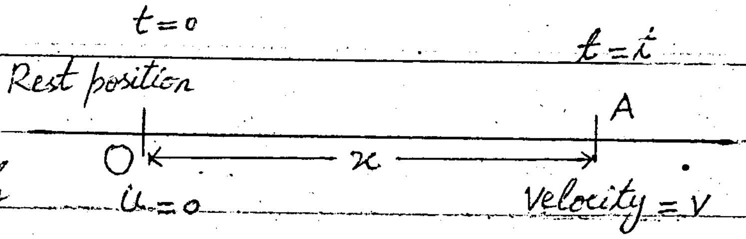
where the constants A and B are determined by using the boundary conditions.

Example P-153

Find the distance travelled and the velocity attained by a particle moving in a straight line, at any time 't' if it starts from rest at $t=0$ and is subject to an acceleration

$$t^2 + \sin t + e^t$$

<u>Solution</u>



Considering a particle which starts moving from rest from point 'O'. Let after a time 't' the particle reaches at point 'A' after covering a distance 'x' and its acceleration becomes $a = t^2 + \sin t + e^t$ and it attains the velocity 'V'.

Now $a = t^2 + \sin t + e^t$

$$\frac{dV}{dt} = t^2 + \sin t + e^t$$

$$dV = (t^2 + \sin t + e^t)\,dt$$

Integrating both the sides

$$\int dV = \int t^2 dt + \int \sin t\, dt + \int e^t dt$$

$$V = \frac{t^3}{3} - \cos t + e^t + A \quad \text{——— (1)}$$

Initially when $t=0$, $u = 0 = V$

$$0 = 0 - 1 + 1 + A$$

$$\Rightarrow A = 0$$

Put $A = 0$ in (1)

$V = \frac{1}{3}t^3 - \cos t + e^t$ is the velocity of the particle after time 't'.

Now

$$\frac{dx}{dt} = \frac{1}{3}t^3 - \cos t + e^t \qquad \because V = \frac{dx}{dt}$$

$$dx = \left(\frac{1}{3}t^3 - \cos t + e^t\right)dt$$

Integrating both the sides

$$\int dx = \frac{1}{3}\int t^3 dt - \int \cos t\, dt + \int e^t dt$$

$$x = \frac{1}{3}\frac{t^4}{4} - \sin t + e^t + B$$

$$x = \frac{1}{12}t^4 - \sin t + e^t + B \quad \text{——— (2)}$$

Initially when $t=0$, $x=0$

$$0 = 0 - 0 + 1 + B$$

$$\Rightarrow B = -1$$

Put $B=-1$ in (2)

$$x = \frac{1}{12}t^4 - \sin t + e^t - 1$$ is the distance travelled by the particle after time 't'.

(ii) Velocity dependent acceleration

If acceleration of a particle depends upon velocity, then such an acceleration is called velocity dependent acceleration.

Consider

$$v\frac{dv}{dx} = a(v)$$

or $$dx = \frac{v}{a(v)}dv$$

Integrating $$\int dx = \int \frac{v}{a(v)}dv$$

$$x = \int \frac{v}{a(v)}dv + C$$

acceleration $= \frac{dv}{dt}$

$$= \frac{dv}{dx}\cdot\frac{dx}{dt}$$

$$= \frac{dv}{dx}\cdot v$$

$$= v\frac{dv}{dx}$$

where Constant 'C' can be determined when velocity of the particle is known for some value of x.

Also $$\frac{dv}{dt} = a(v)$$

$$dt = \frac{dv}{a(v)}$$

$$\int dt = \int \frac{dv}{a(v)}$$

$$t = \int \frac{dv}{a(v)} + D$$

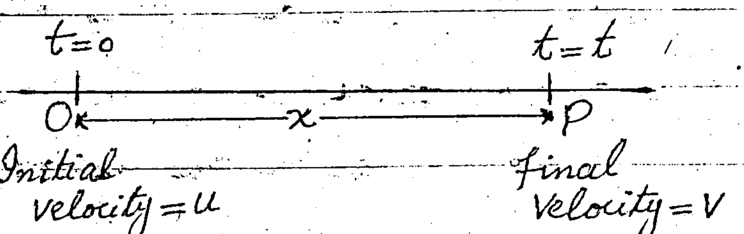
where Constant 'D' can be determined if the velocity at some instant is known.

Example P-154

A particle moves in a straight line with an acceleration kv^3. If its initial velocity is u, find the velocity and the time spent when the particle has travelled a distance 'x'.

Solution

Considering a particle which starts moving



from the point 'O' with initial velocity 'u'. Let after a time 't' the particle reaches at point 'P' after covering a distance 'x' and attains a velocity 'v'

Now

$$\text{Acceleration} = kv^3$$

$$v\frac{dv}{dx} = kv^3$$

$$\frac{dv}{v^2} = k\,dx$$

Integrating both the sides

$$\int v^{-2}\,dv = k\int dx$$

$$-\frac{1}{v} = kx + A \quad \text{———— (1)}$$

Initially when $x = 0$, $v = u$

$$-\frac{1}{u} = k(0) + A$$

$$\Rightarrow A = -\frac{1}{u}$$

Put $A = -\frac{1}{u}$ in (1)

$$-\frac{1}{v} = kx - \frac{1}{u}$$

$$\frac{1}{v} = \frac{1 - kux}{u} \quad \text{———— (2)}$$

So $v = \dfrac{u}{1-kux}$, is the velocity of the particle when it is at a distance 'x' from 'O'.

Again Acceleration $= kv^3$

$$\frac{dv}{dt} = kv^3$$

$$\frac{dv}{v^3} = k\,dt$$

Integrating both sides

$$\int v^{-3}dv = k\int dt$$

$$\frac{v^{-2}}{-2} = kt + B$$

$$-\frac{1}{2v^2} = kt + B \quad \text{———} (3)$$

Initially when $t=0$, $v=u$

$$-\frac{1}{2u^2} = k(0) + B$$

$$B = -\frac{1}{2u^2}$$

Put $B = -\frac{1}{2u^2}$ in (3)

$$-\frac{1}{2v^2} = kt - \frac{1}{2u^2}$$

$$kt = \frac{1}{2u^2} - \frac{1}{2v^2}$$

$$kt = \frac{1}{2}\left[\frac{1}{u^2} - \frac{1}{v^2}\right]$$

$$t = \frac{1}{2k}\left[\frac{1}{u^2} - \frac{1}{v^2}\right] \quad \text{———} (4)$$

using (2) in (4)

$$t = \frac{1}{2k}\left[\frac{1}{u^2} - \left(\frac{1-kux}{u}\right)^2\right]$$

$$= \frac{1}{2k}\left[\frac{1}{u^2} - \frac{(1-kux)^2}{u^2}\right]$$

$$= \frac{1}{2k}\cdot\frac{1}{u^2}\left[1-(1+k^2u^2x^2-2kux)\right]$$

$$= \frac{1}{2ku^2}(\cancel{1}-\cancel{1}-k^2u^2x^2+2kux)$$

$$= \frac{1}{2ku^2}\,kux(2-kux)$$

$t = \frac{x}{2u}(2-kux)$ is the time taken by the particle to cover a distance 'x'.

Distance dependent acceleration

If acceleration of a particle depends upon distance, then such an acceleration is called distance dependent acceleration.

Since $v\dfrac{dv}{dx} = a(x)$

then $v\,dv = a(x)\,dx$

Integrating $\int v\,dv = \int a(x)\,dx$

$$\frac{v^2}{2} = \int a(x)\,dx + C$$

$$\frac{v^2}{2} = f(x) + C \qquad \text{where } f(x) = \int a(x)\,dx$$

$\times 2$ $$v^2 = 2f(x) + 2C$$

$$v = \pm\sqrt{2f(x) + D} \qquad \text{where } 2C = D$$

and $$\frac{dx}{dt} = \pm\sqrt{2f(x)+D}$$

$$\Rightarrow \quad dt = \pm\frac{dx}{\sqrt{2f(x)+D}}$$

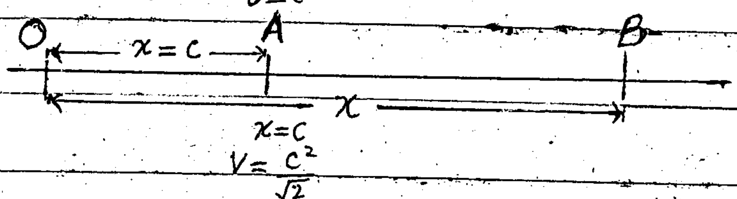
$$\int dt = \pm\int\frac{dx}{\sqrt{2f(x)+D}}$$

$$t = \pm\int\frac{dx}{\sqrt{2f(x)+D}} + E$$

Example Page-156

Discuss the motion of a particle moving in a straight line with an acceleration x^3 where 'x' is the distance of the particle from a fixed point 'O' on the line, if it starts at $t=0$ from a point $x = c$ with velocity $\dfrac{c^2}{\sqrt{2}}$.

Solution



Considering a particle which is moving in a straight line with an acceleration x^3

where 'x' is the distance of the particle from fixed point 'O' on the line. Let the particle starts motion from point 'A' at $t=0$ with velocity $\frac{c^2}{\sqrt{2}}$.

Here $$v\frac{dv}{dx} = x^3$$

$$v\,dv = x^3\,dx$$

Integrating $$\int v\,dv = \int x^3\,dx$$

$$\frac{v^2}{2} = \frac{x^4}{4} + C \quad \text{———} \quad (1)$$

Initially when $x = c$, $v = \frac{c^2}{\sqrt{2}}$

$$\frac{1}{2}\left(\frac{c^2}{\sqrt{2}}\right)^2 = \frac{c^4}{4} + D$$

$$\cancel{\frac{c^4}{4}} = \cancel{\frac{c^4}{4}} + D$$

$$D = 0$$

Put $C = 0$ in (1)

$$\frac{v^2}{2} = \frac{x^4}{4}$$

$\times 2$

$$v^2 = \frac{1}{2}x^4$$

$v = \frac{1}{\sqrt{2}}x^2$ is the velocity of the particle when it is at a distance x from point 'O'.

Again $$v = \frac{1}{\sqrt{2}}x^2$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{2}}x^2$$

$$\frac{\sqrt{2}}{x^2}\,dx = dt$$

Integrating $$\int dt = \int \frac{\sqrt{2}}{x^2}\,dx$$

$$t = -\frac{\sqrt{2}}{x} + D \quad \text{———} \quad (2)$$

Initially when $t = 0$, $x = c$

$$0 = -\frac{\sqrt{2}}{c} + D$$

$$D = \frac{\sqrt{2}}{c}$$

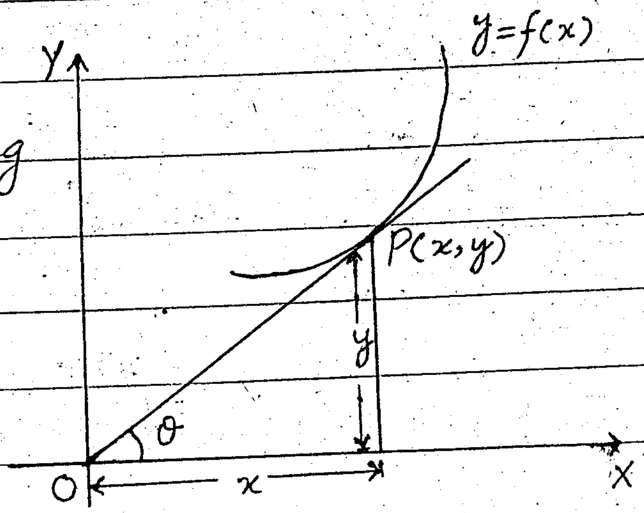
Put $D = \frac{\sqrt{2}}{c}$ in (2)

then $t = -\frac{\sqrt{2}}{x} + \frac{\sqrt{2}}{c}$

$t = \sqrt{2}\left(\frac{1}{c} - \frac{1}{x}\right)$ is the time spent by the particle to cover the distance 'x'.

Graphical Methods

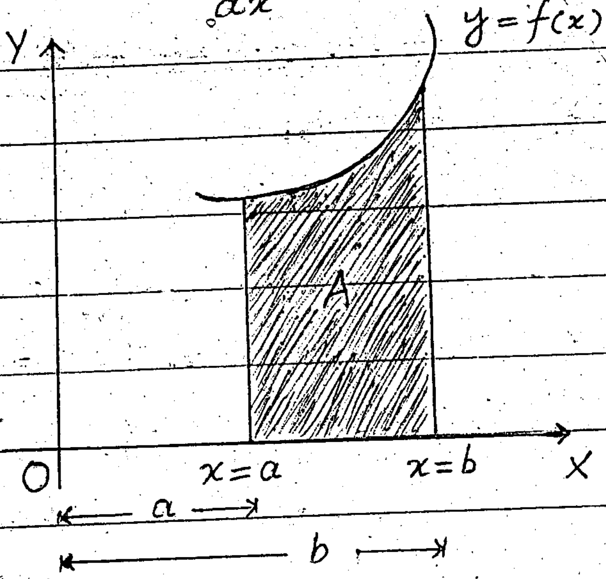
Considering a particle which is moving along the cartesian curve $y = f(x)$



Let at any instant the particle is at point $P(x,y)$. Draw a tangent to the curve at $P(x,y)$ then

Slope of the tangent line $= \frac{dy}{dx}$

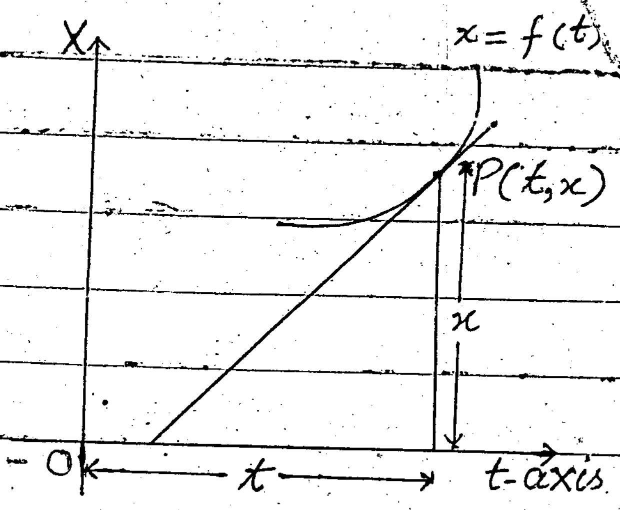
Let 'A' be the area under the curve $y = f(x)$ and in between the lines $x = a$ and $x = b$ then



$$A = \int_{x=a}^{x=b} y\, dx$$

Space time Curve

Considering a particle which is moving along the space time curve $x = f(t)$. Let at any instant the particle is at point $P(t, x)$.



Draw a tangent line to this curve at point P.

The slope of the tangent line $= \frac{dx}{dt}$

$= \text{velocity}$

$\therefore$ The velocity at any point P = Slope of the tangent line at the point P to space time curve

Clearly when a particle moves along a space time curve, then at any instant its velocity is given by the slope of the tangent line at that point.

Let the particle is moving with constant velocity

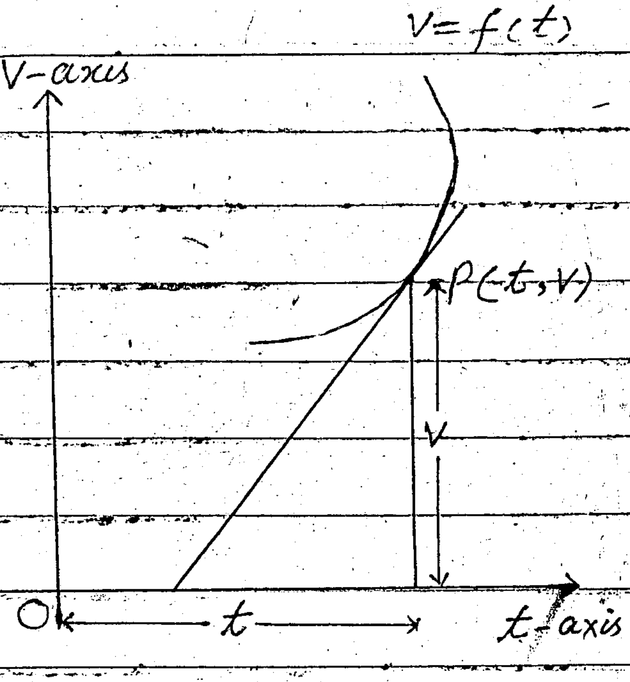
then $\frac{dx}{dt} = c$

$dx = c\,dt$

Integrating $\int dx = c \int dt$

$x = ct + A$ is the first degree equation in 'x' and 't' representing a straight line.

Clearly if a particle moves in a space time plane with constant velocity, its motion will be rectilinear motion.



Velocity time Curve

Considering a particle which is moving along the velocity time curve $V = f(t)$. Let at any instant 't' the particle is at point $P(t, V)$.

Draw a tangent line to this curve through the point $P(t, V)$, then

Slope of the tangent line $= \frac{dv}{dt}$

$=$ acceleration

$\therefore$ Acceleration at any Point P of the particle $=$ Slope of the tangent line at point P

Clearly when a particle moves along the velocity time curve, then at any point its acceleration is equal to the slope of the tangent line at that point.

Let the particle is moving with constant acceleration then $\frac{dv}{dt} = a$ (Constant)

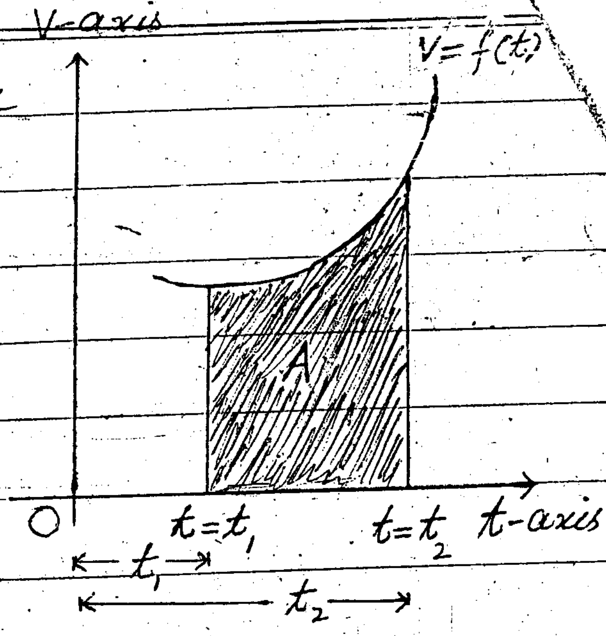
$$dv = a\,dt$$

Integrating $\int dv = a \int dt$

$V = at + B$ is the first degree equation in 'v' and 't' representing a st. line.

Clearly if a particle moves in a velocity time plane with constant acceleration, its motion will be rectilinear motion

Let 'A' be the area under the velocity time curve $V=f(t)$ then



$$A=\int v\,dt$$
$$=\int \frac{dx}{dt}\,dt$$
$$=\int dx$$
$$=|x|_{t_1}^{t_2}$$
$$= x(t_2)-x(t_1)$$

where $x(t_2)-x(t_1)$ is the distance travelled by by the particle in the time interval (t_1, t_2)

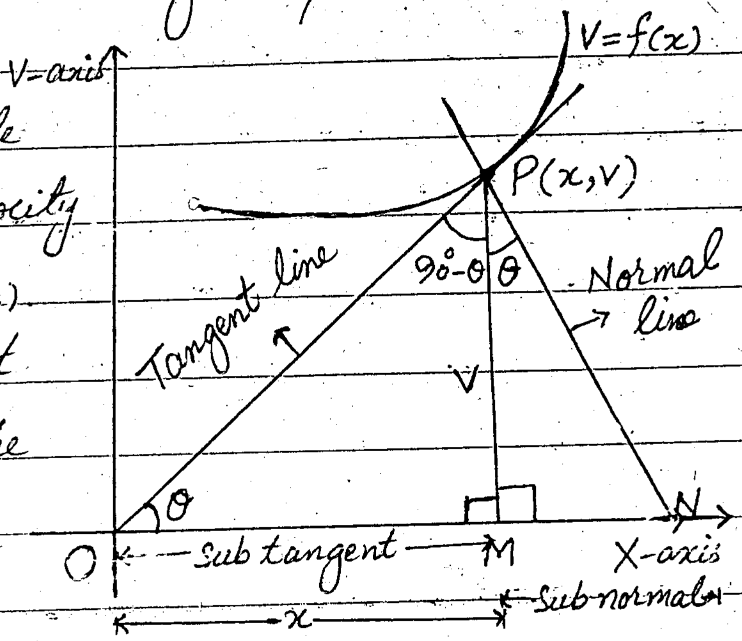
Velocity space curve

Question

Show that the length of the subnormal at any point of the velocity space curve gives the corresponding acceleration of the particle.

Solution

Considering a particle moving along the velocity space curve $V=f(x)$. Let at any instant the particle is at the point $P(x,V)$. Draw tangent and normal to this curve at point $P(x,V)$.



Slope of the tangent line $= \frac{dv}{dx} = \tan\theta$

Now from right angled triangle PMN

$$\frac{MN}{PM} = \tan\theta$$

$$MN = PM \cdot \tan\theta$$

Put $PM = V$, $\tan\theta = \frac{dv}{dx}$

$$MN = v\frac{dv}{dx}$$

Length of subnormal = acceleration

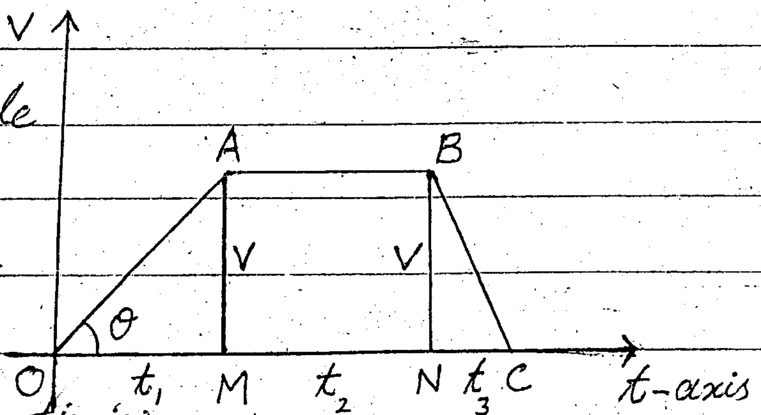
Clearly the length of the subnormal at any point of the velocity space curve, gives the corresponding acceleration of the particle.

Example P-158

A particle starts from rest from 'O' with constant acceleration 'a'. When its velocity acquires a certain value 'v' it moves uniformly and then its velocity starts decreasing with a constant retardation 2a till it comes to rest. Find the distance travelled by the particle, if the time taken from rest to rest is 't'.

Solution

Considering a particle that starts moving from rest from the point 'O' and move with constant acceleration 'a'.



and attains a velocity 'v' in time 't_1', then it moves uniformly for the time 't_2', then its velocity starts decreasing at the rate of '2a' and it comes to rest at point C after further time 't_3'
Let 't' be the time taken by the particle from rest to rest, then

$$t = t_1 + t_2 + t_3 \quad \text{———} \quad (1)$$

Now the slope of the line OA in the velocity time graph is the acceleration and the slope of the line BC is the retardation of the particle. Thus we have

$$\frac{v}{t_1} = a \quad \text{and} \quad \frac{v}{t_3} = 2a$$

Hence $t_1 = \frac{v}{a}$ and $t_3 = \frac{v}{2a}$

From (1)

$$t_2 = t - t_1 - t_3$$
$$= t - \frac{v}{a} - \frac{v}{2a}$$
$$t_2 = t - \frac{3v}{2a}$$

The distance travelled in time 't' is, therefore, given by

$$x = \Delta OAM + \text{Area of the rectangle } ABNM + \Delta BNC$$
$$= \frac{1}{2} t_1 v + t_2 v + \frac{1}{2} t_3 v$$
$$= \frac{1}{2} v \left(t_1 + 2t_2 + t_3 \right)$$
$$= \frac{1}{2} v \left(\frac{v}{a} + 2\left(t - \frac{3v}{2a}\right) + \frac{v}{2a} \right)$$
$$= \frac{1}{2} v \left(\frac{v}{a} + 2t - \frac{3v}{a} + \frac{v}{2a} \right)$$
$$= \frac{1}{2} v \left(2t - \frac{3v}{2a} \right) \text{ is the required distance.}$$

Example-1; P-160

A stone is let fall freely from a height of 100 ft. Find the time that it takes and the velocity that it acquires on reaching the ground.

Solution

Considering a stone that is dropped freely from a height of 100 ft.

For finding the final velocity we apply the formula

$$v^2 - u^2 = 2gs$$

Here $u = 0$, $g = 32\ ft/sec^2$, $x = 100\ ft$

$$v^2 - 0^2 = 2(32)(100)$$

$$v^2 = 6400$$

$$v = 80\ ft./sec$$

To find the time spent by the particle in reaching the ground we use the formula

$$v = u + gt$$

Here $v = 80\ ft./sec$, $u = 0$, $g = 32\ ft./sec$

$$80 = 0 + 32t$$

$$32t = 80$$

$\div 32$

$$t = \frac{80}{32} = \frac{5}{2}\ sec$$

Example-2; P-160

A particle projected vertically upwards at $t=0$ with a velocity u, passes a point at a height 'h' at $t=t_1$ and $t=t_2$. Show that

$$t_1 + t_2 = \frac{2u}{g} \text{ and } t_1 t_2 = \frac{2h}{g}$$

Solution

Considering a particle which is projected vertically upwards at $t=0$ with a velocity u, and passes a point at a height 'h' at $t=t_1$ and $t=t_2$. The distance 'x' travelled by the particle in time t, is given by the formula

$$x = ut - \frac{1}{2}gt^2$$

$$h = ut - \frac{1}{2}gt^2$$

$\times 2$

$$2h = 2ut - gt^2$$

$$gt^2 - 2ut + 2h = 0$$

It is quadratic equation in 't' so there will be two roots say t_1 and t_2

Here $a = g$, $b = -2u$, $c = 2h$

$\therefore$ Sum of the roots $= -\frac{b}{a}$

$$t_1 + t_2 = -\frac{(-2u)}{g}$$

$$= \frac{2u}{g}$$

and Product of the roots $= \frac{c}{a}$

$$t_1 t_2 = \frac{2h}{g}$$

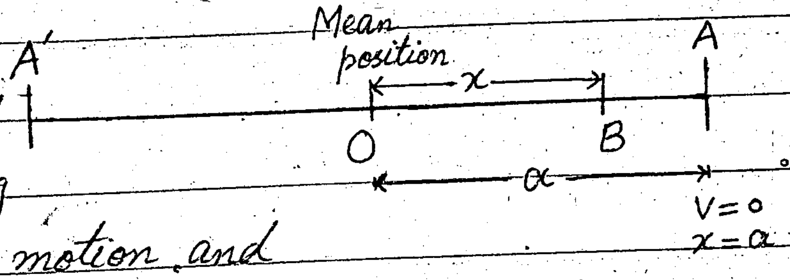
Simple Harmonic Motion

The motion of a body in straight line in which the acceleration of the body is directed towards its mean position and is directly proportional to its displacement from the mean position is called simple harmonic motion. The body performing simple harmonic motion is called simple harmonic oscillator (S.H.O).

Question

Derive an expression for the velocity and distance travelled by a particle at any instant, when it is executing Simple harmonic motion.

Solution



Considering a body which is performing simple harmonic motion and is moving away from the mean position. Let at any instant 't' the particle is at point B whose distance from the mean position is 'x'.

According to definition of simple harmonic motion.

$$\text{Acceleration} \propto -x$$

$$v\frac{dv}{dx} = -\lambda x$$

$$v\,dv = -\lambda x\,dx$$

where 'λ' is the constant of proportionality and the -ve sign indicates that the acc. is directed towards the mean position.

Integrating both the sides

$$\int v\,dv = -\lambda \int x\,dx$$

$$\frac{v^2}{2} = -\lambda \frac{x^2}{2} + C \quad \text{———— (1)}$$

when $x = a$, $v = 0$

$$0 = -\frac{\lambda a^2}{2} + C$$

$$\Rightarrow C = \frac{\lambda a^2}{2}$$

Put $C = \frac{\lambda a^2}{2}$ in (1)

$$\frac{v^2}{2} = -\frac{\lambda x^2}{2} + \frac{\lambda a^2}{2}$$

×2

$$v^2 = \lambda a^2 - \lambda x^2$$

$$V^2 = \lambda(a^2 - x^2)$$

$V = \sqrt{\lambda(a^2 - x^2)}$ is the velocity expression for S.H.O when it is at a distance 'x' from the mean position.

Now $V = \sqrt{\lambda}\,\sqrt{a^2 - x^2}$

$$\frac{dx}{dt} = \sqrt{\lambda}\,\sqrt{a^2 - x^2}$$

$$\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\lambda}\,dt$$

Integrating both the sides

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\lambda}\int dt$$

$$\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\,t + D \quad \text{——— (2)}$$

when $t = 0$, $x = a$

$$\sin^{-1}\left(\frac{a}{a}\right) = \sqrt{\lambda}\,(0) + D$$

$$\sin^{-1}(1) = D$$

$$\frac{\pi}{2} = D$$

Put $D = \frac{\pi}{2}$ in (2)

$$\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\,t + \frac{\pi}{2}$$

$$\frac{x}{a} = \sin\left(\frac{\pi}{2} + \sqrt{\lambda}\,t\right)$$

$$\frac{x}{a} = \cos\sqrt{\lambda}\,t$$

$x = a\cos\sqrt{\lambda}\,t$ is the distance travelled expression by S.H.O in time 't' (time is measured when the particle is at rest at point A)

Special Case Expression for the distance of S.H.O when time is measured from 'O' (Mean position)

From (2) we have

$$\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\, t + D$$

when $x = 0$, $t = 0$

$$\sin^{-1}(0) = \sqrt{\lambda}(0) + D$$

$$\Rightarrow D = 0 \qquad \because \sin^{-1}(0) = 0$$

we have

$$\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\, t$$

$$\frac{x}{a} = \sin\sqrt{\lambda}\, t$$

$x = a \sin\sqrt{\lambda}\, t$ is the distance travelled by S.H.O in time 't' (when time is measured from 'O').

The nature of Simple Harmonic Motion

Amplitude of motion

The maximum displacement of the body from the mean position is called amplitude of motion.

Expression for maximum velocity of S.H.O

The velocity of S.H.O will be maximum at the mean position i.e when $x = 0$

Since $V = \sqrt{\lambda}\sqrt{a^2 - x^2}$

For $V_{max.}$ Put $x = 0$

$$V_{max.} = \sqrt{\lambda}\sqrt{a^2 - 0^2}$$

$$= \sqrt{\lambda}\sqrt{a^2}$$

$V_{max.} = \sqrt{\lambda}\, a$ is the expression for maximum velocity of S.H.O.

Time period of motion

The time taken by S.H.O to Complete one vibration is called its time period. It is usually denoted by T.

we Know that

$$x = a\cos\sqrt{\lambda}\, t$$

we can write

$$x = a\cos(\sqrt{\lambda}\,t + 2\pi) = a\cos\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right)$$

also $$= a\cos(\sqrt{\lambda}\,t + 4\pi) = a\cos\sqrt{\lambda}\left(t + \frac{4\pi}{\sqrt{\lambda}}\right)$$

and $$= a\cos(\sqrt{\lambda}\,t + 6\pi) = a\cos\sqrt{\lambda}\left(t + \frac{6\pi}{\sqrt{\lambda}}\right)$$

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So

$$x = a\cos\sqrt{\lambda}\,t = a\cos\sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right) = a\cos\sqrt{\lambda}\left(t + \frac{4\pi}{\sqrt{\lambda}}\right) \ldots\ldots$$

Clearly 'x' repeats itself after the times

$$t,\ t + \frac{2\pi}{\sqrt{\lambda}},\ t + \frac{4\pi}{\sqrt{\lambda}},\ \ldots\ldots$$

Clearly 'x' repeats itself after the time interval $\left(t + \frac{2\pi}{\sqrt{\lambda}}\right) - t$ which is known as time period of motion

So $$T = \left(t + \frac{2\pi}{\sqrt{\lambda}}\right) - t$$

$$T = \frac{2\pi}{\sqrt{\lambda}}$$ is the time period of motion.

Frequency of oscillation

The number of vibrations completeed by S.H.O in one second is called its frequency. It is equal to the reciprocal of time period

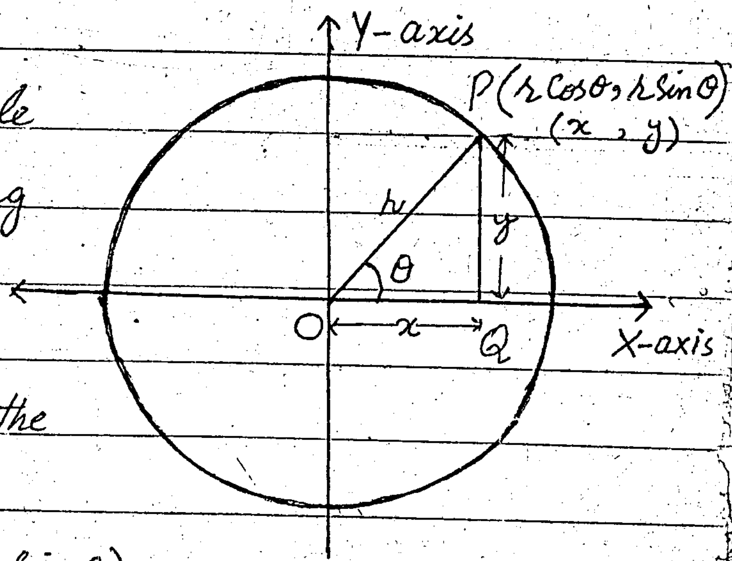
$$\text{frequency} = \nu = \frac{1}{T}$$

Geometrical Representation of S.H.M

Question

Show that when a particle moves along a circle, its projection moving along the diameter of the cirle performs simple harmonic motion.

Solution



Considering a particle which is moving along a circle

$$x^2+y^2=r^2$$

Let at any time 't' the particle is at point

$$P(x,y) = P(r\cos\theta, r\sin\theta)$$

Now

$$x = r\cos\theta \quad \text{———} \quad (1)$$

diff. w.r.t. 't'

$$\frac{dx}{dt} = r(-\sin\theta)\frac{d\theta}{dt}$$

$$v = -r\omega\sin\theta$$

$\frac{d\theta}{dt} = \omega$ (angular velocity)

again diff. w.r.t. 't'

$$\frac{dv}{dt} = -r\omega\cos\theta\frac{d\theta}{dt}$$

$$a = -r\omega^2\cos\theta$$

$$= -\omega^2(r\cos\theta)$$

$$= -\omega^2 x \qquad \text{by (1)}$$

$$\Rightarrow \quad a \propto x$$

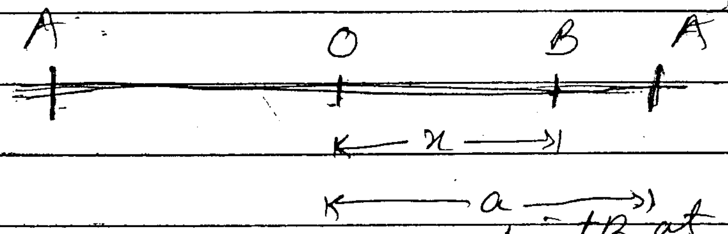
Clearly the acceleration of projection is directly proportional to the displacement of projection and the -ve sign indicates that the acceleration is directed towards the mean position.

∴ The projection of the particle moving along the diameter of the circle performs S.H.M.

<u>Example P-166</u> A particle performs S.H.M is at distance x from the centre of motion and moving away from it. Find the time in which it will be again at the same point moving in the opposite direction if the time period of motion is T and (i) the ~~amplitude~~ of motion is 'a' and (ii) the velocity of the particle is v.

<u>Solution</u>

Consider a particle which is performing



Simple Harmonic motion and is at point B at a distance x from the centre of motion. We have to find the time when the particle will again at the same point B moving in the opposite direction. Let the amplitude of motion be 'a' and the time period of motion be T.

Let the time taken by the particle to cover the distance BA' be t, then

$$\text{Required time} = t + t$$
$$= 2t$$

We know that

$$x = a\cos\sqrt{\lambda}\,t$$

$$\frac{x}{a} = \cos\sqrt{\lambda}\,t$$

or $\sqrt{\lambda}\, t = \cos^{-1}\frac{x}{a}$

or $t = \frac{1}{\sqrt{\lambda}} \cos^{-1}\left(\frac{x}{a}\right)$ ——— (1)

we know that

Time period $= \frac{2\pi}{\sqrt{\lambda}}$

or $T = \frac{2\pi}{\sqrt{\lambda}}$

or $\sqrt{\lambda} = \frac{2\pi}{T}$

(1) $\Rightarrow$ $t = \frac{1}{\frac{2\pi}{T}} \cos^{-1}\frac{x}{a}$

or $t = \frac{T}{2\pi} \cos^{-1}\frac{x}{a}$

$\times 2$ $\quad 2t = \frac{T}{\pi} \cos^{-1}\frac{x}{a}$

Required time $= \frac{T}{\pi} \cos^{-1}\frac{x}{a}$ ——— (2)

Now we shall find required time in terms of velocity v.

As $v = \sqrt{\lambda(a^2 - x^2)}$

$v^2 = \lambda(a^2 - x^2)$

$\frac{v^2}{\lambda} = a^2 - x^2$

or $a^2 = \frac{v^2}{\lambda} + x^2$

As $T = \frac{2\pi}{\sqrt{\lambda}}$

$T^2 = \frac{4\pi^2}{\lambda}$

or $\lambda = \frac{4\pi^2}{T^2}$

$$a^2 = \frac{v^2}{\frac{4\pi^2}{T^2}} + x^2$$

$$= \frac{v^2T^2}{4\pi^2} + x^2$$

$$a^2 = \frac{v^2T^2 + 4\pi^2x^2}{4\pi^2}$$

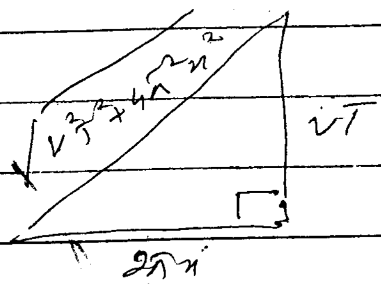
$$a = \frac{\sqrt{v^2T^2 + 4\pi^2x^2}}{2\pi} \quad \text{—— (3)}$$

using (3) in (2)

$$\text{Required time} = \frac{T}{\pi}\cos^{-1}\left(\frac{x}{\frac{\sqrt{v^2T^2+4\pi^2x^2}}{2\pi}}\right)$$

$$= \frac{T}{\pi}\cos^{-1}\left(\frac{2\pi x}{\sqrt{v^2T^2+4\pi^2x^2}}\right)$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi x}\right)$$



$$H^2 = B^2 + P^2$$

$$P^2 = H^2 - B^2$$

$$= v^2T^2 + \cancel{4\pi^2x^2} - \cancel{4\pi^2x^2}$$

$$P^2 = v^2T^2$$

$$P = vT$$

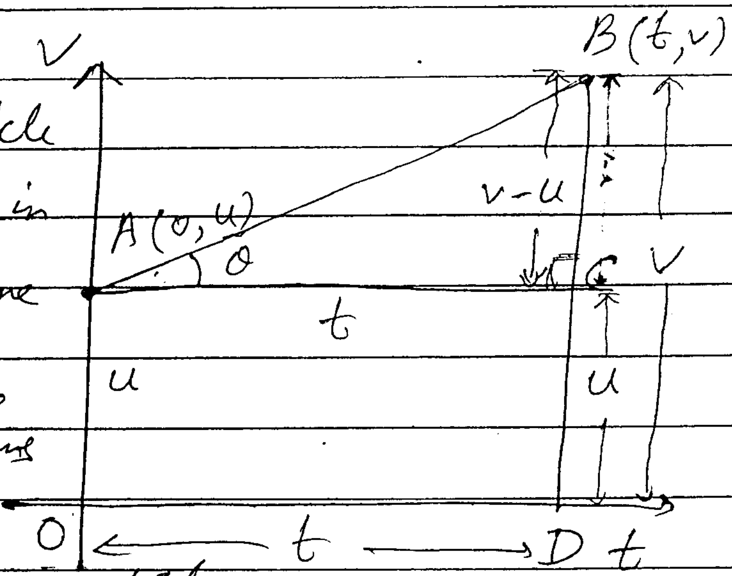
End

Exercises Set 8 P-167 30

Q-no.1 obtain the equations of motion by Graphical Method.

Sol

Considering a particle which is moving in velocity time plane with constant acceleration. Then motion of the particle will be rectilinear.



Also

Acc. of the particle = Slope of the line AB

$$a = \tan\theta$$

$$a = \frac{v-u}{t}$$

$\times t$

$$v - u = at \quad \text{———(1)}$$

$$v = u + at \quad \text{———(2)}$$

is the first equation of motion

Also Distance travelled by the particle = Area under the curve i.e

x = Area of triangle ABC + Area of rectangle ODCA

$= \frac{1}{2} t(v-u) + u.t$

$= \frac{1}{2} t.at + ut \quad \because v-u = at$ by (1)

or $\therefore x = ut + \frac{1}{2}at^2$ is the 2nd equation of motion. ——(3)

from (1) $t = \frac{v-u}{a}$ ——(4)

using (4) in (3)

$x = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$

$= \frac{uv-u^2}{a} + \frac{1}{2}\cancel{a}\,\frac{v^2+u^2-2uv}{a^{\cancel{2}}}$

$= \frac{uv-u^2}{a} + \frac{v^2+u^2-2uv}{2a}$

$\times 2a$

$2ax = \cancel{2uv} - 2u^2 + v^2 + u^2 - \cancel{2uv}$

$2ax = v^2 - u^2$ is the 3rd equation of motion.

32

Q.no.2 A particle moving in a St. line starts from rest and is accelerated uniformly to attain a velocity of 60 m/h in 4 seconds. Find the acceleration of motion and the distance travelled by the particle in the last 3 seconds.

Sol Consider a particle which is moving in st. line start from rest and is accelerated uniformly to attain a velocity of 60 m/h in 4 seconds.

$V = 60$ mile/h

$= \frac{\cancel{60} \times 1760 \times 3}{\cancel{60} \times \cancel{60}_{\,\,\,\,\,29}}$ 1 mile = 1760 yd

$= 88$ ft/sec

we know that

$v = u + at$

$88 = 0 + a \times 4$

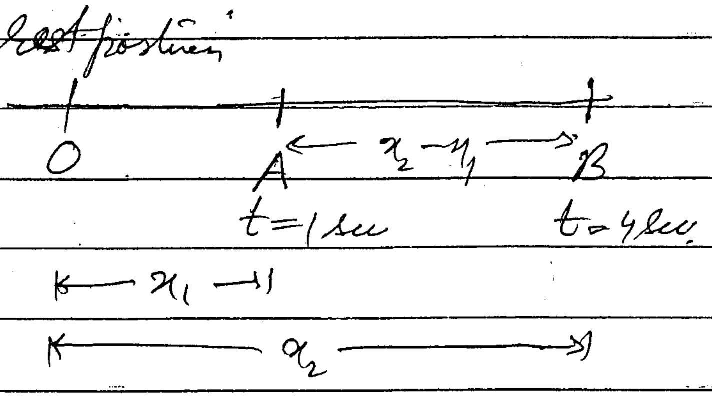
$4a = 88$

$a = \frac{88}{4} = 22$ ft/sec^2

i.e $a = 22$ ft/sec^2

33

Let the particle starts moving from point O and reaches at point A



in 1 sec after covering a distance x_1

then $x = ut + \frac{1}{2}at^2$

$$x_1 = 0 \times 1 + \frac{1}{2}(22)(1)^2$$

$$x_1 = 11 \text{ ft}.$$

Let the particle reaches at point B in 4 seconds after covering a distance x_2

then $x = ut + \frac{1}{2}at^2$

$$x_2 = 0 \times 4 + \frac{1}{\cancel{2}}(\overset{11}{\cancel{22}}) \times (4)^2$$

$$x_2 = 0 + 11 \times 16$$

$$x_2 = 176 \text{ ft}$$

Distance covered in the last 3 seconds $= x_2 - x_1$

$= 176 - 11$

$= 165$ ft.

Ex. No. 3 Two particle start simultaneously from a point O and move in a st. line one with a velocity of 45 mph and an acceleration of 2 ft/sec² and the other with a velocity of 90 mph and a retardation of 8 ft/sec². Find the time after which the velocities of the particle will be same and the distance of O from the point where they meet again.

Sol: Consider two particles P_1 and P_2 start moving simultaneously from point O and move in st. line

Particle P_1	Particle P_2
$u_1 = 45$ mph	$u_2 = 90$ mph
$= \frac{45 \times 1760 \times 3}{60 \times 60}$	$= \frac{90 \times 1760 \times 3}{60 \times 60}$
$= 66$ ft/sec	$= 132$ ft/sec
$a = 2$ ft/sec²	$a = -8$ ft/sec²

Let after a time 't' the velocity of both particle be same say v.

Particle P_1 | **Particle P_2**

$v = u + at$

$v = 66 + 2t$

$v = u + at$

$v = 132 + (-8)t$

$v = 132 - 8t$

$$\cancel{v} = 66 + 2t$$
$$-\cancel{v} = -132 + 8t$$

$$0 = -66 + 10t$$

$$10t = 66$$

$$t = \frac{66}{10}$$

$$t = 6.6 \text{ sec}$$

So after $t = 6.6$ velocity of both particles will be same.

Let 'x' be the distance of the particles from point O from the point where they meet again after time t'

Particle P_1 | **Particle P_2**

$x = ut + \frac{1}{2}at^2$

$x = 66 \times t' + \frac{1}{\cancel{2}}(\cancel{2})(t')^2$

$x = 66t' + (t')^2$ —— (1)

$x = ut + \frac{1}{2}at^2$

$x = 132t' + \frac{1}{2}(-8)(t')^2$

$x = 132t' - 4(t')^2$

$$x = 66t' + t'^2$$
$$x = 132t' - 4t'^2$$
$$0 = -66t' + 5t'^2$$

$$5t'^2 - 66t' = 0$$

$$t'(5t' - 66) = 0$$

$t' = 0$ is the time when they were together at the start /

$$5t' - 66 = 0$$
$$5t' = 66$$
$t' = \frac{66}{5}$ is the time when they meet again after a distance x from O

Put $t' = \frac{66}{5}$ in (1)

$$x = 66\left(\frac{66}{5}\right) + \left(\frac{66}{5}\right)^2$$

$$= \frac{(66)^2}{5} + \left(\frac{66}{5}\right)^2 = \frac{(66)^2}{5} + \frac{(66)^2}{25}$$

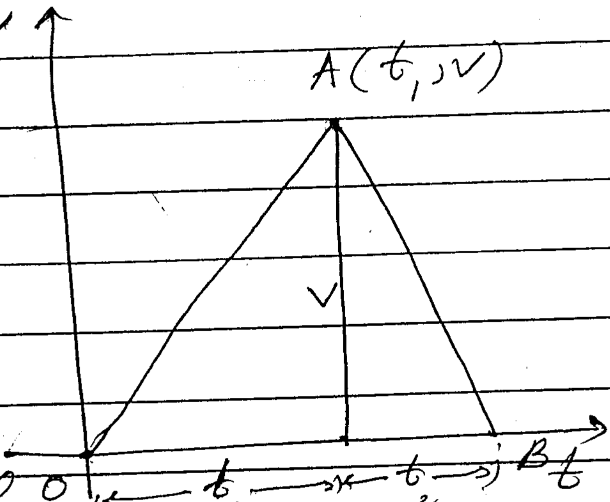
$$= \frac{5(66)^2 + (66)^2}{25}$$

$$= \frac{21780 + 4356}{25}$$

$$= 1045.44\ \text{ft. Ans.}$$

Q. No. 4 A particle moving along a st. line starts from rest and is accelerated uniformly till it attains a velocity v. The motion is then retarded and the particle comes to rest after traversing a total distance x. If the acceleration is 'f' find the retardation and the total time taken by the particle from rest to rest.

Sol:



Consider a particle which is moving along a st. line starts from rest at $t=0$ and is accelerated uniformly in time t_1 till it attains the velocity v. The motion is then retarded and particle comes to rest after further time t_2 after covering a total distance x.

Now

Distance travelled by the particle from rest to rest = Area under the curve (i.e triangle OAB)

$$x = \frac{1}{2}(t_1+t_2)\cdot v$$

$$\text{or} \quad t_1+t_2 = \frac{2x}{v} \qquad (1)$$

Time taken by particle from rest to rest $= \frac{2x}{v}$ Ans.

Also

Acc. = Slope of line OA

$$f = \frac{v}{t_1}$$

$$\text{or } t_1 = \frac{v}{f} \quad \text{——— (2)}$$

Also Retardation = Slope of the line AB

$$r = \frac{v}{t_2}$$

$$\text{or } t_2 = \frac{v}{r} \quad \text{——— (3)}$$

using (2) & (3) in (1)

$$\frac{v}{f} + \frac{v}{r} = \frac{2x}{v}$$

$$\frac{v}{r} = \frac{2x}{v} - \frac{v}{f}$$

$$\frac{v}{r} = \frac{2xf - v^2}{vf}$$

$\div v$

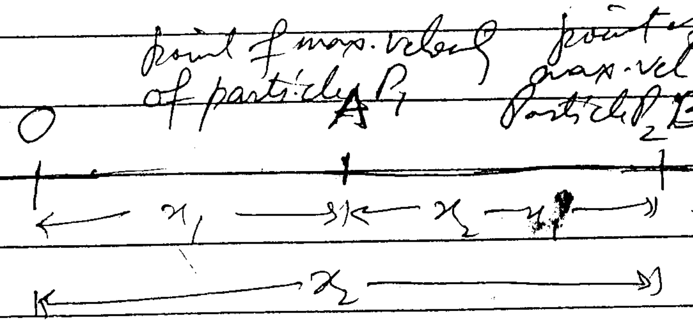
$$\frac{1}{r} = \frac{2xf - v^2}{v^2 f}$$

$$\text{or } r = \frac{v^2 f}{2xf - v^2} \quad \text{Ans.}$$

Q no. 5 Two particles travell along a st. line both start at the same time and are accelerated uniformly at different rates. The motion is such that when a particle attain the max. valocity its motion is retarted uniformly. The two particles come to rest simultaneously at a distance x from the start point. If the acceleration of the first is 'a' and that of second is $\frac{1}{2}a$ find the distance b/w the point where the two particles attain their max. velocities.

Sol

Consider two particle P_1 and P_2 which starts



moving from O and move with different rates. Let the particle P_1 move with acc. 'a' and particle P_2 move with acc. '$\frac{1}{2}a$'.

The particle P_1 & P_2 attain their max. velocities V at point A and B after covering distances x_1 & x_2 respectively.

Now

Particle P_1	Particle P_2
$v^2 - u^2 = 2ax$	$v^2 - u^2 = 2ax$
$v^2 - 0^2 = 2ax_1$	$v^2 - 0^2 = 2 \cdot \frac{1}{2} a \cdot x_2$
$2ax_1 = v^2$	$v^2 = ax_2$
$x_1 = \frac{v^2}{2a}$	$x_2 = \frac{v^2}{a}$

Distance b/w the point where both particles attain their max. velocities $= |AB|$

$= x_2 - x_1$

$= \frac{v^2}{g} - \frac{v^2}{2g}$

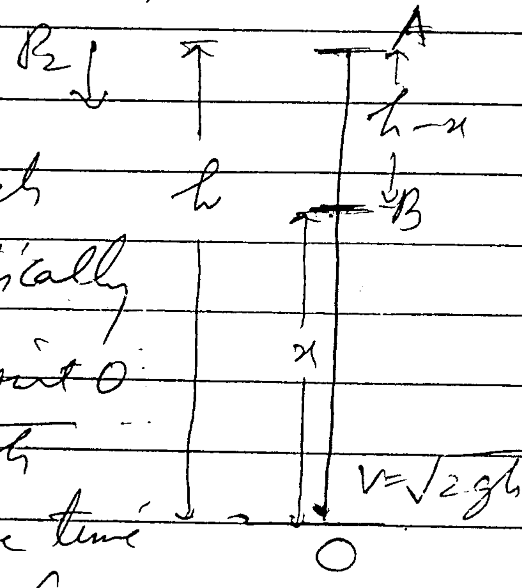
$= \frac{2v^2 - v^2}{2g}$

$= \frac{v^2}{2g}$

v. imp

Q.No.6 A particle is projected vertically upward with a velocity $\sqrt{2gh}$ and another is let fall from a height h at the same time. Find the height of the point where they meet each other.

Sol



Consider a particle P_1 which is projected vertically upwards from point O with velocity $\sqrt{2gh}$ and at the same time another is let fall from point A ~~at~~ from a height h.

Let both particle meet eachother at

point B at a height x from point O.
The time ~~time~~ taken by both particle to reach at point B will be same say t.

Particle P_1

$$x = ut + \frac{1}{2}(-g)t^2$$

$$x = \sqrt{2gh}\; t - \frac{1}{2}gt^2 \quad \text{---(1)}$$

Particle P_2

$$x = ut + \frac{1}{2}gt^2$$

$$h - x = 0 \times t + \frac{1}{2}gt^2$$

$$h - x = \frac{1}{2}gt^2 \quad \text{---(2)}$$

(1) + (2) gives

$$\cancel{x} = \sqrt{2gh}\; t - \cancel{\frac{1}{2}gt^2}$$

$$h - \cancel{x} = \quad + \cancel{\frac{1}{2}gt^2}$$

$$h = \sqrt{2gh}\; t$$

$$\text{or } t = \frac{h}{\sqrt{2gh}}$$

$$= \sqrt{\frac{h^2}{2gh}}$$

$$= \sqrt{\frac{h}{2g}}$$

Put $t = \sqrt{\frac{h}{2g}}$ in (1)

$$x = \sqrt{2gh}\sqrt{\frac{h}{2g}} - \frac{1}{2}g\left(\sqrt{\frac{h}{2g}}\right)^2$$

$$x = \sqrt{\frac{2gh^2}{2g}} - \frac{1}{2} g \frac{h}{2g}$$

$$= h - \frac{h}{4}$$

$$= \frac{4h-h}{4}$$

$$= \frac{3h}{4}$$

or $x = \frac{3}{4}h$ is the height at which both the particles meet each other.

~~Q. no. 7 Two particles are projected simultaneously in the vertically upwards directions with velocities $\sqrt{2gh}$ and $\sqrt{2gk}$ $(k > h)$. After a time t when two particles are still in flight another particle is projected upward with a velocity u. Find the condition so that the 3rd particle may meet the first two during their upward flight.~~

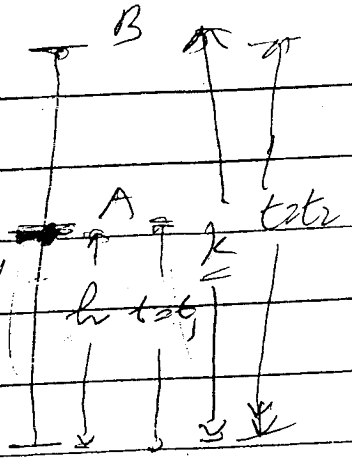
Sol Let max. height attained by first particle be H.

$v^2 - u^2 = 2(g)x$	$v = 0$
$0 - (\sqrt{2gh})^2 = 2(-g)H$	$u = \sqrt{2gh}$
	$a = -g$
$-2gh = -2gH$	$x = H$

$\div(-2g)$

or $H = h$

Similarly max. height attained by the 2nd particle will be k.



Also $k > h$ (given)

Let t_1 be the time taken by the first particle to attain its max height h.

Then $v = u + (-g)t$

$$0 = \sqrt{2gh} - gt_1$$

$$gt_1 = \sqrt{2gh}$$

$$t_1 = \frac{\sqrt{2gh}}{g}$$

or $t_1 = \sqrt{\frac{2gh}{g^2}}$

$$= \sqrt{\frac{2h}{g}}$$

Let t_2 be the time taken by second particle to attain its max. height k.

Then $t_2 = \sqrt{\frac{2k}{g}}$

Since $h < k$ so $t_1 < t_2$

Let after a time 't' the 3rd particle

is projected vertically upwards with velocity u and attains its max. height x

Now the 3rd particle will catch the 1st two particles during their upwards motion if

(i) $t < t_1$ i.e. $t < \sqrt{\frac{2h}{g}}$

(ii) height attained by the 3rd particle in time $(t_1 - t)$ must be greater than k.

Now for 3rd particle

$$x = ut + \frac{1}{2}(-g)t^2$$

$$x = u(t_1 - t) - \frac{1}{2} g (t_1 - t)^2$$

Now $x > k$

$$u(t_1 - t) - \frac{1}{2} g (t_1 - t)^2 > k$$

$$u(t_1 - t) > k + \frac{1}{2} g (t_1 - t)^2$$

$\div (t_1 - t)$
$$u > \frac{k}{t_1 - t} + \frac{1}{2} g (t_1 - t)$$

Put $t_1 = \sqrt{\frac{2h}{g}}$

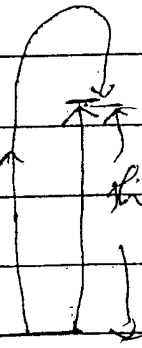
Thus
$$u > \frac{k}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2} g \left(\sqrt{\frac{2h}{g}} - t\right)$$

is the required condition.

Q.No.9. A particle is projected vertically upwards after a time t another particle is sent up from the same point with the same velocity and meets the first at height h during the downward flight of the first. Find the velocity of projection.

Sol

Let initial velocity of both particles be u



Let Time of flight of first particle be t_1 so time of flight of 2nd particle will be $t_1 - t$.

Particle P_1

$$x = ut + \frac{1}{2}(-g)t^2$$

$$h = ut_1 - \frac{1}{2}gt_1^2 \quad \text{--- (1)}$$

Particle P_2

$$x = ut + \frac{1}{2}(-g)t^2$$

$$h = u(t_1 - t) - \frac{1}{2}g(t_1 - t)^2 \quad \text{--- (2)}$$

(1) − (2) gives

$$\cancel{h} = ut_1 - \frac{1}{2}gt_1^2$$

$$-\cancel{h} = -u(t_1 - t) + \frac{1}{2}g(t_1 - t)^2$$

$$0 = \cancel{ut_1} - \frac{1}{2}gt_1^2 - \cancel{ut_1} + ut + \frac{1}{2}g(t_1^2 + t^2 - 2tt_1)$$

$$0 = -\cancel{\tfrac{1}{2}gt_1^2} + ut + \cancel{\tfrac{1}{2}gt_1^2} + \tfrac{1}{2}gt^2 - gtt_1$$

$$gtt_1 = ut + \tfrac{1}{2}gt^2$$

$\div gt$

$$t_1 = \frac{u}{g} + \frac{1}{2}t \qquad (3)$$

using (3) in (1)

$$h = u\left(\frac{u}{g} + \frac{1}{2}t\right) - \frac{1}{2}g\left(\frac{u}{g} + \frac{1}{2}t\right)^2$$

$$= \frac{u^2}{g} + \frac{ut}{2} - \frac{1}{2}g\left(\frac{u^2}{g^2} + \frac{1}{4}t^2 + \frac{ut}{g}\right)$$

$$= \frac{u^2}{g} + \cancel{\frac{ut}{2}} - \frac{u^2}{2g} - \frac{gt^2}{8} - \cancel{\frac{1}{2}ut}$$

$$= \frac{u^2}{2g} - \frac{gt^2}{8}$$

$\times 8g$

$$8gh = 4u^2 - g^2t^2$$

$$8gh + g^2t^2 = 4u^2$$

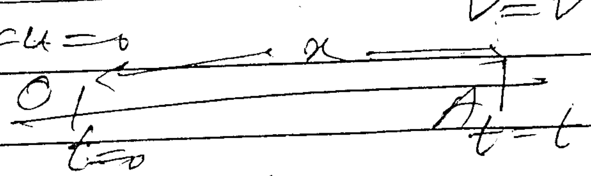
$$u^2 = \frac{8gh + g^2t^2}{4}$$

$$u = \frac{\sqrt{8gh + g^2t^2}}{2}$$

is the velocity of projection

Q. No. 10 Discuss the motion of particle moving in a st. line if it starts from rest at $t=0$ and its acceleration is equal to (i) t^n (ii) a const + b sin t.

Sol
Consider a particle which starts from rest at $t=0$ and moves in a straight line and its acc. is given by



$$\text{acc.} = t^n$$

$$\frac{dv}{dt} = t^n$$

$$dv = t^n dt$$

$$\int dv = \int t^n dt$$

$$v = \frac{t^{n+1}}{n+1} + C \quad \text{---(1)}$$

when $t=0$, $v=0$

$$0 = 0 + C$$

$$C = 0$$

$$(1) \Rightarrow v = \frac{t^{n+1}}{n+1}$$

$$\frac{dx}{dt} = \frac{t^{n+1}}{n+1}$$

$$\int dx = \int \frac{t^{n+1}}{n+1}\, dt$$

$$x = \frac{1}{n+1} \frac{t^{n+2}}{n+2} + D \quad \text{—— (2)}$$

Initially when $t = 0$, $x = 0$

$$0 = 0 + D$$

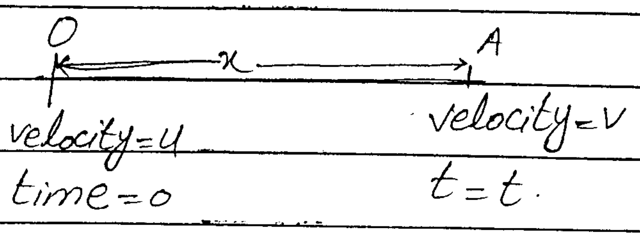
$$D = 0$$

$$(2) \rightarrow \quad x = \frac{t^{n+2}}{(n+1)(n+2)} \text{ is distance}$$

Travelled by the particle in time t.

Q No:11: A partical starts with a velocity u and moves in a straight line if it suffer a retardation equal to the square of its velocity find the distance travelled by the partical in a time t.

Sol Consider a partical which start from point O with velocity u and suffer a retardation equal to square of its velocity. After a time t the partical reaches at point A after covering a distance



Hence $a = -v^2$

$$\frac{dv}{dt} = -\cancel{dt}v^2$$

$$\int \frac{dv}{v^2} = -\int dt$$

$$\frac{v^{-1}}{-1} = -t + B$$

$$\frac{-1}{v} = -t + B$$

Intialy when $t=0$, $v=u$

$$-\frac{1}{u} = 0 + B$$

$$B = -\frac{1}{u}$$

put $B = -\frac{1}{u}$

$$-\frac{1}{v} = -t - \frac{1}{u}$$

$$\frac{1}{v} = t + \frac{1}{u}$$

$$\frac{1}{v} = \frac{ut+1}{u}$$

$$v = \frac{u}{ut+1}$$

$$\frac{dx}{dt} = \frac{u}{ut+1}$$

$$\int dx = \int \frac{u}{ut+1}\,dt$$

$$x = \ln(ut+1) + C \longrightarrow (2)$$

Intialy when $t=0$, $x=0$

$$0 = \ln(0+1) + C$$

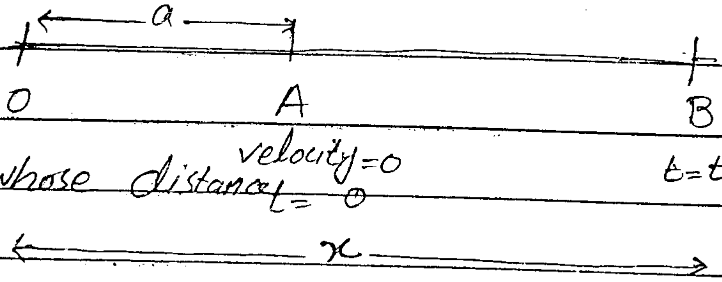
$$0 = 0 + C$$

$$C = 0$$

$(2) \Rightarrow x = \ln(ut+1)$ is the distance travelled by a partical in time t.

Qno:12: Discuss the motion of a partical moving in a straight line if it start from rest at a distance 'a' from a point O and move with a acceleration equal to μ times its distance from O.

Consider a partical which start from rest from point A whose distance from point O is 'a'.



Let after a time the partical reach at point B after Covering a distance x from O.

According to question

$$a\mu = \mu x$$

$$v\frac{dv}{dx} = \mu x$$

$$\int v\,dv = \mu\int x\,dx$$

$$\frac{v^2}{2} = \mu\frac{x^2}{2} + C \longrightarrow (1)$$

Intialy when $x = a$, $v = 0$.

$$0 = \mu\frac{a^2}{2} + C$$

$$C = -\mu\frac{a^2}{2}$$

$$(1) \Rightarrow \frac{v^2}{2} = \mu\frac{x^2}{2} - \mu\frac{a^2}{2}$$

$$v^2 = \mu x^2 - \mu a^2$$

$$v^2 = \mu(x^2 - a^2)$$

$$v = \sqrt{\mu(x^2 - a^2)}$$

velocity of the partical after the ~~time~~ distance

$$\frac{dx}{dt} = \sqrt{\mu(x^2 - a^2)}$$

$$\frac{dx}{dt} = \sqrt{\mu}\sqrt{x^2 - a^2}$$

$$\frac{1}{\sqrt{x^2 + a^2}}\,dx = \sqrt{\mu}\,dt.$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dt = \sqrt{\mu}\int dt$$

$\cosh^{-1}\frac{x}{a} = \sqrt{u}\,t + D \longrightarrow (2)$

Intialy when $t=0$, $x=a$.

$\cosh^{-1}\frac{a}{a} = \sqrt{u}(0) + D$

$\cosh^{-1}(1) = D$

$D = 0$.

i.e $D=0$

$(2) \Rightarrow \cosh^{-1}\frac{x}{a} = \sqrt{u}\,t + 0$.

$\cosh^{-1}\frac{x}{a} = \sqrt{u}\,t$.

$\frac{x}{a} = \cosh\sqrt{u}\,t$.

$x = a\cosh\sqrt{u}\,t$.

is the distance traveled by the partical in time t.

Q No: 13: A partical moving in a straight line start with velocity u and has acceleration v^3, where v is the velocity of the partical at time t. Find the velocity and the time as function of the distance travelled by the partical.

Consider a partical starts moving from ~~velocity = u~~ point O with intial velocity u at $t=0$ and moves with acceleration v^3. Let after a time t the partica A after Covering a distan x.

x

~~velocity = v~~

$t=0$ $\qquad$ $t=t$

According to question

$\cancel{V}AU = V^3$

52

$$v\frac{dv}{dx} = v^3$$

$$\int \frac{v\,dv}{v^3} = \int dx$$

$$\int v^{-2}dv = \int dx$$

$$\frac{v^{-1}}{-1} = x + B.$$

$$\frac{-1}{v} = x + B.$$

intialy when $x = 0$, $v = u$.

$$\frac{-1}{u} = 0 + B.$$

$$B = -\frac{1}{u}$$

cal

$$① \Rightarrow \quad -\frac{1}{v} = x - \frac{1}{u}$$

$$\frac{1}{v} = \frac{1}{u} - x$$

$$\frac{1}{v} = \frac{1-ux}{u}$$

$$v = \frac{u}{1-ux}$$

$$\frac{dx}{dt} = \frac{u}{1-ux}$$

$$\int (1-ux)\,dx = \int u\,dt$$

$$\int dx - \int ux\,dx = u\int dt$$

$$x - \frac{ux^2}{2} = ut + D \longrightarrow ②$$

y=v

Intialy when $t = 0$, $x = 0$

ation

$$0 = 0 - (u)(0) + D.$$

$$\rightarrow D = 0.$$

$$② \quad x - \frac{ux^2}{2} = ut + 0$$

$$ut = -\frac{ux^2}{2} + x$$

$$ut = x\left(-\frac{ux}{2} + 1\right).$$

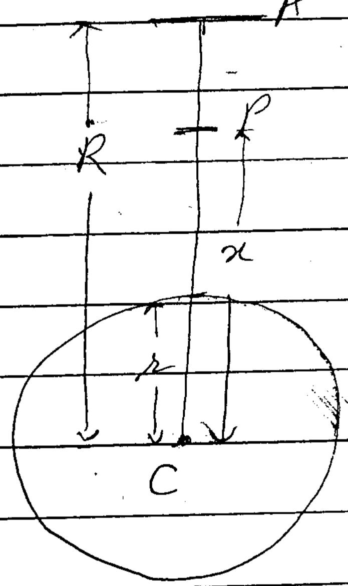
$$ut = x\left(\frac{-ux+2}{2}\right)$$

$$t = \frac{x}{2u}(-ux+2)$$

$$t = \frac{x}{2u}(2-ux).$$

is the time spending by the partical for Covering distance x.

Q.no.14: The acceleration of a particle falling freely under the gravitational pull is equal to $\frac{k}{x^2}$ where x is the distance of the particle from the centre of the earth. Find the velocity of the particle if it is let fall from an altitude R ~~and~~ striking on the surface of the earth if the radius of earth is 'r' and the air offer no resistance to motion.



Sol Consider a particle which is falling freely under gravitational pull from an altitude R. Let at any instant the particle is at point P whose distance from centre of earth is x. Let 'r' be radius of earth and air offer no resistance to the motion.

According to question

$Acc. = \frac{-k}{x^2}$ (Acceleration is against the direction in which x-increases)

$$v\frac{dv}{dx} = -\frac{k}{x^2}$$

$$v\,dv = -\frac{k}{x^2}\,dx$$

$$\int v\,dv = -k\int x^{-2}\,dx$$

$$\frac{v^2}{2} = -k\,\frac{x^{-1}}{-1} + B$$

$$\frac{v^2}{2} = \frac{k}{x} + B \quad \text{——— (1)}$$

Initially when $x = R$, $v = 0$

$$0 = \frac{k}{R} + B$$

So $B = -\frac{k}{R}$

$$(1) \Rightarrow \frac{v^2}{2} = \frac{k}{x} - \frac{k}{R}$$

$$v^2 = 2\left(\frac{k}{x} - \frac{k}{R}\right)$$

$$v^2 = 2k\left(\frac{1}{x} - \frac{1}{R}\right)$$

$v = \sqrt{2k\left(\frac{1}{x} - \frac{1}{R}\right)}$ is the velocity —— (2) of the particle when it is at a distance x from the centre of the earth.

In order to find the velocity on the surface of earth put $x = r$ in (2)

$v = \sqrt{2k\left(\frac{1}{r} - \frac{1}{R}\right)}$ Ans.

Q. No. 15 A particle describes S.H.M with the frequency N. If the greatest velocity is v find the amplitude and max. value of acceleration of the particle. Also show that the velocity v at a distance x from the centre of motion is given by $v = 2\pi N\sqrt{a^2 - x^2}$ where a is the amplitude.

Sol Consider a particle which is describing simple harmonic motion

Frequency of the particle = N (given)

But Frequency of the particle $= \frac{\sqrt{\lambda}}{2\pi}$

so $N = \frac{\sqrt{\lambda}}{2\pi}$.

or $\sqrt{\lambda} = 2\pi . N$ ——— (1)

Max. velocity of the particle = v (given)

But Max. velocity of the particle $= \sqrt{\lambda}\, a$

so $v = \sqrt{\lambda}\, a$

or $a = \frac{v}{\sqrt{\lambda}}$

$a = \frac{v}{2\pi N}$ by (1)

is the amplitude of motion.

we know that **56**

Acc. $\propto$ Displacement from mean position.

Max. Acc. $\propto$ Max. Displacement from mean position (amplitude)

Max. value of acceleration $= \lambda a$

$= 4\pi^2 N^2 \cdot \frac{v}{2\pi N}$ by (1) & (2)

$= 2\pi N v$

Consider the particle is moving ~~etc~~ with velocity v and at any instant the particle is at a distance x from the mean position then

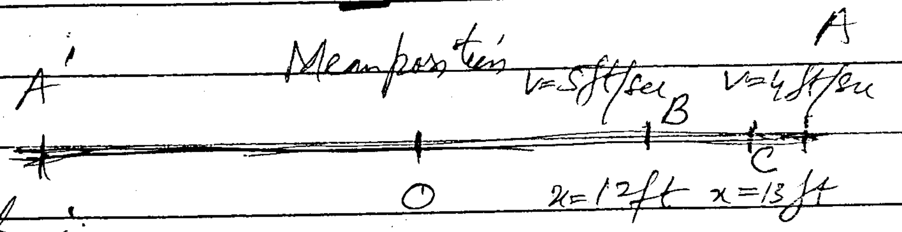
$$v = \sqrt{\lambda}\sqrt{a^2 - x^2}$$

$$v = 2\pi N\sqrt{a^2 - x^2} \quad \text{by (1)}$$

is the required velocity.

Q. No. 16 A particle describing S.H.M. has velocities 5 ft/sec and 4 ft/sec when its distance from the centre of motion is 12 ft and 13 ft respectively. Find its time period of motion.

Sol



Consider a particle which is describing S.H.M and moving away from the mean position. The velocities of the ~~are 5 ft/sec and 4 ft/sec when it is at a distance of 12 ft and 13 ft from~~ the mean position.

~~At point B~~

$x = 12$ ft

$v = 5$ ft/sec

we know that

~~$v = \sqrt{\lambda(a^2 - x^2)}$~~

$v^2 = \lambda(a^2 - x^2)$

$v^2 = \lambda a^2 - \lambda x^2$ ——— (*)

$(5)^2 = \lambda a^2 - \lambda(12)^2$

$25 = \lambda a^2 - 144\lambda$ ——— (1)

At point C

$x = 13$ ft

$v = 4$ ft/sec

we know that

$v^2 = \lambda a^2 - \lambda x^2$ by (*)

$(4)^2 = \lambda a^2 - \lambda(13)^2$

$16 = \lambda a^2 - 169\lambda$ ——— (2)

(1) − (2) gives.

$$25 = \cancel{\lambda a^2} - 144\lambda$$
$$16 = \cancel{\lambda a^2} - 169\lambda$$
$$9 = 25\lambda$$

$$\lambda = \frac{9}{25}$$

$$\& \sqrt{\lambda} = \frac{3}{5}$$

We know that

$$\text{Time period of motion} = \frac{2\pi}{\sqrt{\lambda}}$$

$$= \frac{2\pi}{\frac{3}{5}}$$

$$= \frac{10\pi}{3}$$

Q.No.17 The maximum velocity that a particle executing S.H.M of amplitude 'a' attains is 'v'. If it is disturbed in such a way that its max. velocity becomes nv, find the change in amplitude and the time period of motion.

Sol Consider a particle which is executing S.H.M. Let 'a' be the amplitude of motion

Max. velocity of the particle when its amplitude is $a = v$ (given)

But Max. velocity of the particle when its amplitude is $a = \sqrt{\lambda}\, a$

$$\therefore \quad v = \sqrt{\lambda}\, a$$

$$\text{or} \quad a = \frac{v}{\sqrt{\lambda}} \quad \text{——— (1)}$$

Let the particle is disturbed and its amplitude becomes ~~a~~ a'

Max. velocity of particle when its amplitude is $a' = nv$ (given)

But Max. velocity of the particle when its amplitude is $a' = \sqrt{\lambda}\, a'$

$$\therefore \; nv = \sqrt{\lambda}\, a'$$

$$\text{or} \quad a' = \frac{nv}{\sqrt{\lambda}} \quad \text{——— (2)}$$

Change in amplitude $= a' - a$

$$= \frac{nv}{\sqrt{\lambda}} - \frac{v}{\sqrt{\lambda}} \quad \text{by (1) \& (2)}$$

$$= \frac{v}{\sqrt{\lambda}}(n-1)$$

$$= a(n-1) \quad \text{by (1)}$$

Since their is no change in $\sqrt{\lambda}$ so their will be no change in ~~amplitude~~ time period of motion.

V. Imp

Q.No.18 A point describes S.H.M in such a way that its velocity and acc. at point P are u and f resp. and the Corresponding quantities at

v

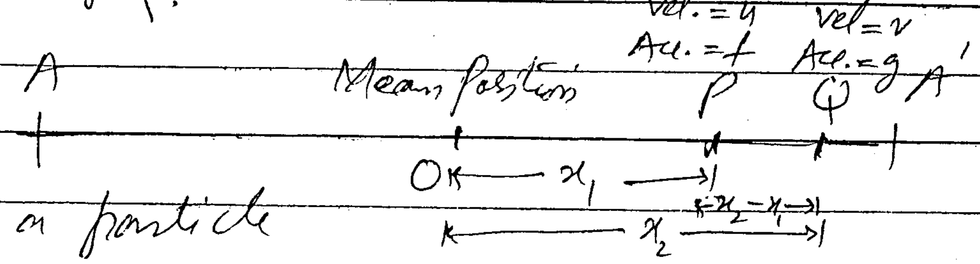
v'

u

a^2

another point Q are v and g. Find the distance PQ.

Sol



Consider a particle which is describing S.H.M and is moving away from mean position. The velocity and acc. of the particle at point P are u and f resp. and the corresponding quantities at another point Q are v and g resp.

At point P

$$v = \sqrt{\lambda(a^2-x^2)}$$

$$v^2 = \lambda(a^2-x^2)$$

$$u^2 = \lambda(a^2-x_1^2)$$

$$u^2 = \lambda a^2 - \lambda x_1^2 \quad \text{—(1)}$$

Acc. ∝ Displacement

$$f \propto x_1$$

$$f = -\lambda x_1 \quad \text{—(2)}$$

At Point Q

$$v^2 = \lambda(a^2-x^2)$$

$$v^2 = \lambda(a^2-x_2^2)$$

$$v^2 = \lambda a^2 - \lambda x_2^2 \quad \text{—(3)}$$

Acc. ∝ Displace

$$g \propto x_2$$

$$g = -\lambda x_2 \quad \text{—(4)}$$

(1) − (3)

$$u^2 = \cancel{\lambda a^2} - \lambda x_1^2$$

$$v^2 = \cancel{\lambda a^2} - \lambda x_2^2$$

$$u^2 - v^2 = \lambda(x_2^2 - x_1^2)$$

$$u^2 - v^2 = \lambda(x_2 + x_1)(x_2 - x_1) \quad \text{—(5)}$$

$(2)+(4) \Rightarrow f = -\lambda x_1$

$$g = -\lambda x_2$$

$$f+g = -\lambda(x_1+x_2)$$

$$-(f+g) = \lambda(x_2+x_1) \quad \text{——— (6)}$$

using (6) in (5)

$$u^2-v^2 = -(f+g)(x_2-x_1)$$

or $$x_2-x_1 = \frac{u^2-v^2}{-(f+g)}$$

or $$x_2-x_1 = \frac{v^2-u^2}{f+g}$$

$$|PQ| = \frac{v^2-u^2}{f+g}$$

Q.No.19 If a point P moves with a velocity v given by $v^2 = n^2(ax^2+2bx+c)$ show that P executes a S.H.M. Find the center amplitude and time period of motion.

Sol Consider a point P, that moves with velocity v given by

$$v^2 = n^2(ax^2+2bx+c) \quad \text{——— (1)}$$

diff w.r.t. x

$$2(v)^1\frac{dv}{dx} = n^2(2ax+2b)$$

$$2v\frac{dv}{dx} = 2an^2\left(x+\frac{b}{a}\right)$$

$\div 2$

$$v\frac{dv}{dx} = \cancel{2}an^2\left(x+\frac{b}{a}\right)$$

Let ~~$v\frac{dv}{dt}$~~ $an^2 = \lambda$ and $x+\frac{b}{a} = X$

then $v\dfrac{dv}{dx} = \lambda X$

$$Acc. = \lambda X$$

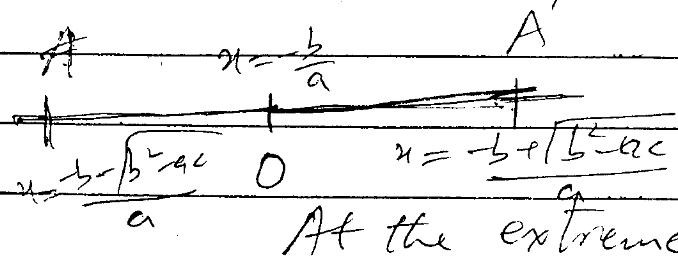
Acc. $\propto X$ (displacement)

Thus P executes S.H.M.

At the centre $X = 0$

$$x+\frac{b}{a} = 0$$

$$x = -\frac{b}{a}$$



At the extreme positions velocity of the particle will be zero

So (1) $\Rightarrow$ $n^2(ax^2+2bx+c)=0$ $\because v=0$

$\div n^2$

$$ax^2+2bx+c=0$$

Now $x = \dfrac{-2b \pm\sqrt{(2b)^2-4(a)(c)}}{2a}$

$$x = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$$

$$= \frac{-2b \pm 2\sqrt{b^2 - ac}}{2a}$$

$$= \frac{\cancel{2}\left(-b \pm \sqrt{b^2 - ac}\right)}{\cancel{2}a}$$

$$x = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

Amplitude $(OA') = \frac{-b + \sqrt{b^2 - ac}}{a} - \left(-\frac{b}{a}\right)$

$$= \frac{-b + \sqrt{b^2 - ac}}{a} + \frac{b}{a}$$

$$= \frac{\cancel{-b} + \sqrt{b^2 - ac} + \cancel{b}}{a}$$

$$= \frac{\sqrt{b^2 - ac}}{a}$$

As $\lambda = an^2$

$$\sqrt{\lambda} = n\sqrt{a}$$

Time period of motion $= \frac{2\pi}{\sqrt{\lambda}}$

$$= \frac{2\pi}{n\sqrt{a}}$$

end