Rectilinear motion

The motion of a body along a straight line is called rectilinear motion.

Motion with constant acceleration

Question

Derive the expression for velocity and distance covered by a particle in time \( t \) when it is moving with constant acceleration.

Solution

Considering a particle which is moving with constant acceleration along a straight line.

Let at \( t = 0 \) the particle is at point \( O \) and is moving with velocity \( u \). After a time \( t \) the particle is at point \( P \) at a distance \( x \) from \( O \) and attains a velocity \( v \).

we know that

\[
\frac{dx}{dt} = v
\]

\[
diff \text{ w.r.t. } t'
\]

\[
\frac{d^2x}{dt^2} = a
\]

\[
\frac{d^2x}{dt^2} = \alpha
\]

Integrating w.r.t. \( t \)

\[
\int \frac{d^2x}{dt^2} \ dt = \alpha \int dt
\]

\[
\frac{dx}{dt} = \alpha t + A
\]
\begin{align*}
V &= at + A \quad \text{(or } \frac{dx}{dt} = V) \quad (1) \\
\text{Initially when } t = 0, \ V &= U \\
\Rightarrow \quad u &= u(0) + A \\
\Rightarrow \quad A &= U \\
\text{From (1) we have} \\
V &= at + u \\
\text{or } \ V &= u + at \quad \text{is the velocity of the} \\
\text{particle at any time } t. \\
\text{Again } \quad V &= u + at \\
\text{then } \quad \frac{dx}{dt} &= u + at \\
\Rightarrow \quad \frac{dx}{dt} &= (u + at) \ dt \\
\text{Integrating: } \quad \int \frac{dx}{dt} \ dt &= \int (u + at) \ dt \\
\int dx &= u \ dt + at^2 \ dt \\
\Rightarrow \quad x &= ut + at^2 + B \quad (2) \\
\text{Initially when } t = 0, \ x &= 0 \\
\Rightarrow \quad 0 &= u(0) + a(0) + B \\
\Rightarrow \quad B &= 0 \\
\text{So, from (2) we have} \\
x &= ut + at^2 + 0 \\
\text{or } \ x &= ut + \frac{1}{2} at^2 \text{ is the distance} \\
\text{covered by the particle in time } t. \\
\text{Question} \\
\text{Drive the expression } V^2 - u^2 = 2ax \\
\text{where } u \text{ and } v \text{ are velocities of a particle at } t = 0 \text{ and } t = t \text{, sup.} \\
\text{`a' is the acceleration and } x \text{ is the distance covered in time } t.
Solution

we know that

\[ v = u + at \]  \hspace{1cm} (1) \\
\[ x = ut + \frac{1}{2}at^2 \]  \hspace{1cm} (2)

Now from (1) \[ t = \frac{v - u}{a} \]

Using the value of \( t \) in (2)

\[ x = u \left( \frac{v - u}{a} \right) + \frac{1}{2} \frac{a}{a^2} (v-u)^2 \]

\[ x = u \left( \frac{v - u}{a} \right) + \frac{1}{2} \frac{a}{a^2} (v-u)^2 \]

\[ 2ax = 2uv - 2u^2 + v^2 + u^2 = 2uv \]

\[ 2ax = v^2 - u^2 \] as required

Example (1-15)

Find the distance travelled by the particle moving in a straight line with uniform acceleration in the \( n \)th unit of time.

Solution

Considering a particle which is moving in a straight line with uniform acceleration. Let \( x_1 \) and \( x_2 \) be the distances travelled by the particle in the first \( n \) and \( n-1 \) units of time respectively.

Then by using the formula

\[ x = ut + \frac{1}{2}at^2 \]

we have

\[ x_1 = u(n) + \frac{1}{2}an^2 \]  \hspace{1cm} (1) \\
\[ x_2 = u(x-1) + \frac{1}{2}a(n-1)^2 \]  \hspace{1cm} (2)
Hence the distance travelled in \( n \)th unit of time is given by

\[
x_1 - x = \frac{1}{2}(u(n+1) + u(n)^2) - \frac{1}{2}(u(n-1) + u(n-2) + \frac{1}{2}a(n-1)^2)
\]

\[
= \frac{1}{2}u(n^2 - n) - un + u - \frac{1}{2}a(n^2 + 1 - 2n)
\]

\[
= u + \frac{1}{2}a(n^2 - n^2 - 1 + 2n)
\]

\[
= u + \frac{1}{2}a(2n-1)
\]

**Motion with Variable Acceleration**

(i) **Time dependent acceleration**

If acceleration of a particle depends upon time, then such an acceleration is called time dependent acceleration.

Since \( \frac{d^2x}{dt^2} = a(t) \)

Integrating both the sides

\[
\frac{dx}{dt} = \int a(t)\,dt + A
\]

\[
x = \int b(t)\,dt + At + B
\]

where the constants \( A \) and \( B \) are determined by using the boundary conditions.

**Example P-153**

Find the distance travelled and the velocity attained by a particle moving in a straight line, at any time \( t \), if it starts from rest at \( t=0 \) and is subject to an acceleration \( x^2 + \sin t + e^t \).
Solution

Considering a particle which starts moving from rest from point '0'. Let after a time 't' the particle reaches at point 'A', after covering a distance 'x' and its acceleration becomes \( a = t^2 + \sin t + e^t \) and it attains the velocity 'v'.

Now, \( a = t^2 + \sin t + e^t \)
\[
\frac{dv}{dt} = t^2 + \sin t + e^t
\]
\[
\int dv = \int (t^2 + \sin t + e^t) \, dt
\]
Integrating both the sides

\[
\int dv = \frac{1}{3} t^3 + \cos t + e^t + A
\]

Initially when \( t = 0 \), \( v = 0 = v \)
\[
0 = 0 - 1 + 1 + A
\]
\[
\Rightarrow A = 0
\]

Put \( A = 0 \) in (1)

\[
v = \frac{1}{3} t^3 - \cos t + e^t \]

is the velocity of the particle after time 't'.

Now
\[
\frac{dx}{dt} = \frac{1}{3} t^3 - \cos t + e^t \quad \Rightarrow v = \frac{dx}{dt}
\]
\[
\int dx = \frac{1}{3} \int t^3 \, dt - \int \cos t \, dt + \int e^t \, dt
\]
\[
x = \frac{1}{12} t^4 - \sin t + e^t + B
\]
\[
x = \frac{1}{12} t^4 - \sin t + e^t + B \quad (2)
\]
Initially when \( t = 0 \), \( \mathbf{x} = 0 \)

\[
0 = c - 0 + 1 + B \\
\Rightarrow B = -1
\]

Put \( B = -1 \) in (2)

\[
x = \frac{1}{12} t^4 - \sin t + e^t - 1
\]

is the distance traveled by the particle after time \( t \).

(ii) Velocity dependent acceleration

If acceleration of a particle depends upon velocity, then such an acceleration is called velocity dependent acceleration.

Considers

\[
\frac{v}{d\alpha} = \alpha(v) \\
\Rightarrow d\alpha = \frac{v}{\alpha(v)} \, dv
\]

Integrating

\[
\int d\alpha = \int \frac{v}{\alpha(v)} \, dv
\]

\[
x = \int \frac{v}{\alpha(v)} \, dv + C
\]

Where constant \( C \) can be determined when velocity of the particle is known for some value of \( x \).

Also

\[
\frac{d\mathbf{v}}{dt} = \alpha(v) \\
\Rightarrow dt = \frac{d\mathbf{v}}{\alpha(v)}
\]

\[
\int dt = \int \frac{d\mathbf{v}}{\alpha(v)}
\]

\[
t = \int \frac{d\mathbf{v}}{\alpha(v)} + D
\]

Where constant \( D \) can be determined if the velocity at some instant is known.
Example 7-154

A particle moves in a straight line with an acceleration \( kv^3 \). If its initial velocity is \( v \), find the velocity and the time spent when the particle has travelled a distance \( x \).

Solution

Considering a particle which starts moving from the point '0' with initial velocity '\( u \)'. Let after a time '\( t \)' the particle reaches at point '\( P \)' after covering a distance '\( x \)' and attains a velocity '\( v \)'.

Now, Acceleration = \( kv^3 \)

\[ v \frac{dv}{dx} = kv^3 \]

\[ \frac{dv}{v^2} = k \, dx \]

Integrating both the sides

\[ \int \frac{dv}{v^2} = k \int dx \]

\[ \frac{-1}{v} = kx + A \quad (1) \]

Initially when \( x = 0 \), \( v = u \)

\[ -\frac{1}{u} = k(0) + A \]

\[ \Rightarrow A = -\frac{1}{u} \]

Put \( A = -\frac{1}{u} \) in \( (1) \)

\[ \frac{-1}{v} = kx - \frac{1}{u} \]

\[ \frac{1}{v} = \frac{1 - kux}{u} \quad (2) \]

So, \( \frac{v}{1 - kux} \) is the velocity of the particle when it is at a distance '\( x \)' from '0'.
Again, acceleration = $k v^3$

\[
\frac{dv}{dt} = k v^3
\]

\[
\frac{dv}{V^3} = k dt
\]

Integrating both sides:

\[
\int V^{-3} dv = k \int dt
\]

\[
\frac{V^{-2}}{-2} = kt + B
\]

\[
-\frac{1}{2V^2} = kt + B \quad \text{(3)}
\]

Initially, when $t = 0$, $V = u$

\[
-\frac{1}{2u^2} = k(0) + B
\]

\[
B = -\frac{1}{2u^2}
\]

Put $B = -\frac{1}{2u^2}$ in (3),

\[
-\frac{1}{2V^2} = kt - \frac{1}{2u^2}
\]

\[
k t = \frac{1}{2} u^2 - \frac{1}{2} V^2
\]

\[
k t = \frac{1}{2} \left[ \frac{1}{u^2} - \frac{1}{V^2} \right]
\]

\[
x = \frac{1}{2K} \left[ \frac{1}{u^2} - \frac{1}{V^2} \right] \quad \text{(4)}
\]

Using (2) in (4)

\[
x = \frac{1}{2K} \left[ \frac{1}{u^2} - \left( \frac{1 - ku^2}{u^2} \right) \right]
\]

\[
= \frac{1}{2K} \left[ \frac{1}{u^2} - \left( \frac{1 - ku^2}{u^2} \right) \right]
\]

\[
= \frac{1}{2K} \left[ 1 - \left( 1 + k^2 u^2 x^2 - 2 ku x \right) \right]
\]

\[
= \frac{1}{2Ku^2} \left[ ku x \left( 2 - ku x \right) \right]
\]

\[
x = \frac{x}{2u} \left( 2 - ku x \right) \text{ is the time taken by the particle to cover a distance } x.
\]
Distance dependent acceleration

If acceleration of a particle depends upon distance, then such an acceleration is called distance dependent acceleration.

Since \( \frac{dv}{dx} = a(x) \)
then \( v \frac{dx}{dx} = a(x) dx \)

Integrating \( \int v \frac{dx}{dx} = \int a(x) dx \)

\( \frac{v^2}{2} = \int a(x) dx + C \)

where \( f(x) = \int a(x) dx \)

\( x^2 \)

\( \frac{v^2}{2} = f(x) + C \)

\( v = \pm \sqrt{2f(x)+D} \) where \( C = D \)

\( \frac{dx}{dt} = \pm \sqrt{2f(x)+D} \)

\( \Rightarrow \frac{dt}{dx} = \frac{\pm \sqrt{2f(x)+D}}{dx} \)

\( \int dt = \pm \int \frac{dx}{\sqrt{2f(x)+D}} \)

\( t = \pm \int \frac{dx}{\sqrt{2f(x)+D}} + E \)

Example Page 156

Discuss the motion of a particle moving in a straight line with an acceleration \( x^3 \)
where 'x' is the distance of the particle from a fixed point 'O' on the line, if it starts at \( t=0 \) from a point \( x = c \) with velocity \( \frac{c^2}{\sqrt{2}} \).

Solution

Considering a particle which is moving in a straight line with an acceleration \( x^3 \)
where \( x \) is the distance of the particle from fixed point 'O' on the line. Let the particle starts motion from point 'A' at \( t=0 \) with velocity \( \frac{c^2}{\sqrt{2}} \).

Here

\[
\frac{v}{dx} = x^3
\]

\[
v = x^2 dx
\]

Integrating \( \int v \, dv = \int x^3 \, dx \)

\[
\frac{v^2}{2} = \frac{x^4}{4} + C \quad (1)
\]

Initially when \( x=c, v = \frac{c^2}{\sqrt{2}} \)

\[
\frac{1}{2} \left( \frac{c^2}{\sqrt{2}} \right)^2 = \frac{c^4}{4} + D
\]

\[
\frac{c^4}{4} = \frac{c^4}{4} + D
\]

\[
D = 0
\]

Put \( c=0 \) in (1).

\[
\frac{v^2}{2} = \frac{x^4}{4}
\]

\[
\frac{v}{x} = \frac{x^2}{4}
\]

\[
V = \frac{1}{\sqrt{2}} \frac{x^2}{2} \text{ is the velocity of the particle when it is at a distance } x \text{ from point 'O'.}
\]

Again

\[
v = \frac{1}{\sqrt{2}} x^2
\]

\[
\frac{dx}{dt} = \frac{1}{\sqrt{2}} \frac{x^2}{2}
\]

\[
\frac{\sqrt{2}}{x^2} \, dx = dt
\]

Integrating \( \int dt = \int \frac{\sqrt{2}}{x^2} \, dx \)

\[
t = -\frac{\sqrt{2}}{x} + D \quad (2)
\]

Initially when \( t=0, x = c \)

\[
0 = -\frac{\sqrt{2}}{c} + D
\]
\[ D = \frac{\sqrt{2}}{C} \]

Put \( D = \frac{\sqrt{2}}{C} \) in (2)

Then
\[
\tau = \frac{\sqrt{2}}{x} + \frac{\sqrt{2}}{\tau} = \frac{1}{\tau} \left( \frac{1}{C} - \frac{1}{x} \right)
\]

is the time spent by the particle to cover the distance \( x \).

**Graphical Methods**

Considering a particle which is moving along the Cartesian curve
\( y = f(x) \):

Let at any instant the particle is at point \( P(x, y) \). Draw a tangent to the curve at \( P(x, y) \) then:

Slope of the tangent line = \( \frac{dy}{dx} \)

Let \( A \) be the area under the curve \( y = f(x) \) and in between the lines \( x = a \) and \( x = b \):

Then
\[
A = \int_{x=a}^{x=b} y \, dx
\]
Space time curve

Considering a particle which is moving along the space time curve.

\( x = f(t) \): Let at any instant the particle is at point \( P(t, x) \).

Draw a tangent line to this curve at point \( P \).

The slope of the tangent line is \( \frac{dx}{dt} \) = velocity.

\( \therefore \) The velocity at any point \( P \) = slope of the tangent line at the point \( P \) of space time curve.

Clearly, when a particle moves along a space time curve, then at any instant its velocity is given by the slope of the tangent line at that point.

Let the particle is moving with constant velocity then

\[ \frac{dx}{dt} = c \]

\[ dx = c \, dt \]

Integrating \( \int dx = \int c \, dt \)

\( x = ct + A \) is the first degree equation in \( x \) and \( t \) representing a straight line.

Clearly, if a particle moves in a space time plane with constant velocity, its motion will be rectilinear motion.
Velocity time curve

Considering a particle
which is moving along
the velocity time curve
\( v = f(t) \). Let at any
instant \( t \), the particle
is at point \( P(t, v) \).

Draw a tangent line
to this curve through the point \( P(t, v) \), then
slope of the tangent line = \( \frac{dv}{dt} \).

Acceleration = \( \frac{dv}{dt} \).

Acceleration at any point \( P \) of the particle line at point \( P \).

Clearly when a particle moves along the velocity
time curve, then at any point its acceleration
is equal to the slope of the tangent line at
that point.

Let the particle is moving with constant acceler-
tion then \( \frac{dv}{dt} = a \) (Constant)

\( \frac{dv}{dt} = a \, dt \)

Integrating \( \int dv = \int a \, dt \)

\( v = at + B \) is the first
degree equation in \( v \) and \( t \) representing a straight
line.

Clearly if a particle moves in a velocity time plane
with constant acceleration, its motion will be rectilinear.
Let 'A' be the area under the velocity-time curve $V = f(t)$ then

$$A = \int_v^t dt = \int \frac{dx}{dt} dt = \int dx = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

where $x(t_2) - x(t_1)$ is the distance travelled by the particle in the time interval $(t_1, t_2)$

**Velocity space curve**

**Question**

Show that the length of the subnormal at any point of the velocity space curve gives the corresponding acceleration of the particle.

**Solution**

Considering a particle moving along the velocity space curve $V = f(x)$, let at any instant the particle is at the point $P(x, V)$. Draw tangent and normal to this curve at point $P(x, V)$.
Slope of the tangent line = \( \frac{dv}{dx} = \tan \theta \)

Now from right angled triangle PMN,

\[
\frac{MN}{PM} = \tan \theta = \frac{MN}{PM} \cdot \tan \alpha
\]

Put \( PM = V \), \( \tan \theta = \frac{dv}{dx} \)

\[
MN = V \frac{dv}{dx}
\]

Length of subnormal = acceleration

Clearly the length of the subnormal at any point of the velocity space curve, gives the corresponding acceleration of the particle.

Example P-158

A particle starts from rest from 'O' with constant acceleration 'a'. When its velocity acquires a certain value 'v' it moves uniformly and then its velocity starts decreasing with a constant retardation 'a' till it comes to rest.

Find the distance travelled by the particle, if the time taken from rest to rest is 't'.

Solution

Considering a particle that starts moving from rest from the point 'O' and move with constant acceleration 'a'.

\[ t_1, t_2, t_3 \]

\[ O, M, N, C \]

\[ V, V \]

\[ A, B \]
and attains a velocity \( v \) in time \( t \), then it moves uniformly for the time \( t_2 \), then its velocity starts decreasing at the rate of \( 2a \) and it comes to rest at point \( C \) after further time \( t_3 \).

Let \( t \) be the time taken by the particle from rest to rest, then

\[
t = t_1 + t_2 + t_3 \quad (1)
\]

Now the slope of the line OA in the velocity-time graph is the acceleration and the slope of the line BC is the retardation of the particle. Thus we have

\[
\frac{v}{t_1} = a \quad \text{and} \quad \frac{v}{t_3} = 2a
\]

Hence \( t_1 = \frac{v}{a} \) and \( t_3 = \frac{v}{2a} \)

From (1)

\[
t_2 = t - t_1 - t_3
\]

\[
= t - \frac{v}{a} - \frac{v}{2a}
\]

\[
t_2 = t - \frac{3v}{2a}
\]

The distance travelled in time \( t \) is, therefore,

given by

\[
x = \triangle OAM + \text{Area of the rectangle } ABNM + \triangle BNC
\]

\[
= \frac{1}{2} t_1 v + t_2 v + \frac{1}{2} t_3 v
\]

\[
= \frac{1}{2} v \left( t_1 + 2 t_2 + t_3 \right)
\]

\[
= \frac{1}{2} v \left( \frac{v}{a} + 2 \left( t - \frac{3v}{2a} \right) + \frac{v}{2a} \right)
\]

\[
= \frac{1}{2} v \left( \frac{v}{a} + 2 t - 3 \frac{v}{2a} + \frac{v}{2a} \right)
\]

\[
= \frac{1}{2} v \left( 2 t - \frac{3v}{2a} \right) \text{ is the required distance}
\]
Example-1: P-160

A stone is let fall freely from a height of 100 ft. Find the time that it takes and the velocity that it acquires on reaching the ground.

**Solution**

Considering a stone that is dropped freely from a height of 100 ft.

For finding the final velocity we apply the formula

\[ v^2 - u^2 = 2gs \]

Here, \( u = 0 \), \( g = 32 \text{ ft/sec}^2 \), \( s = 100 \text{ ft} \)

\[ v^2 - 0^2 = 2(32)(100) \]

\[ v^2 = 6400 \]

\[ v = 80 \text{ ft/sec} \]

To find the time spent by the particle in reaching the ground we use the formula

\[ v = u + gt \]

Here \( v = 80 \text{ ft/sec}, u = 0, g = 32 \text{ ft/sec} \)

\[ 80 = 0 + 32t \]

\[ 32t = 80 \]

\[ t = \frac{80}{32} = \frac{5}{2} \text{ sec} \]

Example-2: P-160

A particle projected vertically upwards at \( t=0 \) with a velocity \( u \), passes a point at a height \( h \) at \( t = t_1 \) and \( t = t_2 \). Show that

\[ t_1 + t_2 = \frac{2u}{g} \] and \( t_1, t_2 = \frac{2h}{g} \)
Solution

Considering a particle which is projected vertically upwards at \( t=0 \) with a velocity \( u \), and passes a point at a height \( h \) at \( t=t_1 \), and \( t=t_2 \).

The distance \( x \) travelled by the particle in time \( t \), is given by the formula

\[
x = ut - \frac{1}{2} gt^2
\]

\[
h = ut - \frac{1}{2} gt^2
\]

\[
x^2 = 2h = 2ut - gt^2
\]

\[gt^2 - 2ut + 2h = 0\]

It is quadratic equation in \( t \), so there will be two roots, say \( t \), and \( t_2 \).

Here \( a = g \), \( b = -2u \), \( c = 2h \)

Sum of the roots \( = -\frac{b}{a} \)

\[
t_1 + t_2 = -\left(-\frac{2u}{g}\right) = \frac{2u}{g}
\]

and Product of the roots \( = \frac{c}{a} \)

\[
t_1 \cdot t_2 = \frac{2h}{g}
\]

Simple Harmonic Motion

The motion of a body in a straight line in which the acceleration of the body is directed towards its mean position and is directly proportional to its displacement from the mean position is called simple harmonic motion. The body performing simple harmonic motion is called
Simple harmonic oscillator (S.H.O).

**Question**

Derive an expression for the velocity and distance travelled by a particle at any instant, when it is executing simple harmonic motion.

**Solution**

Considering a body which is performing simple harmonic motion and is moving away from the mean position. Let at any instant \( t \), the particle is at point \( B \) whose distance from the mean position is \( x \).

According to definition of simple harmonic motion

\[
\text{Acceleration } \lambda = -x
\]

\[
v dv = -\lambda x \frac{dx}{dx}
\]

Integrating both the sides

\[
\int v \, dv = -\int \lambda x \, dx
\]

\[
\frac{v^2}{2} = -\lambda \frac{x^2}{2} + C \quad \text{(1)}
\]

When \( x = a, \quad v = 0 \)

\[
0 = -\lambda \frac{a^2}{2} + C
\]

\[
\Rightarrow \quad C = \frac{\lambda a^2}{2}
\]

Put \( C = \frac{\lambda a^2}{2} \) in (1)

\[
\frac{v^2}{2} = -\lambda \frac{x^2}{2} + \frac{\lambda a^2}{2}
\]

\[
\frac{v^2}{2} = \lambda a^2 - \lambda x^2
\]
\[ V^2 = \lambda (a^2 - x^2) \]
\[ V = \sqrt{\lambda (a^2 - x^2)} \]
is the velocity expression for S.H.O. when it is at a distance \( x \) from the mean position.

Now \[ V = \sqrt{\lambda (a^2 - x^2)} \]
\[ \frac{dx}{dt} = \sqrt{\lambda (a^2 - x^2)} \]
\[ \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\lambda} \, dt \]
Integrating both the sides
\[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\lambda} \int dt \]
\[ \sin^{-1} \left( \frac{x}{a} \right) = \sqrt{\lambda} \, t + D \quad (2) \]
when \( t = 0 \), \( x = a \)
\[ \sin^{-1} \left( \frac{a}{a} \right) = \sqrt{\lambda} \, (0) + D \]
\[ \sin^{-1} (1) = D \]
\[ \frac{\pi}{2} = D \]

Put \( D = \frac{\pi}{2} \) in (2)
\[ \sin^{-1} \left( \frac{x}{a} \right) = \sqrt{\lambda} \, t + \frac{\pi}{2} \]
\[ \frac{x}{a} = \sin \left( \sqrt{\lambda} \, t + \frac{\pi}{2} \right) \]
\[ \frac{x}{a} = \cos \sqrt{\lambda} \, t \]
\[ x = a \cos \sqrt{\lambda} \, t \]
is the distance travelled expression by S.H.O. in time \( t \) (time is measured when the particle is at rest at point A)

Special Case: Expression for the distance of S.H.O. when time is measured from 'O' (Mean position)
From (2) we have
\[
\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{a} \cdot t + D
\]
when \( x = 0 \), \( t = 0 \)
\[
\sin^{-1}(0) = \sqrt{a}(0) + D
\]
\[
\Rightarrow D = 0 \quad \therefore \sin^{-1}(0) = 0
\]
we have
\[
\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{a} \cdot t
\]
\[
\frac{x}{a} = \sin \sqrt{a} \cdot t
\]
\[
x = a \sin \sqrt{a} \cdot t \quad \text{is the distance}
\]
travelled by S.H.O in time \( t \) (when time is measured from '0').

The nature of Simple Harmonic Motion

Amplitude of motion

The maximum displacement of the body from the mean position is called amplitude of motion.

Expression for maximum velocity of S.H.O

The velocity of S.H.O will be maximum at the mean position i.e. when \( x = 0 \).

Since \( V = \sqrt{a^2 - x^2} \)

for \( V_{\text{max}} \).

Put \( x = 0 \)

\[
V_{\text{max}} = \sqrt{a^2 - 0^2} = \sqrt{a^2}
\]

\[
V_{\text{max}} = \sqrt{a} \quad \text{is the expression}
\]

for maximum velocity of S.H.O.
Time period of motion

The time taken by S.H.O. to complete one vibration is called its time period. It is usually denoted by $T$.

We know that

$$x = a \cos \sqrt{\frac{k}{m}} t$$

we can write

$$x = a \cos \left( \sqrt{\frac{k}{m}} t + 2\pi \right) = a \cos \sqrt{\frac{k}{m}} \left( t + \frac{2\pi}{\sqrt{\frac{k}{m}}} \right)$$

also

$$x = a \cos \left( \sqrt{\frac{k}{m}} t + 4\pi \right) = a \cos \sqrt{\frac{k}{m}} \left( t + \frac{4\pi}{\sqrt{\frac{k}{m}}} \right)$$

and

$$x = a \cos \left( \sqrt{\frac{k}{m}} t + 6\pi \right) = a \cos \sqrt{\frac{k}{m}} \left( t + \frac{6\pi}{\sqrt{\frac{k}{m}}} \right)$$

So

$$x = a \cos \sqrt{\frac{k}{m}} t = a \cos \sqrt{\frac{k}{m}} \left( t + \frac{2\pi}{\sqrt{\frac{k}{m}}} \right) = a \cos \sqrt{\frac{k}{m}} \left( t + \frac{4\pi}{\sqrt{\frac{k}{m}}} \right)$$

Clearly $x$ repeats itself after the times

$$t, \ t + \frac{2\pi}{\sqrt{\frac{k}{m}}}, \ t + \frac{4\pi}{\sqrt{\frac{k}{m}}}, \ldots$$

Clearly $x$ repeats itself after the time interval \( (t + 2\pi) - t \) which is known as the time period of motion.

So \( T = (t + 2\pi) - t \)

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

is the time period of motion.

**Frequency of oscillation**

The number of vibrations completed by S.H.O. in one second is called its frequency. It is equal to the reciprocal of time period.

$$\text{Frequency} = v = \frac{1}{T}$$
Geometrical Representation of S.H.M

Question

Show that when a particle moves along a circle, its projection moving along the diameter of the circle performs simple harmonic motion.

Solution

Considering a particle which is moving along a circle

\[ x^2 + y^2 = r^2 \]

Let at any time \( t \) the particle is at point \( P(x, y) = P(r \cos \theta, r \sin \theta) \)

Now

\[ x = r \cos \theta \quad \text{(1)} \]

diff. w.r.t. \( t \)

\[ \frac{dx}{dt} = r(-\sin \theta) \frac{d\theta}{dt} \]

\[ v = -r \omega \sin \theta \]

Again diff. w.r.t. \( t \)

\[ \frac{dv}{dt} = -r \omega \cos \theta \frac{d\theta}{dt} \]

\[ a = -r \omega^2 \cos \theta \]

\[ = -\omega^2 (r \cos \theta) \]

\[ = -\omega^2 x \quad \text{by (1)} \]

\[ \Rightarrow a \propto x \]

Clearly the acceleration of projection is directly proportional to the displacement of projection and
the +ve sign indicates that the acceleration
is directed towards the mean position.

The projection of the particle moving along
the diameter of the circle performs S.H.M.
Example P-166 A particle performs S.H.M.
at distance \( x \) from the centre of motion
and moves from it. Find the time in which
it will be again at the same point moving
in the opposite direction if the time period \( T \)
of motion is \( T \), and (i) the amplitude of
motion is \( a \) and (ii) the velocity of the
particle is \( v \).

Solution

Consider a

\[
\text{Consider a particle which is performing simple harmonic motion and is at a distance } x \text{ from the centre of motion. We have to find the time when the particle will again at the same point moving in the opposite direction. Let the amplitude of motion be } a \text{ and the time period of motion be } T. \]

Let the time taken by the particle to cover the distance \( BA \) be \( t \), then

\[
\text{Required time } = t + t = 2t.
\]

We have that

\[
x = a \cos A \, t
\]

\[
\frac{dx}{dt} = \cos A \, t
\]
\[ \sqrt{\Delta t} = \cos^{-1} \frac{a}{A} \]

\[ \Delta t = \frac{1}{\sqrt{A}} \cos^{-1} \left( \frac{a}{A} \right) \quad \text{(1)} \]

From which we have:

\[ T = \frac{2\pi}{\sqrt{A}} \]

\[ A = \frac{2\pi}{T} \]

\[ t = \frac{1}{2\pi} \cos^{-1} \left( \frac{a}{A} \right) \]

\[ 2t = \frac{1}{\pi} \cos^{-1} \frac{a}{A} \quad \text{(2)} \]

The period of the motion is given by:

\[ \text{Period} = \frac{T}{\pi} \cos^{-1} \frac{a}{A} \]

Now we shall find the required time in terms of velocitiy \( v \).

\[ v = \sqrt{A\left(a^2 - x^2\right)} \]

\[ v^2 = A\left(a^2 - x^2\right) \]

\[ v^2 = a^2 - x^2 \]

\[ \frac{v^2}{A} = 1 - \frac{x^2}{A} \]

\[ \therefore \quad x^2 = \frac{v^2}{A} + x^2 \]
\[ a^2 = \frac{v^2}{4\pi^2} + \alpha^2 \]
\[ = \frac{v^2 - \alpha^2}{4\pi^2} + \alpha^2 \]
\[ \alpha^2 = \frac{v^2 \alpha^2 + 4\pi^2 \alpha^2}{4\pi^2} \]
\[ \alpha = \sqrt{v^2 - \alpha^2 + 4\pi^2 \alpha^2} \] \[ \tag{3} \]
\[ \text{using (3)} \rightarrow (2) \]

Required time = \[ \frac{T}{\pi} \cos \left( \frac{\alpha}{\sqrt{v^2 - 4\pi^2 \alpha^2}} \right) \]

\[ = \frac{T}{\pi} \cos \left( \frac{2\pi \alpha}{\sqrt{v^2 - 4\pi^2 \alpha^2}} \right) \]

\[ \Rightarrow \frac{T}{\pi} \tan^{-1} \left( \frac{\pi \alpha}{v^2} \right) \]

End
Exercises Set 8, P-167

Q.M.1. Obtain the equations of motion by Graphical Method.

\[\begin{align*}
A(t, v) & = B(t, v) \\
\text{Considering a particle} & \\
\text{which is moving in} & \\
\text{velocity-time plane} & \\
\text{with constant} & \\
\text{acceleration}. \quad \text{Thus} & \\
\text{motion of the} & \\
\text{particle will be retilinear.} & \\
\text{Also} & \\
\text{Ace of the particle = Slope of the} & \\
\text{Line AB} & \\
\end{align*}\]

\[\begin{align*}
\alpha & = \tan \theta \\
\alpha & = \frac{v-u}{t} \\
\end{align*}\]

\[\begin{align*}
v-u & = at \quad \text{(1)} \\
v & = u + at \quad \text{(2)} \\
\text{is the first equation of motion.} & \\
\text{Also Distance travelled by the particle} & = \text{Area under the curve}\]
31

\[ x = \text{Area of triangle } ABC + \text{Area of } \text{Sectors } \theta \]

\[ = \frac{1}{2} b (v-u) + ut \]

\[ = \frac{1}{2} br + at \quad : \quad v-u = at \text{ by } (1) \]

\[ x = ut + \frac{1}{2} at^2 \quad \text{is the 2nd equation} \]

of motion. \hfill (3)

From (1) \[ t = \frac{v-u}{a} \quad (b) \]

Using (b) \[ x = \left( \frac{v-u}{a} \right) t + \frac{1}{2} a \left( \frac{v-u}{a} \right)^2 \]

\[ = \frac{uv-u^2}{a} + \frac{1}{2} a \frac{v^2+u^2-2uv}{a^2} \]

\[ = \frac{uv-u^2 + v^2+u^2-2uv}{a} \]

\[ = \frac{2uv}{2a} \]

\[ 2ax = 2uv-2u^2 + v^2+u^2 \]

\[ 2ax = u^2-u^2 \quad \text{is the 3rd equation} \]

of motion.
A particle moving in a straight line starts from rest and is accelerated uniformly to attain a velocity of 60 mph in 4 seconds. Find the acceleration of motion and the distance travelled by the particle in the last 3 seconds.

Consider a particle which is moving in a straight line from rest and is accelerated uniformly to attain a velocity of 60 mph in 4 seconds.

\[ v = 60 \text{ mph} \]
\[ = \frac{60 \times 1760 \times 3}{29} \quad \text{mile to ft conversion} \]
\[ = 88 \text{ ft/sec} \]

We know that
\[ v = u + at \]
\[ 88 = 0 + a \times 4 \]
\[ 4a = 88 \]
\[ a = \frac{88}{4} = 22 \text{ ft/sec}^2 \]

i.e., \[ a = 22 \text{ ft/sec}^2 \]
Let the particle start from point 0 and reach point A in 1 second after covering a distance \( x_1 \),

then \( x = ut + \frac{1}{2} at^2 \)

\( x_1 = 0 \times 1 + \frac{1}{2} \times 11 \times (1)^2 \)

\( x_1 = 11 \) ft.

Let the particle reach at point B in 4 seconds after covering a distance \( x_2 \),

then \( x = ut + \frac{1}{2} at^2 \)

\( x_2 = 0 \times 4 + \frac{1}{2} \times 11 \times (4)^2 \)

\( x_2 = 0 + 11 \times 16 \)

\( x_2 = 176 \) ft.

Distance covered in the last 3 seconds

\( = x_2 - x_1 \)

\( = 176 - 11 \)

\( = 165 \) ft.
Ex. 3 Two particles start simultaneously from a point O and move in a straight line. 
One with a velocity of 45 mph and an acceleration of 2 ft/sec² and the 
other with a velocity of 90 mph and a 
retardation of 8 ft/sec². Find the time 
after which the velocities of the particles 
will be same and the distance of O 
from the point where they meet again.

Let the 
consider two particles start moving 

simultaneously from point O and 
moves in straight line

Particle P₁  Particle P₂

\[ \begin{align*}
\text{v}_1 &= 45 \text{ mph} \\
\text{v}_2 &= 90 \text{ mph} \\
\text{a}_1 &= 2 \text{ ft/sec}^2 \\
\text{a}_2 &= -8 \text{ ft/sec}^2
\end{align*} \]

But after a time \( t \), the velocity of 
both particles be same, say \( v \).
\[ \text{Particle } P_1 \quad \text{Particle } P_2 \]

\[ v = u + at \quad v = u + at \]
\[ v = 66 + gt \quad v = 132 + (8)t \]
\[ v = 132 - 8t \]

\[ v = 66 + 2t \]
\[ v = 132 - 8t \]
\[ 0 = -66 + 10t \]
\[ 10t = 66 \]
\[ t = \frac{66}{10} \]
\[ t = 6.6 \text{ sec} \]

So after \( t = 6.6 \) sec, the velocity of both particles will be same.

Let \( x' \) be the distance of the particles from point \( O \) from the point where they meet again after time \( t' \).

\[ \text{Particle } P_1 \quad \text{Particle } P_2 \]
\[ x = u_1 t + \frac{1}{2} a t^2 \quad x = u_2 t + \frac{1}{2} a t^2 \]
\[ x = 6t \left( t + \frac{1}{2} \right) \quad x = 132t + \frac{1}{2} (8t^2) \]
\[ n = 6t^2 + t^2 \quad n = 132t + 4(t^2) \]
\[ x = 66t^2 + t^2 \]
\[ x = 132t^2 - 4t^2 \]
\[ \frac{dx}{dt} = -66t + 5t^2 \]
\[ 5t^2 - 66t = 0 \]
\[ t'(5t - 66) = 0 \]
\[ t = 0 \quad \text{or} \quad 5t = 66 \]

At the start, when they were together at \( t = 66 \) is the time when they meet again. After a distance \( x \) from 0.

\[ \text{Part } t = \frac{66}{5} \text{ in } (\frac{13}{5}) \]
\[ x = 66\left(\frac{66}{5}\right)^2 + \left(\frac{66}{5}\right)^2 \]
\[ = \frac{(66)^2}{5} + \frac{(66)^2}{5} = \frac{(66)^2}{5} + \frac{(66)^2}{5} \]
\[ = \frac{5(66)^2 + (66)^2}{25} \]
\[ = 2178 \text{ ft} + 3356 \]
\[ = 5.53 \text{ ft. Ans} \]
A particle moves along a straight line from rest and is accelerated uniformly till it attains a velocity \( v \). The motion is then retarded and the particle comes to rest after traversing a total distance. If the acceleration is \( a \), find the retarded and the total time taken by the particle from rest to rest.

Consider a particle which is moving along a straight line, starts from rest at \( t = 0 \) and is accelerated uniformly till it attains the velocity \( v \). The motion is then retarded and the particle comes to rest after further time \( t_2 \) after covering a total distance \( x \).

\[
x = \frac{1}{2} (t_1 + t_2) v
\]

or \( t_1 - t_2 = \frac{x}{v} \)  

Time taken by particle from rest to rest = \( \frac{2x}{v} \).
Also
\[ \text{Acc.} = \text{Slope of line } \Delta \overline{v} \]
\[ f = \frac{v}{\tau} \]
or \[ \tau f = v \]
(2) \[ 1 = \frac{\nu}{2} \]
(3)

Also Retardation = Slope of the line \( \Delta x \)
\[ x = \frac{\nu}{t^2} \]
\[ \alpha \tau^2 = \frac{\nu}{2} \]

using (2) & (3) in (1)
\[ \frac{\nu}{f} + \frac{\nu}{2} = \frac{2x}{\nu} \]
\[ \frac{\nu}{f} = \frac{2x - \nu}{\frac{\nu}{2}} \]
\[ \frac{\nu}{f} = \frac{2xf - \nu^2}{\nu} \]
\[ \frac{1}{f} = \frac{2xf - \nu^2}{\nu} \]
\[ \alpha \tau^2 = \frac{\nu^2}{2} \]
\[ 1 = \frac{\nu^2 f}{2} \]
\[ 2xf - \nu^2 = \frac{\nu^2 f}{2} \]
\[ \alpha \tau^2 = \frac{\nu^2}{2} \]

(3)

6.5 Two particles travel along a straight line
beams start at the same time and are
accelerated uniformly at different
rates. The middle is found that when a
particle attain the maximum, its
system is retarded uniformly. The
Two particles come to rest simultaneously at a distance \( x \) from the start. Find the acceleration of the first if \( x \) and that of the second if \( x \) and find the distance by the point where the two particles attain their max. velocities.

Consider two points of particles \( A \) and \( B \), particle \( P \) and \( Q \), which start at \( x \) and move with different rates. Let the particle move with acc. \( a \) and particle \( B \) move in the acc. \( b \).

The particle attain their max. velocities \( v \) at point \( A \) and \( B \) respectively.

\[
\begin{align*}
\text{Particle } P & \\
& \text{ Particle } Q \\
\frac{v^2 - u^2}{2a} & = \frac{v^2 - u^2}{2a} \\
\frac{v^2 - u^2}{2a} & = \frac{v^2 - u^2}{2a} \\
\frac{v^2 - u^2}{2a} & = \frac{v^2 - u^2}{2a} \\
\frac{v^2 - u^2}{2a} & = \frac{v^2 - u^2}{2a}
\end{align*}
\]
Distance from the point where the two particles attain their max velocity

\[ x = y \]

\[ = \frac{u^2 - u_1}{a} 
\]

\[ = \frac{2u^2 - u_1}{2a} \]

\[ = \frac{v^2}{2a} \]

**Q. 20:6** A particle is projected vertically upward with a velocity \( v \) and another is let fall from a height \( h \) at the same time. Find the height of the point where they meet each other.

Consider a particle \( P_1 \) which is projected vertically upward from point \( O \) with velocity \( v \) and at the same time another is let fall from point \( A \) at a height \( h \). Let both particles meet each other at
point B at a height \( h \) from point A.

The time taken by both particles to reach point B will be same say \( t \).

**Particle P₁**

\[ x = ut + \frac{1}{2} (-g)t^2 \]

\[ x = \sqrt{2gh} t - \frac{1}{2} gt^2 \]  

\[ h = x_t = 0xt + \frac{1}{2} gt^2 \]  

\[ h = x = \frac{1}{2} gt^2 \]  

\[ (1) \]

**Particle P₂**

\[ x = ut + \frac{1}{2} (-g)t^2 \]

\[ x = \sqrt{2gh} t - \frac{1}{2} gt^2 \]  

\[ h = x = + \frac{1}{2} gt^2 \]

\[ h = \sqrt{2gh} t \]

\[ \Rightarrow t = \frac{\sqrt{2gh}}{g} \]

\[ t = \frac{\sqrt{h}}{\sqrt{2g}} \]

Put \( t = \frac{\sqrt{h}}{\sqrt{2g}} \) in (1)

\[ x = \sqrt{2gh} \left( \frac{\sqrt{h}}{2g} \right) - \frac{1}{2} \left( \frac{\sqrt{h}}{2g} \right)^2 \]
\[ x = \sqrt{\frac{2gh}{g}} - \frac{1}{2} \frac{b}{2g} \]

\[ = \frac{b}{4} \]

\[ = \frac{4b - b}{4} \]

\[ = \frac{3b}{4} \]

or \( n = \frac{3b}{4} \) is the height at which both the particles meet each other.

\[ \text{Ex. no. 7 Two particles are projected simultaneously in the vertically upwards direction with velocities } \frac{1}{2} \sqrt{gk} \text{ and } \sqrt{gk} \text{ (} k \neq h) \text{. After a time } t \text{ when two particles are still in flight another particle is projected upward with a velocity } u. \]

First the condition so that the 3rd particle may meet the first two during their upward flight.

Let height attained by first particle be \( H \).

\[ v^2 = u^2 - 2gH \]

\[ 0 = \left( \frac{1}{2} \sqrt{gk} \right)^2 - 2(-g)H \]

\[ a = -g \]

\[ 2gh = -2gH \]

\[ x = H \]
\text{Similarly max. height attained by the 2nd particle will be } h.

Also \( k > h \) (given).

Let \( t_1 \) be the time taken by the first particle to attain its max. height \( h \).

then \( v = u + gt_1 \)

\[ 0 = \sqrt{2gh} - gt_1 \]

\[ gt_1 = \sqrt{2gh} \]

\[ t_1 = \frac{\sqrt{2gh}}{g} \]

or \( t_1 = \sqrt{\frac{2h}{g}} \)

Let \( t_2 \) be the time taken by second particle to attain its max. height \( k \).

Then \( t_2 = \sqrt{\frac{2k}{g}} \)

Since \( h < k \) \( \Rightarrow \) \( t_1 < t_2 \)

Let after a time \( t \) the 3rd particle
is projected vertically upwards with velocity \( u \) and attains its max. height atm the 3rd particle will catch the 1st two particles during their upwards motion if \( t < t_1 \) i.e \( t < \sqrt{\frac{k}{g}} \) 

(iii) height attained by the 3rd particle in time \( t_1 \) must be greater than \( k \). 

Now for 3rd particle 

\[ x = ut + \frac{1}{2}gt^2 \]

\[ x = u(t_1 - t) - \frac{1}{2}g(t_1 - t)^2 \]

Now \( x > k \)

\[ u(t_1 - t) - \frac{1}{2}g(t_1 - t)^2 > k \]

\[ u(t_1 - t) > k + \frac{1}{2}g(t_1 - t)^2 \]

\[ u > \frac{k}{t_1 - t} + \frac{1}{2}g(t_1 - t) \]

Thus 

\[ u > \frac{k}{t_1 - t} + \frac{1}{2}g(\sqrt{\frac{2k}{g}} - t) \]

is the required condition.
Qn. 2. A particle is projected vertically upwards after a time to another particle is sent up from the same point with the same velocity and meets the first at height above the ground. Prove Slaght of the first. Find the velocity of projection.

Let initial velocity of both particles be \( u \).

Let time of flight of first particle be \( t \).

Time of flight of second particle will be \( t, t \).

**Particle 1**

\[
x = ut + \frac{1}{2}(-g)t^2
\]

\[
h = ut - \frac{1}{2}gt^2 \tag{1}
\]

**Particle 2**

\[
x = ut + \frac{1}{2}(-g)t^2
\]

\[
h = u(t_1-t) - \frac{1}{2}gt_1(t_1-t)^2 \tag{2}
\]

(1) - (2) gives

\[
h = ut - \frac{1}{2}gt^2
\]

\[
h = u(t_1-t) - \frac{1}{2}gt_1(t_1-t)^2
\]

\[
o = ut_1 - \frac{1}{2}gt_1^2 - ut + ut + \frac{1}{2}gt_1^2 \]

\[
o = \frac{1}{2}gt_1^2
\]
\[ \begin{align*}
0 &= -\frac{1}{2}gt^2 + ut + \frac{1}{2}gt^2 + \frac{1}{2}gt^2 - gt^2 \\
\Rightarrow t &= \frac{u}{g} + \frac{1}{2}t \\
\Rightarrow t_1 &= \frac{u}{g} + \frac{1}{2}t \quad (3) \\
\text{Use } (3) \text{ in } (1) \\
h &= u\left(\frac{u}{g} + \frac{1}{2}t\right) - \frac{1}{2}g\left(\frac{u^2}{g^2} + \frac{u^2}{g^2} + \frac{u}{g}\right) \\
&= \frac{u^2}{g^2} + \frac{u^2}{g} - \frac{1}{2}g\left(\frac{u^2}{g^2} + \frac{u^2}{g^2} + \frac{u}{g}\right) \\
&= \frac{u^2}{g} + \frac{u^2}{2g} - \frac{u^2}{8} - \frac{u}{g} \\
&= \frac{u^2}{2g} - \frac{g^2 t^2}{8} \\
&= 8gt^2 = \frac{g^2 u^2}{8} \\
8gt^2 + g^2 t^2 &= \frac{g^2 u^2}{8} \\
\Rightarrow u^2 &= \frac{8gt^2 + g^2 t^2}{g^2} \\
u &= \sqrt{8gt^2 + g^2 t^2} \\
is \text{ the velocity of projectile.}
\end{align*} \]
Consider a particle which starts from rest at $t = 0$ and moves in a straight line and its acceleration is equal to $a(t) = t^n$. Let $u(t) = t^n$.

\[
\begin{align*}
\frac{dv}{dt} &= t^n \\
\frac{dv}{dt} &= t^n dt \\
\int dv &= \int t^n dt \\
v &= \frac{t^{n+1}}{n+1} + C - (1)
\end{align*}
\]

When $t = 0$, $v = 0$.

$v = \theta + C$

$C = 0$

(1) \Rightarrow \quad v = \frac{t^{n+1}}{n+1}$

\[
\begin{align*}
\int dx &= \int \frac{t^{n+1}}{n+1} dt \\
\int dx &= \int \frac{t^{n+1}}{n+1} dt
\end{align*}
\]
\[ x = \frac{1}{n+1} \quad (2) \]

Initially \( t = 0, x = 0 \),

\[ o = 0 + D \]

\[ D = 0 \]

\[ (0) \rightarrow x = \frac{1}{(n+1)(n+2)} \text{ in distance} \]

Travelling by the path \( \text{in time} t \).

No. 11: A particle starts with a velocity \( u \) and moves in a straight line if it suffers a retardation equal to the square of its velocity. Find the distance travelled by the particle in a time \( t \).

So consider a particle \( o \rightarrow x \rightarrow A \)

which starts from point \( o \) velocity \( u \) and reaches at point \( A \) after covering a distance \( v \).

Hence \( u = v \)

\[
\frac{dv}{dt} = -dv^2
\]

\[
\frac{dv}{v^2} = -dt
\]

\[
v^{-1} = -t + B
\]

\[
-\frac{1}{v} = -t + B
\]
Initially when \( t=0 \), \( v=u \)

\[
-1 = 0 + B
\]

\[
u = B = -1
\]

Put \( B = -1 \)

\[
-1 = -t - 1
\]

\[
\frac{1}{v} = t + \frac{1}{u}
\]

\[
\frac{1}{v} = ut + 1
\]

\[
v = \frac{u}{ut + 1}
\]

\[
\frac{dx}{dt} = \frac{u}{ut + 1}
\]

\[
\int \frac{dx}{u} = \frac{u}{ut + 1} dt
\]

\[
x = \ln(ut+1) + C \rightarrow (2)
\]

Initially when \( t=0 \), \( x=0 \)

\[
0 = \ln(0+1) + C
\]

\[
0 = 0 + C
\]

\[
C = 0
\]

(2) \( \rightarrow \) \( x = \ln(ut+1) \) is the distance travelled by a particle in time \( t \).

Qn. 1: Discuss the motion of a particle moving in a straight line if it starts from rest at a distance \( 'a' \) from a point \( O \) and move with a acceleration equal to \( \mu \) times its distance from \( O \).
Consider a particle which start from rest from point A whose distance from point O in 'a'. Let after a time the particle reach at point B after covering a distance x from O. According to question 

\[ a = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} \]

\[ \int_0^x dv = \int_0^x dx \cdot \frac{dv}{dx} \]

\[ v^2 = \frac{1}{2} u^2 \frac{x^2}{2} + C \]

Initially when \( x = a \), \( v = 0 \)

\[ 0 = \frac{1}{2} u^2 a^2 + C \]

\[ C = -\frac{1}{2} u^2 a^2 \]

\[ \Rightarrow v^2 = \frac{1}{2} u^2 a^2 - \frac{1}{2} u^2 x^2 \]

\[ v^2 = u(a^2 - x^2) \]

\[ v = \sqrt{u(a^2 - x^2)} \]

velocity of the particle after the time-distance

\[ \frac{dx}{dt} = \sqrt{u(x^2 - a^2)} \]

\[ \frac{dx}{dt} = \sqrt{u} \sqrt{x^2 - a^2} \]

\[ \frac{1}{\sqrt{x^2 - a^2}} \]

\[ \frac{1}{\sqrt{x^2 - a^2}} dt = \sqrt{u} dt \]

\[ \int \frac{1}{\sqrt{x^2 - a^2}} dx = \sqrt{u} \int dt \]
\[
\text{Cosh}^{-1} \frac{a}{x} = \sqrt{u(t)} + D
\]

Initially when \( t = 0 \), \( x = a \).

\[
\text{Cosh}^{-1} \frac{a}{x} = \sqrt{u(0)} + D
\]

\[
\text{Cosh}^{-1}(1) = D
\]

\[
D = 0
\]

i.e \( D = 0 \)

\[
\Rightarrow \text{Cosh}^{-1} \frac{a}{x} = \sqrt{u(t)} + 0
\]

\[
\text{Cosh}^{-1} \frac{a}{x} = \sqrt{u(t)}
\]

\[
x = \text{Cosh}^{-1} \sqrt{u(t)}
\]

\[
a x = a \text{Cosh} \sqrt{u(t)}
\]

is the distance traveled by the partial in time \( t \).

**Q No. 13:** A partial moving in a straight line starts with velocity \( u \) and has acceleration \( v^3 \), where \( v \) is the velocity of the partial at time \( t \). Find the velocity and the time as a function of the distance travelled by the partial.

Consider a partial \( f \rightarrow x \) starts moving from velocity \( u \) point \( D \) with initial \( t = 0 \) velocity \( v \) velocity \( u \) at \( t = 0 \) and moves with acceleration \( v^3 \). Let after a time \( t \) the partial A after covering a distance \( x \).

According to question

\[
\frac{dA}{dU} = v^3
\]
\[ \frac{dv}{dx} = v^3 \]
\[ \int \frac{vdv}{v^3} - \int dx \]
\[ \int v^{-2}dv = \int dx \]
\[ v^{-1} = x + B \]
\[ \frac{-1}{v} = x + B.\]

Initially when \( x = 0 \), \( v = u \).
\[ \frac{-1}{v} = 0 + B. \]
\[ \frac{-1}{v} = u. \]
\[ B = -1. \]

Calc.
\[ \frac{dv}{dt} = u \]
\[ \int (1 - ux)dv = \int u dt \]
\[ \int dx = \int u dx - \int u dt \]
\[ x - ux^2 = ut + D \quad \Rightarrow D \]

Initially when \( t = 0 \), \( x = 0 \)
\[ 0 = 0 - (u)(0)^2 + D \]
\[ \Rightarrow D = 0. \]

\[ x = ux^2 - ut + 0 \]
\[ ut = -ux^2 + x \]
\[ ut = x \left( \frac{-ux^2 + 1}{2} \right) \]
\[ u \cdot u = x \left( -u x + \frac{1}{2} \right) \]
\[ t = \frac{x}{u} \left( -u x + 2 \right) \]
\[ t = \frac{x}{2u} (-u x) \]

is the time spending by the particle for covering distance \( x \).

Q. no. 14: The acceleration of a particle falling freely under the gravitational pull is expressed by \( \frac{x}{t^2} \), where \( x \) is the distance of the particle from the centre of the earth. Find the velocity of the particle if it is let fall from an altitude \( R \) and strike on the surface of the earth if the radius of earth is \( R \) and the air offer no resistance to motion.

2) Consider a particle which is falling freely under gravitational pull from an altitude \( R \).

Let at any instant the particle be at point \( P \) whose distance from center of earth is \( x \). Let \( r \) be radius of earth and air offers no resistance to the motion.

According to question...
\[ a = -\frac{k}{x^2} \quad \text{(acceleration is against the direction in which } x \text{ increases,)} \]

\[ \frac{vdv}{dx} = -\frac{k}{x^2} \]

\[ vdv = -\frac{k}{x^2} \, dx \]

\[ \frac{v^2}{2} = -k \frac{x^{-1}}{1} + B \]

\[ \frac{v^2}{2} = \frac{k}{x} + B \quad \text{(1)} \]

Initially when \( x = R, v = 0 \)

\[ 0 = \frac{k}{R} + B \]

\[ \therefore B = -\frac{k}{R} \]

\[ (1) \Rightarrow \frac{v^2}{2} = \frac{k}{x} - \frac{k}{R} \]

\[ v^2 = 2 \left( \frac{k}{x} - \frac{k}{R} \right) \]

\[ v^2 = 2k \left( \frac{1}{x} - \frac{1}{R} \right) \]

\[ v = \sqrt{2k} \left( \frac{1}{x} - \frac{1}{R} \right) \quad \text{is the velocity of the particle when it is at a distance } x \]

from the centre of the earth.

In order to find the velocity on the surface of earth put \( x = R \) in \((2)\)

\[ v = \sqrt{2k} \left( \frac{1}{R} - \frac{1}{R} \right) \quad \text{Ans.} \]
A particle describes $S = \pi a^2$ with the frequency $N$. If the greatest velocity is $v$, find the amplitude and max. value of acceleration of the particle. Also show that the velocity $v$ at a distance $x$ from the center of motion is given by $v = 2\pi N \sqrt{a^2 - x^2}$ where $a$ is the amplitude.

Consider a particle which is describing simple harmonic motion.

Frequent of the particle $= N$ (given)

Then Frequent of the particle $= \frac{\sqrt{A}}{2\pi}$

So $N = \frac{\sqrt{A}}{2\pi}$

or $\sqrt{A} = 2\pi N$ ——— (1)

Max. velocity of the particle $= v$ (given)

Then Max. velocity of the particle $= \sqrt{A} a$

So $v = \sqrt{A} a$

or $a = \frac{v}{\sqrt{A}}$

or $a = \frac{v}{2\pi N}$ ——— (2)

is the amplitude of motion.
we know that

All. \( \ddot{u} \) Displacement from mean position

Max. Ac. \( \ddot{u} \) Max. Displacement from mean position

(Coordinate)

Max. value of acceleration \( \ddot{u} = \ddot{a} \)

\[
\ddot{u} = \frac{v^2}{2\pi N} = \frac{2\pi N v}{2\pi N} = 2\pi N v
\]

Consider the particle is moving with a constant velocity \( v \) and at any instant the particle is at a distance \( x \) from the mean position the

\[
v = \sqrt{a^2 - x^2}
\]

\[
v = 2\pi N \sqrt{a^2 - x^2} \quad \text{by (1)}
\]

is the required velocity

\( 2\pi N \) A particle vibrating \( 5 \) ft/sec has

Velocities \( 5 \) ft/sec and \( 4 \) ft/sec when its
distance from the Centre of motion is 12 ft

and 13 ft respectively. Find its time
period of motion.
Consider a particle which is describing SHM and moving away from the mean position. The velocity at $B$ and $C$ is at a distance of 12 ft and 13 ft from the mean position.

<table>
<thead>
<tr>
<th>At point B</th>
<th>At point C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 12, \text{ft}$</td>
<td>$x = 13, \text{ft}$</td>
</tr>
<tr>
<td>$v = 5, \text{ft/s}$</td>
<td>$v = 4, \text{ft/s}$</td>
</tr>
<tr>
<td>$v = \sqrt{2a, (a^2 - x^2)}$</td>
<td>$v^2 = a^2 - 169, \text{ft}^2$</td>
</tr>
<tr>
<td>$v^2 = a, (a^2 - x^2)$</td>
<td></td>
</tr>
<tr>
<td>$v^2 = a^4 - ax^2$</td>
<td>$16 = a^2 - 169, \text{ft}^2$</td>
</tr>
<tr>
<td>$c^2 = a^2 - \lambda (12)^2$</td>
<td>$(1) - (2)$ gives: $25 = a^2 - 144, \text{ft}^2$</td>
</tr>
<tr>
<td>$25 = a^2 - 144, \text{ft}^2$</td>
<td>$25 = a^2 - 153, \text{ft}^2$</td>
</tr>
<tr>
<td>$16 = a^2 - 169, \text{ft}^2$</td>
<td>$9 = 25, \lambda$</td>
</tr>
</tbody>
</table>
\[ d = \frac{9}{25} \]
\[ \Delta \sqrt{A} = \frac{3}{5} \]

We know that

Time period of motion \( T = \frac{2\pi}{\sqrt{A}} \)

\[ T = \frac{2\pi}{\frac{3}{5}} = \frac{10\pi}{3} \]

Qno. 12: The maximum velocity that a particle executing S.H.M of amplitude \( a \) attains is \( v \). If it is disturbed in such a way that its max. velocity becomes \( \frac{4}{5}v \), find the change in amplitude and the time period of motion.

Consider a particle which is executing S.H.M. Let \( a' \) be the amplitude of motion.

Max. velocity of the particle
when its amplitude is \( a = v \) \( \text{given} \)

But Max. velocity of the particle
when its amplitude is \( a' = \sqrt{A} a' \)

\[ v = \sqrt{A} \]

or \( a = \frac{v}{\sqrt{A}} \) \( \cdots (1) \)
Let the particle be disturbed and its amplitude becomes $a'$. 

Max. velocity of particle when 
its amplitude is $a' = \nu$ (given) 

But Max. velocity of the particle 
when its amplitude is $a' = \sqrt{A} a'$ 

$$\nu = \sqrt{A} a'$$

or $a' = \frac{\nu}{\sqrt{A}}$ \hspace{1cm} (2)

Change in amplitude $= a' - a$

$$= \frac{\nu}{\sqrt{A}} - \frac{\nu}{\sqrt{A}} \cdot b \hspace{1cm} (2-v)$$

$$= \frac{\nu}{\sqrt{A}} \left( 1 - b \right)$$

$$= a \left( 1 - b \right) \hspace{1cm} \text{by} \hspace{0.5cm} (1)$$

Since there is no change in $\sqrt{A}$, so there will be no change in the amplitude of motion. 

Q. no. 18: A point describes S.H.M. in such a way that its velocity and acceleration at point P are $v$ and $a$, respectively, and the corresponding quantities at
and let point \( Q \) as well. Find the distance \( PQ \).

Consider a particle which is describing S.H.M and is moving away from mean position. The velocity and acceleration of the particle at point \( P \) are \( u \) and \( f \), and the corresponding quantities at another point \( Q \) are \( u' \) and \( f' \).

\[\begin{align*}
\text{At point } P & \quad \text{At point } Q \\
\dot{v} &= \sqrt{\lambda (\ddot{a} - \dot{y}^2)} & \frac{\ddot{a}}{\ddot{y}} & \sqrt{\lambda (\ddot{a} - \dot{y}^2)} \\
\dot{u} &= \lambda (\ddot{a} - \dot{y}^2) & \dot{u}' &= \lambda (\ddot{a} - \dot{y}^2) \\
u &= \lambda (\ddot{a} - \dot{y}^2) & \dot{u}' &= \lambda (\ddot{a} - \dot{y}^2) \\
\ddot{u} &= \lambda (\ddot{a} - \dot{y}^2) & \ddot{u}' &= \lambda (\ddot{a} - \dot{y}^2) \\
\end{align*}\]

\begin{align*}
(1) & - (3) \\
u^2 - u'^2 &= \lambda (y_2^2 - y_1^2) \\
u^2 - v^2 &= \lambda (y_2 + \dot{y}_1)(y_2 - \dot{y}_1) \quad (5)
\end{align*}
\[ (2) + (4) \implies f = -\lambda y_1 \]
\[ g = -\lambda y_2 \]
\[ f + g = -\lambda (y_1 + y_2) \]
\[ -(f + g) = \lambda (x_2 + x_1) \quad (6) \]

Using (6) in (5)
\[ u^2 - v^2 = -(f + g)(x_2 - y_1) \]

or
\[ x_2 - y_1 = \frac{u^2 - v^2}{-(f + g)} \]

or
\[ x_2 - y_1 = \frac{v^2 - u^2}{f + g} \]

or
\[ f + g = \frac{v^2 - u^2}{x_2 - y_1} \]

Problem 19: If a point moves with a

velocity \( v \) given by \( v^2 = n^2(\alpha^2 + 2\beta n + \gamma) \)

show that \( P \) executes a S.H.M. Find the center of the track and time period of

motion.

Consider a point \( P \), that moves with

velocity \( v \) given by

\[ v^2 = n^2(\alpha^2 + 2\beta n + \gamma) \quad (1) \]
\[ 2 \left( v \right)^{\frac{1}{n}} \frac{dv}{dn} = n \left( 2ax + 2b \right) \]

\[ 2v \frac{dv}{dn} = 2an^2 \left( x + \frac{b}{a} \right) \]

\[ -2 \quad \frac{dv}{dn} = -an^2 \left( x + \frac{b}{a} \right) \]

Let \( x + \frac{b}{a} = \lambda \) and \( x + \frac{b}{a} = X \)

Then \( \frac{dv}{dx} = \lambda X \)

\[ \text{Acc.} = \lambda X \]

\[ \text{Acc.} \propto X \quad (\text{displacement}) \]

Thus \( P \) executes SHM.

At the centre \( X = 0 \)

\[ x + \frac{b}{a} = 0 \]

\[ x = -\frac{b}{a} \]

At the extreme positions velocity of the particle will be zero.

\[ \text{For } (v) = 0 \quad n^2 \left( ax^2 + 2bx + c \right) = 0 \quad \therefore v = 0 \]

\[ n^2 \quad ax^2 + 2bx + c = 0 \]

\[ x = \frac{-2b \pm \sqrt{(2b)^2 - 4ac}}{2a} \]
\[
\begin{align*}
\text{Amplitude } &\text{ of } y = -b + \sqrt{b^2 - ac} \\
&= \frac{-b + \sqrt{b^2 - ac}}{a} \\
&= \frac{-b + (b^2 - ac + b^2)}{a} \\
&= \frac{\sqrt{b^2 - ac}}{a}
\end{align*}
\]

As \( a = \omega \) 

\[ T = \frac{2\pi}{\omega} \]

\[ T = \frac{2\pi}{\sqrt{a}} \]

End