Superior College Sargodha

Mechanics
Virtual Work
Chapter # 6

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CHAPTER # 6:-

-: Virtual Work:-

Types of motion:-

1- Translational Motion:-

In a translatory motion all the points of the body undergo the same displacement in any interval.

2: Rotational Motion:-

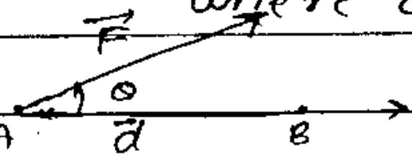
In a rotational motion points on a certain line remain fixed where as the other points moved with different speed.

Work done by a force:-

If a constant force $\vec{F}$ acts on a particle and particle moves from point A to point B. Then work done of force is,

$$W = \vec{F}\cdot\vec{d} \quad \text{where } \vec{d} = \vec{AB}$$

$$W = Fd\cos\theta$$



where θ is the angle b/w line of action of forces and distance AB.

Special Cases:-

1- If. $\theta = 0°$, then

$$w = Fd\cos 0°$$

$$\Rightarrow W = Fd \quad (\text{maximum})$$

2:- If $\theta = 90°$, then

$W = Fd\cos 90^\circ$

$\Rightarrow W = 0$ (no work done).

3. If $\theta = 180^\circ$, then

$W = Fd\cos 180^\circ$

$W = -Fd$ (-ive work done).

* If force is variable then,

work done $W = \int \vec{F} \cdot d\vec{r}$

Where $d\vec{r}$ is infinitesimal displacement of particle in any time t.

Integration is performed over entire path.

In Cartesian components.

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$W = \int \vec{F} \cdot d\vec{r}$

$= \int (x\,dx + y\,dy + z\,dz)$

* The work done by a no. of forces acting on a particle is equal to the work done by their resultant.

If forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$ acts on a particle. Then their work done is

$W = \int \vec{F_1} \cdot d\vec{r} + \int \vec{F_2} \cdot d\vec{r} + \cdots + \int \vec{F_n} \cdot d\vec{r}$

$= \sum_{i=1}^{n} \int F_i \cdot d\vec{r}$

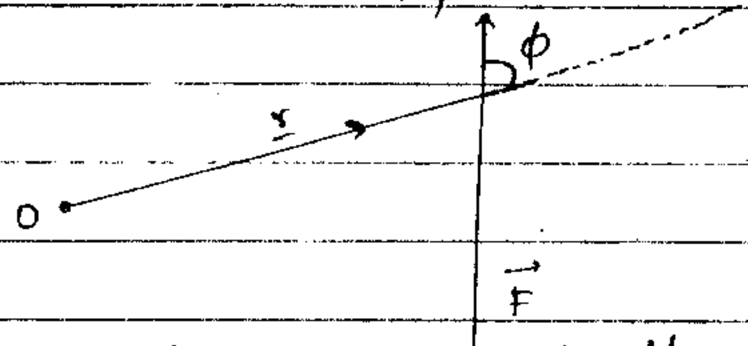
$= \int \sum_{i=1}^{n} F_i \cdot d\vec{r}$

$$W = \int \vec{R} \cdot d\vec{r}$$

Moment of forces $\vec{F}$ about origin is defined as.

$$\text{Mom}_o \vec{F} = \vec{r} \times \vec{F}$$

$$= rF \sin\phi \, \hat{n}$$



where $\hat{n}$ is unit vector ⊥lar to the plane of $\vec{r}$ and $\vec{F}$, $\hat{n}$ is always parallel to axis of Rotation.

$$\text{Mom}_o \vec{F} = \vec{r} \times \vec{F}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ X & Y & Z \end{vmatrix}$$

$$= \underline{i}(yZ - zY) + \underline{j}(Xz - xZ) + \underline{k}(xY - yX)$$

If $\vec{F}$ and $\vec{r}$ lies in xy plane resultant lies along z-axis.

$$\text{Mom}_o \vec{F} = (xY - yX)\underline{k}.$$

V.V.I.P (2015)

Q. State and prove the principle of virtual work done for the single particle?

Ans :- Statement:-

A particle subject to workless constraints is in equilibrium if and only if zero virtual work is done by the applied force in an infinitesimal virtual displacement consistant with constraints.

Proof:- If $\vec{F_a}$ be the total applied forces and $\vec{F_c}$ be the total forces of constraints. if forces are in equilibrium then by the theorem of equilibrium,

$$\vec{F_a} + \vec{F_c} = 0 \longrightarrow (i)$$

We have to prove work done of a applied forces is 0.

if $\delta \vec{r}$ is the infinitesimal virtual distance covered by the particle under the action of forces. Taking dot product of $\delta \vec{r}$ with Equ.(i).

$$(\vec{F_a} + \vec{F_c}) \cdot \delta \vec{r} = 0 \cdot \delta \vec{r}$$

$$\vec{F_a} \cdot \delta \vec{r} + \vec{F_c} \cdot \delta \vec{r} = 0 \longrightarrow (ii)$$

$\because$ the constraints are useless than, They are workless

$$\Rightarrow \vec{F_c} \cdot \delta \vec{r} = 0$$

From Equ.(ii).

$$\vec{F_a} \cdot \delta \vec{r} = 0$$

$\Rightarrow$ Work done by applied forces is zero.

Conversly,

If work done by the applied force is zero i.e

$$\vec{F_a}\cdot\delta\vec{r} = 0 \longrightarrow (iii)$$

Then we prove forces are in equilibrium,

i.e $\vec{F_a} + \vec{F_c} = 0$

$\because$ Constraints are workless then,

$$\vec{F_c}\cdot\delta\vec{r} = 0 \rightarrow (iv)$$

Adding Equ. (iii) and (iv).

$$\vec{F_a}\cdot\delta\vec{r} + \vec{F_c}\cdot\delta\vec{r} = 0$$

$$(\vec{F_a} + \vec{F_c})\cdot\delta\vec{r} = 0$$

$\because \delta\vec{r} \neq 0$

$$\Rightarrow \vec{F_a} + \vec{F_c} = 0$$

Which completes the proof:-

Q:- State and Prove the principle of virtual work done for a set of particles?

Ans: Statement:-

A set of particles, subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces in any arbitrary infinitesimal displacement consistent with the constraints.

Proof:- If $\vec{F}_{ia}$ be the total applied forces and $\vec{F}_{ic}$ be the total forces of constraints.
If forces are in equilibrium then by the theorem of equilibrium

$$\vec{F}_{ia} + \vec{F}_{ic} = 0 \longrightarrow (i) \qquad i = 1, 2, \dots, n.$$

We have to prove work done of a applied force is 0.
if $\delta\vec{r}_i$ is the infinitesimal virtual distance covered by the particle under the action of forces.
Taking dot product of $\delta\vec{r}_i$ with Equ. (i).

$$\sum_{i=1}^{n} (\vec{F}_{ai} + \vec{F}_{ic}) \cdot \delta\vec{r}_i = 0 \cdot \delta\vec{r}_i$$

$$\sum_{i=1}^{n} \vec{F}_{ia} \cdot \delta\vec{r}_i + \sum_{i=1}^{n} \vec{F}_{ic} \cdot \delta\vec{r}_i = 0 \longrightarrow (ii)$$

$\because$ The constraints are useless then they are workless.

$$\Rightarrow \sum_{i=1}^{n} \vec{F}_{ic} \cdot \delta\vec{r}_i = 0$$

From Equ (ii)

$$\sum_{i=1}^{n} \vec{F}_{ia} \cdot \delta\vec{r}_i = 0$$

$\Rightarrow$ Work done by applied forces is zero.

Conversly,
If work done by the applied forces is zero, i.e

$$\sum_{i=1}^{n} \vec{F}_{ia} \cdot \delta\vec{r}_i = 0 \longrightarrow (iii)$$

Then we prove forces are in equilibrium

i.e $\sum_{i=1}^{n} (\vec{F}_{ia} + \vec{F}_{ic}) = 0$

$\because$ Constraints are workless then,

$$\sum_{i=1}^{n} \vec{F}_{ic} \cdot \delta\vec{r}_i = 0 \longrightarrow (iv)$$

Adding Equ. (iii) and (iv).

$$\sum_{i=1}^{n} \vec{F}_{ia} \cdot \delta\vec{r}_i + \sum_{i=1}^{n} \vec{F}_{ic} \cdot \delta\vec{r}_i = 0$$

$$\sum_{i=1}^{n} (\vec{F}_{ia} + \vec{F}_{ic}) \cdot \delta\vec{r}_i = 0$$

$\because \delta\vec{r}_i \neq 0 \Rightarrow \vec{F}_{ia} + \vec{F}_{ic} = 0$

Q:- State and prove principle of virtual work for single rigid body?

Ans:-

Statement:-

A rigid body is subjected to workless constraints is in equilibrium if and only if zero virtual work is done by the applied forces and applied torques in any arbitrary infinitesimal displacement consistent with constraints.

Proof:-

Since all the forces acting on a rigid body can be shifted to any point of body together with a couple is introduce in the system. $\vec{R}$, $\vec{G}$ is the resultant of forces and resultant of torques

$$\vec{R} = \vec{R_a} + \vec{R_c}$$

Here $\vec{R_a}$ is the resultant of all applied forces and $\vec{R_c}$ is the resultant of all constraints forces. Similarly,

$$\vec{G} = \vec{G_a} + \vec{G_c}$$

Here $\vec{G_a}$ is the resultant of applied torques and $\vec{G_c}$ resultant of torques due to constraints forces.

If system is in equilibrium.

$$R = 0$$

$$G = 0$$

if forces do some virtual work $\delta\vec{r}$ and $\delta\vec{\theta}$ which are arbitrary infinitesimal virtual displacement

Taking dot product of $\delta\vec{r}$ with $\vec{R}$ and $\delta\vec{\theta}$ with $\vec{G}$ to get virtual work.

$$\vec{R}\cdot\delta\vec{r} = 0 \qquad \text{and} \qquad \vec{G}\cdot\delta\vec{\theta} = 0$$

$(\vec{R_a} + \vec{R_c})\cdot\delta\vec{r} = 0$	$(\vec{G_a} + \vec{G_c})\cdot\delta\vec{\theta} = 0$
$\vec{R_a}\cdot\delta\vec{r} + \vec{R_c}\cdot\delta\vec{r} = 0$	$\vec{G_a}\cdot\delta\vec{\theta} + \vec{G_c}\cdot\delta\vec{\theta} = 0$

$\because$ Constraints are workless

So,

$\vec{R_c} \cdot \delta\vec{r} = 0$ and $\vec{G_c} \cdot \delta\vec{\theta} = 0$

$\Rightarrow \vec{R_a} \cdot \delta\vec{r} = 0$ and $\vec{G_a} \cdot \delta\vec{\theta} = 0$

$\Rightarrow$ virtual work of applied forces and applied torques is zero.

Conversly,

If virtual work of applied forces and applied torques is zero.

i.e $\vec{R_a} \cdot \delta\vec{r} = 0$ and $\vec{G_a} \cdot \delta\vec{\theta} = 0$

We have to proved forces in equilibrium

$R = 0$; $G = 0$

Adding virtual work of constraints

$\vec{R_c} \cdot \delta\vec{r} = 0$; $\vec{G_c} \cdot \delta\vec{\theta} = 0$

$\Rightarrow \vec{R_a} \cdot \delta\vec{r} + \vec{R_c} \cdot \delta\vec{r} = 0$ and $\vec{G_a} \cdot \delta\vec{r} + \vec{G_c} \cdot \delta\vec{\theta} = 0$

$\Rightarrow (\vec{R_a} + \vec{R_c}) \cdot \delta\vec{r} = 0$ and $(\vec{G_a} + \vec{G_c}) \cdot \delta\vec{\theta} = 0$

$\because \delta\vec{r} \neq 0$, so, $\delta\theta \neq 0$

$\Rightarrow \vec{R_a} + \vec{R_c} = 0$; $\Rightarrow \vec{G_a} + \vec{G_c} = 0$

$R = 0$; $G = 0$

Hence proved.

Example # 01:-

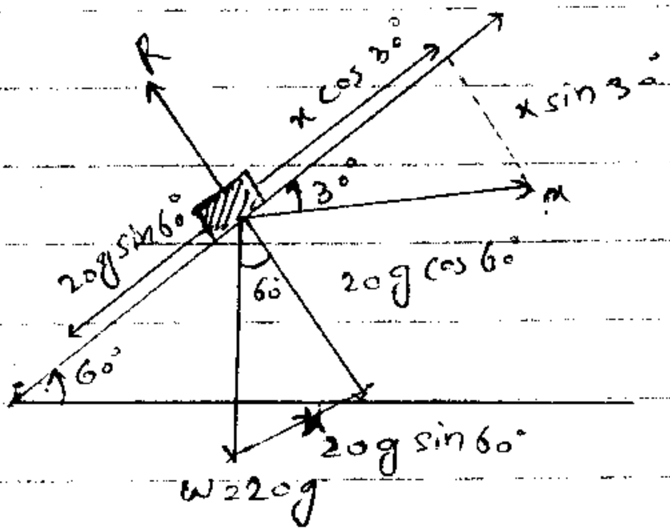
A particle of mass 20 lb. is supported on a smooth plane inclined at 60° to the horizontal by a force of magnelude x poundals which makes an angle of 30° with the plane. Find x and also the reaction of the plane on the particle?

Solution:

mass of the particle $m = 20\,lb$

Then weight of particle $w = mg$

$w = 20g$.



First we consider virtual displacement δs up the plane: Equ. of virtual work

$$x\cos 30^\circ \delta s - 20g\sin 60^\circ \delta s = 0$$

$$(x\cos 30^\circ - 20g\sin 60^\circ) \times \delta s = 0$$

$\because \delta s \neq 0$, As δs is a infinitesimal displacement

$$\Rightarrow x\cos 30^\circ - 20g\sin 60^\circ = 0$$

$$x = \frac{20g\sin 60^\circ}{\cos 30^\circ}$$

$$x = \frac{20g \cdot \cancel{\frac{\sqrt{3}}{2}}}{\cancel{\sqrt{3}/2}} = 20g$$

$x = 20$ lb. weight.

if δy is the virtcle displacement along R

$$R\,\delta y - 20g\cos 60^\circ\,\delta y - x\sin 30^\circ\,\delta y = 0$$

$$\Rightarrow \left(R - \cancel{20}^{10}g\cdot\frac{1}{2} - x\cdot\frac{1}{2}\right)\delta y = 0$$

$\because \delta y \neq 0$, As δy is a infinitesimal displacement.

$$R = 10g + \frac{x}{2} \qquad \because x = 20g$$

$$= 10g + \frac{20g}{2}$$

$$= 10g + 10g$$

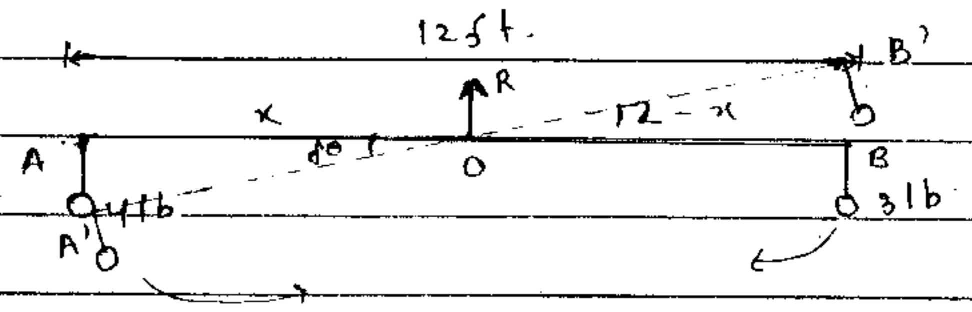
$$R = 20g.$$

$= 20$ lb. weight.

Example # 02:-

A light thin rod, 12 ft. long, can turn in vertical plane about one of its points which is attached to a pivot. If weights of 3lb. and 4lb. are suspended from its ends, it rests in a horizontal position. Find the position of the pivot and its reaction on the rod?

Solution:-



Consider a rod AB of the length 12 ft. O be the pivot. If $AO = x$, $BO = 12 - x$

R be the normal reaction of pivot on Rod if δy is infinitesimal virtual displacement,

Equation of virtual work.

$R \cdot \delta y - 4\delta y - 3\delta y = 0$

$(R - 4 - 3)\delta y = 0$

$(R - 7)\delta y = 0$

$\because \delta y \neq 0$, As a δy is arbitrary infinitesimal displacement.

$\Rightarrow R - 7 = 0 \Rightarrow \boxed{R = 7}$

if $\delta\theta$ is angular displacement of the rod about pivot O.

Moments of forces are $4x$ and $-3(12-x)$

$4x\delta\theta - 3(12-x)\delta\theta = 0$

$(4x - 36 + 3x)\delta\theta = 0$

$\because \delta\theta \neq 0$, As $\delta\theta$ is arbitrary angular displacement

$4x - 36 + 3x = 0$

$7x = 36$

$\Rightarrow \boxed{x = \frac{36}{7}} \Rightarrow x = 5\frac{1}{7}$ ft.

position of pivot from A, $x = 5\frac{1}{7}$ ft.

position of pivot from B, $12 - x =$

$\Rightarrow 12 - x = 12 - \frac{36}{7}$

$= \frac{84 - 36}{7}$

$= \frac{48}{7}$

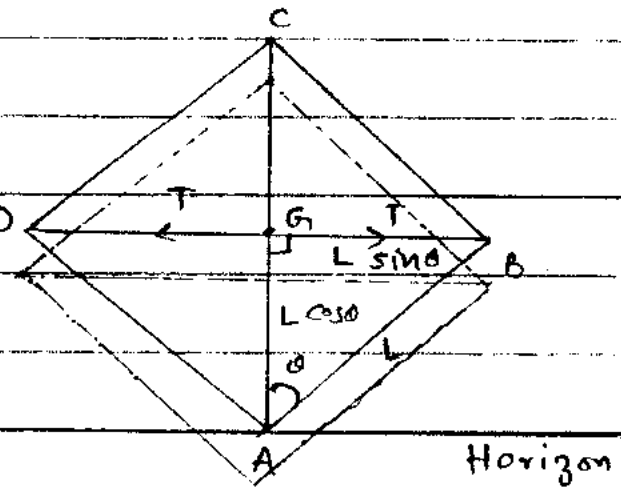
$= 6\frac{6}{7}$ ft.

Ans

Example # 03:-

Four equal uniform rods are smoothly jointed to form a rhombus ABCD, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in shape, with the measure of angle BAC equal to θ, by a light string joining B and D. Find the tension in the string?

Solution:-



Four equal uniform rods are smoothly jointed in the form of rhombus ABCD which is placed in a vertical plane with AC vertical A resting on horizontal plane

Given $\angle BAC = \theta$

if T be the tension in string BD.

if L be the length of each rod.

Equation of virtual work is

$$-4w\,\delta(AG) - T\delta(BD) = 0$$

where w is the weight of each rod and $4w$ is the weights of 4 rods which acts at G where G is the point of intersection of diagonals AC and BD.

From fig. $AG = L\cos\theta$

$BG = L\sin\theta$

$BD = 2(BG) = 2L\sin\theta$

$$-4w\,\delta(L\cos\theta) - T\delta(2L\sin\theta) = 0$$

$$4wL\sin\theta\,\delta\theta - 2TL\cos\theta\,\delta\theta = 0$$

$$(4wL\sin\theta - 2TL\cos\theta)\delta\theta = 0$$

$\because \delta\theta \neq 0$, As $\delta\theta$ is arbitrary infinitesimal Angular displacement

$$\cancel{4}^{2}w\cancel{L}\sin\theta = \cancel{2}T\cancel{L}\cos\theta$$

$$2w\sin\theta = T\cos\theta$$

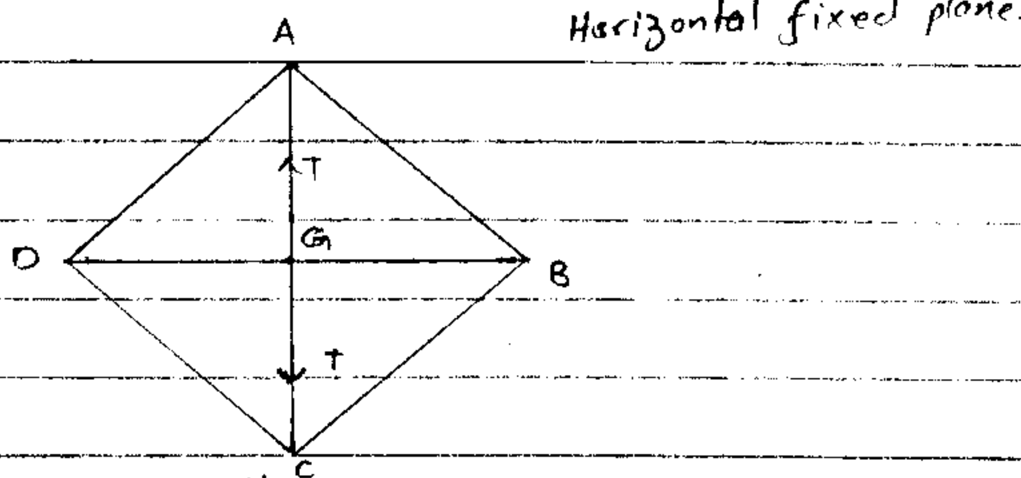
$$T = 2w\frac{\sin\theta}{\cos\theta}.$$

$$T = 2w\tan\theta \qquad \text{Ans}$$

EXERCISE Set 6:-

Q:No.2:- Four equal heavy uniform rods are freely jointed to form a rhombus ABCD which is freely suspended from A, and kept in shape of square by an inextensible string connecting A and C. show that tension in the string is 2W, where W is the weight of one rod?

Solution:- Since W is the weight of each rod. So weights of 4 rods is 4W acts at G. where G is the point of intersection of diagonals AC and BD.



if $AC = 2y$, then $AG = y$

As diagonals of rhombus bisect eachother. Let δy be the infinitesimal virtual displacement along AC.

$$4w\,\delta(AG) - T\delta(AC) = 0$$

$$4w\,\delta y - T\delta(2y) = 0$$

$$4w\,\delta y - 2T\delta y = 0$$

$$(4w - 2T)\delta y = 0$$

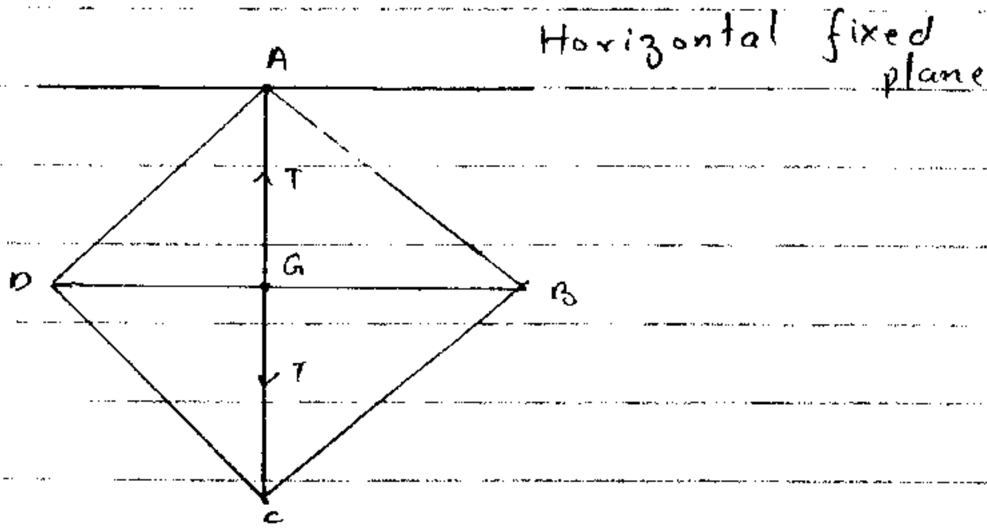
$$\because \delta y \neq 0 \implies \overset{2}{\cancel{4}}w = \cancel{2}T$$

$$\boxed{T = 2w}$$

Ans

Q:No:8:- Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended from the joint A, and is kept in shape by an inextensible string AC. Prove that the tension of the string is equal to half the whole weight?

Solution:- Since w be the wieght of rods forming parallelogram ABCD acts at G. where G is the point of intersection of diagonals AC and BD.



if $AC = 2y$, then $AG = y$

As diagonals of parallelogram bisect each other

let δy be the infinitesimal virtual displacement along AC.

$$W\delta(AG) - T\delta(AC) = 0$$

$$W\delta y - T\delta(2y) = 0$$

$$W\delta y - 2T\delta y = 0$$

$$(W - 2T)\,\delta y = 0$$

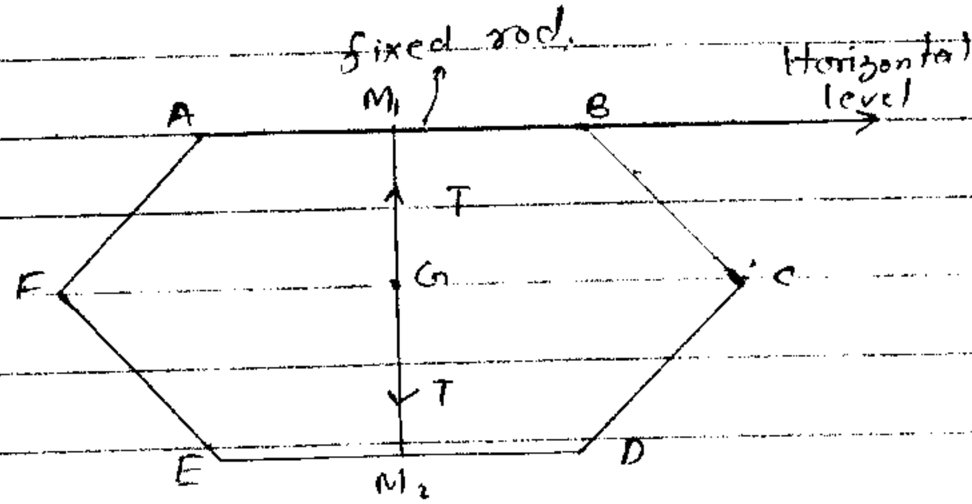
$\because \delta y \neq 0 \Rightarrow W - 2T = 0$

$$W = 2T$$

$$\Rightarrow T = \frac{W}{2} \quad Ans$$

Example # 05:- ~~six equal~~ Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon. The rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string. Prove that its tension is 3W?

Solution:-



$\because$ W is the weight of each rod. So, weight of 6 rods is 6W which acts at G. (Centre of hexagon ABCDEF).

if $M_1M_2 = 2y$ then $M_1G = y$

where M_1 and M_2 be mid points of rod AB and DE resp.

M_1M_2 be the string.

$6w\delta(M_1G) - T\delta(M_1M_2) = 0$

$6w\delta(y) - T\delta(2y) = 0$

$6w\delta y - 2T\delta y = 0$

$(6w - 2T)\delta y = 0$

$\because \delta y \neq 0$, As a δy is infinitesimal verticle displacement.

$6w - 2T = 0$

$2T = 6w$

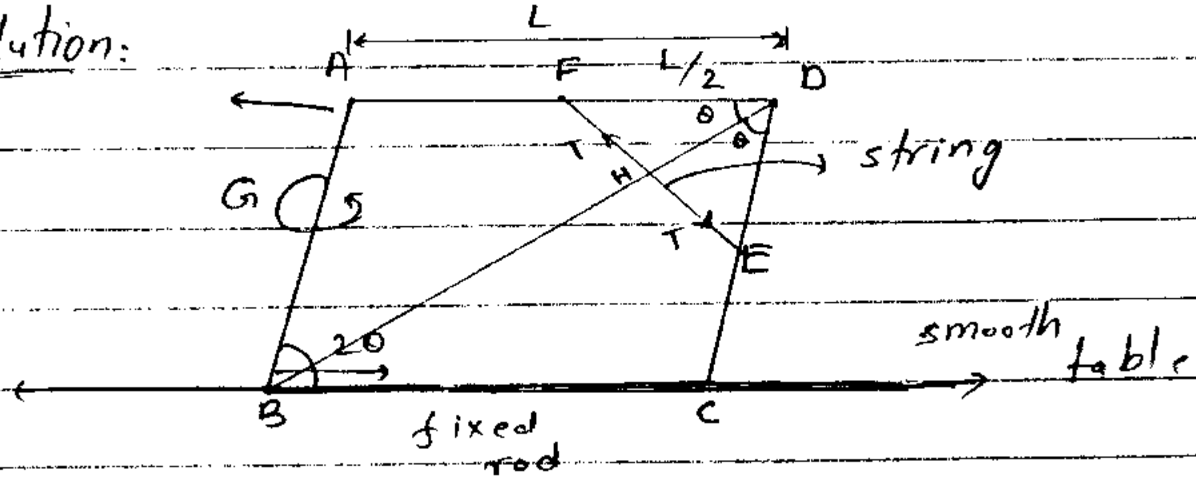
$T = 3W$ Ans proved.

v.v.v.vip :- **Q: No: 13:-**

A rhombus ABCD of smoothly jointed rods, rests on a smooth table with the rod BC fixed in position. The middle points of AD, DC are connected by a string which is kept taut by a couple G applied to the rod AB. Prove that the tension of the string is

$$\frac{2G}{AB} \cdot \frac{1}{\cos(\frac{1}{2}\hat{ABC})} ?$$

Solution:



Consider rhombus ABCD of freely jointed rods. Rod BC is fixed in position of a smooth table. If T be the tension in the string EF where E and F be the mid points of rod CD and AD resp. A couple G is applied at rod AB.

Let $\angle ABC = 2\theta$, then $\angle ADH = \angle CDH = \theta$

"L" be the length of each side of rhombus.

Equation of virtual work.

$$G\delta(2\theta) - T\delta(EF) = 0 \longrightarrow (i)$$

In ΔFDH

$$FH = \frac{L}{2}\sin\theta$$

$$\Rightarrow EF = 2FH = 2\left(\frac{L}{2}\sin\theta\right)$$

$$EF = L\sin\theta.$$

Using in equ. (i)

$$2G\delta\theta - T\delta(L\sin\theta) = 0$$

$$2G\delta\theta - TL\cos\theta\,\delta\theta = 0$$

$$(2G - TL\cos\theta)\delta\theta = 0$$

$\because \delta\theta \neq 0$, As $\delta\theta$ is arbitrary infinitesimal angular displacement.

$$2G - TL\cos\theta = 0$$

$$2G = TL\cos\theta$$

$$\frac{2G}{L\cos\theta} = T$$

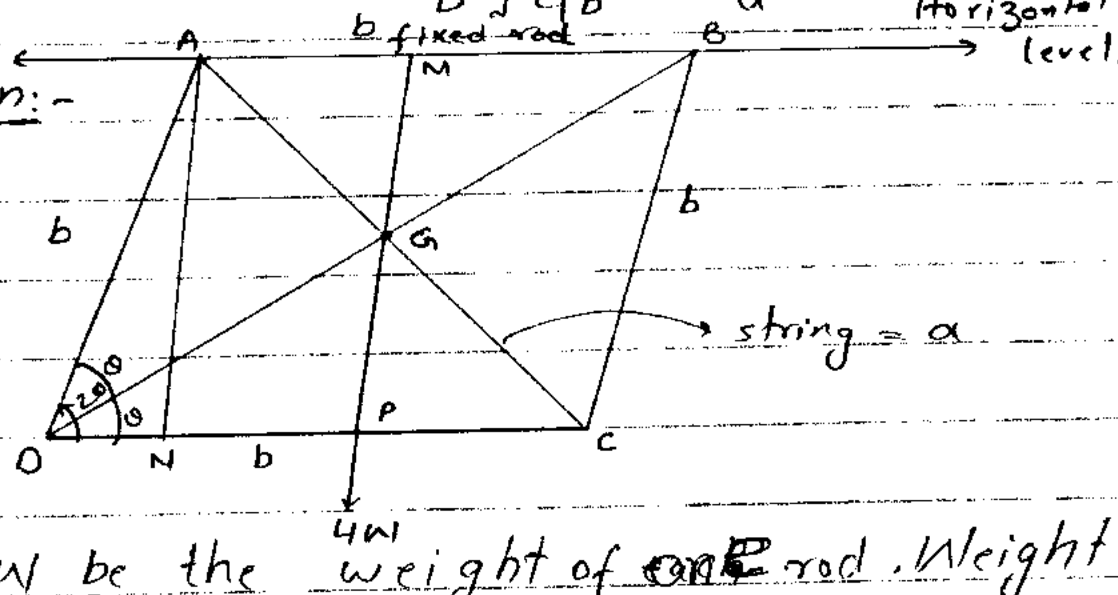
$\because L = AB$ and $\angle ABC = 2\theta \Rightarrow \frac{1}{2}\angle ABC = \theta$

$$\frac{2G}{AB}\cdot\frac{1}{\cos(\frac{1}{2}ABC)} = T \quad \text{proved.}$$ /Ist part.

v.v.vip :- Q:No:9:- (2010)

A string, of length a, forms the shorter diagonal of a rhombus formed by four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension in the string is $\dfrac{2W(2b^2-a^2)}{b\sqrt{4b^2-a^2}}$?

Solution:-



$\because$ W be the weight of ~~one~~ rod. Weight of 4 rods is 4W which acts at G. Where G is point of intersection of diagonals AC and BO.

AC is the shorter diagonal of length a along which string is connected.

AB is fixed horizontal rod.

$\angle ADC = 2\theta$, $\angle ADG = \angle CDG = \theta$

Equation of virtual work is,

$$4W\delta(MG) - T\delta(AC) = 0 \longrightarrow (i)$$

From fig,

$$MG = \frac{1}{2} AN$$

As $MG \parallel AN$ and G is mid pt of MP.

$$MG = \frac{1}{2}(AD\sin 2\theta) \quad \because \Delta ADN.$$

$$= \frac{1}{2} b\sin 2\theta$$

$$AG = AD\sin\theta \quad \because \text{In } \Delta ADG.$$

$$AG = b\sin\theta$$

$$AC = 2(AG)$$

$$AC = 2b\sin\theta$$

put $AC = a$

$$a = 2b\sin\theta.$$

$$\frac{a}{2b} = \sin\theta$$

Using in equation (i)

$$4W\delta\left(\frac{1}{2} b\sin 2\theta\right) - T\delta(2b\sin\theta) = 0$$

$$4W\frac{1}{\cancel{2}} b \cancel{2}\cos 2\theta\,\delta\theta - T2b\cos\theta\,\delta\theta = 0$$

$$(4Wb\cos 2\theta - 2Tb\cos\theta)\delta\theta = 0$$

$\because \delta\theta \neq 0$, As $\delta\theta$ is infinitesimal displacement.

$$4Wb\cos 2\theta - 2Tb\cos\theta = 0$$

$$^{2}\cancel{4}W\cancel{b}\cos 2\theta = \cancel{2}T\cancel{b}\cos\theta$$

$$2W\cos 2\theta = T\cos\theta$$

$$\frac{2w\cos 2\theta}{\cos\theta} = T$$

$$T = \frac{2w\,(1-2\sin^2\theta)}{\sqrt{1-\sin^2\theta}}$$

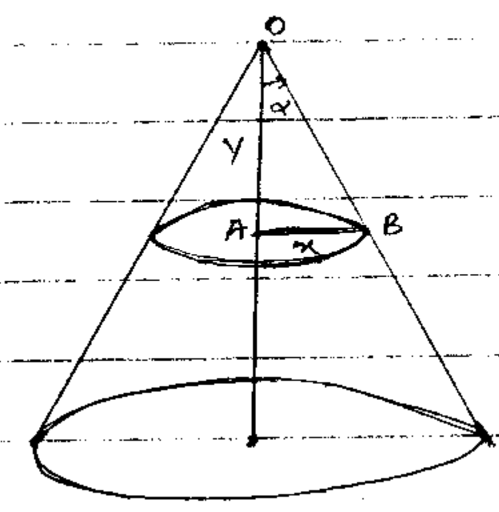
$$= \frac{2w\left(1-\cancel{2}\cdot\frac{a^2}{_2\cancel{4}b^2}\right)}{\sqrt{1-\frac{a^2}{4b^2}}}$$

$$= \frac{2w\left(\frac{2b^2-a^2}{\cancel{2}b^{\cancel{2}}}\right)}{\sqrt{\frac{4b^2-a^2}{\cancel{2}\cancel{b}}}}$$

$$T = \frac{2w\,(2b^2-a^2)}{b\sqrt{4b^2-a^2}}$$ proved.

Example #04:- A heavy elastic string whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle has measure α. If W be the weight and λ the modulus of the string prove that it will be in equilibrium when in the form of a circle of radius $a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right)$

Solution:-



Let the radius of circle formed by the string in equilibrium position be x.
y be the depth of plane formed by the string.

In ΔOAB,

$$\frac{x}{y} = \tan\alpha$$

$$x = y\tan\alpha \longrightarrow (i)$$

The tension in string is by hooke's law of tension,

$$T = \lambda\left(\frac{\text{Change in length}}{\text{original length}}\right)$$

$$= \lambda\left(\frac{2\pi x - 2\pi a}{2\pi a}\right).$$

$$T = \lambda\left(\frac{x-a}{a}\right) \longrightarrow (ii)$$

if δy is the downward virtual distance given by the string.

Equation of virtual work.

$$W\delta y - T\delta(2\pi x) = 0$$

$$W\delta y - 2T\pi\delta x = 0$$

∵ From Equ. (i).

$$\delta x = \tan\alpha\,\delta y$$

$$\Rightarrow W\delta y - 2T\pi\tan\alpha\,\delta y = 0$$

$$(W - 2T\pi\tan\alpha)\,\delta y = 0$$

∵ $\delta y \neq 0$, As δy is infinitesimal displacement

$$\Rightarrow W - 2T\pi\tan\alpha = 0$$

$\Rightarrow \quad W = 2T\pi \tan\alpha.$

$$= 2 \cdot \lambda\left(\frac{x-a}{a}\right)\pi \tan\alpha.$$

$$\frac{aw}{2\pi\lambda \tan\alpha} = x - a$$

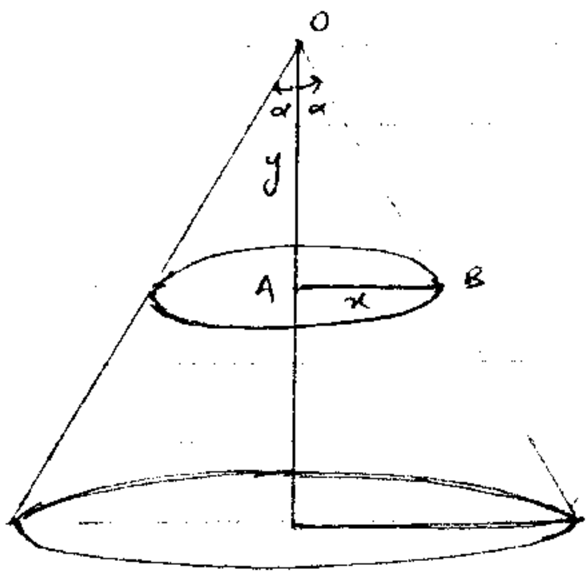
$$a + \frac{aw}{2\pi\lambda \tan\alpha} = x$$

$$a\left(1 + \frac{w}{2\pi\lambda}\cot\alpha\right) = x$$

$\therefore \quad x =$ as required proved.

Q: No: 11:- Two equal particles are connected by two given weightless strings, which are placed like a necklace on a smooth cone whose axis is vertical and whose vertex is uppermost. Show that the tension of each string is $\frac{W}{\pi}\cot\alpha$, where W is the weight of each particle and 2α the measure of the vertical angle of the cone?

Solution:-



Let the radius of circle formed by the necklace in equilibrium position be x.
W be the weight of each particle. Two particles of weight is 2W. Y be the depth of the plane formed by the necklace.

In ΔOAB.

$$\frac{x}{y} = \tan\alpha$$

$$x = y\tan\alpha \longrightarrow (i)$$

if δy is the downward virtual distance given by the necklace.

Equ. of virtual work

$$2W\delta(y) - T\delta(2\pi x) = 0$$

$$2W\delta y - 2T\pi\delta x = 0 \longrightarrow (ii)$$

From Equ. (i)

$$\delta x = \tan\alpha\,\delta y \quad \text{. using in (ii)}$$

$$2W\delta y - 2T\pi\tan\alpha\,\delta y = 0$$

$$(2W - 2T\pi\tan\alpha)\delta y = 0$$

$\because \delta y \neq 0$, As δy is infinitesimal virtual displacement

$$2W - 2T\pi\tan\alpha = 0$$

$$\cancel{2}W = \cancel{2}T\pi\tan\alpha$$

$$W = T\pi\tan\alpha$$

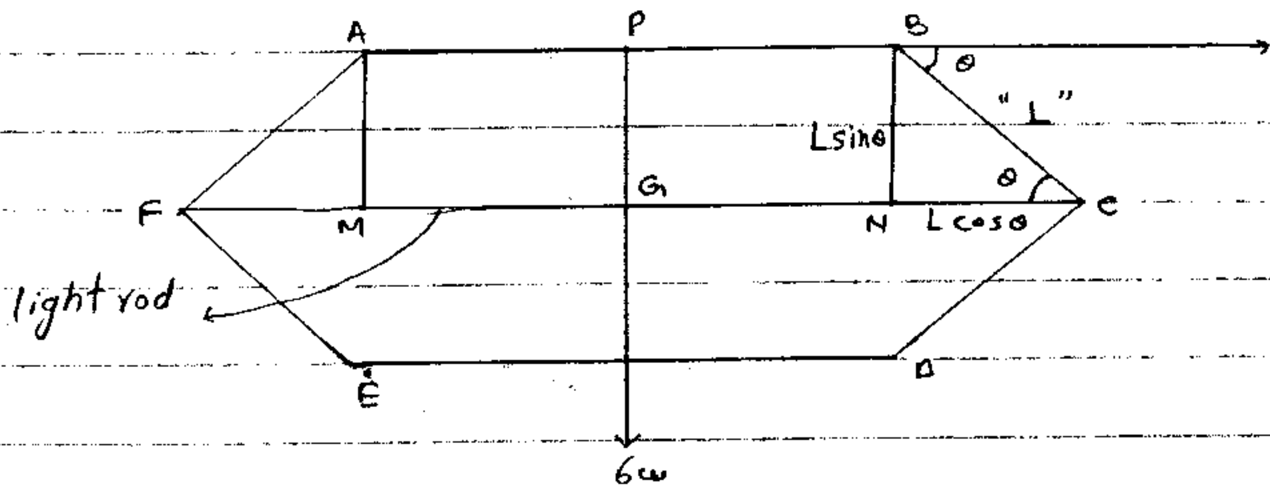
$$\frac{W}{\pi\tan\alpha} = T$$

$$\frac{W}{\pi}\cot\alpha = T$$

proved.

Q:No:03:- Six equal uniform rods AB, BC, CD, DE, EF, FA each of the weight W, are frely jointed to form a regular hexagon. The rod AB is fixed in a horizontal position, and the shape of the hexagon is maintained by a light rod jointing C and F. Show that the thrust is in this rod is $\sqrt{3}W$?

Solution:-



$\because$ W is the weight of each rod so, weight of 6 rods is 6w which acts at G mid point of hexagon ABCDEF, AB is fixed horizontal rod and θ be the angle which the rod BC makes with horizontal.

"L" be the length of each rod.

if "T" be the thrust in rod.

Equation of virtual work:-

$$6w\,\delta(PG) + T\delta(CF) = 0 \longrightarrow (i)$$

From fig. $PG = AM = BN = L\sin\theta$

$CF = CN + NM + MF$ $\quad \because AB = NM$, $MF = CN$

$CF = CN + AB + CN$

$CF = 2CN + AB$

$CF = 2L\cos\theta + L$

using in (i)

$6w\,\delta(L\sin\theta) + T\,\delta(2L\cos\theta + L) = 0$

$6wL\cos\theta\,\delta\theta + 2T\,\delta\theta\,L(-\sin\theta) = 0$

$(6wL\cos\theta - 2TL\sin\theta)\,\delta\theta = 0$

$\because \delta\theta \neq 0$, As $\delta\theta$ is infinitesimal angular displacement.

$6wL\cos\theta - 2TL\sin\theta = 0$

$\overset{3}{\cancel{6}}w\cancel{L}\cos\theta = \cancel{2}T\cancel{L}\sin\theta$

$3w\cot\theta = T$

$\because \theta = 60^\circ$ in regular hexagon.

$3w\cot 60^\circ = T$

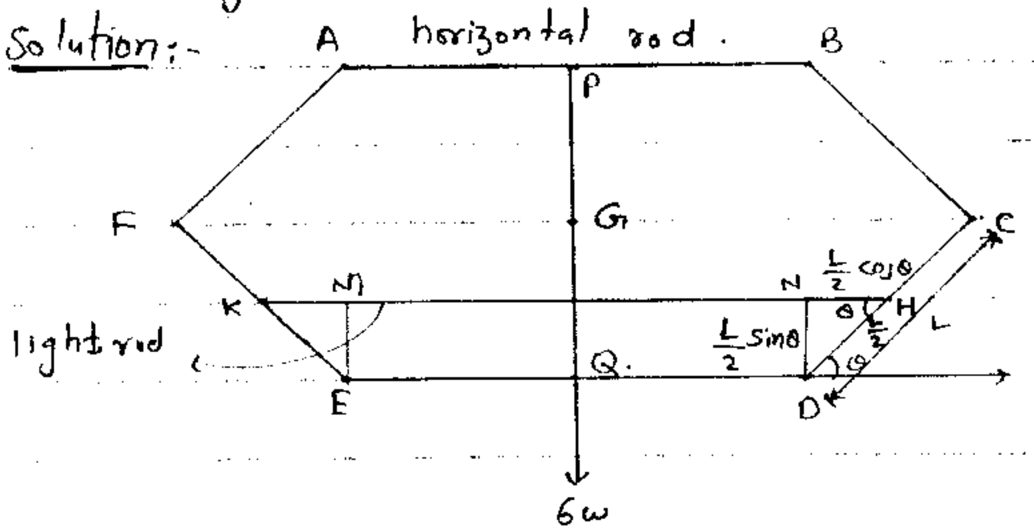
$3w \cdot \frac{1}{\sqrt{3}} = T$

$\cancel{\sqrt{3}}\sqrt{3}\,w \cdot \frac{1}{\cancel{\sqrt{3}}} = T$

$\sqrt{3}\,w = T$ is proved.

Q:No:04:- A hexagon ABCDEF, consisting of six equal heavy rods of weight W, freely jointed together, hangs in a vertical plane with AB horizontal, and the frame is kept in the form of a regular hexagon by a light rod connecting the mid-points of CD and EF. Show that the thrust in the light rod is $2\sqrt{3}\,W$?

Solution:-



∵ W be the weight of each rod so, weight of 6 rod is 6w which acts at G centre of hexagon ABCDEF if T be the thrust in rod HK where H and K be the mid points of rod CD and EF resp. "L" be the length of each rod and θ be the angle which the rod CD makes with horizontal.

Equ. of virtual work is,

$$6w\,\delta(PG) + T\,\delta(HK) = 0 \longrightarrow (i).$$

From fig:

$$PG = GQ = 2DN$$

$$= 2\cdot\frac{L}{2}\sin\theta$$

$$= L\sin\theta.$$

$$HK = HN + NM + MK$$

∵ ~~NM = DE~~ HN = MK

$$= HN + DE + HN$$

$HK = 2HN + DE$

$= \cancel{2}.\frac{L}{\cancel{2}}\cos\theta + L$

$HK = L\cos\theta + L$

Using in Equ. (i)

$6w\,\delta(L\sin\theta) + T\,\delta(L\cos\theta + L) = 0$

$6wL\cos\theta\,\delta\theta + TL(-\sin\theta\,\delta\theta) = 0$

$(6wL\cos\theta - TL\sin\theta)\delta\theta = 0$

$\because \delta\theta \neq 0$, As $\delta\theta$ is infinitesimal displacement.

$\Rightarrow \quad 6wL\cos\theta - TL\sin\theta = 0$

$6w\cancel{L}\cos\theta = T\cancel{L}\sin\theta$

$\frac{6w\cos\theta}{\sin\theta} = T$

$w\,2\cdot3\cot\theta = T \qquad \because \theta = 60°$

$w\,2\cdot3\cdot\cot 60° = T$

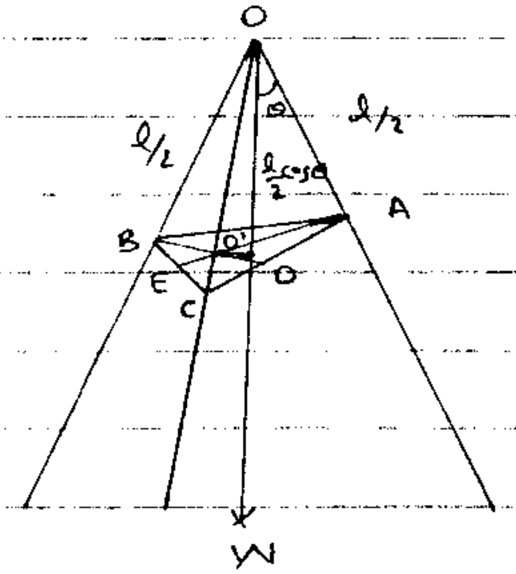
$w\,2\cdot\sqrt{3}\,\cancel{\sqrt{3}}\cdot\frac{1}{\cancel{\sqrt{3}}} = T$

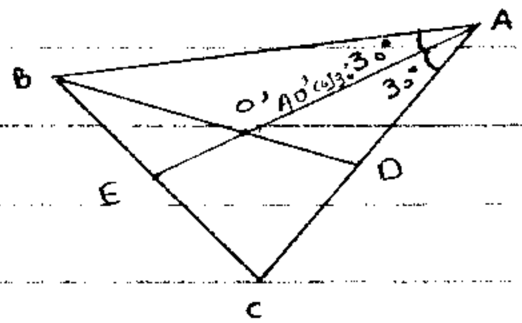
$2\sqrt{3}\,w = T$ proved

Example # 06:- A weightless tripod, consisting of three legs of equal length l, smoothly jointed at the vertex, stands on a smooth horizontal plane. A weight W hangs from the apex. The tripod is prevented from collapsing by three inextensible strings, each of length $l/2$, jointing the mid-points of the legs. Show that the tension in each string is $\frac{\sqrt{2}}{3\sqrt{3}}w$?

(1±3).

Solution:-





Let ABC be the triangle formed by the strings as shown in fig.

$OA = OB = OC = AB = BC = AC = l/2$

OABC is regular tetrahedron.

If the line along with w acts meet the plane of ABC at O' (centroid of $\triangle ABC$) which is the point of concurrency of medians BD and AE resp.

Equation of virtual work is,

$-w\,\delta(2OO') - 3T\,\delta(AC) = 0 \longrightarrow (i)$

"O" is the height of apex from horizontal.

Height of apex = 2 OO', As O' is mid point.

if θ is the angle which the vertical line OO' makes with each leg.

$OO' = \frac{l}{2}\cos\theta$

$AO' = \frac{l}{2}\sin\theta$

AC = 2AD , As D is mid pt of AC.

$AC = 2AO' \cos 30^\circ$

$= \cancel{2} \frac{l}{\cancel{2}} \sin\theta \cos 30^\circ$

$AC = l \sin\theta \frac{\sqrt{3}}{2}$

$\because AC = \frac{l}{2}$

$\frac{\cancel{l}}{\cancel{2}} = \cancel{l} \sin\theta \frac{\sqrt{3}}{\cancel{2}}$

$\frac{1}{\sqrt{3}} = \sin\theta \longrightarrow (ii)$

~~Using in Equ. (i)~~

$\cancel{-w\delta\left(\cancel{2} \frac{l}{\cancel{2}} \cos\theta\right) - 3T\delta\left(l \sin\theta \frac{\sqrt{3}}{2}\right) = 0}$

$-wl(-\sin\theta\,\delta\theta) - 3Tl\frac{\sqrt{3}}{2}(\cos\theta\,\delta\theta) = 0$

$\left(wl\sin\theta - \frac{3\sqrt{3}}{2} Tl\cos\theta\right)\delta\theta = 0$

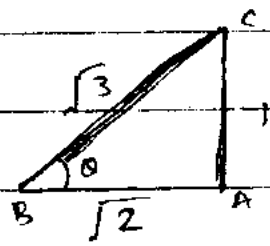
$\because \delta\theta \neq 0$, As $\delta\theta$ is infinitesimal displacement

$Wl\sin\theta - \frac{3\sqrt{3}}{2} Tl\cos\theta = 0$

$W\cancel{l}\sin\theta = \frac{3\sqrt{3}}{2} T\cancel{l}\cos\theta$

$\frac{2W\sin\theta}{3\sqrt{3}\cos\theta} = T$

$\frac{2w}{3\sqrt{3}} \tan\theta = T$



From (ii)

$\tan\theta = \frac{1}{\sqrt{2}}$

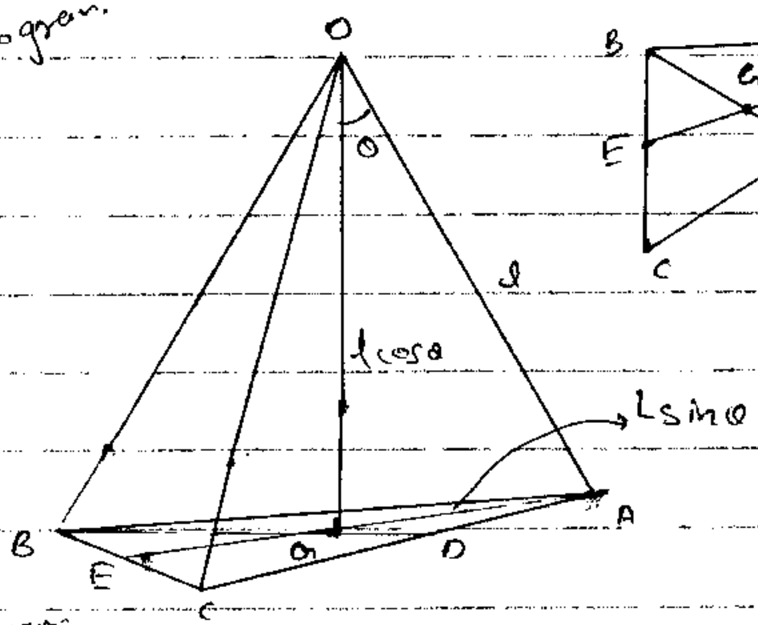
$\frac{2w}{3\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = T$

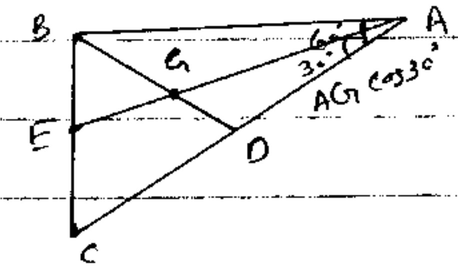
$\frac{\sqrt{2}}{3\sqrt{3}} w = T$ proved.

Q:No:5:- Three equal rods, each of weight W, are freely jointed together at one extremity of each to form a tripod, and rest with their other extremities on a smooth horizontal plane, each rod inclined at an angle of measure θ to the vertical, equilibrium being maintained by three equal light strings each joining two of these extremities. Prove that the tension in each string is $\frac{W\tan\theta}{2\sqrt{3}}$?

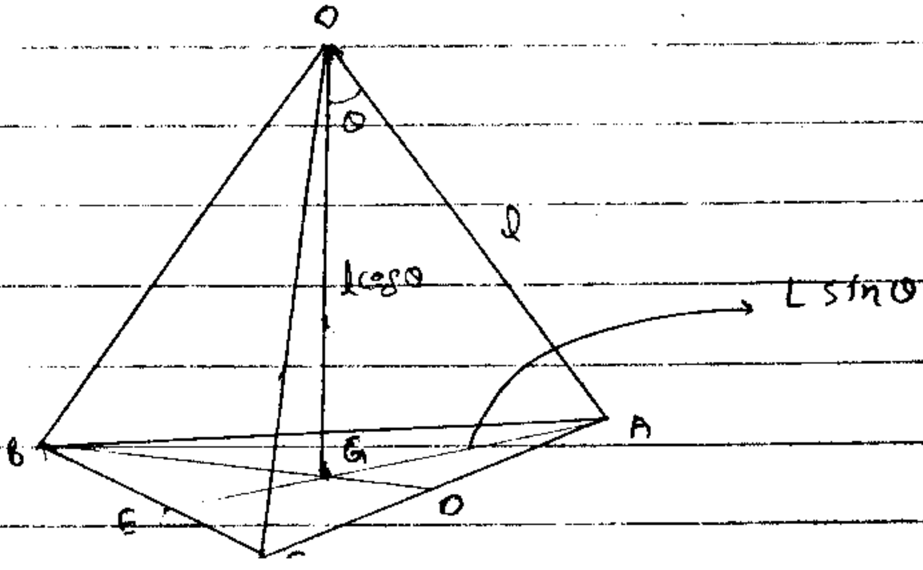
Solution:—

1st diagram.





2nd diagram



$\because$ W is the weight of each rod so weight of 3 rods is 3w which acts at mid point of OG i.e at $\frac{OG}{2}$ where OG is the verticle line through O and G is the point where is the vertical line meets with plane of ABC which is the point of intersection of medians BD and AE.

If l be the length of each rod and θ be the angle which each rod makes with vertical line OG. $\because AB = BC = CA = x$.

T be the tension in each string.

$$-3W\delta\left(\frac{OG}{2}\right) - 3T\delta(AC) = 0 \longrightarrow (i)$$

From fig. $OG = l\cos\theta$

$AG = l\sin\theta$

$AC = 2AD$

$= 2AG\cos 30^\circ$ $\because$ In ΔAGD.

$AC = \cancel{2}\, l\sin\theta \frac{\sqrt{3}}{\cancel{2}}$

$AC = \sqrt{3}\, l\sin\theta$

using in (i)

$$-3w\,\delta\left(\frac{l\cos\theta}{2}\right) - 3T\,\delta(\sqrt{3}\, l\sin\theta) = 0$$

$$\frac{3wl}{2}\sin\theta\,\delta\theta - 3\sqrt{3}\,Tl\cos\theta\,\delta\theta = 0$$

$$\left(\frac{3wl}{2}\sin\theta - 3\sqrt{3}\,Tl\cos\theta\right)\delta\theta = 0$$

$\because \delta\theta \neq 0$, As $\delta\theta$ is a infinitesimal displacement

$$\frac{3Wl}{2}\sin\theta - 3\sqrt{3}Tl\cos\theta = 0$$

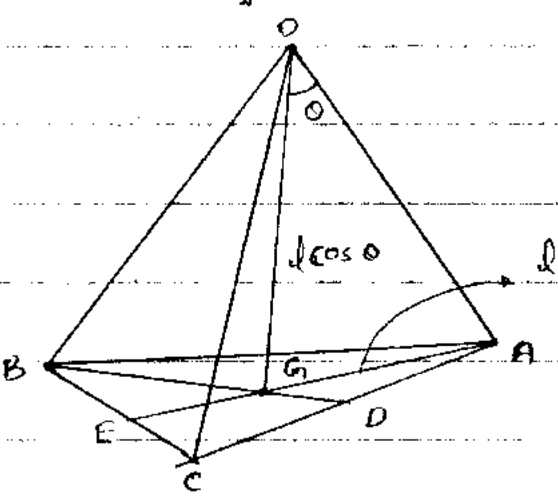
$$\cancel{3}W\frac{\cancel{l}}{2}\sin\theta = \cancel{3}\sqrt{3}T\cancel{l}\cos\theta$$

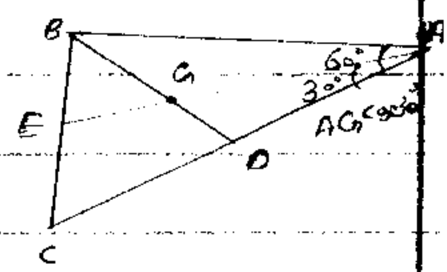
$$\frac{W}{2\sqrt{3}}\cdot\frac{\sin\theta}{\cos\theta} = T$$

$$\frac{W\tan\theta}{2\sqrt{3}} = T \quad \text{proved.}$$

Q. No: 06:- Six equal uniform rods freely jointed at their extremities from a tetrahedron. If this tetrahedron is placed with one face on a smooth horizontal table, prove that the thrust along a horizontal rod is $\frac{W}{2\sqrt{6}}$, where W is weight of a rod?

Solution:-





$\because$ W is the weight of each rod so weight of 3 rods (which are inclined in variable plane) is $3W$ which acts at mid point of OG i.e at $\frac{OG}{2}$ (weight of rods is horizontal plane plays no role to displace position of tetrahedron).

If l be the length of each rod.

$OA = OB = OC = AB = BC = AC = l$

T be the thrust in each horizontal rod.
θ be the angle which each rod makes with OG.

$$-3W\,\delta\left(\frac{OG}{2}\right) - 3T\,\delta(AC) = 0 \longrightarrow (i)$$

From fig:- $OG = l\cos\theta$.

$$AG = l\sin\theta.$$

$$AC = 2AD$$

$$AC = 2AG\cos 30^\circ \qquad \because \text{In } \Delta AGD$$

$$= \cancel{2}\,l\sin\theta\,\frac{\sqrt{3}}{\cancel{2}}$$

$\boxed{\because AC = l}$ $AC = \sqrt{3}\,l\sin\theta \Rightarrow \cancel{l} = \sqrt{3}\,\cancel{l}\sin\theta$.

$$\sin\theta = \frac{1}{\sqrt{3}} \longrightarrow (ii)$$

using in (i).

$$-3w\,\delta\left(\frac{l\cos\theta}{2}\right) - 3T\,\delta(\sqrt{3}\,l\sin\theta) = 0$$

$$\frac{3w}{2}\,l\sin\theta\,\delta\theta - 3\sqrt{3}\,T\,l\cos\theta\,\delta\theta = 0$$

$$\left(\frac{3Wl}{2}\sin\theta - 3\sqrt{3}\,T\,l\cos\theta\right)\delta\theta = 0$$

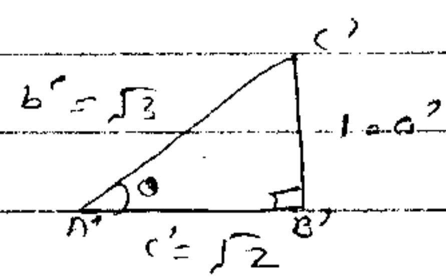
$\because \delta\theta \neq 0$, As $\delta\theta$ is a infinitesimal displacement.

$$\frac{3Wl}{2}\sin\theta - 3\sqrt{3}\,T\,l\cos\theta = 0$$

$$\frac{\cancel{3}W\cancel{l}\sin\theta}{2} = \cancel{3}\sqrt{3}\,T\,\cancel{l}\cos\theta$$

$$\frac{W}{2\sqrt{3}}\tan\theta = T \longrightarrow (iii)$$

~~From (ii) $\sin\theta = \frac{1}{\sqrt{3}}$~~



By P. Theorem:

$$(\sqrt{3})^2 = (1)^2 + c'^2$$

$$\sqrt{3-1} = c' \Rightarrow c' = \sqrt{2}$$

$\tan\theta = \frac{1}{\sqrt{2}}$ put in (iii)

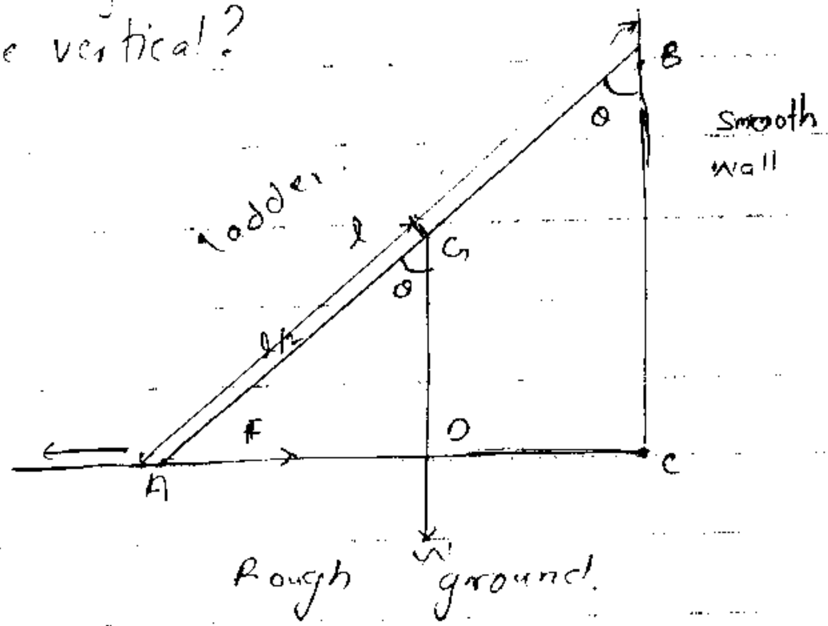
$$\frac{W}{2\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = T$$

$$\frac{W}{2\sqrt{3\times 2}} = T$$

$$\frac{W}{2\sqrt{6}} = T \quad \text{proved}$$

Q:No: 01:- A uniform ladder rests with its upper end against a smooth vertical wall and its foot on rough horizontal ground. Show that the force of friction at the ground is $\frac{1}{2}W\tan\theta$, where W is the weight of ladder and θ is its inclination with the vertical?

Solution:-



Let l be the length of ladder AB. End B is attached with smooth wall and end A on rough horizontal ground. Weight W of ladder acts at G (mid point of AB) $AG = l/2$.

∴ θ be the inclination of ladder with verticle.

F be the force friction.

Equation of virtual work

$-W\,\delta(OG) - F\,\delta(AC) = 0 \longrightarrow (i)$

In ΔADG

$OG = \frac{l}{2}\cos\theta$.

In ΔABC.

$AC = l\sin\theta$

using in (i)

$-W\,\delta\left(\frac{l}{2}\cos\theta\right) - F\,\delta(l\sin\theta) = 0$

$\frac{Wl}{2}\sin\theta\,\delta\theta - Fl\cos\theta\,\delta\theta = 0$

$\left(\frac{Wl}{2}\sin\theta - Fl\cos\theta\right)\delta\theta = 0$

$\because \delta\theta \neq 0$, so

$\Rightarrow \frac{Wl}{2}\sin\theta - Fl\cos\theta = 0$

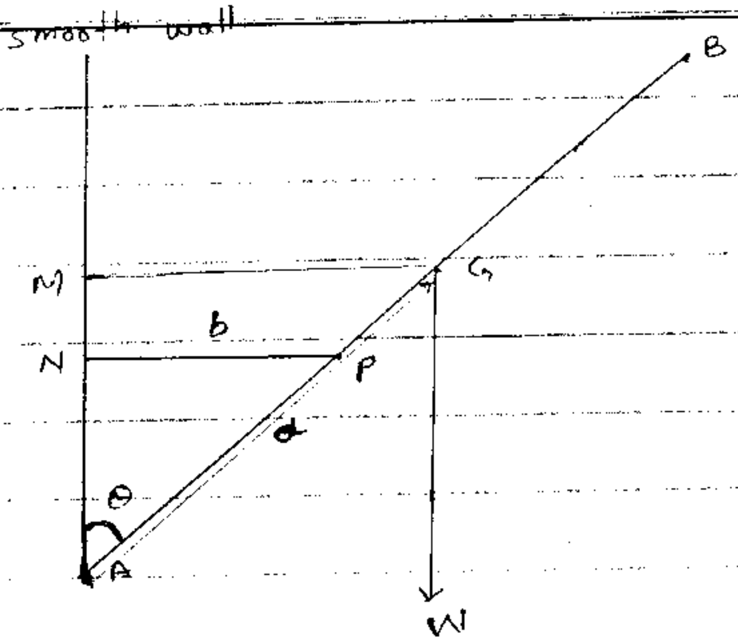
$\frac{W\cancel{l}\sin\theta}{2} = F\cancel{l}\cos\theta$.

$\frac{1}{2}W\frac{\sin\theta}{\cos\theta} = F$

$\frac{1}{2}W\tan\theta = F$ proved

v.p (2012)

Q: No: 10: A uniform rod of length 2a rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that, in the position of equilibrium, the beam is inclined to the wall at an angle $\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$?



Given $AB = 2a$ length of rod.

$AG = a$ where G is the point where W acts.

$NP = b$ Distance of peg P from wall

$\theta =$ angle of rod with wall.

Equation of virtual work.

$$-W\,d(MN) = 0 \longrightarrow (i)$$

From fig.

$$MN = AM - AN \longrightarrow (ii)$$

In $\triangle AMG$

$$AM = AG\cos\theta$$

$$= a\cos\theta$$

In $\triangle ANP$

$$\tan\theta = \frac{b}{AN}$$

$$AN = \frac{b}{\tan\theta}$$

$$AN = b\cot\theta$$

Both values using in (ii).

$MN = a\cos\theta - b\cot\theta$

using in Equ. (i).

$-W\delta(a\cos\theta - b\cot\theta) = 0$

$-W[-a\sin\theta\delta\theta + b\,\text{cosec}^2\theta\,\delta\theta] = 0$

$\delta\theta\,[W a\sin\theta - Wb\,\text{cosec}^2\theta] = 0$

$\because \delta\theta \neq 0$

$\Rightarrow W a\sin\theta - W b\,\text{cosec}^2\theta = 0$

$\cancel{W}a\sin\theta = \cancel{W}b\,\text{cosec}^2\theta$

$a\sin\theta = \dfrac{b}{\sin^2\theta}$

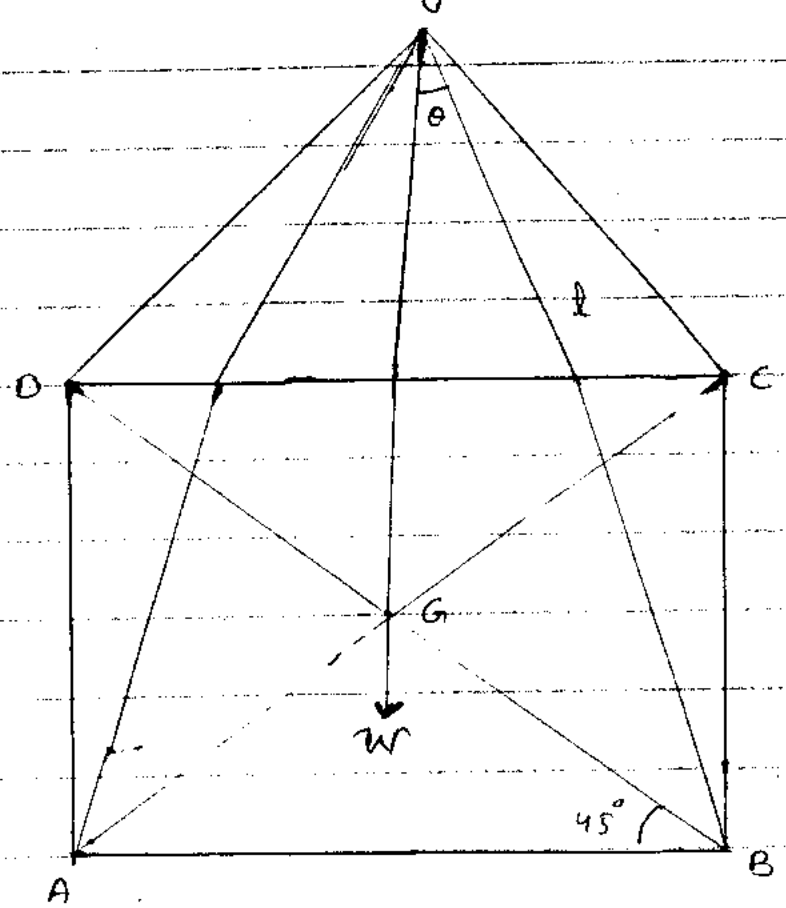
$\sin^3\theta = \dfrac{b}{a}$

$\sin\theta = \left(\dfrac{b}{a}\right)^{1/3}$

$\theta = \sin^{-1}\left(\dfrac{b}{a}\right)^{1/3}$ Ans

Q:No: 07:- Four equal uniform rods, each of weight w, are connected at one end of each by means of a smooth jointed, and the other ends rest on a smooth table and are connected by equal strings. A weight W is suspended from the joint show that the tension in each string is.

$\left(\dfrac{W+2w}{4}\right)\dfrac{a}{\sqrt{4l^2-2a^2}}$ Where l is the length of each rod and a is the length of each string?



∵ l be the length of each rod.

i.e $OA = OB = OC = OD = l$

strings $AB = BC = CD = AD = a$

W is the weight of each rod.

$4W$ weight of 4 rod acts at mid point of OG i.e at $\frac{OG}{2}$ where OG is verticle line through O and G is the point of intersection of diagonals AC and BD.

θ is the angle which each rod makes with verticle OG.

T be the tension in each string W is the weight attached to OG.

Equ. of virtual work.

$$-4W\delta\left(\frac{OG}{2}\right) - W\delta(OG) - 4T\delta(AB) = 0 \rightarrow (i)$$

In ΔBGO

$$OG = l\cos\theta$$
$$BG = l\sin\theta.$$

Then, $BD = 2BG = 2l\sin\theta$

In ΔABD.

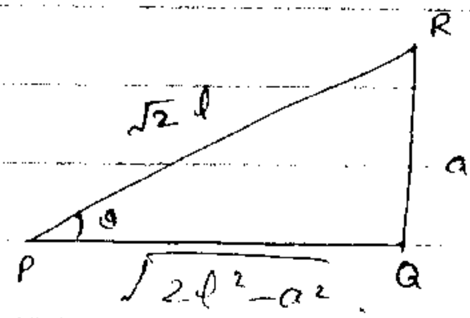
$$AB = BD\cos 45^\circ$$
$$= 2l\sin\theta \cdot \frac{1}{\sqrt{2}}$$
$$AB = \sqrt{2}\,l\sin\theta$$

$\because AB = a$

$$a = \sqrt{2}\,l\sin\theta.$$

$$\frac{a}{\sqrt{2}\,l} = \sin\theta.$$

$\therefore$



$$(\sqrt{2}\,l)^2 = a^2 + (PQ)^2$$
$$2l^2 - a^2 = (PQ)^2$$
$$\sqrt{2l^2 - a^2} = PQ \longrightarrow$$

$$\tan\theta = \frac{a}{\sqrt{2l^2 - a^2}}. \longrightarrow (ii)$$

using values in Equ. (i)

$$-4w\,\delta\left(\frac{l\cos\theta}{2}\right) - W\delta(l\cos\theta) - 4T\delta(\sqrt{2}\,l\sin\theta) = 0$$

$$\frac{\overset{2}{\cancel{4}}w}{\cancel{2}}\, l\sin\theta\,\delta\theta + \frac{Wl}{1}\sin\theta\,\delta\theta - \frac{4\sqrt{2}\,Tl\cos\theta\,\delta\theta}{1} = 0$$

$$(2wl\sin\theta + Wl\sin\theta - 4\sqrt{2}\,Tl\cos\theta)\,\delta\theta = 0$$

$\because \delta\theta \neq 0$

$$\Rightarrow 2wl\sin\theta + Wl\sin\theta - 4\sqrt{2}\,lT\cos\theta = 0$$

$$2wl\sin\theta + Wl\sin\theta = 4\sqrt{2}\,Tl\cos\theta$$

$\div$ ing by "l".

$$\frac{(2w + W)\sin\theta}{4\sqrt{2}\cos\theta} = T$$

$$\frac{(2w + W)}{4\sqrt{2}}\tan\theta = T$$

$$\frac{(2w + W)}{4\sqrt{2}}\,\frac{a}{\sqrt{2l^2 - a^2}} = T$$

$$\frac{(2w + W)\,a}{4\sqrt{4l^2 - 2a^2}} = T$$

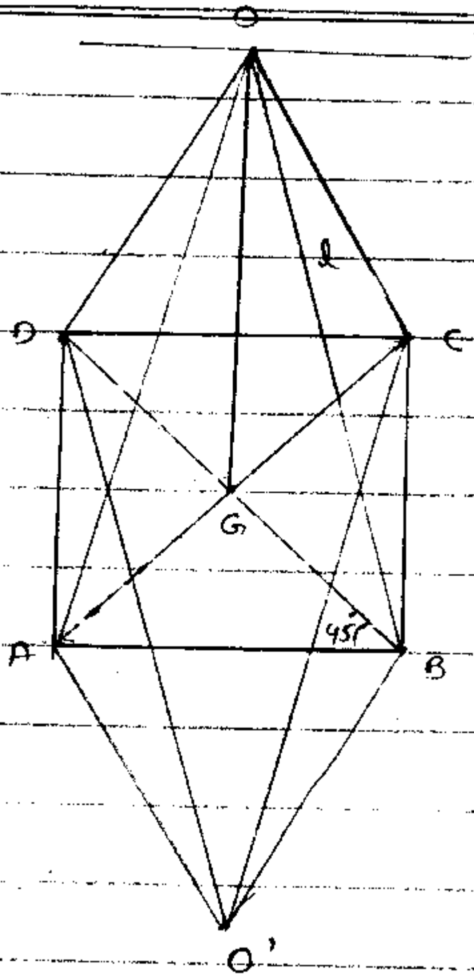
proved

Q#12: v.i.p (2014)

A regular octahedron formed of twelve equal rods, each of weight w, freely jointed together is suspended from one corner. Show that the thrust in each horizontal rod is

$$\frac{3}{2}\sqrt{2}\,w?$$

Solution:-



∴ W be the weight of each rod, 12 w is the weight of 12 rod which acts at mid point a of octahedron

if l be the length of each rod.

$OA = OB = OC = OD = AB = BC = CD = AD = O'A = O'B = O'C = O'D = l$

θ be the angle which rods in upper half makes with verticle OG. where G is the point of intersection of diagonals BD and AC.

if T be the thrust of each horizontal rod.