M.Tahir Aziz Tehleel

CHAPTER No. 2

Shahzad Ahmad Khan
M.Sc.(BZU),B.Ed.(AIOU)
دکنکس شہزاد صاحب

Composition of forces

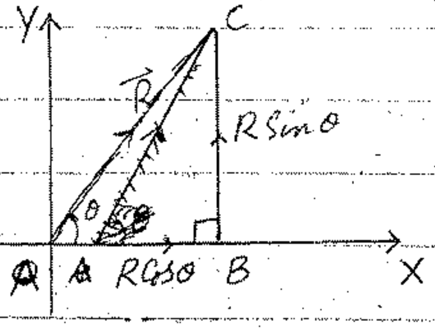
1

The basic problem of statics consist in the reduction of a given system of forces to the simplest system which will be equivalent to the original system, such a reduction is called composition of forces.

Resolved parts of a force

Considering a force $\vec{R}$ which makes an angle 'θ' with x-axis. Let the two directions in which we resolve the force $\vec{R}$ be mutually perpendicular, then each component is called the resolved part of $\vec{R}$ in the specified direction. The resolved parts of $\vec{R}$ along OX and OY are

$$R\cos\theta \quad \text{and} \quad R\sin\theta$$



If $\hat{i}$ and $\hat{j}$ are unit vectors along OX and OY respectively then

$$\vec{AC} = \vec{AB} + \vec{BC}$$

$$\vec{R} = (R\cos\theta)\hat{i} + (R\sin\theta)\hat{j}$$

or $$\vec{R} = X\hat{i} + Y\hat{j}$$

Question

Find the magnitude and inclination of a force when its resolved parts are given.

Solution

Let $\vec{R}$ be the force whose resolved parts along the axes are X and Y, then

$$\vec{R} = X\hat{i} + Y\hat{j}$$

$$\therefore |\vec{R}| = \sqrt{X^2 + Y^2}$$

$$\text{or } R = \sqrt{X^2 + Y^2}$$

is the magnitude of the force $\vec{R}$

Let $\vec{R}$ makes an angle 'θ' with X-axis then

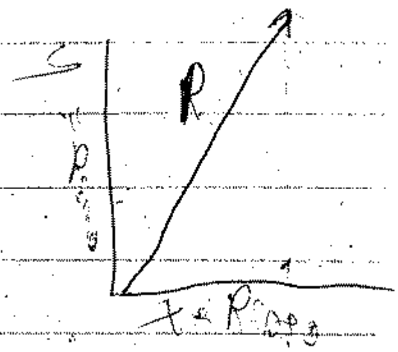
$$X = R\cos\theta \text{ and } Y = R\sin\theta$$

and

$$\frac{R\sin\theta}{R\cos\theta} = \frac{Y}{X}$$

$$\tan\theta = \frac{Y}{X}$$

$$\text{or } \theta = \tan^{-1}\frac{Y}{X}$$



Concurrent forces

Forces whose line of action intersect at one point are called concurrent forces. Such forces always have a resultant through the point of concurrence.

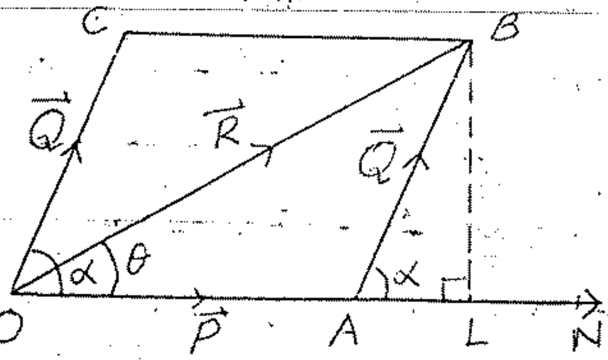
Question

Find the magnitude and direction of the resultant of two concurrent forces $\vec{P}$ and $\vec{Q}$ inclined at an angle α with eachother.

Solution

Considering two forces $\vec{P}$ and $\vec{Q}$ represented by vector lines $\vec{OA}$ and $\vec{OC}$ respectively act at point O and are inclined at an angle α with eachother.

Complete a ||gm OABC. Let $\vec{R}$ be the resultant of the forces $\vec{P}$ and $\vec{Q}$ represented by



the vector line $\overrightarrow{OB}$ (By triangle law of vectors)

Draw a perpendicular from point B on $\overrightarrow{ON}$ which meets it at point L. In this way we get two right angled triangles OLB and ALB

From right angled triangle OLB, according to Pythagoras theorem

$$|\overrightarrow{OB}|^2 = |\overrightarrow{OL}|^2 + |\overrightarrow{LB}|^2$$

$$|\vec{R}|^2 = (|\overrightarrow{OA}| + |\overrightarrow{AL}|)^2 + |\overrightarrow{LB}|^2$$

$$R^2 = (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$$

$$R^2 = P^2 + Q^2\cos^2\alpha + 2PQ\cos\alpha + Q^2\sin^2\alpha$$

$$= P^2 + Q^2(\cos^2\alpha + \sin^2\alpha) + 2PQ\cos\alpha$$

$$= P^2 + Q^2(1) + 2PQ\cos\alpha$$

$$= P^2 + Q^2 + 2PQ\cos\alpha$$

From right angled triangle ALB

$$\frac{|\overrightarrow{AL}|}{|\overrightarrow{AB}|} = \cos\alpha$$

$$\frac{|\overrightarrow{AL}|}{Q} = \cos\alpha$$

$$|\overrightarrow{AL}| = Q\cos\alpha$$

similarly

$$|\overrightarrow{LB}| = Q\sin\alpha$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

is the magnitude of the resultant of two forces $\vec{P}$ and $\vec{Q}$ inclined at an angle α with eachother.

Let the resultant $\vec{R}$ of the concurrent force $\vec{P}$ and $\vec{Q}$ makes an angle θ with the horizental

then from right angled triangle OLB

$$\tan\theta = \frac{|\overrightarrow{BL}|}{|\overrightarrow{OL}|}$$

$$= \frac{|\overrightarrow{BL}|}{|\overrightarrow{OA}|+|\overrightarrow{AL}|} \qquad \text{as } |\overrightarrow{OL}| = |\overrightarrow{OA}|+|\overrightarrow{AL}|$$

$$= \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

$$\text{or} \quad \theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right)$$

is the expression for the direction of the resultant force.

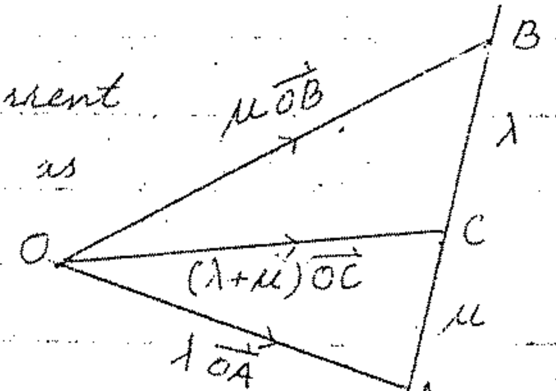
If two concurrent forces are represented by $\lambda\overrightarrow{OA}$ and $\mu\overrightarrow{OB}$. Then their resultant is given by $(\lambda+\mu)\overrightarrow{OC}$ where C divids AB such that AC : CB = μ : λ

Imp (λ, μ) Theorem

Statement If $\lambda\overrightarrow{OA}$ and $\mu\overrightarrow{OB}$ be two concurrent forces, then their resultant is a force $(\lambda+\mu)\overrightarrow{OC}$, where 'C' is a point on $\overrightarrow{AB}$ s.t $|\overrightarrow{AC}|:|\overrightarrow{CB}| = \mu : \lambda$

Proof

Considering two concurrent forces $\lambda\overrightarrow{OA}$ and $\mu\overrightarrow{OB}$ as shown. From vector triangle OCA,



$$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA}$$

$$\times\lambda \quad \lambda\overrightarrow{OA} = \lambda\overrightarrow{OC} + \lambda\overrightarrow{CA} \quad \text{——— (1)}$$

From vector triangle OCB

Force

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$\times \mu$

$\mu\overrightarrow{OB} = \mu\overrightarrow{OC} + \mu\overrightarrow{CB}$ ——— (2)

Adding (1) and (2)

$\lambda\overrightarrow{OA} + \mu\overrightarrow{OB} = \lambda\overrightarrow{OC} + \lambda\overrightarrow{CA} + \mu\overrightarrow{OC} + \mu\overrightarrow{CB}$

$= (\lambda+\mu)\overrightarrow{OC} + \lambda\overrightarrow{CA} + \mu\overrightarrow{CB}$ ——— (3)

Now according to statement

$|\overrightarrow{AC}| : |\overrightarrow{CB}| = \mu : \lambda$

or $\dfrac{|\overrightarrow{AC}|}{|\overrightarrow{CB}|} = \dfrac{\mu}{\lambda}$

or $\lambda\overrightarrow{AC} = \mu\overrightarrow{CB}$

$-\lambda\overrightarrow{CA} = \mu\overrightarrow{CB}$

or $\lambda\overrightarrow{CA} + \mu\overrightarrow{CB} = 0$ ——— (4)

Using (4) in (3) we get

$\lambda\overrightarrow{OA} + \mu\overrightarrow{OB} = (\lambda+\mu)\overrightarrow{OC} + 0$

$\lambda\overrightarrow{OA} + \mu\overrightarrow{OB} = (\lambda+\mu)\overrightarrow{OC}$

P is the proof of (λ, μ) Theorem.

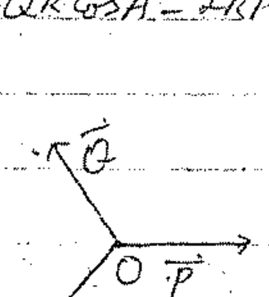
Example 1 P-19

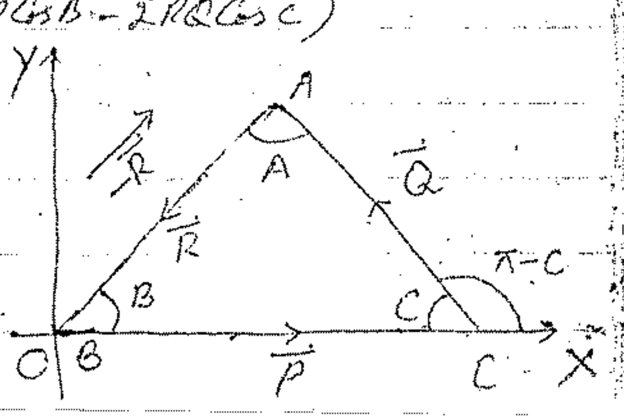
Forces $\vec{P}, \vec{Q}, \vec{R}$ act at a point parallel to the sides of a triangle ABC taken in the same order. Show that the magnitude of the resultant is

$(P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C)^{1/2}$

Sol

Considering the forces $\vec{P}, \vec{Q}, \vec{R}$ acting at pt 'O'





along the sides of a triangle in the same order.
Resolving the forces along the coordinate axes.

$$X = P\cos 0 + Q\cos(\pi - C) - R\cos B$$

$$= P(1) - Q\cos C - R\cos B \qquad \because \cos(\pi - C) = -\cos C$$

$$X = P - Q\cos C - R\cos B \qquad \sin(\pi - C) = \sin C$$

and $$Y = P\sin 0 + Q\sin(\pi - C) - R\sin B \qquad \sin(2\pi - C) = -\sin C$$

$$= P(0) + Q\sin C - R\sin B \qquad \cos(2\pi - C) = +\cos C$$

$$Y = Q\sin C - R\sin B$$

Let "R'" be the resultant of the forces.
then

$$R' = \sqrt{X^2 + Y^2}$$

$$R' = \left[(P - Q\cos C - R\cos B)^2 + (Q\sin C - R\sin B)^2\right]^{1/2}$$

$$= \left[P^2 + Q^2\cos^2 C + R^2\cos^2 B - 2PQ\cos C + 2QR\cos B\cos C - 2PR\cos B + Q^2\sin^2 C + R^2\sin^2 B - 2QR\sin B\sin C\right]^{1/2}$$

$$= \left[P^2 + Q^2(\cos^2 C + \sin^2 C) + R^2(\cos^2 B + \sin^2 B) + 2QR(\cos B\cos C - \sin B\sin C) - 2PQ\cos C - 2RP\cos B\right]^{1/2}$$

$$= \left(P^2 + Q^2(1) + R^2(1) + 2QR\cos(B + C) - 2PQ\cos C - 2RP\cos B\right)^{1/2}$$

$$= \left[P^2 + Q^2 + R^2 + 2QR(-\cos A) - 2PQ\cos C - 2RP\cos B\right]^{1/2}$$

$$\because A + B + C = \pi$$

$$B + C = \pi - A$$

$$\cos(B + C) = \cos(\pi - A)$$

$$= -\cos A$$

$$R' = \left(P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C\right)^{1/2}$$

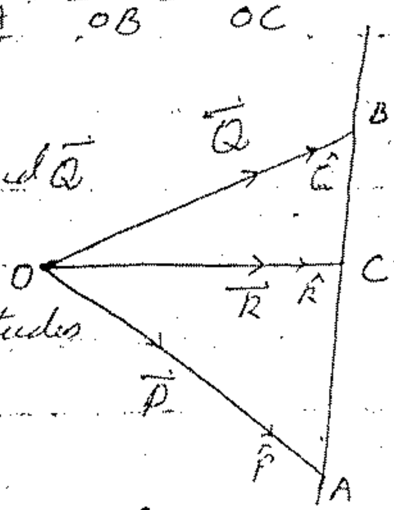
as required

⑦

Example 2

Forces $\vec{P}$ and $\vec{Q}$ act at a point 'O' and their resultant is $\vec{R}$. If any transversal cuts the line of action of the forces in the points A, B, C respectively, Prove that $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$

Sol



Considering the forces $\vec{P}$ and $\vec{Q}$ act at a point O and are represented by magnitudes and directions by vector lines $\overrightarrow{OA}$ and $\overrightarrow{OB}$ resp.

Let $\vec{R}$ be their resultant force represented by vector line $\overrightarrow{OC}$.

Then

$$\vec{P} = |\vec{P}|\,\hat{P}$$

and $\vec{Q} = |\vec{Q}|\,\hat{Q}$ where $\hat{P}$ and $\hat{Q}$ are unit vectors along forces $\vec{P}$ and $\vec{Q}$ respectively.

Since $\vec{R}$ is the resultant of $\vec{P}$ and $\vec{Q}$, so

$$\vec{R} = |\vec{R}|\cdot\hat{R}$$

Now

$$\hat{P} = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|}, \quad \hat{Q} = \frac{\overrightarrow{OB}}{|\overrightarrow{OB}|} \text{ and } \hat{R} = \frac{\overrightarrow{OC}}{|\overrightarrow{OC}|}$$

Now

$$\vec{P} = |\vec{P}|.\frac{\overrightarrow{OA}}{|\overrightarrow{OA}|}$$

$$\vec{P} = \left(\frac{P}{OA}\right)\overrightarrow{OA} \qquad \because |\vec{P}| = P, |\overrightarrow{OA}| = OA$$

so $\vec{Q} = \left(\frac{Q}{OB}\right)\vec{OB}$

and $\vec{R} = \left(\frac{R}{OC}\right)\vec{OC}$

Since $\left(\frac{P}{OA}\right)\vec{OA}$ and $\left(\frac{Q}{OB}\right)\vec{OB}$ are Concurrent forces so applying the (λ, μ) Theorem

$$\left(\frac{P}{OA}\right)\vec{OA} + \left(\frac{Q}{OB}\right)\vec{OB} = \left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC}$$

$$\vec{P} + \vec{Q} = \left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC} \quad \text{--- (1)}$$

Also $\vec{P} + \vec{Q} = \vec{R}$ (according to question)

$$\vec{P} + \vec{Q} = \left(\frac{R}{OC}\right)\vec{OC} \quad \text{--- (2)}$$

from (1) and (2)

$$\left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC} = \left(\frac{R}{OC}\right)\vec{OC}$$

Equating the Coefficients of $\vec{OC}$ we get

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$$ as Required.

Imp **Lamy's-Theorem** If a particle is in equilibrium under the action of three concurrent forces, Then magnitude of each force (is proportional to the sine of the angle between the other two forces)

has proportional to the sine of angle b/w the other two.

LAMI'S THEOREM F_1, F_2, F_3 act at a point are in equilib

Then $\frac{F_1}{\ldots}$ — ①

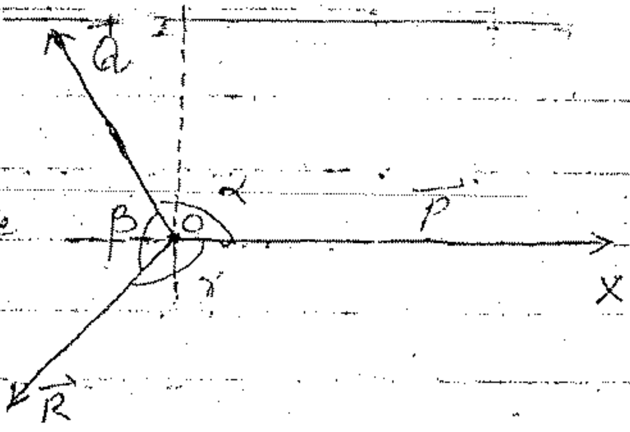
uct of ① with F_1 we get

(9) Ch-2

$\vec{A}\times\vec{r}$

Proof

Considering a body at pt O, and is in equilibrium under the action of three concurrent forces $\vec{P}, \vec{Q}$ and $\vec{R}$



Resolving the forces along the coordinate axes.

$$X = P\cos 0° + Q\cos\alpha + R\cos(\alpha+\beta)$$
$$= P(1) + Q\cos\alpha + R\cos\gamma$$
$$= P + Q\cos\alpha + R\cos\gamma$$

$\alpha+\beta+\gamma = 2\pi$
$\alpha+\beta = 2\pi-\gamma$
$\cos(\alpha+\beta) = \cos(2\pi-\gamma) = \cos\gamma$

$$Y = P\sin 0° + Q\sin\alpha + R\sin(\alpha+\beta)$$
$$= P(0) + Q\sin\alpha - R\sin\gamma$$
$$= Q\sin\alpha - R\sin\gamma$$

$\alpha+\beta+\gamma = 2\pi$
$\alpha+\beta = 2\pi-\gamma$
$\sin(\alpha+\beta) = \sin(2\pi-\gamma) = -\sin\gamma$

Since particle is in equilibrium

so $R = 0$

$\Rightarrow X = Y = 0$

$\because R = \sqrt{X^2+Y^2}$
Put $R=0$: $0 = \sqrt{X^2+Y^2}$
$X^2+Y^2 = 0$
$\Rightarrow X=0, Y=0$

$$X=0 \Rightarrow P + Q\cos\alpha + R\cos\gamma = 0 \quad \text{—— ①}$$

$$Y=0 \Rightarrow Q\sin\alpha - R\sin\gamma = 0$$

or $R\sin\gamma = Q\sin\alpha$

$\div \sin\alpha\sin\gamma$

$$\frac{R}{\sin\alpha} = \frac{Q}{\sin\gamma} \quad \text{—— ②}$$

Also from (2)

$$Q = \frac{R\sin\gamma}{\sin\alpha} \quad \text{———— (3)}$$

using (3) in (1)

$$P + \frac{R\sin\gamma}{\sin\alpha}\cos\alpha + R\cos\gamma = 0$$

$$P + R\left(\frac{\sin\gamma\cos\alpha}{\sin\alpha} + \cos\gamma\right) = 0$$

$$P + R\left(\frac{\sin\gamma\cos\alpha + \cos\gamma\sin\alpha}{\sin\alpha}\right) = 0$$

$$P + R\,\frac{\sin(\alpha+\gamma)}{\sin\alpha} = 0$$

$\alpha + \beta + \gamma = 2\pi$

$\alpha + \gamma = 2\pi - \beta$

$\sin(\alpha+\gamma) = \sin(2\pi - \beta)$

$= -\sin\beta$

$$P - R\,\frac{\sin\beta}{\sin\alpha} = 0$$

$\times \sin\alpha$

$$P\sin\alpha - R\sin\beta = 0$$

or $\quad R\sin\beta = P\sin\alpha$

$\div \sin\alpha\sin\beta$

$$\frac{R}{\sin\alpha} = \frac{P}{\sin\beta} \quad \text{———— (4)}$$

Now

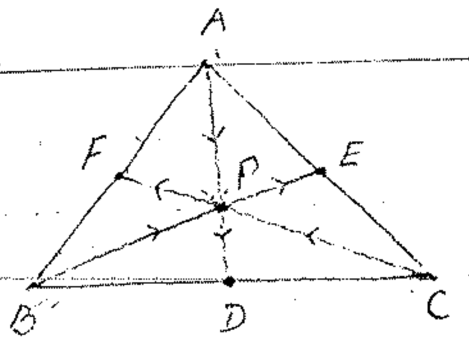
$$\frac{R}{\sin\alpha} = \frac{R}{\sin\alpha} = \frac{R}{\sin\alpha}$$

$$\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha} \quad \text{by (2) and (4)}$$

is the proof of Lamy's Theorem.

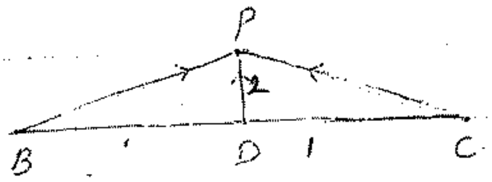
<u>Example 1 P-22</u>

P is any point in the plane of a triangle ABC and D, E, F are the middle points of its sides, Prove that the forces $\overrightarrow{AP}, \overrightarrow{BP}, \overrightarrow{CP}, \overrightarrow{PD}, \overrightarrow{PE}$ & $\overrightarrow{PF}$ are in equilibrium tion



Considering the forces $\overrightarrow{AP}, \overrightarrow{BP}, \overrightarrow{CP}, \overrightarrow{PD}, \overrightarrow{PE}$ and $\overrightarrow{PF}$ acting as shown in fig.

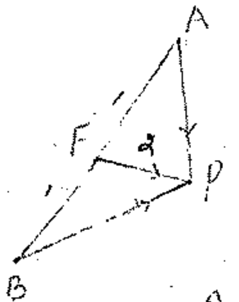
To prove The forces are in equilibrium



Applying the (λ, μ) Theorem to the Concurrent forces $\overrightarrow{BP}$ and $\overrightarrow{CP}$

$$1\overrightarrow{BP} + 1\overrightarrow{CP} = (1+1)\overrightarrow{DP}$$

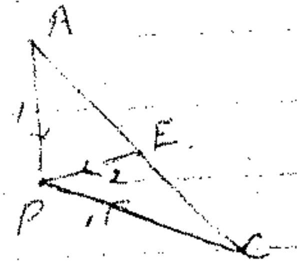
$$\overrightarrow{BP} + \overrightarrow{CP} = 2\overrightarrow{DP} \quad \text{——— (1)}$$



Now applying (λ, μ) Theorem to the Concurrent forces $\overrightarrow{AP}$ and $\overrightarrow{BP}$

$$1\overrightarrow{AP} + 1\overrightarrow{BP} = (1+1)\overrightarrow{FP}$$

$$\overrightarrow{AP} + \overrightarrow{BP} = 2\overrightarrow{FP} \quad \text{——— (2)}$$



Now applying (λ, μ) Theorem to the Concurrent forces $\overrightarrow{AP}$ and $\overrightarrow{CP}$

$$1\overrightarrow{AP} + 1\overrightarrow{CP} = (1+1)\overrightarrow{EP}$$

$$\overrightarrow{AP} + \overrightarrow{CP} = 2\overrightarrow{EP} \quad \text{——— (3)}$$

Adding (1), (2) and (3)

$$\overrightarrow{BP} + \overrightarrow{CP} = 2\overrightarrow{DP}$$

$$\overrightarrow{AP} + \overrightarrow{BP} = 2\overrightarrow{FP}$$

$$\overrightarrow{AP} + \overrightarrow{CP} = 2\overrightarrow{EP}$$

$$2\overrightarrow{AP} + 2\overrightarrow{BP} + 2\overrightarrow{CP} = 2\overrightarrow{DP} + 2\overrightarrow{FP} + 2\overrightarrow{EP}$$

$\div 2$ $\quad \overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} = \overrightarrow{DP} + \overrightarrow{FP} + \overrightarrow{EP}$

$$\overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} = -\overrightarrow{PD} - \overrightarrow{PF} - \overrightarrow{PE}$$

$\vec{AP} + \vec{BP} + \vec{CP} + \vec{PD} + \vec{PE} + \vec{PF} = 0$

$\therefore \Sigma F = 0$

Sum of the forces acting on the system is zero

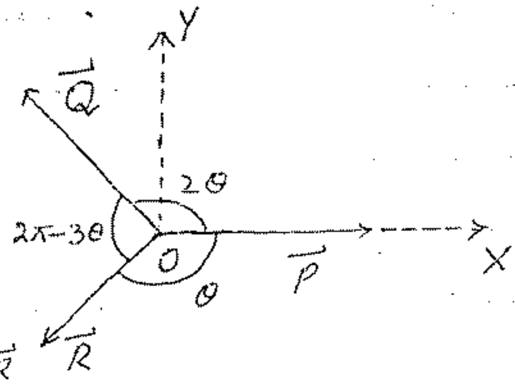
So the system is in equilibrium.

Imp **Example 2, P-23**

Three forces $\vec{P}, \vec{Q}, \vec{R}$, acting at a point are in equilibrium and the angle between $\vec{P}$ and $\vec{Q}$ is double of the angle between $\vec{P}$ and $\vec{R}$. Prove that.

$$R^2 = Q(Q-P)$$

Sol Considering a particle placed at 'O' is in equilibrium under the action of three concurrent forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$



Let the angle between $\vec{P}$ and $\vec{R}$ is θ, Then the angle between $\vec{P}$ and $\vec{Q}$ will be 2θ.

Applying the Lamy's theorem

$$\frac{P}{\sin(2\pi-3\theta)} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$

$$\frac{P}{-\sin 3\theta} = \frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta} \qquad \because \sin(2\pi-3\theta) = -\sin 3\theta$$

Now

$$\frac{P}{-\sin 3\theta} = \frac{Q}{\sin\theta}$$

$P\sin\theta = -Q\sin 3\theta$

$P\sin\theta + Q\sin 3\theta = 0$ —— (1)

$$\frac{Q}{\sin\theta} = \frac{R}{\sin 2\theta}$$

$Q\sin 2\theta = R\sin\theta$

$Q(2\sin\theta\cos\theta) = R\sin\theta$

$\div 2Q\sin\theta$

$\cos\theta = \dfrac{R}{2Q}$ —— (2)

we know that

$$\sin 3\theta = \sin(2\theta+\theta)$$

$$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$$

$$= 2\sin\theta\cos\theta\cos\theta + (2\cos^2\theta - 1)\sin\theta$$

$$\sin 3\theta = 2\sin\theta\cos^2\theta + \sin\theta(2\cos^2\theta-1) \quad \text{——— (3)}$$

$\sin 3\theta = [3\sin\theta - \ldots$

using (3) in (1)

$$P\sin\theta + Q(2\sin\theta\cos^2\theta + \sin\theta(2\cos^2\theta-1)) = 0$$

$$P\sin\theta + Q\sin\theta(2\cos^2\theta + 2\cos^2\theta - 1) = 0$$

$\div \sin\theta$

$$P + Q(4\cos^2\theta - 1) = 0 \quad \text{——— (4)}$$

using (2) in (4)

$$P + Q\left(4\left(\frac{R}{2Q}\right)^2 - 1\right) = 0$$

$$P + Q\left(\frac{4R^2}{4Q^2} - 1\right) = 0$$

$$P + \frac{R^2}{Q} - Q = 0$$

$$\frac{R^2}{Q} = Q - P$$

$\times Q$

$$R^2 = Q(Q-P) \quad \text{as required.}$$

$$P\sin\theta + Q(3\sin\theta - 4\sin^3\theta) = 0$$

$$P\sin\theta + Q\sin\theta(3 - 4\sin^2\theta) = 0$$

$$P + Q(3 - 4\sin^2\theta) = 0$$

$$P + Q[3 - 4(1-\cos^2\theta)] = 0$$

$$P + Q[3 - 4 + 4\cos^2\theta]$$

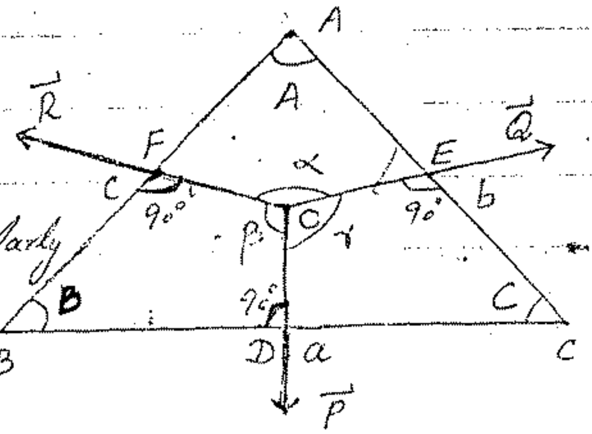
$$P + Q[4\cos^2\theta - 1]$$

Example 3 P-24

Three forces act perpendicularly to the sides of a triangle at their middle points and are proportional to the sides, prove that they are in equilibrium.

Solution

Considering the forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ act perpendicularly to the sides of a triangle ABC



Exp 3 P-24- Notes P B

Let a, b, c are the length of the sides of triangle ABC.

By the condition (forces are proportional to the sides) at their middle points D, E and F respectively as shown in figure.

Since forces are proportional to the sides so

$P \propto a$ | $Q \propto b$ | $R \propto c$

$P = ka$ | $Q = kb$ | $R = kc$

where 'k' is the constant of proportionality.

Now according to Sine Law of triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\times k$

$$\frac{ka}{\sin A} = \frac{kb}{\sin B} = \frac{kc}{\sin C}$$

Put $ka = P$, $kb = Q$, $kc = R$

$$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C} \quad \text{——— (1)}$$

From quadrilateral BDOF

$$B + 90^\circ + \beta + 90^\circ = 360^\circ$$

$$B + \beta + 180^\circ = 360^\circ$$

$$B = 360^\circ - 180^\circ - \beta$$

$$B = 180^\circ - \beta$$

Similarly from quadrilaterals AFOE and CDOE we have $A = 180^\circ - \alpha$ and $C = 180^\circ - \gamma$

Put $A = 180^\circ - \alpha$, $B = 180^\circ - \beta$, $C = 180^\circ - \gamma$ in (1)

$$\frac{P}{\sin(180^\circ - \alpha)} = \frac{Q}{\sin(180^\circ - \beta)} = \frac{R}{\sin(180^\circ - \gamma)}$$

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

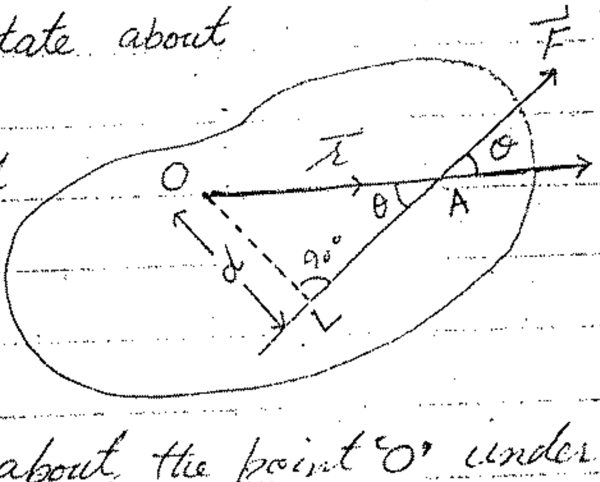
Thus magnitude of each force is proportional to the sine of the angle between other two forces so Lamy's theorem is satisfied. Hence the system is in equilibrium.

Moment of a force (Torque)

The tendency of a force to rotate a body about some fixed point is called the moment of a force.

Expression for the moment of a force

Considering a rigid body which can rotate about the fixed point 'O'.
Let a force $\vec{F}$ is applied on the rigid body at point A, whose position vector w.r.t 'O' is $\vec{r}$.



Let the body rotates about the point 'O' under the action of this force. Let $\vec{M}$ be the moment of this force then

$$\vec{M} = \vec{r} \times \vec{F}$$
$$= |\vec{r}||\vec{F}| \sin\theta \, \hat{n}$$
$$= rF \sin\theta \, \hat{n}$$

where $rF\sin\theta$ represents the magnitude of moment vector $\vec{M}$ and $\hat{n}$ represents the direction of moment vector.

Again

$$\vec{M} = rF\sin\theta\,\hat{n}$$

$$|\vec{M}| = (rF\sin\theta)\,|\hat{n}|$$

$$M = rF\sin\theta \cdot 1 \qquad \because |\hat{n}| = 1$$

$$= rF\sin\theta$$

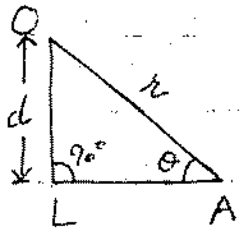
Now draw a perpendicular from point 'O' on the the line of action of the force $\vec{F}$ which meets it at the point L.

Again

$$M = rF\sin\theta$$
$$= F(r\sin\theta)$$
$$= Fd$$

from $\triangle OLA$

$$\frac{d}{r} = \sin\theta$$
$$d = r\sin\theta$$



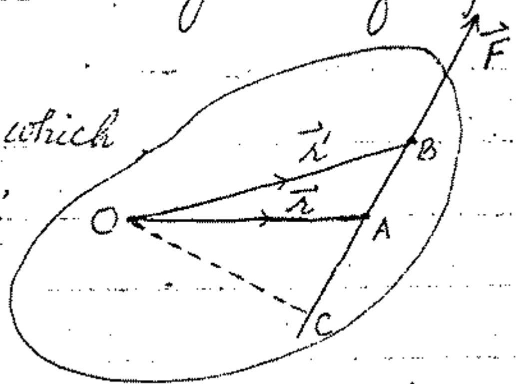
is the expression of the magnitude of the moment of the force.

Question

Prove that the moment of the force is independent of the choice of the point (point of application of the force) on the line of action of the force.

Proof

Considering a rigid body which rotates about the point 'O'. Let a force $\vec{F}$ is applied on the rigid body at point A whose position vector w.r.t 'O' is $\vec{r}$.



Let $\vec{M}$ be the moment of this force then

$$\vec{M} = \vec{r} \times \vec{F}$$

Let the same force $\vec{F}$ is applied on the rigid body at point B, whose position vector w.r.t 'O' is $\vec{r}'$.
Let $\vec{M}'$ be the moment of this force then

$$\vec{M}' = \vec{r}' \times \vec{F}$$

Put $\vec{r}' = \vec{r} + \vec{AB}$

$$\vec{M}' = (\vec{r} + \vec{AB}) \times \vec{F}$$
$$= \vec{r} \times \vec{F} + \vec{AB} \times \vec{F}$$

From vector triangle OAB
$$\vec{OB} = \vec{OA} + \vec{AB}$$
$$\vec{r}' = \vec{r} + \vec{AB}$$

Put $\vec{r} \times \vec{F} = \vec{M}$ and $\vec{AB} \times \vec{F} = 0$ ($\because$ $\vec{AB}$ and $\vec{F}$ are Collinear)

$$\vec{M}' = \vec{M} + 0$$

$\therefore$ $\vec{AB} \times \vec{F} = 0$ ∥ to each oth.

So $\vec{M}' = \vec{M}$

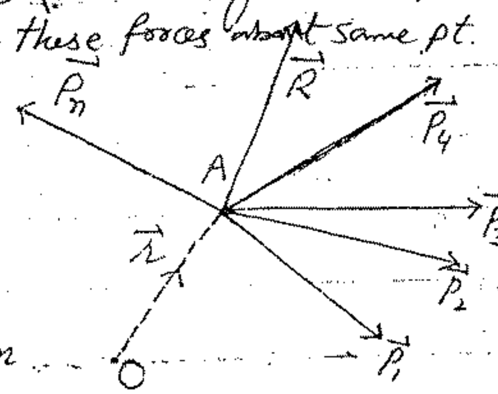
Moment of force acting at point B = Moment of force acting at point A

Clearly the moment of the force about some fixed point is independent of the choice of the point on the line of action of the force.

Imp Varignon's Theorem ✓

Statement. The moment about a point 'O' of the resultant of a system of concurrent forces is equal to the sum of the moments of the (various forces about the same point 'O') of these forces about same pt.

Proof



Considering a system of Concurrent forces $\vec{P}_1, \vec{P}_2, \vec{P}_3, \dots, \vec{P}_n$ acting at point A, whose position vector w.r.t 'O' is $\vec{r}$.

Let $\vec{R}$ be the resultant of the forces then,

$$\vec{R} = \vec{P_1} + \vec{P_2} + \vec{P_3} + \cdots + \vec{P_n} \quad \text{——— (1)}$$

Now

$$\left.\begin{aligned} &\text{Moment of } \vec{P_1} \text{ about 'O'} = \vec{G_1} = \vec{r} \times \vec{P_1} \\ &\;\;\text{"} \quad \text{"} \quad \vec{P_2} \quad \text{"} \quad \text{"} \;\; = \vec{G_2} = \vec{r} \times \vec{P_2} \\ &\;\;\text{"} \quad \text{"} \quad \vec{P_3} \quad \text{"} \quad \text{"} \;\; = \vec{G_3} = \vec{r} \times \vec{P_3} \\ &- - - - - - - - - - - - \\ &- - - - - - - - - - - - \\ &\text{Moment of } \vec{P_n} \text{ about 'O'} = G_n = \vec{r} \times \vec{P_n} \end{aligned}\right\} \text{——— (2)}$$

Let 'G' be the moment of the resultant force $\vec{R}$ about 'O' then

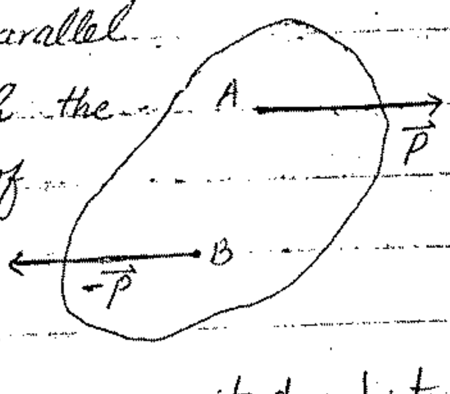
$$\begin{aligned} \vec{G} &= \vec{r} \times \vec{R} \\ &= \vec{r} \times (\vec{P_1} + \vec{P_2} + \vec{P_3} + \cdots + \vec{P_n}) \\ &= \vec{r} \times \vec{P_1} + \vec{r} \times \vec{P_2} + \vec{r} \times \vec{P_3} + \cdots + \vec{r} \times \vec{P_n} \\ &= \vec{G_1} + \vec{G_2} + \vec{G_3} + \cdots + \vec{G_n} \quad \text{by (2)} \end{aligned}$$

Hence the proof.

Couple

Definition: Two equal, unlike, parallel forces, whose line of action is not same form a couple.

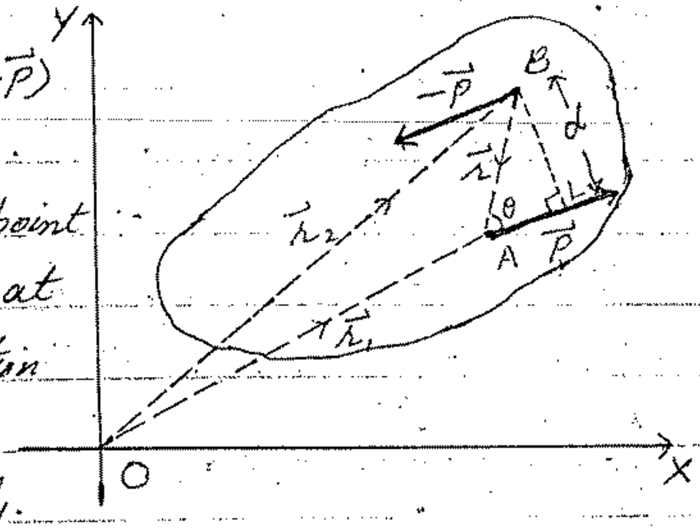
Considering two equal, unlike, parallel forces $\vec{P}$ and $-\vec{P}$ acting through the points A and B, whose line of action is not same, so they form a Couple $(\vec{P}, -\vec{P})$.



A pair of forces $(\vec{F}, -\vec{F})$ of same magnitude but opposite direction acting on a rigid body form a couple. When couple acts on a body it rotate the body.

[Ex]pression for magnitude of the moment of a Couple

Considering a couple $(\vec{P}, -\vec{P})$ acting on a rigid body. The force $\vec{P}$ acts at point 'A' and force $-\vec{P}$ acts at point 'B' whose position vector w.r.t. 'O' are $\vec{r}_1$ and $\vec{r}_2$ respectively.



Let $\vec{G}$ be the moment of this couple, then according to Varignon's theorem.

$$\vec{G} = \vec{r}_1 \times \vec{P} + \vec{r}_2 \times (-\vec{P})$$
$$= \vec{r}_1 \times \vec{P} - \vec{r}_2 \times \vec{P}$$
$$= (\vec{r}_1 - \vec{r}_2) \times \vec{P}$$

From vector triangle OAB

$$\vec{OA} = \vec{OB} + \vec{BA}$$
$$\vec{r}_1 = \vec{r}_2 + \vec{r}$$
$$\vec{r}_1 - \vec{r}_2 = \vec{r}$$

Put $\vec{r}_1 - \vec{r}_2 = \vec{r}$

$$\vec{G} = \vec{r} \times \vec{P}$$
$$= |\vec{r}||\vec{P}| \sin\theta \cdot \hat{n}$$
$$= rP \sin\theta \cdot \hat{n}$$
$$|\vec{G}| = (rP\sin\theta)|\hat{n}|$$
$$G = rP\sin\theta \cdot 1 \qquad \because |\hat{n}| = 1$$
$$= rP\sin\theta$$

Now draw a perpendicular from point B on the line of action of force $\vec{P}$ which meets it at point L (BL = d is called arm of the couple). In this way we get a right angled triangle ALB.

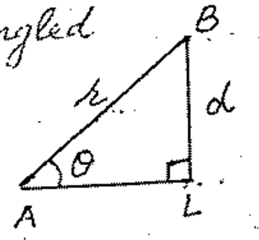
Again $G = rP\sin\theta$

$= P(r\sin\theta)$

Put $r\sin\theta = d$

$G = Pd$

From right angled triangle ALB



$\frac{d}{r} = \sin\theta$

$d = r\sin\theta$

is the expression for the magnitude of the moment of the couple $(\vec{P}, -\vec{P})$

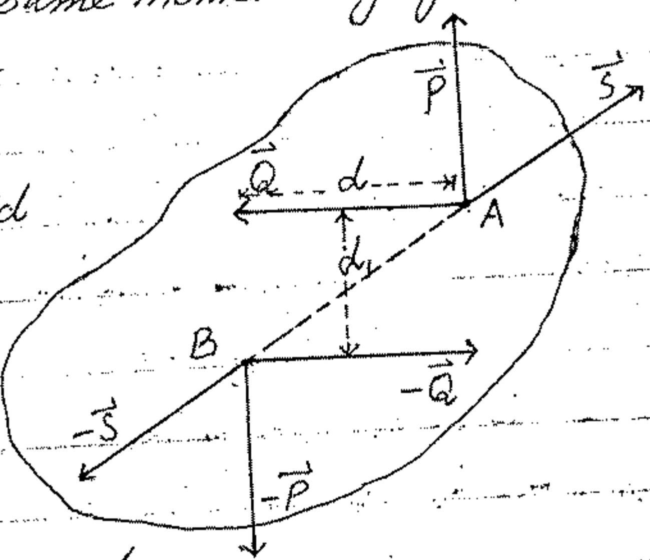
Equivalent couples

Theorem

The effect of a couple upon a rigid body remains unaltered, if it is replaced by any other couple of the same moment lying in the same plane.

Proof

Considering a rigid body upon which a couple $(\vec{P}, -\vec{P})$ with arm 'd' is acting. Let 'G' be the magnitude of the moment of this couple then,



$$G = Pd \quad \text{——} \quad (1)$$

Resolving the forces $\vec{P}$ and $-\vec{P}$ into component forces $\vec{S}, \vec{Q}, -\vec{S}$ and $-\vec{Q}$. Now $\vec{S}$ through A and $-\vec{S}$ through point B are equal in magnitude opposite in direction and their line of action is same, so they cancel

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$(\vec{F_1}, \vec{F_2})$ be a couple. Let A and B be points on the $\vec{F}$ and $-\vec{F}$ respectively. We want to replace the couple by ... couple. For this we draw to lines AC and BD is desire

Ch-2

the effect of eachother.

Now we are left with two forces $\vec{Q}$ through A and $-\vec{Q}$ through B. These two forces are equal in magnitude opposite in direction and their line of action is not same so they form a couple $(\vec{Q}, -\vec{Q})$. Let 'd_1' be the arm of this couple and let G_1 be the magnitude of the moment of this couple. Then

$$G_1 = Qd_1 \text{ ——— } (2)$$

Now we prove that $G = G_1$

Taking the moment of the forces about point B. According to Varignon's theorem

$$Pd = S(0) + Qd_1$$

$$G = G_1 \quad \text{by (1) and (2)}$$

BA × F = BA × (S + Q)

BA × S + BA × Q

0 + BA × Q

= BA × Q

Moment of the couple $(\vec{P}, -\vec{P})$ = Moment of couple $(\vec{Q}, -\vec{Q})$

Hence the proof.

Imp. Composition of Couples.

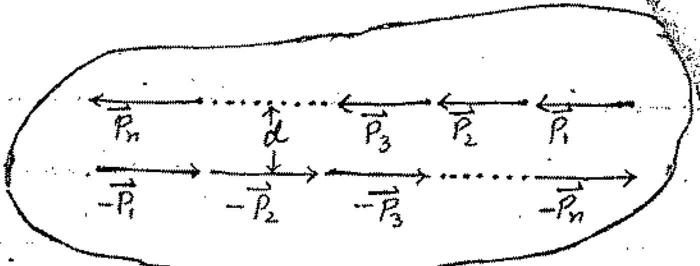
Theorem.

The coplaner couples of moments $G_1, G_2, G_3, \ldots, G_n$ are equivalent to a single couple lying in the same plane, whose moment G is given by

$$G = G_1 + G_2 + G_3 + \cdots + G_n$$

Proof.

Considering a system of Coplanar couples



$(\vec{P_1}, -\vec{P_1}), (\vec{P_2}, -\vec{P_2}), (\vec{P_3}, -\vec{P_3}), \ldots, (\vec{P_n}, -\vec{P_n})$

with common arm 'd' acting on a rigid body.

Let $G_1, G_2, G_3, \ldots, G_n$ be the magnitudes of the moments of these couples respectively.

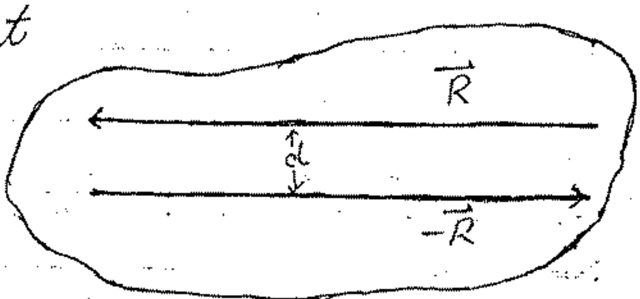
then
$$\left.\begin{aligned} G_1 &= P_1 d \\ G_2 &= P_2 d \\ G_3 &= P_3 d \\ &\ldots \\ &\ldots \\ G_n &= P_n d \end{aligned}\right\} \text{———} (1)$$

Take +ve couple always

Let $\vec{R} = \vec{P_1} + \vec{P_2} + \vec{P_3} + \cdots + \vec{P_n}$ ——— (2)

$-\vec{R} = (-\vec{P_1}) + (-\vec{P_2}) + (-\vec{P_3}) + \cdots + (-\vec{P_n})$

Now the new resultant Couple is $(\vec{R}, -\vec{R})$ with arm d. Let 'G' be the magnitude of moment of this couple



then $G = Rd$

$= (P_1 + P_2 + P_3 + \cdots + P_n) d$ by (2)

$= P_1 d + P_2 d + P_3 d + \cdots + P_n d$

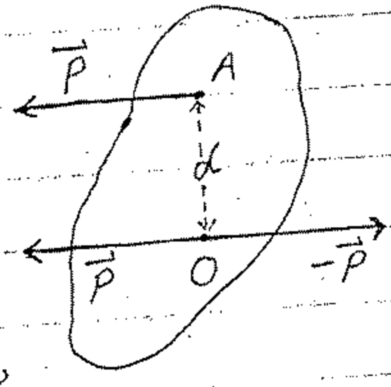
$G = G_1 + G_2 + G_3 + \cdots + G_n$ by (1)

Hence the proof.

A for

Theorem

A force $\vec{P}$ acting on a rigid body can be moved to a point 'O' of a rigid body provided a couple is added whose moment is equal to the moment of $\vec{P}$ about 'O'.



Proof:

A force $\vec{P}$ is acting at point 'A' of a rigid body. Let G be the magnitude of the moment of this force about 'O'.

Then $G = Pd$

We want to move this force $\vec{P}$ through the point 'O'.
We introduce two forces $\vec{P}$ and $-\vec{P}$ through point 'O'
(These forces cancel the effect of eachother). Now the force $\vec{P}$ through 'A' and $-\vec{P}$ through 'O' are equal in magnitude, opposite in direction and their line of action is not same so they form a couple $(\vec{P}, -\vec{P})$ of moment Pd about the point 'O'.
Now we are left with a force $\vec{P}$ acting through the point 'O'.

$\therefore$ A force $\vec{P}$ acting on a rigid body can be moved to any point 'O' of the rigid body provided a couple is added, whose moment is equal to the moment of $\vec{P}$ about the point 'O'.

Theorem (Converse of the previous theorem)

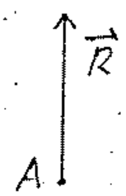
A single force and a couple acting in the same plane upon a rigid body are equivalent to a single force, acting in the direction parallel to its origional direction.

Proof.

Let $\vec{R}$ be a force through the point 'A' and a coplanar couple $(\vec{F}, -\vec{F})$ with arm 'd' are acting on a rigid body. Let G be the magnitude of the moment of this couple then

$$G = Fd \quad \text{——} \quad (1)$$

Replace this couple $(\vec{F}, -\vec{F})$ by a new couple $(\vec{R}, -\vec{R})$ with d' as d as its arm, then

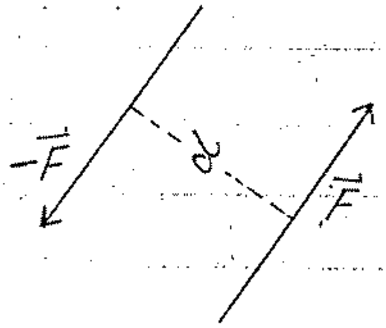


$$G = Rd' = Fd \quad \text{by (1)}$$

$$Rd' = Fd$$

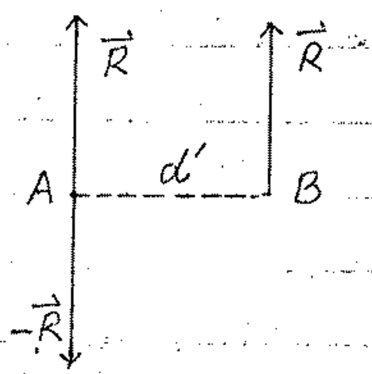
$$\therefore d' = \frac{Fd}{R}$$

$$= \frac{G}{R}$$



Transfer the new couple $(\vec{R}, -\vec{R})$ such that the force $-\vec{R}$ acts at A while $\vec{R}$ ats at B where $AB = d'$

The forces $\vec{R}$ and $-\vec{R}$ acting at A cancel the effect of eachother we are left with the only force $\vec{R}$ acting at B and in the direction parallel to its origional direction. Hence the proof.



ABC be a triangle represented by

Ch-2

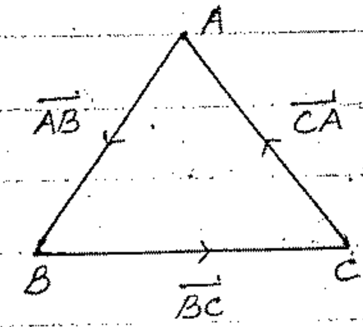
Example 1 P-31

If three forces are represented by magnitude direction and position by the sides of a triangle taken in order, they are equal to a couple. The moment of this couple is equal to the twice the area of the triangle.

Solution

Considering the forces $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ which are represented by magnitude direction and position by the sides of a triangle ABC taken in order.

To prove (i) Three forces are equal to a couple (ii) The moment of this couple is equal to twice the area of triangle ABC.

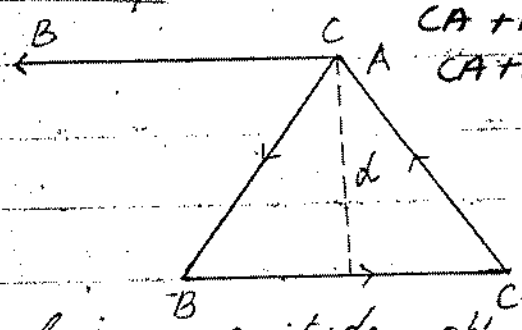


From head to tail rule

$\overrightarrow{CB}$ is the resultant of the forces $\overrightarrow{CA}$ and $\overrightarrow{AB}$ which must pass through the point 'A'

$BC + CA + AB = 0$
$CA + AB = -BC$
$CA + AB = CB$

Now we are left with the two forces



(i) $\overrightarrow{BC}$ (through B and C)

(ii) $\overrightarrow{CB}$ (through point A)

These two forces are equal in magnitude, opposite in direction and their line of action is not same so they form a couple $(\overrightarrow{BC}, -\overrightarrow{BC})$. Let 'd' be the arm of this couple and G be the magnitude of the moment of this couple, then

Ch-2

$$G = |\overrightarrow{BC}| \cdot d \quad \text{——— (1)}$$

Now Area of triangle ABC $= \frac{1}{2} |\overrightarrow{BC}| d$

$= \frac{1}{2} G$ by (1)

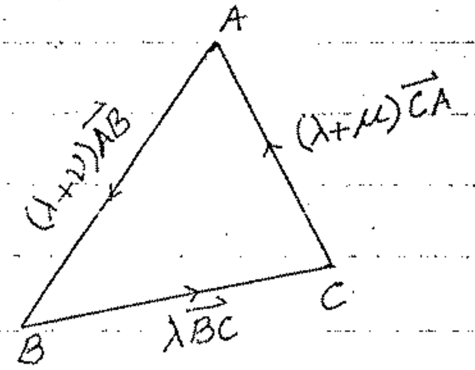
$\Rightarrow G = 2$ (Area of triangle ABC)

as required.

Example 2 P-3

Forces act along the sides $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ of a triangle. Show that they are equivalent to a Couple only if the forces are proportional to the sides.

Solution



Let $\vec{P} = \lambda\overrightarrow{BC}$, $\vec{Q} = (\lambda+\mu)\overrightarrow{CA}$ and $\vec{R} = (\lambda+\nu)\overrightarrow{AB}$ be the three forces acting along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ respectively of a triangle ABC.

Let $\vec{P} + \vec{Q} + \vec{R} =$ a couple

then $\lambda\overrightarrow{BC} + (\lambda+\mu)\overrightarrow{CA} + (\lambda+\nu)\overrightarrow{AB} =$ a Couple

$\lambda\overrightarrow{BC} + \lambda\overrightarrow{CA} + \mu\overrightarrow{CA} + \lambda\overrightarrow{AB} + \nu\overrightarrow{AB} =$ a Couple

$\lambda(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}) + \mu\overrightarrow{CA} + \nu\overrightarrow{AB} =$ a Couple

Since we know that the forces $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ form a Couple (By previous example)

So λ(a couple) $+ \mu\overrightarrow{CA} + \nu\overrightarrow{AB} =$ a couple

a couple $+ \mu\overrightarrow{CA} + \nu\overrightarrow{AB} =$ a couple

$\mu\overrightarrow{CA} + \nu\overrightarrow{AB} =$ a couple $-$ a couple

$\mu\overrightarrow{CA} + \nu\overrightarrow{AB} = 0$

If $\mu\overrightarrow{CA} + \nu\overrightarrow{AB} = 0$

then $\mu\overrightarrow{CA} = 0$ and $\nu\overrightarrow{AB} = 0$

Since $\overrightarrow{CA} \neq 0$ and $\overrightarrow{AB} \neq 0$

$\Rightarrow \mu = \nu = 0$

Now		
$\vec{P} = \lambda\overrightarrow{BC}$	$\vec{Q} = (\lambda+\mu)\overrightarrow{CA}$	$\vec{R} = (\lambda+\nu)\overrightarrow{AB}$
	Put $\mu = 0$	Put $\nu = 0$
	$\vec{Q} = (\lambda+0)\overrightarrow{CA}$	$\vec{R} = (\lambda+0)\overrightarrow{AB}$
	$\vec{Q} = \lambda\overrightarrow{CA}$	$\vec{R} = \lambda\overrightarrow{AB}$

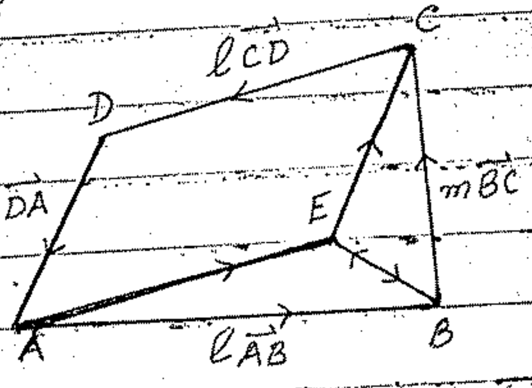
where λ is the constant of proportionality

$\therefore \vec{P} \propto \overrightarrow{BC}$ | $\vec{Q} \propto \overrightarrow{CA}$ | $\vec{R} \propto \overrightarrow{AB}$

Clearly three forces acting along the sides of a triangle are equal to a couple if they are proportional to the sides of a triangle.

Example-3; P-32

If the forces $l\overrightarrow{AB}, m\overrightarrow{BC}, l\overrightarrow{CD}$ and $m\overrightarrow{DA}$ acting along the sides of a quadrilateral are equivalent to a couple. Show that either $l = m$ or ABCD is a parallelogram.



Solution. Considering the forces $l\overrightarrow{AB}, m\overrightarrow{BC}, l\overrightarrow{CD}$ & $m\overrightarrow{DA}$ acting along the quadrilateral ABCD.

Also $l\overrightarrow{AB} + m\overrightarrow{BC} + l\overrightarrow{CD} + m\overrightarrow{DA} =$ Couple (given)

To prove: Either $l = m$ or ABCD is a parallelogram

Let ABCD is not a parallelogram. Take a poin in the quadrilateral such that AECD is a ||gm. Considering a force $\vec{AB}$ acting along the side AB. We want to introduce this force along AE, this is possible only if a couple is added.

So, Force $\vec{AB}$ (along AB) = Force $\vec{AB}$ (along AE) + a couple

$\vec{AB} = \vec{AE} + \vec{EB}$ + a couple | From vector triangle AEB $\vec{AB} = \vec{AE} + \vec{EB}$

$\times l$

$l\vec{AB} = l\vec{AE} + l\vec{EB} + l$(a couple)

$l\vec{AB} = l\vec{AE} + l\vec{EB}$ + a couple

Now considering the force $\vec{BC}$ acting along the side BC. We want to introduce this force along EC, this is possible only if a couple is added.

So, Force $\vec{BC}$ (along BC) = Force $\vec{BC}$ (along EC) + a couple

$\vec{BC} = \vec{BE} + \vec{EC}$ + a couple | From vector triangle BEC $\vec{BC} = \vec{BE} + \vec{EC}$

$\times m$

$m\vec{BC} = m\vec{BE} + m\vec{EC} + m$(a couple)

$m\vec{BC} = m\vec{BE} + m\vec{EC}$ + a couple

Now

$l\vec{AB} = l\vec{AE} + l\vec{EB}$ + a couple

$m\vec{BC} = m\vec{BE} + m\vec{EC}$ + a couple

$l\vec{CD} = l\vec{CD}$

$m\vec{DA} = m\vec{DA}$

$l\vec{AB} + m\vec{BC} + l\vec{CD} + m\vec{DA} = l(\vec{AE} + \vec{CD}) + m(\vec{EC} + \vec{DA}) + l\vec{EB} + m\vec{BE}$ + a couple + a couple

Now the forces $\vec{AE}$ and $\vec{CD}$ are equal in magnitude, opposite in direction and their line of action is not same, so form a couple, similarly $\vec{EC}$ and $\vec{DA}$ form a couple

Also $\vec{EB} = -\vec{BE}$

So,

$$l\vec{AB} + m\vec{BC} + l\vec{CD} + m\vec{DA} = l(\text{a couple}) + m(\text{a couple}) + l(-\vec{BE}) + m\vec{BE} + \text{a couple}$$

$$\text{A Couple} = \text{a couple} + \text{a couple} + \vec{BE}(m-l) + \text{a couple}$$

$$\text{A Couple} = \text{a couple} + \vec{BE}(m-l)$$

$$\Rightarrow \vec{BE}(m-l) = \text{a couple} - \text{a couple}$$

$$\Rightarrow \vec{BE}(m-l) = 0$$

Either $\vec{BE} = 0$ or $m - l = 0$

$\Rightarrow B \equiv E$ or $l = m$

Now $ABCD$ = quadrilateral

Put $B = E$

$AECD$ = quadrilateral

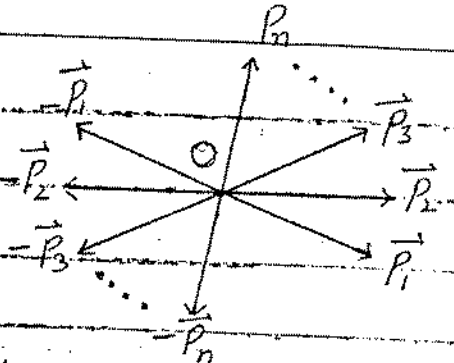
But $AECD$ = parallelogram. Thus $ABCD$ is ||gm

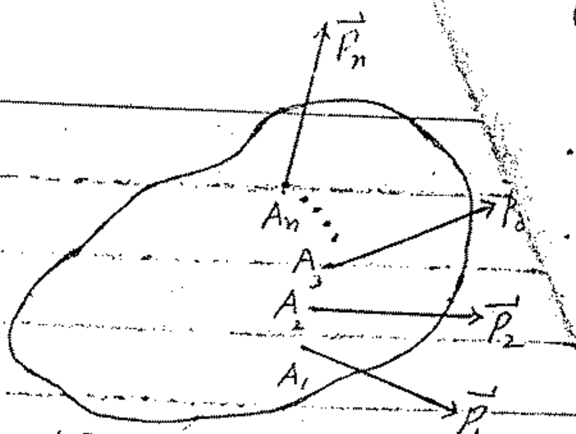
$\therefore$ If the forces $l\vec{AB}, m\vec{BC}, l\vec{CD}, m\vec{DA}$ acting along the sides of a quadrilateral are equivalent to a couple then either $l = m$ or $ABCD$ is a parallelogram.

Question: Reduce a system of coplanar forces into a single force $\vec{R}$ and a single couple of moment G.

Answer:

Considering a system of coplanar forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \ldots, \vec{P_n}$ acting through the points $A_1, A_2, A_3, \ldots, A_n$ respectively of a rigid body.





<u>To prove</u> This system of coplanar forces can be reduced into a single force $\vec{R}$ and a single couple of moment G.

Introduce the forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \dots, \vec{P_n}$ and the forces $-\vec{P_1}, -\vec{P_2}, -\vec{P_3}, \dots, -\vec{P_n}$ through the point 'O'.

Now the forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \dots, \vec{P_n}$ through the points $A_1, A_2, A_3, \dots, A_n$ respectively and the forces $-\vec{P_1}, -\vec{P_2}, -\vec{P_3}, \dots, -\vec{P_n}$ through the point 'O' form the couples $(\vec{P_1}, -\vec{P_1}), (\vec{P_2}, -\vec{P_2}), (\vec{P_3}, -\vec{P_3}), \dots, (\vec{P_n}, -\vec{P_n})$ say of the moments $G_1, G_2, G_3, \dots, G_n$.

These coplanar couples can be composed into a single couple of moment G

Then $G = G_1 + G_2 + G_3 + \dots + G_n$

Now we are left with the concurrent forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \dots, \vec{P_n}$ acting through the point 'O'

These concurrent forces can be composed into a single force $\vec{R}$

i.e $\vec{R} = \vec{P_1} + \vec{P_2} + \vec{P_3} + \dots + \vec{P_n}$

$\therefore$ A system of coplanar forces can be reduced into a single force $\vec{R}$ and a single couple of moment G.

Special Cases

(i) $\vec{R}=0, G=0$ The system of coplanar forces is said to be in complete equilibrium.

(ii) $\vec{R}=0, G\neq 0$ The system of coplanar forces is equal to a single couple of moment G

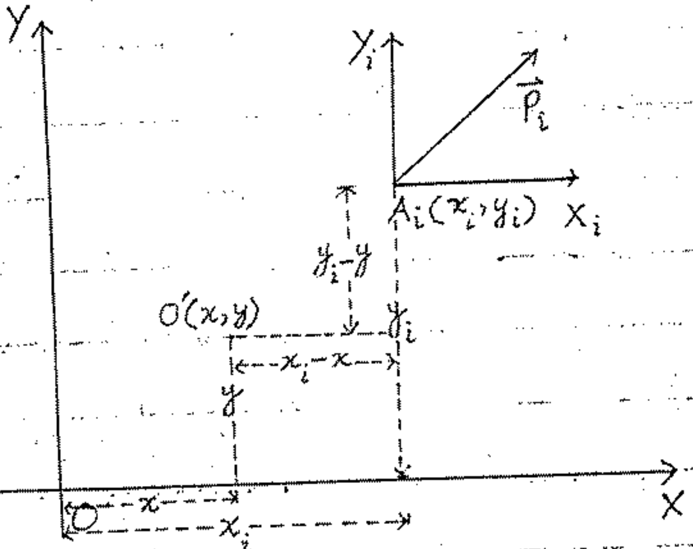
(iii) $\vec{R}\neq 0, G=0$ The system of coplanar forces is equal to a single force $\vec{R}$

(iv) $\vec{R}\neq 0, G\neq 0$ The system of coplanar forces is reduced into a single force R and a single couple of moment G.

Imp **Question** Find the equation of line of a action of the resultant of system of coplanar forces acting on a rigid body.

Answer

Considering a system of coplanar forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \ldots\ldots, \vec{P_n}$ acting through the points $A_1, A_2, A_3, \ldots\ldots, A_n$ of a rigid body.



Let the forces are generally represented by a force $\vec{P_i}$ acting through the point $A_i(x_i, y_i)$.

where $i = 1, 2, 3, \ldots, n$.

Let the system of coplanar forces is reduced into a single force $\vec{R}$ and a single couple of moment G about 'O'.

Then $R = \sqrt{X^2 + Y^2}$

where $X = \sum_{i=1}^{n} X_i = X_1 + X_2 + X_3 + \cdots + X_n$

$Y = \sum_{i=1}^{n} Y_i = Y_1 + Y_2 + Y_3 + \cdots + Y_n$

Also $G = \sum_{i=1}^{n} G_i = G_1 + G_2 + G_3 + \cdots + G_n$

Let G_i be the moment of the genral force $\vec{P_i}$ about 'O', then according to Varignon's Theorem

$$G_i = Y_i x_i + (-X_i) y_i$$

$$G_i = Y_i x_i - X_i y_i$$

$$\sum_{i=1}^{n} G_i = \Sigma (Y_i x_i - X_i y_i)$$

Put $\sum_{i=1}^{n} G_i = G$

then $G = \Sigma (Y_i x_i - X_i y_i)$ ——— ①

Let the system of coplanar forces is reduced into a single force $\vec{R}'$ and a single couple of moment G' at the point O'.

Let G_i' be the moment of the force $\vec{P_i}$ about 'O''.

Then according to varignon's theorem.

$$G_i' = Y_i (x_i - x) + (-X_i)(y_i - y)$$

$$= Y_i x_i - Y_i x - X_i y_i + X_i y$$

$$= (Y_i x_i - X_i y_i) + (X_i y - Y_i x)$$

Applying Σ on both sides

$$\Sigma G_i' = \Sigma (Y_i x_i - X_i y_i) + \Sigma (X_i y - Y_i x)$$

Put $\Sigma G_i' = G'$ and $\Sigma (Y_i x_i - X_i y_i) = G$ by ①

$$\therefore G' = G + \Sigma (X_i y - Y_i x)$$

$$= G + (\Sigma X_i) y - (\Sigma Y_i) x$$

Put $\Sigma X_i = X$ and $\Sigma Y_i = Y$

then $G' = G + yX - xY$ ——— (2)

Let the system of coplanar forces is reduced into a single force $\vec{R}'$ at 'O', Then $G' = 0$

Thus from (2) we have

$$G - xY + yX = 0$$

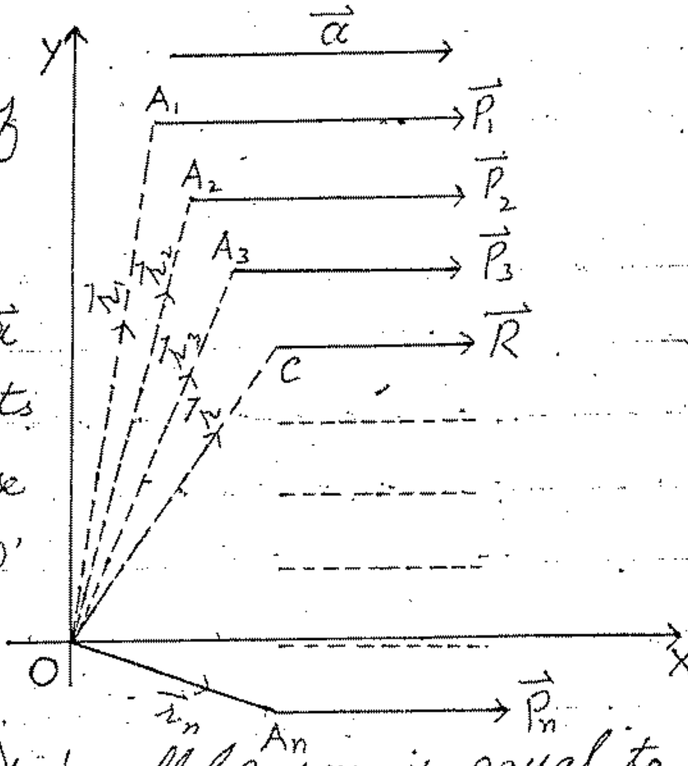
This is the first degree equation in 'x' and 'y' represents a st. line also represents the resultant force $\vec{R}'$

$\therefore$ $G - xY + yX = 0$ represents the equation of line of action of the resultant force.

<u>Question</u>: Show that a system of parallel forces can be reduce into a single force $\vec{R}$ and a single couple of moment $\vec{G}$.

<u>Answer</u>:

Considering a system of parallel forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \ldots\ldots, \vec{P_n}$ parallel to the vector $\vec{a}$ acting through the points $A_1, A_2, A_3, \ldots\ldots, A_n$ whose position vectors w.r.t. 'O' are $\vec{r_1}, \vec{r_2}, \vec{r_3}, \ldots\ldots, \vec{r_n}$ respectively.



<u>To prove</u>: The system of parallel forces is equal to a single force $\vec{R}$ and a single couple of moment $\vec{G}$

Since the forces are parallel to the vector $\vec{a}$, then by definition of parallel forces.

$$\left.\begin{aligned} \vec{P_1} &= K_1\vec{a} \\ \vec{P_2} &= K_2\vec{a} \\ \vec{P_3} &= K_3\vec{a} \\ &\cdots \\ &\cdots \\ \vec{P_n} &= K_n\vec{a} \end{aligned}\right\} \longrightarrow (1)$$

where $K_1, K_2, K_3, \dots, K_n$ are some constants.

Let $\vec{R}$ be the resultant of the forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, \dots, \vec{P_n}$

then $\vec{R} = \vec{P_1} + \vec{P_2} + \vec{P_3} + \dots + \vec{P_n}$

$= K_1\vec{a} + K_2\vec{a} + K_3\vec{a} + \dots + K_n\vec{a}$ by (1)

$= (K_1 + K_2 + K_3 + \dots + K_n)\vec{a}$

Put $K_1 + K_2 + K_3 + \dots + K_n = K$

$\vec{R} = K\vec{a}$

Clearly the resultant force $\vec{R}$ of the system of parallel forces is also parallel to vector $\vec{a}$.

Now taking the moment of the forces about point 'O'.

Let $\vec{G}$ be the moment of the resultant force, then according to Varignon's theorem

$\vec{G} = \vec{r_1} \times \vec{P_1} + \vec{r_2} \times \vec{P_2} + \vec{r_3} \times \vec{P_3} + \dots + \vec{r_n} \times \vec{P_n}$

$= \vec{r_1} \times K_1\vec{a} + \vec{r_2} \times K_2\vec{a} + \vec{r_3} \times K_3\vec{a} + \dots + \vec{r_n} \times K_n\vec{a}$ by (1)

$= (K_1\vec{r_1} + K_2\vec{r_2} + K_3\vec{r_3} + \dots + K_n\vec{r_n}) \times \vec{a}$

Multiplying and dividing by K

then $\vec{G} = \dfrac{(K_1\vec{r_1} + K_2\vec{r_2} + K_3\vec{r_3} + \dots + K_n\vec{r_n})}{K} \times K\vec{a}$

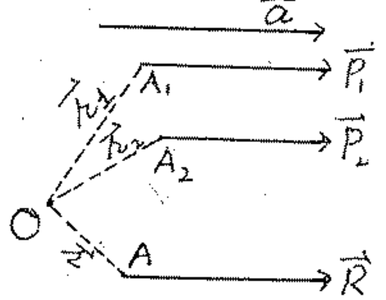
Put $\underline{K_1\vec{r}_1 + K_2\vec{r}_2 + K_3\vec{r}_3 + \cdots + K_n\vec{r}_n} = \vec{r}$ and $K\vec{a} = \vec{R}$

then $\vec{G} = \vec{r} \times \vec{R}$

Clearly a system of parallel forces can be reduced into a single force $\vec{R}$ and a single couple of moment $\vec{G}$.

Question: Show that a system of two parallel forces can be reduced into a single force $\vec{R}$ and a single couple of moment $\vec{G}$.

Answer:



Considering two parallel forces $\vec{P}_1$ and $\vec{P}_2$ acting through the points A_1 and A_2 whose position vectors w.r.t. O are $\vec{r}_1$ and $\vec{r}_2$ resp. are parallel to a vector $\vec{a}$.

To prove: The system of two parallel forces is equal to a single force $\vec{R}$ and a single couple of moment $\vec{G}$.

Since the forces are parallel to the vector $\vec{a}$, so by definition of parallel forces

$$\left.\begin{aligned} \vec{P}_1 &= K_1\vec{a} \\ \vec{P}_2 &= K_2\vec{a} \end{aligned}\right\} \text{———} (1)$$

Let $\vec{R}$ be the resultant force of the forces $\vec{P}_1$ and $\vec{P}_2$

then $\vec{R} = \vec{P}_1 + \vec{P}_2$

$$= K_1\vec{a} + K_2\vec{a}$$

$$= (K_1 + K_2)\vec{a}$$

Put $K_1 + K_2 = K$

then $\vec{R} = K\vec{a}$

$\therefore$ The resultant of two parallel forces is a force which is also parallel to the parallel force $\vec{a}$.

Now talking the moment of the forces about the the point 'O'; Let $\vec{G}$ be the moment of the resultant force, then according to Varignon's theorem.

$$\vec{G} = \vec{r}_1 \times \vec{P}_1 + \vec{r}_2 \times \vec{P}_2$$
$$= \vec{r}_1 \times K_1\vec{a} + \vec{r}_2 \times K_2\vec{a} \qquad \text{by (1)}$$
$$= (K_1\vec{r}_1 + K_2\vec{r}_2) \times \vec{a}$$

Multiplying and dividing by K

$$\vec{G} = \frac{(K_1\vec{r}_1 + K_2\vec{r}_2)}{K} \times K\vec{a}$$

Put $\frac{(K_1\vec{r}_1 + K_2\vec{r}_2)}{K} = \vec{r}$ and $K\vec{a} = \vec{R}$

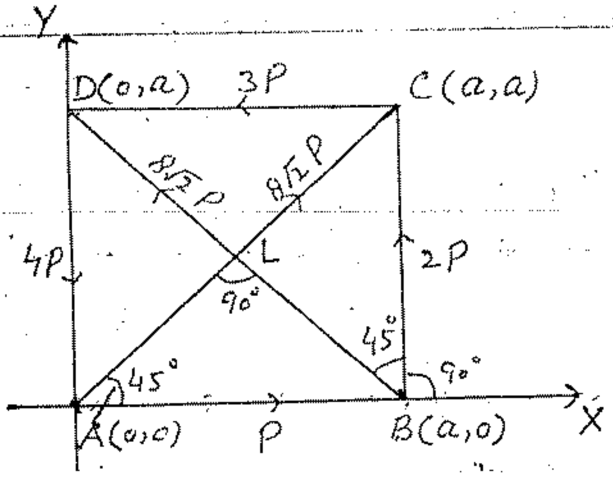
$$\vec{G} = \vec{r} \times \vec{R}$$

Clearly a system of two parallel forces can be reduced into a single force $\vec{R}$ and a single couple of moment $\vec{G}$. Ans

sol Imp **Example-1; P-39**. Forces of magnitude P, 2P, 3P, 4P act respectively along the sides AB, BC, CD, DA of a square ABCD of side 'a' and forces each of magnitude $8\sqrt{2}P$ act along the diagonals BD, AC. Find the magnitude of the resultantant force and the distance of its line of action from A.

Solution

Considering the forces of magnitude $P, 2P; 3P, 4P$ acting respectively along the sides AB, BC, CD, DA of a square ABCD each of side 'a' and



forces each of magnitude $8\sqrt{2}P$ acting along the diagonals BD and AC.

Resolving the forces along the coordinate axes.

$$X = P\cos 0° + 2P\cos 90° - 3P\cos 0° - 4P\cos 90° + 8\sqrt{2}P\cos 135° + 8\sqrt{2}P\cos 45°$$

$$= P(1) + 2P(0) - 3P(1) - 4P(0) + \cancel{8\sqrt{2}P\left(\tfrac{-1}{\sqrt{2}}\right)} + \cancel{8\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\right)}$$

$$= P - 3P$$

$$= -2P$$

$$Y = P\sin 0° + 2P\sin 90° - 3P\sin 0° - 4P\sin 90° + 8\sqrt{2}P\sin 135° + 8\sqrt{2}P\sin 45°$$

$$= P(0) + 2P(1) - 3P(0) - 4P(1) + 8\sqrt{2}P\left(\tfrac{1}{\sqrt{2}}\right) + 8\sqrt{2}P\left(\tfrac{1}{\sqrt{2}}\right)$$

$$= 2P - 4P + 8P + 8P$$

$$= 14P$$

Let 'R' be the magnitude of the resultant force.

then $R = \sqrt{X^2 + Y^2}$

$$= \sqrt{(-2P)^2 + (14P)^2}$$

$$= \sqrt{4P^2 + 196P^2}$$

$$= \sqrt{200P^2}$$

$$= \sqrt{100 \times 2P^2}$$

$$= 10\sqrt{2}P$$

is the magnitude of the resultant force

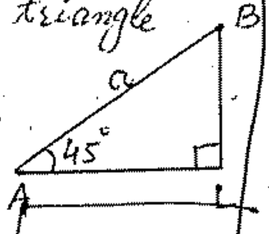
Now taking the moment of the forces about point A.
Let 'G' be the moment of the resultant force R
then according to Varignon's theorem

G = sum of moment of the forces about A
$G = 2P(AB) + 3P(AD) + 8\sqrt{2}P(AL)$

$$G = 2P(a) + 3P(a) + 8\sqrt{2}\,P(|\overrightarrow{AL}|)$$
$$= 5Pa + 8\sqrt{2}\,P(|\overrightarrow{AL}|)$$
$$= 5Pa + 8\cancel{\sqrt{2}}\,P \cdot \frac{a}{\cancel{\sqrt{2}}}$$
$$= 5Pa + 8Pa$$
$$= 13\,Pa$$

From right triangle ALB

$$\frac{|\overrightarrow{AL}|}{a} = \cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$|\overrightarrow{AL}| = \frac{a}{\sqrt{2}}$$



Now equation of line of action of the resultant is

$$G - xY + yX = 0$$
$$13Pa - 14Px - 2Py = 0$$
$$\div P \qquad 13a - 14x - 2y = 0$$

Let 'd' be the distance of the line of action of the resultant force from Point A, then

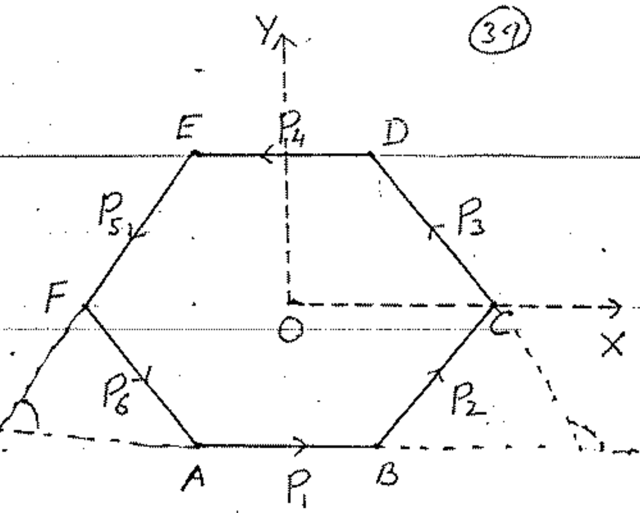
$$d = \frac{|13a - 14x - 2y|}{\sqrt{(-14)^2 + (-2)^2}}$$

A(0,0)
$x = y = 0$

$$= \frac{|13a - 14(0) - 2(0)|}{\sqrt{196+4}}$$
$$= \frac{13a}{\sqrt{200}}$$
$$= \frac{13a}{10\sqrt{2}} \quad \text{Ans.}$$

Imp

Example-2; P-39 Forces $P_1, P_2, P_3, P_4, P_5, P_6$ act along the sides of a regular hexagon taken in order. Show that they will be in equilibrium if $\Sigma P = 0$ and $P_1 - P_4 = P_3 - P_6 = P_5 - P_2$



Solution

Considering the forces $P_1, P_2, P_3, P_4, P_5, P_6$ acting along the sides of a regular hexagon taken in order.

Resolving the forces along the coordinate axes.

$$X = P_1\cos 0^\circ + P_2\cos 60^\circ + P_3\cos 120^\circ - P_4\cos 0^\circ - P_5\cos 60^\circ - P_6\cos 120^\circ$$

$$= P_1(1) + P_2\left(\tfrac{1}{2}\right) + P_3\left(-\tfrac{1}{2}\right) - P_4(1) - P_5\left(\tfrac{1}{2}\right) - P_6\left(-\tfrac{1}{2}\right)$$

$$= P_1 + \tfrac{1}{2}P_2 - \tfrac{1}{2}P_3 - P_4 - \tfrac{1}{2}P_5 + \tfrac{1}{2}P_6$$

$$Y = P_1\sin 0^\circ + P_2\sin 60^\circ + P_3\sin 120^\circ - P_4\sin 0^\circ - P_5\sin 60^\circ - P_6\sin 120^\circ$$

$$= P_1(0) + P_2\left(\tfrac{\sqrt{3}}{2}\right) + P_3\left(\tfrac{\sqrt{3}}{2}\right) - P_4(0) - P_5\left(\tfrac{\sqrt{3}}{2}\right) - P_6\left(\tfrac{\sqrt{3}}{2}\right)$$

$$= \tfrac{\sqrt{3}}{2}(P_2 + P_3 - P_5 - P_6)$$

Now taking the moment of the forces about 'O'. Let 'G' be the magnitude of the moment of resultant force R then according to varignon's theorem

$$G = P_1 d + P_2 d + P_3 d + P_4 d + P_5 d + P_6 d$$

$$= (P_1 + P_2 + P_3 + P_4 + P_5 + P_6)d$$

$= (\Sigma P)d$ where 'd' is the length of the perpendicular from 'O' to the sides of the hexagon.

The forces are in equilibrium if

$$G = 0;\ R = 0 \quad \text{i.e } X = 0,\ Y = 0$$

$$G = 0 \implies (\Sigma P)d = 0$$

$$\implies \Sigma P = 0 \quad (\text{as } d \neq 0) \quad \text{as required}$$

$$X = 0 \implies P_1 + \tfrac{1}{2}P_2 - \tfrac{1}{2}P_3 - P_4 - \tfrac{1}{2}P_5 + \tfrac{1}{2}P_6 = 0$$

$$P_1 + \tfrac{1}{2}P_2 - \tfrac{1}{2}(P_3 - P_6) - P_4 - \tfrac{1}{2}P_5 = 0 \quad \text{—— (1)}$$

and $Y=0 \Rightarrow \frac{\sqrt{3}}{2}(P_2+P_3-P_5-P_6)=0$

$\Rightarrow P_2+P_3-P_5-P_6=0$

$\Rightarrow P_3-P_6=P_5-P_2$ ——— (2)

Using (2) in (1)

$P_1+\frac{1}{2}P_2-\frac{1}{2}(P_5-P_2)-P_4-\frac{1}{2}P_5=0$

$P_1+\frac{1}{2}P_2-\frac{1}{2}P_5+\frac{1}{2}P_2-P_4-\frac{1}{2}P_5=0$

$P_1+P_2-P_4-P_5=0$

$P_1-P_4=P_5-P_2$ ——— (3)

Now $P_5-P_2=P_5-P_2=P_5-P_2$

$P_1-P_4=P_3-P_6=P_5-P_2$ as required.

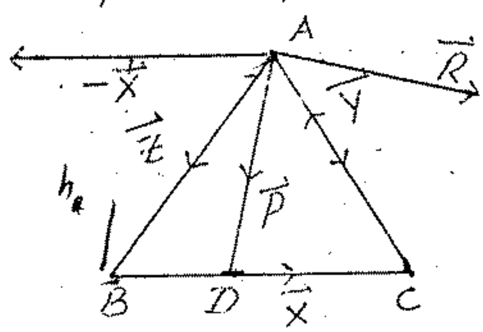
Imp <u>Example-3; P-40</u>

Prove that any system of forces in a plane is equal to three suitably chosen forces $\vec{X}, \vec{Y}, \vec{Z}$ acting along the sides BC, CA, AB of a given triangle in the plane. Prove that if the system is equivalent to a couple G, then $\frac{X}{BC}=\frac{Y}{CA}=\frac{Z}{AB}=\frac{G}{2\Delta}$

where Δ is the area of triangle ABC.

<u>Solution</u> Let the system of forces is reduced into a single force $\vec{R}$ and a single couple at point A.

We replace the couple by a force $\vec{X}$ through BC and an equal and opposite force $-\vec{X}$ at point A. If h_A is $\perp$r from point A to BC, then



$$G_A = X \cdot h_A$$

The concurrent forces $\vec{R}$ and $-\vec{X}$ are equal to a single force $\vec{P}$ at A. Let the line of action of force $\vec{P}$ cut BC at point D. we can choose λ and μ so that:

$$\vec{P} = (\lambda+\mu)\vec{AD}$$

Then by (λ, μ) theorem

$$(\lambda+\mu)\vec{AD} = \lambda\vec{AB} + \mu\vec{AC}$$

$$\text{s.t} \quad BD:DC = \mu:\lambda$$

Now the system of forces is reduced into three forces $\vec{X}$ (through BC), $\vec{Y} = \mu\vec{CA}$ and $\vec{Z} = \lambda\vec{AB}$

Since the system is equivalent to a couple of moment G, so the sum of the moment of the forces about any point (A, B or C) will be G.

Taking the moment of the forces about 'A'.	Taking the moment of the forces about 'B'.	Taking the moment of the forces about 'C'.
$G = Xh_A + Y(0) + Z(0)$	$G = X(0) + Y\cdot h_B + Z(0)$	$G = X(0) + Y(0) + Zh_C$
$G = X\cdot h_A$	$G = Yh_B$	$G = Zh_C$

where h_A, h_B, h_C are the heights of the triangle from points A, B and C respectively. So $Xh_A = Yh_B = Zh_C = G$

Now area of triangle ABC is given by

$\Delta = \frac{1}{2}BC\cdot h_A$	$\Delta = \frac{1}{2}CA\cdot h_B$	$\Delta = \frac{1}{2}\cdot AB\cdot h_C$
$\Rightarrow h_A = \frac{2\Delta}{BC}$	$\Rightarrow h_B = \frac{2\Delta}{CA}$	$\Rightarrow h_C = \frac{2\Delta}{AB}$

Again $Xh_A = Yh_B = Zh_C = G$

then $$X\cdot\frac{2\Delta}{BC} = Y\frac{2\Delta}{CA} = Z\cdot\frac{2\Delta}{AB} = G$$

$\div 2\Delta$ $$\frac{X}{BC} = \frac{Y}{CA} = \frac{Z}{AB} = \frac{G}{2\Delta}$$ as required

Exercise Set 2

Q No 1

If two forces P and Q act at such an angle that their resultant $R = P$, show that if P is doubled the new resultant is at right angles to Q.

Solution

Considering two forces P and Q inclined at an angle α. Let 'R' be the magnitude of their resultant force then,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \quad (1)$$

Here $R = P$

$$P = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

squaring both the sides

$$P^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

$$P^2 - P^2 = Q^2 + 2PQ\cos\alpha$$

$$0 = Q(Q + 2P\cos\alpha)$$

Since $Q \neq 0$

so $Q + 2P\cos\alpha = 0$ ——— (1)

Let 'P' is doubled and the new resultant makes an angle 'θ' with Q.

then $\tan\theta = \dfrac{2P\sin\alpha}{Q + 2P\cos\alpha}$

$\tan\theta = \dfrac{2P\sin\alpha}{0}$ by (1)

$\tan\theta = \infty$

$\Rightarrow \theta = 90^\circ$

$\Rightarrow$ The new resultant (when P is doubled) is at right angles to Q.

(1) $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

$P^2 = P^2 + Q^2 + 2PQ\cos\alpha$

$Q^2 + 2PQ\cos\alpha = 0$

$Q(Q + 2P\cos\alpha) = 0$

$Q + 2P\cos\alpha = 0$

By theorem of resolved parts

R_x = Sum of the resolved parts along x-axis

$= Q\cos 0^\circ + P\cos\alpha$

$= Q + P\cos\alpha$

$\tan\theta = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$

If 'θ' is the angle of resultant R (of P & Q) with P.

and $\tan\theta = \dfrac{P\sin\alpha}{Q + P\cos\alpha}$

If 'θ' is the angle of resultant R (of P & Q) with Q.

(2)

$R_y = Q\sin 0 + P\sin\alpha$

$= 0 + P\sin\alpha$

$= P\sin\alpha$

If $P = 2P$

Then $R_x = Q + 2P\cos\alpha$

$R_y = 0 + 2P\sin\alpha$

$\theta = \tan^{-1}\dfrac{R_y}{R_x} = \tan^{-1}\dfrac{2P\sin\alpha}{Q + 2P\cos\alpha}$

$= \tan^{-1}\left(\dfrac{2P\sin\alpha}{0}\right)$

$= \tan^{-1}(\infty)$

$\theta = \dfrac{\pi}{2}$

Q No. 2

The greatest resultant that two forces can have is of magnitude P and the least is of magnitude Q. Show that when they act at an angle α their resultant is of magnitude

$$\sqrt{P^2\cos^2\alpha/_2 + Q^2\sin^2\alpha/_2}$$

Solution

Considering two forces P_1 and P_2 acting at an angle α, then magnitude of their resultant 'R' is given by

$$R = \sqrt{P_1^2 + P_2^2 + 2P_1P_2\cos\alpha} \quad \text{——— (1)}$$

For R_{max}: Put max. value of $\cos\alpha$; i.e $\cos\alpha = 1$ in (1)

$$R_{max} = \sqrt{P_1^2 + P_2^2 + 2P_1P_2(1)}$$

$$P = \sqrt{(P_1+P_2)^2} \qquad \because R_{max} = P$$

$$P = P_1 + P_2 \quad \text{——— (2)}$$

For R_{min}: Put min. value of $\cos\alpha$; i.e $\cos\alpha = -1$ in (1)

$$R_{min} = \sqrt{P_1^2 + P_2^2 + 2P_1P_2(-1)}$$

$$Q = \sqrt{(P_1-P_2)^2} \qquad \because R_{min} = Q$$

$$Q = P_1 - P_2 \quad \text{——— (3)}$$

Adding (2) and (3)

$$P = P_1 + \cancel{P_2}$$
$$Q = P_1 - \cancel{P_2}$$
$$P + Q = 2P_1$$
$$\Rightarrow P_1 = \frac{P+Q}{2}$$

Subtracting (3) from (2)

$$P = \cancel{P_1} + P_2$$
$$-Q = \cancel{P_1} \mp P_2$$
$$P - Q = 2P_2$$
$$\Rightarrow P_2 = \frac{P-Q}{2}$$

Using the values of P_1 and P_2 in (1)

$$R = \sqrt{\left(\frac{P+Q}{2}\right)^2 + \left(\frac{P-Q}{2}\right)^2 + 2\left(\frac{P+Q}{2}\right)\left(\frac{P-Q}{2}\right)\cos\alpha}$$

$$R=\sqrt{\frac{P^2+Q^2+2PQ}{4}+\frac{P^2+Q^2-2PQ}{4}+2\left(\frac{P^2-Q^2}{4}\right)\cos\alpha}$$

$$=\sqrt{\frac{1}{4}\left(P^2+Q^2+\cancel{2PQ}+P^2+Q^2-\cancel{2PQ}+2P^2\cos\alpha-2Q^2\cos\alpha\right)}$$

$$=\sqrt{\frac{1}{4}\left(2P^2+2P^2\cos\alpha+2Q^2-2Q^2\cos\alpha\right)}$$

$$=\sqrt{\frac{1}{4}\left[2P^2(1+\cos\alpha)+2Q^2(1-\cos\alpha)\right]}$$

Put $1+\cos\alpha=2\cos^2\alpha/_2$ and $1-\cos\alpha=2\sin^2\alpha/_2$

$$=\sqrt{\frac{1}{4}\left[2P^2\cdot 2\cos^2\alpha/_2+2Q^2\cdot 2\sin^2\alpha/_2\right]}$$

$$=\sqrt{\frac{1}{4}\left(4P^2\cos^2\alpha/_2+4Q^2\sin^2\alpha/_2\right)}$$

$$=\sqrt{\frac{\cancel{4}}{\cancel{4}}\left(P^2\cos^2\alpha/_2+Q^2\sin^2\alpha/_2\right)}$$

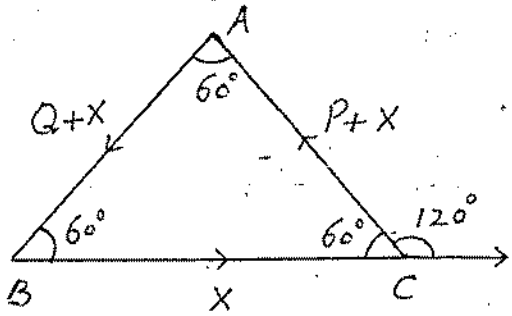
$$=\sqrt{P^2\cos^2\alpha/_2+Q^2\sin^2\alpha/_2}$$

as required.

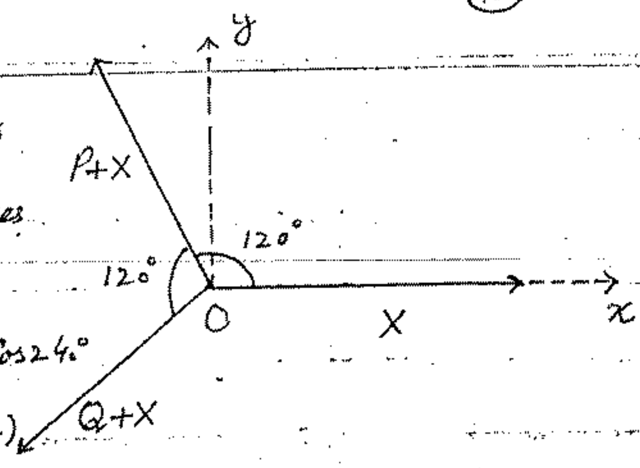
Q.No 3; P

Forces X, $P+X$, $Q+X$ act at a point in the directions of the sides of an equilateral triangle taken one way round. Show that they are equivalent to two forces P and Q acting at an angle of $120°$.

Solution;



Considering the forces X, $P+X$, $Q+X$ acting at point O parallel to the sides BC, CA, AB respectively of an equilateral triangle ABC. Resolving the forces along the Coordinate axes.



Let X' and Y' are sums of resolved parts of forces along ox and oy, then

$$X' = X\cos 0^\circ + (P+X)\cos 120^\circ + (Q+X)\cos 240^\circ$$
$$= X(1) + (P+X)(-\tfrac{1}{2}) + (Q+X)(-\tfrac{1}{2})$$
$$= X - \tfrac{1}{2}P - \tfrac{1}{2}X - \tfrac{1}{2}Q - \tfrac{1}{2}X$$
$$= X - X - \tfrac{1}{2}(P+Q)$$
$$= -\tfrac{1}{2}(P+Q)$$

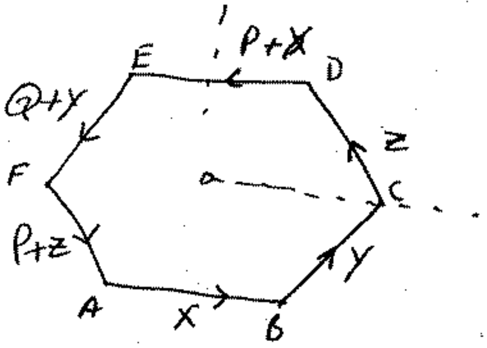
$$Y' = X\sin 0^\circ + (P+X)\sin 120^\circ + (Q+X)\sin 240^\circ$$
$$= X(0) + (P+X)\left(\tfrac{\sqrt{3}}{2}\right) + (Q+X)\left(-\tfrac{\sqrt{3}}{2}\right)$$
$$= \tfrac{\sqrt{3}}{2}(P+X-Q-X)$$
$$= \tfrac{\sqrt{3}}{2}(P-Q)$$

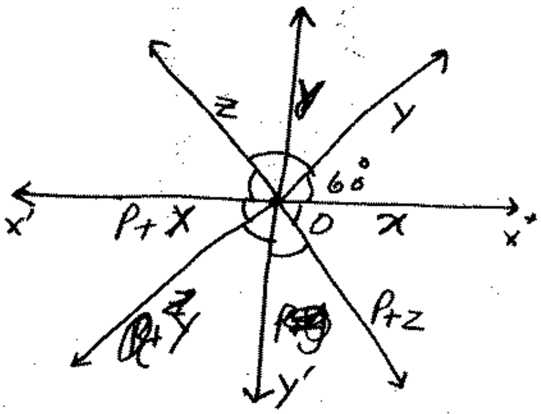
Let 'R' be the magnitude of the resultant force then

$$R = \sqrt{X'^2 + Y'^2}$$
$$= \sqrt{\left(-\tfrac{1}{2}(P+Q)\right)^2 + \left(\tfrac{\sqrt{3}}{2}(P-Q)\right)^2}$$
$$= \sqrt{\tfrac{1}{4}(P^2+Q^2+2PQ) + \tfrac{3}{4}(P^2+Q^2-2PQ)}$$
$$= \sqrt{\tfrac{1}{4}(P^2+Q^2+2PQ+3P^2+3Q^2-6PQ)}$$
$$= \sqrt{\tfrac{1}{4}(4P^2+4Q^2-4PQ)}$$
$$= \sqrt{\tfrac{1}{\cancel{4}}\cdot\cancel{4}(P^2+Q^2-PQ)}$$
$$= \sqrt{P^2+Q^2+2PQ(-\tfrac{1}{2})}$$
$$= \sqrt{P^2+Q^2+2PQ\cos 120^\circ}$$

Hence the system of forces is equivalent to two forces P & Q acting at an angle of 120°

Q.4 Forces $X, Y, Z, P+X, Q+Y, P+Z$ are acting at a pt in the direction of the side of a regular hexagon Taken one way round. Show that resultant in the force $P+Q$ in the direction $Q+Y$.





Resolving in the components

$$X' = X\cos 0° + Y\cos 60° + Z\cos 120° + (P+X)\cos 180 + (Q+Y)\cos 240 + (P+Z)\cos 300$$

$$= X + Y\left(\tfrac{1}{2}\right) + Z\left(-\tfrac{1}{2}\right) + (P+X)(-1) + (Q+Y)\left(-\tfrac{1}{2}\right) + (P+Z)\left(\tfrac{1}{2}\right)$$

$$= \cancel{X} + \tfrac{1}{2}\cancel{Y} - \tfrac{1}{2}\cancel{Z} - P - \cancel{X} - \tfrac{1}{2}Q - \tfrac{1}{2}\cancel{Y} + \tfrac{1}{2}P + \tfrac{1}{2}\cancel{Z}$$

$$= -\tfrac{1}{2}(P+Q)$$

$$Y' = X\sin 0 + Y\sin 60° + Z\sin 120° + (P+X)\sin 180° + (Q+Y)\sin 240 + (P+Z)\sin 300°$$

$$= X(0) + Y\tfrac{\sqrt{3}}{2} + Z\left(\tfrac{\sqrt{3}}{2}\right) + (P+X)(0) + (Q+Y)\left(-\tfrac{\sqrt{3}}{2}\right) + (P+Z)\left(-\tfrac{\sqrt{3}}{2}\right)$$

$$= \tfrac{\sqrt{3}}{2}\left(\cancel{Y} + \cancel{Z} - Q - \cancel{Y} - P - \cancel{Z}\right)$$

$$= -\tfrac{\sqrt{3}}{2}(P+Q)$$

Let $R = \sqrt{X'^2 + Y'^2} = \sqrt{\left[-\tfrac{1}{2}(P+Q)\right]^2 + \left[-\tfrac{\sqrt{3}}{2}(P+Q)\right]^2} = \sqrt{\tfrac{1}{4}(P+Q)^2 + \tfrac{3}{4}(P+Q)^2}$

$$= \sqrt{(P+Q)^2\left(\tfrac{1}{4}+\tfrac{3}{4}\right)} = \sqrt{(P+Q)^2} = P+Q$$

Let θ is the angle that the resultant for makes with x axis

then $\tan\theta = \dfrac{Y}{X} = \dfrac{-\frac{\sqrt{3}}{2}\cancel{(P+Q)}}{-\frac{1}{2}\cancel{(P+Q)}} = \sqrt{3}$

$\theta = \tan^{-1}\sqrt{3} = 240°$

be res ... of force then

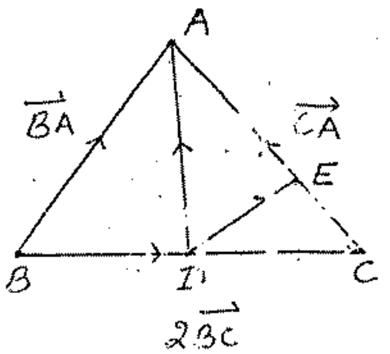
Ch-2

Q No 5.

Forces $2\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{BA}$ act along the sides of a triangle ABC. Show that their resultant is $6\overrightarrow{DE}$, where D bisect BC and E is a point on CA s.t $CE = \frac{1}{3}CA$

Solution

Considering the forces $2\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{BA}$ acting along the sides of a triangle ABC, as shown. Applying (λ, μ) theorem to the concurrent forces $\overrightarrow{BA}$ and $\overrightarrow{CA}$



$$\overrightarrow{BA} + \overrightarrow{CA} = (1+1)\overrightarrow{DA}$$ (where D is the mid point of the side BC.)

$$\overrightarrow{BA} + \overrightarrow{CA} = 2\overrightarrow{DA} \quad \text{—(2)}$$

Adding $2\overrightarrow{BC}$ both sides

$$\overrightarrow{BA} + \overrightarrow{CA} + 2\overrightarrow{BC} = 2\overrightarrow{DA} + 2\overrightarrow{BC} \quad \text{—(1)}$$

Since $BD = DC$

So $\overrightarrow{BD} = \overrightarrow{DC}$

Thus $2\overrightarrow{DC} = \overrightarrow{BC}$

$\times 2$ $\quad 4\overrightarrow{DC} = 2\overrightarrow{BC}$

Put $2\overrightarrow{BC} = 4\overrightarrow{DC}$ in (1)

$$\overrightarrow{BA} + \overrightarrow{CA} + 2\overrightarrow{BC} = 2\overrightarrow{DA} + 4\overrightarrow{DC}$$

$$R = (2+4)\overrightarrow{DE}$$ Applying (λ, μ) theorem to

$$R = 6\overrightarrow{DE}$$ the concurrent forces $\overrightarrow{DA}$ & $\overrightarrow{DC}$

where E is a point on CA s.t $CE : EA = 2:4$

i.e, $\frac{CE}{EA} = \frac{2}{4}$

$4CE = 2EA$

$4CE + 2CE = 2EA + 2CE$ (Adding 2CE both sides)

Resultant

$R = \cancel{2BC - AC}A$

$= 2BC + CA + BA$

Put $BA + CA = 2DA$

$R = 2\overrightarrow{BC} + 2DA$

$BC = 2DC$

$\frac{CE}{EA} = \frac{1}{2}$

$2CE = EA$

$\cancel{2CE + CE = EA + CE}$

$3\overline{CE} = \overline{CA}$

$\overline{CE} = \frac{1}{3}\overline{CA}$

Let R resultant of forces

$R = 2BC + CA + BA$

$2BC + (CA + BA)$

$= \cancel{2BC \; AC \; AB}$

$\cancel{2BC - (AC + AB)}$ — (1)

Apply λ, μ Theorem)

$2DC + DA = (2+1)DE = 3DE$ — (3)

By (1) Put $(CA + BA)$

$\cancel{R = 2B}$

$R = 2BC + 2DA = 2(BC + DA)$

Put $(BC + DA) = 2(2DC + DA)$

$R = 2(3DE) = 2(3DE)$

$R = 6DE$

$$6CE = 2(EA + CE)$$

$$6CE = 2CA \qquad \because EA + CE = CA$$

$\div 6$

$$CE = \frac{1}{3} CA$$

Hence resultant of the forces is $6\overrightarrow{DE}$ and

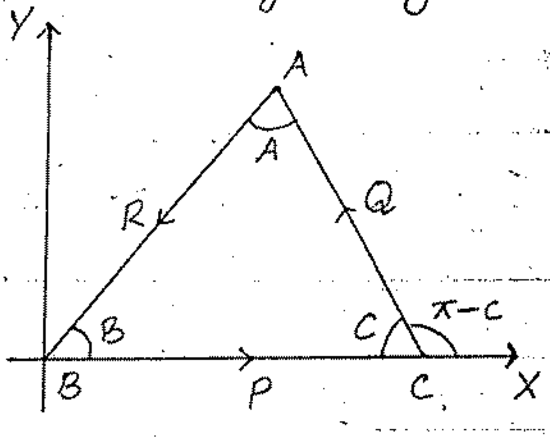
$$CE = \frac{1}{3} CA$$

Q. No. 6

Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC. Find the condition that their resultant is parallel to BC and find its magnitude.

Solution

Considering the forces P, Q, R acting along the sides BC, CA, AB of a triangle ABC.



Resolving the forces along the coordinate axes

$$X = P\cos 0^\circ + Q\cos(\pi - C) + (-R)\cos B$$

$$= P(1) + Q(-\cos C) - R\cos B$$

$$= P - Q\cos C - R\cos B$$

$$Y = P\sin 0^\circ + Q\sin(\pi - C) - R\sin B$$

$$= P(0) + Q\sin C - R\sin B$$

$$= Q\sin C - R\sin B$$

Let 'R'' be the resultant force which makes an angle 'θ' with X-axis (or BC)

Then $\tan\theta = \dfrac{Y}{X}$

$$= \frac{Q\sin C - R\sin B}{P - Q\cos C - R\cos B}$$

For resultant 'R'' to be parallel to BC, Put $\theta = 0$

$$\tan 0 = \frac{Q\sin C - R\sin B}{P - Q\cos C - R\cos B}$$

$$\frac{0}{1} = \frac{Q\sin C - R\sin B}{P - Q\cos C - R\cos B}$$

$$\Rightarrow Q\sin C - R\sin B = 0$$

$$Q\sin C = R\sin B \quad \text{——— ①}$$

By Sine law of triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \Big| \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\Rightarrow \sin B = \frac{b\sin A}{a} \qquad \Big| \qquad \Rightarrow \sin C = \frac{c\sin A}{a}$$

Using the values of $\sin B$ and $\sin C$ in ①

$$Q\frac{c\sin A}{a} = R\frac{b\sin A}{a}$$

$$\times \frac{a}{\sin A} \qquad Qc = Rb$$

is the condition under which the resultant is parallel to BC.

Now we shall find the magnitude of the resultant force.

$$R' = \sqrt{X^2 + Y^2}$$

Since the resultant is parallel to BC or X-axis, so

Put $Y = 0$

Thus $R' = \sqrt{X^2 + 0}$

$$R' = X$$

so $R' = P - Q\cos C - R\cos B$

Put $R = \dfrac{Q\sin C}{\sin B}$ by (1)

$$R' = P - Q\cos C - \frac{Q\sin C}{\sin B}\cos B$$

$$= P - Q\left(\cos C + \frac{\sin C\cos B}{\sin B}\right)$$

$$= P - Q\left(\frac{\sin B\cos C + \cos B\sin C}{\sin B}\right)$$

$$= P - Q\,\frac{\sin(B+C)}{\sin B}$$

$$= P - Q\,\frac{\sin A}{\sin B}$$

$$= P - Q\,\frac{a}{b}$$

is the magnitude of the resultant force.

$A+B+C = \pi$

$B+C = \pi - A$

$\sin(B+C) = \sin(\pi - A)$

$= \sin A$

By Law of Sines

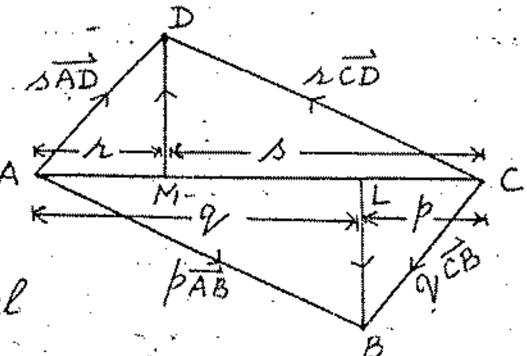
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{\sin A}{\sin B} = \frac{a}{b}$$

Q.W

If the forces $p\overrightarrow{AB}, q\overrightarrow{CB}, r\overrightarrow{CD}, s\overrightarrow{AD}$ act along the sides of a plane quadrilateral are in equilibrium. Show that $pr = qs$

<u>Solution</u>



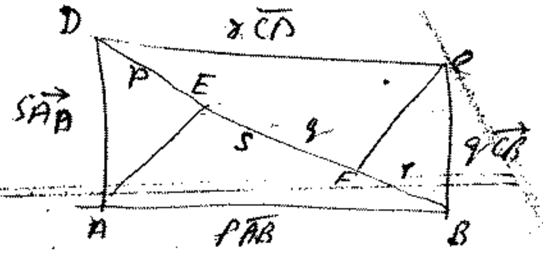
Considering the forces $p\overrightarrow{AB}, q\overrightarrow{CB}, r\overrightarrow{CD}$ & $s\overrightarrow{AD}$ acting along the sides of a plane quadrilateral ABCD. The system is in equilibrium under the action of these forces.

by A.U.

$p\vec{AB}+s\vec{AD}=(p+s)\vec{AE}$ where E lies on BD such that $\frac{BE}{ED}=\frac{s}{p}$ —①

Again $q\vec{CB}+r\vec{CD}=(q+r)\vec{CF}$ where F on BD such that $\frac{BF}{FD}=\frac{r}{q}$ —②

Since forces in equilib. therefore E and F must coincide so by ① and ②



To prove $\quad pr = qs$

Applying (λ, μ) theorem to the concurrent forces $p\vec{AB}$ and $q\vec{CB}$

$$p\vec{AB}+q\vec{CB}=(p+q)\vec{LB}$$

$$\text{s.t. } AL:LC = q:p$$

$$\text{i.e. } \frac{AL}{LC}=\frac{q}{p} \quad \text{—— ①}$$

Also applying (λ, μ) theorem to the concurrent forces $s\vec{AD}$ and $r\vec{CD}$

$$s\vec{AD}+r\vec{CD}=(s+r)\vec{MD}$$

$$\text{s.t. } AM:MC = r:s$$

$$\text{i.e. } \frac{AM}{MC}=\frac{r}{s} \quad \text{—— ②}$$

Now we are left with the two forces $(p+q)\vec{LB}$ and $(r+s)\vec{MD}$. Since the system is in equilibrium under the action of these two forces, this is possible only when these forces are equal in magnitude opposite in direction and their line of action must be same. This is possible only when 'L' and 'M' are not different from eachother. i.e $L=M$

Put $L=M$ in ①

$$\frac{AM}{MC}=\frac{q}{p} \quad \text{—— ③}$$

From ② and ③

$$\frac{r}{s}=\frac{q}{p}$$

$$\Rightarrow pr = qs \quad \text{as required.}$$

$\frac{s}{p}=\frac{r}{q}$

$pr = sq$

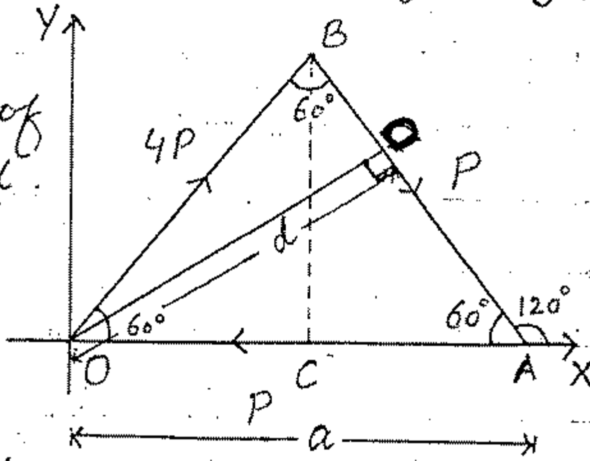
$\frac{BE}{ED}=\frac{BF}{FD}$

Q. No. 8

OAB is an equilateral triangle of side 'a'; C is the mid-point of OA. Forces 4P, P and P act along the sides OB, BA and AO respectively. If OA and OY (parallel to BC) are taken as x- and y-axes, prove that the resultant of the forces is 3P and the equation of its line of action is $3y = \sqrt{3}(3x + a)$

Solution

Considering the forces 4P, P and P acting along the sides OB, BA and AO of an equilateral triangle OAB of side 'a'. Since 'C' is the mid point of OA, so BC is ⊥ to OA.



Taking OA as X-axis and OY parallel to BC as Y-axis.

Resolving the forces along the Coordinate axes.

$X = -P\cos 0° - P\cos 120° + 4P\cos 60°$

$= -P(1) - P(-\frac{1}{2}) + 4P(\frac{1}{2})$

$= -P + \frac{1}{2}P + 2P$

$= P + \frac{1}{2}P$

$= \frac{3}{2}P$

$Y = -P\sin 0° - P\sin 120° + 4P\sin 60°$

$= -P(0) - P(\frac{\sqrt{3}}{2}) + 4P(\frac{\sqrt{3}}{2})$

$= \frac{\sqrt{3}}{2}(-P + 4P)$

$= \frac{3\sqrt{3}}{2}P$

Let 'R' be the magnitude of the resultant force

then $R = \sqrt{X^2+Y^2}$

$= \sqrt{(\frac{3}{2}P)^2 + (\frac{3\sqrt{3}}{2}P)^2}$

$= \sqrt{\frac{9}{4}P^2 + \frac{27}{4}P^2}$

$= \sqrt{\frac{1}{4}(9P^2+27P^2)}$

$= \sqrt{\frac{1}{4}\cdot 36P^2}$

$= \sqrt{9P^2}$

$= 3P$ as required

Let 'G' be moment of the resultant force about O. then according to Varignon's theorem "

$G = -P(d)$

$= -P.\frac{\sqrt{3}}{2}a$

$= -\frac{\sqrt{3}}{2}Pa$

From right triangle ODA

$\frac{d}{a} = \sin 60°$

$\frac{d}{a} = \frac{\sqrt{3}}{2}$

$d = \frac{\sqrt{3}}{2}a$

Now equation of line of action of the resultant is

$G - xY + yX = 0$

$-\frac{\sqrt{3}}{2}Pa - \frac{3\sqrt{3}}{2}Px + \frac{3}{2}Py = 0$

$\times -\frac{2}{P}$

$\sqrt{3}\,a + 3\sqrt{3}\,x - 3y = 0$

$\Rightarrow \quad 3y = \sqrt{3}\,a + 3\sqrt{3}\,x$

or $\quad 3y = \sqrt{3}\,(a+3x)$

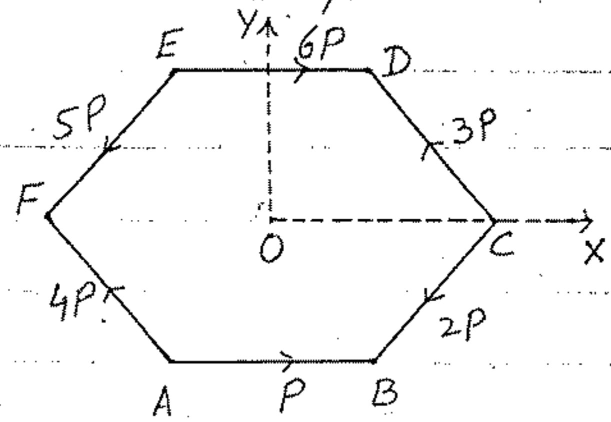
or $\quad 3y = \sqrt{3}\,(3x+a)$

as required.

Q. No. 9

Forces P, 2P, 3P, 6P, 5P and 4P act respectively along the sides AB, CB, CD, ED, EF and AF of a regular hexagon of side 'a'. The sense of the forces being indicated by the order of the letters. Prove that the six forces are equivalent to a couple.

Solution



Considering the forces P, 2P, 3P, 6P, 5P and 4P acting respectively along the sides AB, CB, CD, ED, EF and AF of a regular hexagon of side 'a'.

We reduce the system of forces into a single force R and a single couple of moment G about 'O'. If the system is equivalent to a couple then

$$R = 0 \quad i.e \quad X = 0 \text{ and } Y = 0$$

$$\text{and } G \neq 0$$

Now resolving the forces along the coordinate axes.

$$X = P\cos 0^\circ - 2P\cos 60^\circ + 3P\cos 120^\circ + 6P\cos 0^\circ - 5P\cos 60^\circ + 4P\cos 120^\circ$$

$$= P(1) - 2P\left(\frac{1}{2}\right) + 3P\left(-\frac{1}{2}\right) + 6P(1) - 5P\left(\frac{1}{2}\right) + 4P\left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}(2P - 2P - 3P + 12P - 5P - 4P)$$

$$= \frac{1}{2}(0)$$

$$= 0$$

$$Y = P\sin 0^\circ - 2P\sin 60^\circ + 3P\sin 120^\circ + 6P\sin 0^\circ - 5P\sin 60^\circ + 4P\sin 120^\circ$$

$$= P(0) - 2P\left(\frac{\sqrt{3}}{2}\right) + 3P\left(\frac{\sqrt{3}}{2}\right) + 6P(0) - 5P\left(\frac{\sqrt{3}}{2}\right) + 4P\left(\frac{\sqrt{3}}{2}\right)$$

$Y = \frac{\sqrt{3}}{2}(-2P + 3P - 5P + 4P)$

$= \frac{\sqrt{3}}{2}(0)$

$= 0$

Since $X = Y = 0$ so $R = 0$

Now taking the moment of the forces about 'O'.
Let 'G' be magnitude of the moment of the resultant force. Then according to Varignon's theorem.

$G = Pd - 2Pd + 3Pd - 6Pd + 5Pd - 4Pd$

$= -3Pd \neq 0$ (where 'd' is the length of the perpendicular from O to the sides of the hexagon)

Since $R = 0$ and $G \neq 0$

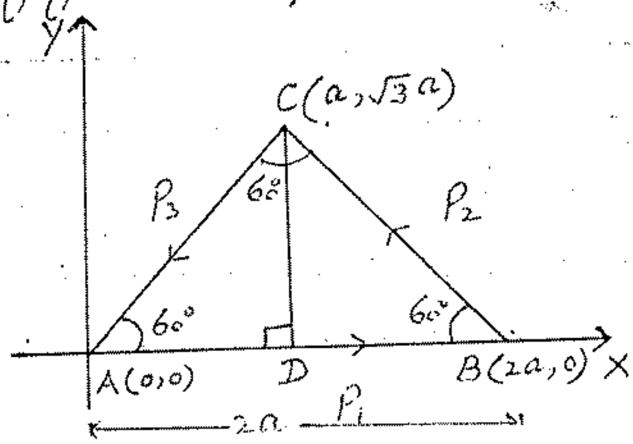
$\therefore$ The forces are equivalent to a couple.

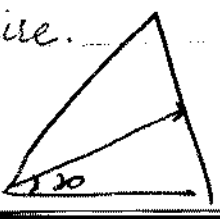
Q. No. 19

A system of forces act on a plate in the form of an equilateral triangle of side $2a$. The moments of the forces about the three vertices are G_1, G_2, G_3 respectively. Find the magnitude of the resultant.

Solution

Let the system of forces be equivalent to three forces P_1, P_2, P_3 acting along the sides of an equilateral triangle ABC of side $2a$ as shown, in the figure.





Let 'D' be the mid point of the side AB.

Let G_1, G_2, G_3 be the moments of these forces about A, B, C respectively.

From right angled triangle ADC

$\frac{CD}{AC} = \sin 60°$

$CD = AC \cdot \sin 60°$

$= 2a \cdot \frac{\sqrt{3}}{2}$

$= \sqrt{3}\, a$

We know that

$G' = G - xY + yX$ at $A(0,0)$

$G_1 = G - (0)Y + (0)X$ $\quad G' = G_1$

$\Rightarrow G = G_1$

Also

$G' = G - xY + yX$ at $B(2a, 0)$

$G_2 = G_1 - (2a)Y + (0)X$ $\quad G' = G_2$ and $G = G_1$

or $2aY = G_1 - G_2$

$Y = \frac{1}{2a}(G_1 - G_2)$

and

$G' = G - xY + yX$ $\quad C(a, \sqrt{3}a)$

$G_3 = G_1 - (a)\frac{1}{2a}(G_1 - G_2) + (\sqrt{3}a)X$ $\quad G' = G_3,\ G = G_1,\ Y = \frac{1}{2a}(G_1 - G_2)$

$\times 2$

$2G_3 = 2G_1 - G_1 + G_2 + 2\sqrt{3}\, aX$

$2G_3 = G_1 + G_2 + 2\sqrt{3}\, aX$

or $2\sqrt{3}\, aX = 2G_3 - G_1 - G_2$

or $X = \frac{1}{2\sqrt{3}\, a}(2G_3 - G_1 - G_2)$

Let "R" be the magnitude of the resultant force.

then $R = \sqrt{X^2 + Y^2}$

$= \sqrt{\left[\frac{1}{2\sqrt{3}\, a}(2G_3 - G_1 - G_2)\right]^2 + \left[\frac{1}{2a}(G_1 - G_2)\right]^2}$

$= \sqrt{\frac{1}{12a^2}(4G_3^2 + G_1^2 + G_2^2 - 4G_1G_3 + 2G_1G_2 - 4G_2G_3) + \frac{1}{4a^2}(G_1^2 + G_2^2 - 2G_1G_2)}$

$$R = \sqrt{\frac{1}{12a^2}\left(4G_3^2 + G_1^2 + G_2^2 - 4G_1G_3 + 2G_1G_2 - 4G_2G_3 + 3G_1^2 + 3G_2^2 - 6G_1G_2\right)}$$

$$= \sqrt{\frac{1}{12a^2}\left(4G_1^2 + 4G_2^2 + 4G_3^2 - 4G_1G_2 - 4G_2G_3 - 4G_1G_3\right)}$$

$$= \sqrt{\frac{4}{12a^2}\left(G_1^2 + G_2^2 + G_3^2 - G_1G_2 - G_2G_3 - G_1G_3\right)}$$

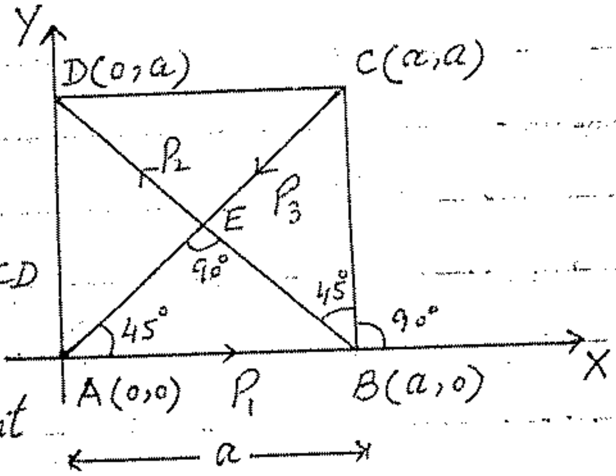
$$= \sqrt{\frac{G_1^2 + G_2^2 + G_3^2 - G_1G_2 - G_2G_3 - G_1G_3}{3a^2}}$$ is the magitude of the resultant force.

Q. No. 11

A Couple of moment 'G' acts on a square board ABCD of side 'a'. Replace the couple by the forces acting along AB, BD, CA.

Solution

Considering a couple of moment G is acting on a square board ABCD whose each side is 'a'.



Let the couple of moment 'G' is replaced by the forces P_1, P_2, P_3 along the sides AB, BD, CA respectively.

Resolving the forces along the coordinate axes.

$$X = P_1 \cos 0^\circ + P_2 \cos 135^\circ - P_3 \cos 45^\circ$$

$$= P_1(1) + P_2\left(-\frac{1}{\sqrt{2}}\right) - P_3\left(\frac{1}{\sqrt{2}}\right)$$

$$= P_1 - \frac{1}{\sqrt{2}}P_2 - \frac{1}{\sqrt{2}}P_3$$

$$Y = P_1 \sin 0^\circ + P_2 \sin 135^\circ - P_3 \sin 45^\circ$$
$$= P_1(0) + P_2\left(\frac{1}{\sqrt{2}}\right) - P_3\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}(P_2 - P_3)$$

Since the system of forces is equal to a single couple of moment 'G' so $G \neq 0$, $R = 0$ i.e $X = Y = 0$

Now

$$X = 0 \Rightarrow P_1 - \frac{1}{\sqrt{2}}P_2 - \frac{1}{\sqrt{2}}P_3 = 0$$
$$P_1 - \frac{1}{\sqrt{2}}(P_2 + P_3) = 0$$
$$P_1 = \frac{1}{\sqrt{2}}(P_2 + P_3) \quad \text{———} \quad (1)$$

and

$$Y = 0 \Rightarrow \frac{1}{\sqrt{2}}(P_2 - P_3) = 0$$
$$\times\sqrt{2} \quad P_2 - P_3 = 0$$
$$\text{or} \quad P_2 = P_3 \quad \text{———} \quad (2)$$

Taking the moment of the forces about 'A'. Let 'G' be the moment of the resultant force, then according to Varignon's theorem

$$G = P_1(0) + P_2|\overrightarrow{AE}| + P_3(0)$$
$$G = P_2|\overrightarrow{AE}|$$
$$G = P_2\frac{a}{\sqrt{2}}$$
$$\text{or} \quad P_2 = \frac{\sqrt{2}G}{a}$$
$$\text{So} \quad P_3 = \frac{\sqrt{2}G}{a} \quad \because P_2 = P_3 \text{ by } (2)$$

From right triangle AEB

$$\frac{|\overrightarrow{AE}|}{a} = \cos 45^\circ$$
$$|\overrightarrow{AE}| = a\cos 45^\circ$$
$$= a \cdot \frac{1}{\sqrt{2}}$$

Using the values of P_2, P_3 in (1)

$$P_1 = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}G}{a} + \frac{\sqrt{2}G}{a}\right) = \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2}G}{a} = \frac{2G}{a}$$

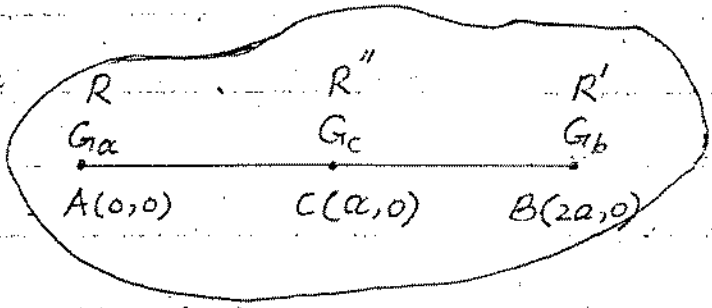
so $P_1 = \dfrac{2G}{a}$ and $P_2 = P_3 = \dfrac{\sqrt{2}G}{a}$

Q. No. 12

A, B are two points in a lamina on which a system of forces coplanar with it are acting, and when the forces are reduced to a single force at each of these points and a couple, the moments of the couples are G_a and G_b respectively. Prove that when reduction is made to a force at the mid point of AB and a couple, the moment of the couple is $\frac{1}{2}(G_a+G_b)$.

Solution

Consider a system of coplanar forces acting on a lamina. Let



the system of forces is reduced into a single force R and a single couple of moment G_a at point A(0,0)

We know that

$G' = G - xY + yX$ at A(0,0)

$G_a = G - (0)Y + (0)X$ $\quad G' = G_a$

$\Rightarrow G = G_a$

Let the system of forces is reduced into a single force R' and a single couple of moment G_b at point B(2a,0).

Now $G' = G - xY + yX$ at $(2a, 0)$

$G_b = G_a - (2a)Y + (0)X$ $\qquad G' = G_b$ and $G = G_a$

$2aY = G_a - G_b$

$\Rightarrow Y = \frac{1}{2a}(G_a - G_b)$

Let the system of coplanar forces is reduced into a single force R" and a single couple of moment G_c at mid-point $C(a, 0)$ of AB.

We know that

$G' = G - xY + yX$ at $C(a, 0)$

$G_c = G_a - (a)\cdot\frac{1}{2a}(G_a - G_b) + (0)X$ $\qquad G' = G_c$ and $G = G_a$, $Y = \frac{1}{2a}(G_a - G_b)$

$= G_a - \frac{1}{2}(G_a - G_b)$

$= \frac{1}{2}(2G_a - G_a + G_b)$

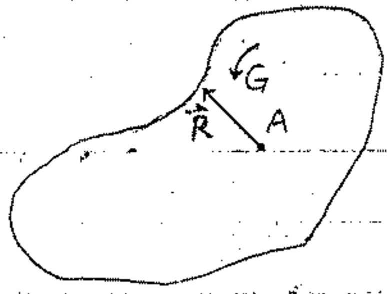
$= \frac{1}{2}(G_a + G_b)$

as required.

Q.No.13

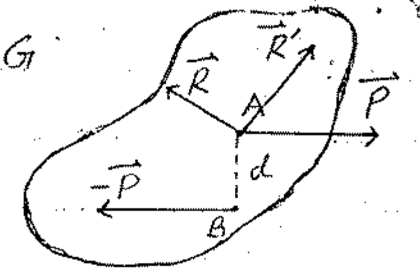
Show that a system of coplanar forces may be represented in an infinite number of ways by two forces one through each of two fixed points in their plane.

Solution



Considering a system of coplanar forces acting on a rigid body. Let the system of coplanar forces is reduced into a single force $\vec{R}$ and a single couple of moment 'G' at 'A'.

Let the couple of moment G is replaced by another couple $(\vec{P}, -\vec{P})$ such that moments of both the couples are same i.e. $G = Pd$



where 'd' is the arm of the couple.

Now the forces $\vec{R}$ and $\vec{P}$ are concurrent at A and can be composed into a force R' (R' is the resultant of $\vec{R}$ and $\vec{P}$).

Now we are left with the two forces

(i) $\vec{R}'$ through A (ii) $-\vec{P}$ through B

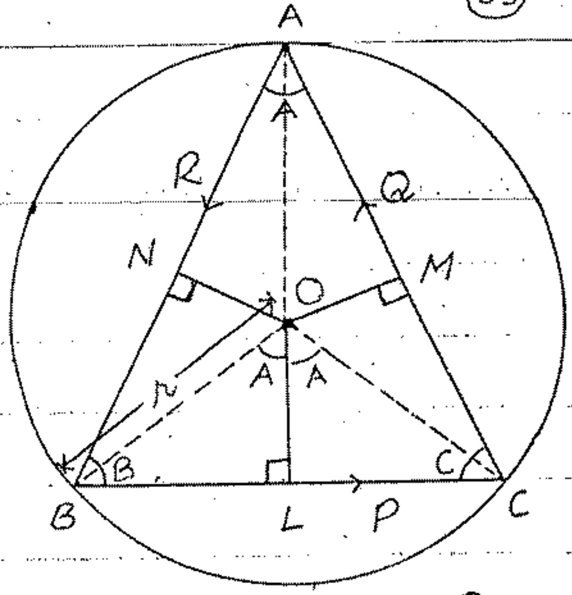
Since the choice of 'A' and 'B' is arbitrary, so the system of coplanar forces may be represented in an infinite number of ways by two forces one through each of two fixed points in their plane.

Q. No. 14 Imp

Three forces P, Q, R acting along the sides BC, CA, AB respectively of a triangle ABC. Prove that if $P\cos A + Q\cos B + R\cos C = 0$ then the line of action of the resultant passes through the circumcentre of the triangle.

Solution

Considering three forces P, Q, R acting along the sides BC, CA, AB of a triangle ABC as shown in figure.



Let the right bisectors of the sides meet at point 'O' (Circumcentre)
Join 'O' with A, B and C.
OA, OB or OC is called
is called circumradius
(denoted by r) of triangle

Let $\quad P\cos A + Q\cos B + R\cos C = 0$ ——— (1)

<u>To prove</u> The line of action of the resultant passes through the circumcentre 'O' of the triangle ABC.

Let 'R′' be the resultant of the three forces and $G = R'd$ moment of the resultant force about 'O'.
Taking the moment of the forces about 'O'. Then according to Varignon's theorem.

$$R'd = P|\overrightarrow{OL}| + Q|\overrightarrow{OM}| + R|\overrightarrow{ON}|$$

$R'd = P(r\cos A) + Q(r\cos B) + R(r\cos C)$
$\quad = r(P\cos A + Q\cos B + R\cos C)$
$\quad = r(0) \quad$ by (1)

So $R'd = 0$

as $R' \neq 0$ then $d = 0$

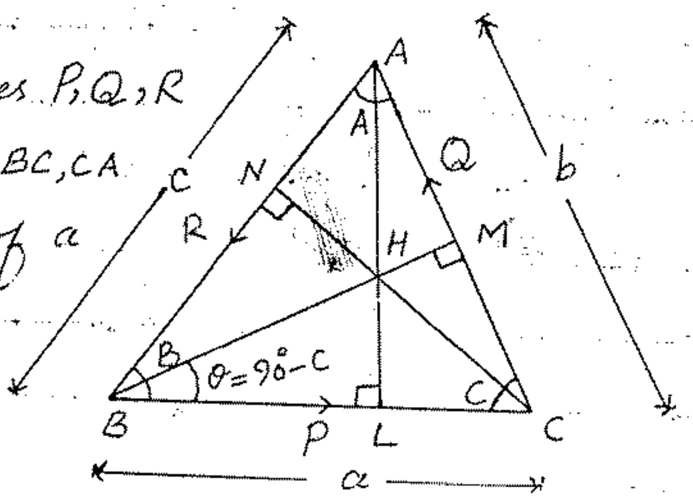
So the line of action of the resultant 'R′' passes through the circumcentre 'O' of the triangle ABC.

From right angled triangle OLB
$\dfrac{|\overline{OL}|}{r} = \cos A$
$|\overline{OL}| = r\cos A$
Similarly from triangles OMC & ONA
$|\overrightarrow{OM}| = r\cos B$
& $|\overrightarrow{ON}| = r\cos C$

<u>Q. No. 15</u> Three forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC. Prove that if $P\sec A + Q\sec B + R\sec C = 0$ then the line of action of the resultant passes through the orthocentre of the triangle.

<u>Solution</u>

Let the three forces P, Q, R act along the sides BC, CA and AB respectively of a triangle ABC.



Let the perpendiculars AL, BM, CN from the vertices A, B, C on opposite sides meet at point H. We call H orthocentre of the triangle.

Let $P\sec A + Q\sec B + R\sec C = 0$ ——— ①

<u>To prove</u> The line of action of the resultant passes through the orthocentre 'H' of the triangle ABC.

From triangle BMC.

$$\theta + 90^\circ + C = 180^\circ$$

$$\theta = 180^\circ - 90^\circ - C$$

$$\theta = 90^\circ - C$$

From triangle BLH

$$\frac{HL}{BL} = \tan(90^\circ - C)$$

$$\frac{HL}{BL} = \cot C$$

$$HL = BL\cot C \quad \text{——— ②}$$

From right angled triangle ABL

$\frac{BL}{c} = \cos B$

$BL = c\cos B$ ——— (3)

Using (3) in (2)

$HL = c\cos B\cot C$

$= c\cos B \cdot \frac{\cos C}{\sin C}$

$HL = \frac{c}{\sin C}\cos B\cos C$ ——— (4)

By Law of Sines

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K$ (say) ——— (5)

Using (5) in (4)

$$\left.\begin{aligned} HL &= K\cos B\cos C \\ \text{Similarly } HM &= K\cos A\cos C \\ \text{and } HN &= K\cos A\cos B \end{aligned}\right\} \text{——— (6)}$$

Let 'R'' be the resultant of the three forces and $G = R'd$ be the moment of the resultant force about 'H'. Taking the moment of the forces about 'H'. Then according to Varignon's theorem

$R'd = P(HL) + Q(HM) + R(HN)$

$= P(K\cos B\cos C) + Q(K\cos A\cos C) + R(K\cos A\cos B)$ by (6)

$= K\cos A\cos B\cos C\left(\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C}\right)$

$= K\cos A\cos B\cos C\,(P\sec A + Q\sec B + R\sec C)$

$= K\cos A\cos B\cos C\,(0)$ by (1)

$R'd = 0 \Rightarrow R' \neq 0$ so $d = 0$. Hence the line of action of the resultant passes through the orthocentre 'H' of triangle ABC.

End