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Orbital Motion

Central Force

If a particle is moving under the action of a force which is always directed towards or away from a fixed point such a force is called a central force.
The fixed point is called Centre of Force and is usually taken as origin.
The central force may be Attractive or Repulsive as it is directed towards or away from fixed point.
The central force at a pt is is a function of distance of that point from the centre of force. This functional relationship is called Law of Force.
The path described by the particle moving under a central force is called the central Orbit.

Thus $\vec{F}$ is a central force iff $\vec{r}\times\vec{F} = 0$

$$\because \vec{r}\times\vec{F} = r\hat{r}\times F\hat{r} = rF\,\hat{r}\times\hat{r} = 0$$

where $\vec{F} = F(r)\,\hat{r}$
$\hat{r}$ is unit vector in the direction of p.v $\vec{r}$.
$\vec{F}$ is fn of r (distance)

Examples

① Motion of earth around the sun, takes place under a force which is attractive and is alway directed towards the sun.
② Motion of planet round the sun.
③ Motion of electron about the nucleus in atom.

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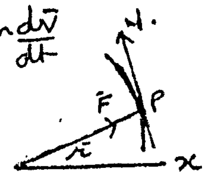
Th <u>The orbit of a particle under a central force is necessarily a plane curve.</u>

Proof Let $\bar{F}$ be a central force acting on a particle of mass 'm'. Let origin 'O' be the centre of force, then

$$\vec{r} \times \vec{F} = 0 \qquad (\because \bar{F} \,\&\, \bar{r} \text{ acts along same direction})$$

$$\Rightarrow \vec{r} \times m\frac{d\vec{v}}{dt} = 0 \qquad \because \bar{F} = m\bar{a} = m\frac{d\bar{v}}{dt}$$



$$\Rightarrow \bar{r} \times \frac{d\bar{v}}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}(\bar{r} \times \bar{v}) = 0$$

Integrating $\bar{r} \times \bar{v} = \text{const vector}.$

$\Rightarrow$ The normal to the plane formed by $\bar{r} \,\&\, \bar{v}$ has a constant direction. This is only possible when particle moves along a plane curve.

$$\because \frac{d}{dt}(\bar{r} \times \bar{v}) = \frac{d\bar{r}}{dt} \times \bar{v} + \bar{r} \times \frac{d\bar{v}}{dt}$$
$$= \bar{v} \times \bar{v} + \bar{r} \times \frac{d\bar{v}}{dt}$$
$$\frac{d}{dt}(\bar{r} \times \bar{v}) = 0 + \bar{r} \times \frac{d\bar{v}}{dt}$$

$\bar{v}$ is along tangent to orbit

<u>IInd Statement of this Th.</u>

"Motion under a Central Force is always in a Plane."

<u>Magnitude of Angular Momentum 'H'</u>

We know $\bar{V} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{s}$ (radial & Transverse comp of velocity)

Momentum $= m\bar{v}$ (m: mass, $\bar{v}$: velo)

$$\therefore \bar{H} = m\,\bar{r} \times \bar{V}$$

Proof

$$= m\bar{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{s})$$
$$= m r\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{s})$$
$$= m r\dot{r}(\hat{r}\times\hat{r}) + mr^2\dot{\theta}\,\hat{r}\times\hat{s}$$
$$\bar{H} = 0 + mr^2\dot{\theta}\,\hat{r}\times\hat{s}$$
$$H = |\bar{H}| = mr^2\dot{\theta} \qquad \because |\hat{r}\times\hat{s}| = 1$$
$$\frac{|\bar{H}|}{m} = r^2\dot{\theta}$$
$$h = r^2\dot{\theta}$$

where $h = \dfrac{|H|}{m}$ is angular momentum of particle of unit mass.

<u>Angular Momentum</u>

$\bar{H} = \bar{r} \times m\bar{V}$
$= m\,\bar{r}\times\bar{V}$

m = mass of moving particle
$\bar{V}$ = velocity of " at 't'
$\bar{r}$ = p.vector of " at 't'

As 'm' is const
Also $\bar{r}\times\bar{v}$ is const
$\therefore \bar{H}$ is const
Hence magnitude of Angular momentum is const.

The orbit described under a central attractive force varying directly as the distance is an <u>Ellipse</u> having centre at the centre of force.

<u>Proof</u>. Let the plane of the orbit be xy-plane. {∵ the orbit is a plane curve. ∴ it may be xy-plane}

then $\bar{F} \propto -\bar{r}$ (−ve sign ∵ attractive force)

$$\bar{F} = -K^2\bar{r} \quad \text{——(1)}$$

(K^2 is const of proportionality)

$$m\bar{a} = -K^2\bar{r}$$

∵ $\bar{F} = m\bar{a}$, $\bar{a} = \ddot{\bar{r}}$

$$m\ddot{\bar{r}} = -K^2\bar{r}$$

$\ddot{\bar{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

$\bar{r} = x\hat{i} + y\hat{j}$

$$m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) = -K^2(x\hat{i} + y\hat{j})$$

$\frac{K^2}{m} = \text{const}, K^2$

$$\ddot{x} = -K^2x \quad , \quad \ddot{y} = -K^2y$$

$$\ddot{x} + K^2x = 0 \quad , \quad \ddot{y} + K^2y = 0$$

$$(D^2+K^2)x = 0 \quad , \quad (D^2+K^2)y = 0$$

$$m^2+K^2 = 0 \quad , \quad m^2+K^2 = 0$$

$$m = \pm Ki \quad , \quad m = \pm Ki$$

The General Sol of these

$$x = e^0(A\cos Kt + B\sin Kt) \quad \text{——(2)}$$

$$y = e^0(C\cos Kt + D\sin Kt) \quad \text{——(3)}$$

×(2) by C $\quad Cx = AC\cos Kt + BC\sin Kt$

×(3) by A & Subtract $\quad Ay = AC\cos Kt + AD\sin Kt$

$$Cx - Ay = (BC-AD)\sin Kt$$

$$\sin Kt = \frac{Cx-Ay}{BC-AD} \quad \text{——(4)}$$

×(2) by D $\quad xD = AD\cos Kt + BD\sin Kt$

×(3) by B $\quad By = BC\cos Kt + BD\sin Kt$

$$xD - By = (AD-BC)\cos Kt$$

$$\cos Kt = \frac{xD-By}{AD-BC} \quad \text{——(5)}$$

Squaring & Adding (4) & (5)

$$\left(\frac{Cx-Ay}{BC-AD}\right)^2 + \left(\frac{xD-By}{AD-BC}\right)^2 = 1$$

∵ $\sin^2 Kt + \cos^2 Kt = 1$

∵ $(AD-BC)^2 = (BC-AD)^2$

$$(Cx-Ay)^2 + (xD-By)^2 = (AD-BC)^2$$

which is eq of central conic and is Ellipse, A, B, C, D can be determined by initial conditions.

Available at
www.mathcity.org

Prove that $r^2\dot\theta = h = \text{const}$

Proof Let (r,θ) be the position of the particle on the orbit and F be the central attractive force then

$$-\bar F = m\bar a \qquad \because \bar a = (\ddot r - r\dot\theta^2)\hat r + (2\dot r\dot\theta + r\ddot\theta)\hat s$$

$$= m[(\ddot r - r\dot\theta^2)\hat r + (2\dot r\dot\theta + r\ddot\theta)\hat s]$$

$$= m[(\ddot r - r\dot\theta^2)\hat r + \frac{1}{r}(2r\dot r\dot\theta + r^2\ddot\theta)\hat s] \qquad \times \text{ \& } \div \text{ by } 'r'$$

$$-\bar F = m[(\ddot r - r\dot\theta^2)\hat r + \frac{1}{r}\frac{d}{dt}(r^2\dot\theta)\hat s]$$

$$-F_r\hat r - F_\theta\hat s = m(\ddot r - r\dot\theta^2)\hat r + \frac{m}{r}\frac{d}{dt}(r^2\dot\theta)\hat s$$

where F_r & F_θ are components of central force $\bar F$, along & perpendicular to radius vector resp.

$$\left.\begin{aligned} -F_r &= m(\ddot r - r\dot\theta^2) \\ \text{\&} \quad -F_\theta &= \frac{m}{r}\frac{d}{dt}(r^2\dot\theta) \end{aligned}\right\}.$$

Since $\bar F$ is central attractive force directed towards 'O', So $\bar F = -ve$ and the central force is always directed along radius vector

So $F_\theta = 0$. $\quad \because F_\theta$ is a $\perp$ to radius vector

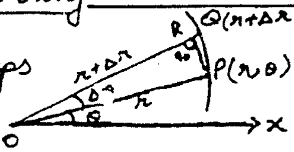
$$\therefore \frac{m}{r}\frac{d}{dt}(r^2\dot\theta) = 0$$

$$\frac{d}{dt}(r^2\dot\theta) = 0$$

Integrating $r^2\dot\theta = \text{const} = h(\text{say})$

Th The Areal Speed of a particle moving under a central force is constant.

Proof Suppose the radius vector $\bar{OP}$ sweeps an area ΔA in time Δt



The area swept by the radius of a particle in unit time is called Areal Speed

then
$$\Delta A = \text{Area of } OPQ$$
$$= \frac{1}{2}(OQ)(PR)$$
$$= \frac{1}{2}(r+\Delta r)r\sin\Delta\theta$$
$$= \frac{1}{2}(r^2\sin\Delta\theta + r\cdot\Delta r\cdot\sin\Delta\theta)$$

Note OPQ is app a Δ because Q is very near to P
$$\therefore \frac{PR}{OP} = \sin\Delta\theta$$
$$\boxed{PR = r\sin\Delta\theta}$$
$\because \Delta\theta$ is very small so $\sin\Delta\theta \approx \Delta\theta$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\left(r^2\frac{\Delta\theta}{\Delta t} + r\cdot\Delta r\cdot\frac{\Delta\theta}{\Delta t}\right)$$

Taking limit as $\Delta t\to 0$, $\Delta r\to 0$, $\Delta\theta\to 0$

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot\theta + r\cdot 0\cdot\dot\theta$$

$$\text{Areal Speed} = \frac{1}{2}r^2\dot\theta$$

$$\text{Areal Speed} = \frac{1}{2}h \qquad \because r^2\dot\theta = h$$

OR $\quad 2(\text{Areal Speed}) = h$

'h' is equal to twice the Areal Speed.

~~ive~~ The Differential Eq of Orbit in Polar Form.

Derive the Differential Eq of the motion of particle of orbit under central force per unit mass.

OR Derive the eq $h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=f(u)$

Proof. Let $\bar{F}$ be the attractive central force on the particle then

$$-F=ma$$

$$-(F_r\hat{r}+F_\theta\hat{s})=m(\ddot{r}-r\dot{\theta}^2)\hat{r}+m(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{s}$$

$$-F_r\hat{r}=m(\ddot{r}-r\dot{\theta}^2)\hat{r}$$

$$-F_\theta\hat{s}=m(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{s}$$

Since F_θ is $\perp$ to radius vector i.e. transverse component of $\bar{F}$, so $F_\theta=0$ & $\bar{F}$ is along the radius vector directed towards O so $-\bar{F}=-(F_r\hat{r}+0)$

$\because -F=-F_r$

$\because r^2\dot{\theta}=h$
$\Rightarrow \dot{\theta}=\frac{h}{r^2}$

$$\therefore -F=m(\ddot{r}-r\dot{\theta}^2)$$

$$=m\left(\ddot{r}-r\frac{h^2}{r^4}\right)$$

$$-F=m\left(\ddot{r}-\frac{h^2}{r^3}\right) \quad \text{--- (1)}$$

Let $u=\frac{1}{r} \Rightarrow r=\frac{1}{u}$

$$\frac{dr}{dt}=\dot{r}=-\frac{1}{u^2}\frac{du}{dt}$$

$$=-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$=-r^2\frac{du}{d\theta}\cdot\dot{\theta}$$

$$\dot{r}=-h\frac{du}{d\theta} \qquad \because r^2\dot{\theta}=h$$

$$\frac{d^2r}{dt^2}=\ddot{r}=\frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

$$=-h\frac{d}{dt}\left(\frac{du}{d\theta}\right)$$

$$=-h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$=-h\frac{d^2u}{d\theta^2}\cdot\dot{\theta}$$

$$=-h\frac{d^2u}{d\theta^2}\cdot\left(\frac{h}{r^2}\right)$$

$$=-h^2\frac{d^2u}{d\theta^2}\cdot\left(\frac{1}{r^2}\right)$$

$$\because u^2=\frac{1}{r^2} \qquad \ddot{r}=-h^2u^2\frac{d^2u}{d\theta^2} \quad \text{--- (2)}$$

using (2) in (1)

$$-F=m\left(-h^2u^2\frac{d^2u}{d\theta^2}-h^2u^3\right)$$

$$\cancel{-}F=\cancel{-}mh^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

$$\frac{F}{m}=h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

$$f=h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

is the required D.Eq of the orbit in polar coord. where f is attractive central force per unit mass.

Note when the central force is Repulsive then D.Eq of orbit is

$$-f=h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

Derive the Diff eq of orbit in Pedal Form

Let $P(r,\theta)$ be a point on the orbit described by a pa[rticle] of mass 'm' moving under the central attractive force.

We know

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

(p, r) are called Pedal Coordina[tes]

p = Length of perpendicular from orig[in] to the tangent to the orbit.

$$\because r = \frac{1}{u},\ \frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

$$\frac{1}{p^2} = u^2 + u^4\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \text{——} \ (1)$$

Diff (1) w.r.t 'r'

$$-\frac{2}{p^3}\frac{dp}{dr} = \frac{d}{dr}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$= \frac{d}{d\theta}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]\frac{d\theta}{dr}$$

$$= \left[2u\frac{du}{d\theta} + 2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2}\right]\frac{d\theta}{dr}$$

$$= 2\cancel{\frac{du}{d\theta}}\left[u + \frac{d^2u}{d\theta^2}\right]\left(-u^2\cancel{\frac{d\theta}{du}}\right)$$

$$r = \frac{1}{u}$$

$$\left(\begin{aligned}&\because \frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} \\ &\therefore \frac{d\theta}{dr} = -u^2\frac{d\theta}{du}\end{aligned}\right.$$

$$\cancel{-2}\frac{1}{p^3}\frac{dp}{dr} = \cancel{-2}\left(u + \frac{d^2u}{d\theta^2}\right)(u^2)$$

$$\frac{1}{p^3}\frac{dp}{dr} = \frac{f}{h^2}$$

$$\left(\begin{aligned}&\because f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) \\ &\therefore \frac{f}{h^2} = u^2\left(u + \frac{d^2u}{d\theta^2}\right)\end{aligned}\right.$$

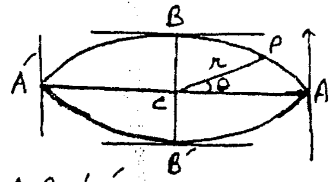
$$\boxed{\frac{h^2}{p^3}\frac{dp}{dr} = f}$$

Pedal eq of motion of orbit where $f = \frac{F}{m}$

h = twice the Areal speed

Apse

An apse is a pt on a central orbit at which the radius vector drawn from the centre of the force is max or min. At AA' & BB' on the elliptic orbit the radius vector from centre C of the force is max or min.



A, B, A', B' are apses.

Apsidal Distance

The length (magnitude) of radius vector at an apse is known as apsidal distance. CA & CA'

Apse Line

The line joining an apse to the centre of force is called an apse line. AA', BB' are apse lines.

Apsidal Angle

The angle between two consecutive apse line is called an apsidal angle. Angle between AA' & BB' is $\frac{\pi}{2}$

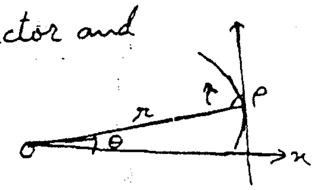
Theorem Analytical Condition for a pt $P(r,\theta)$ to be an Apse is

(centre of force as origin)

$$\frac{dr}{d\theta} = 0 \quad \text{or} \quad \frac{du}{d\theta} = 0$$

OR The radius vector is perpendicular to the tangent at an Apse

Proof Let ϕ is the angle between radius vector and tangent at P



then since $\tan\phi = \dfrac{r}{dr/d\theta}$

but $\dfrac{dr}{d\theta} = 0 \Rightarrow \tan\phi = \dfrac{r}{0}$

$\Rightarrow \tan\phi = \infty$

$\Rightarrow \boxed{\phi = \frac{\pi}{2}}$

∵ **Note** For Apse r is Max & Min. From calculus we know the radius vector r is Max or Min if $\dfrac{dr}{d\theta} = 0$

Also

Since $r = \dfrac{1}{u} = u^{-1}$

$\dfrac{dr}{d\theta} = 0$

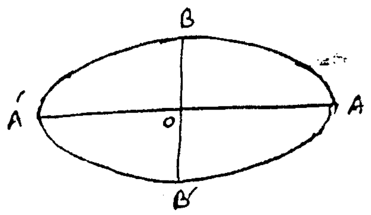
$-\dfrac{1}{u^2}\dfrac{du}{d\theta} = 0 \Rightarrow \dfrac{du}{d\theta} = 0$

Imp

<u>Theorem</u> If a particle describes an <u>Ellipse</u> under central force towards its centre, the orbit has
(i) four apses (ii) two apse lines (iii) two apsidal distances (iv) One apsidal angle

<u>Proof</u>. The eq of orbit, referred to O, the force centre as origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



In polar coordinates

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1$$

Diff w.r.t 'θ'

$$r^2\left(\frac{-2\cos\theta\sin\theta}{a^2} + \frac{2\sin\theta\cos\theta}{b^2}\right) + 2r\frac{dr}{d\theta}\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 0$$

$$r^2\sin 2\theta\left(-\frac{1}{a^2} + \frac{1}{b^2}\right) + 2r\frac{dr}{d\theta}\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 0$$

For an apse $\frac{dr}{d\theta} = 0$

$$\therefore \quad r^2\sin 2\theta\left(-\frac{1}{a^2} + \frac{1}{b^2}\right) = 0$$

$$\Rightarrow \quad \sin 2\theta = 0$$

$$\Rightarrow \quad 2\theta = 0, \pi, 2\pi, 3\pi$$

$$\Rightarrow \quad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

<u>Note</u> If we use $2\sin\theta\cos\theta$ instead of $\sin 2\theta$, then $2\sin\theta\cos\theta = 0$

$\sin\theta\cos\theta = 0$

$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi$.

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

i) So the apses are A, A', B, B' which are extremes of major and minor axes.

ii) Apse lines are OA, OA', OB and OB' i.e AA' & BB'

iii) Apsidal distance $|OA| = |OA'| = a$
$|OB| = |OB'| = b$ two apsidal distances

iv) Apsidal Angles are $\angle AOB = \angle BOA' = \angle A'OB' = \angle B'OA = \frac{\pi}{2}$

So one apsidal angle. i.e 90.

<u>Note</u> If a particle describes an ellipse under an attractive central force directed to one of foci, then there are only two apses, A & A' two apsidal distances and only one apse line AA' and apsidal angle is π.

Show that the orbit described by the planet around sun is a Conic.

or **Polar Eq of the Orbit**

Proof We consider the motion of the planet round the sun and the force is governed by Newton's Law of Gravitation.
If 'M' and 'm' are the mass of Sun and the planet at a distance 'r' apart then they attract each other with a force $\frac{MmG}{r^2}$ where 'G' is constant of gravitation.
Take the sun as the pole, the D.Eq of the orbit is

$$\cancel{m}(h^2u^2)\left[u+\frac{d^2u}{d\theta^2}\right]=G\frac{M\cancel{m}}{r^2}$$

$$h^2\cancel{u^2}\left(u+\frac{d^2u}{d\theta^2}\right)=\mu\cancel{u^2} \qquad \because GM=\mu,\ \because u^2=\frac{1}{r^2}$$

$$u+\frac{d^2u}{d\theta^2}=\frac{\mu}{h^2}$$

$$\frac{d^2u}{d\theta^2}+u-\frac{\mu}{h^2}=0$$

$$\frac{d^2}{d\theta^2}\left(u-\frac{\mu}{h^2}\right)+u-\frac{\mu}{h^2}=0 \qquad \because \frac{d^2u}{d\theta^2}=\frac{d^2}{d\theta^2}\left(u-\frac{\mu}{h^2}\right) \ \because \frac{\mu}{h^2} \text{ is const.}$$

$$\left[D^2+1\right]\left(u-\frac{\mu}{h^2}\right)=0 \qquad D=\frac{d}{d\theta}$$

$$m^2+1=0 \Rightarrow m=\pm i$$

general sol is
$$u-\frac{\mu}{h^2}=(A\cos\theta+B\sin\theta)e^0$$
$$=C\cos(\theta-\theta_0)$$
$$u=\frac{\mu}{h^2}+C\cos(\theta-\theta_0) \quad \text{——— (1)}$$

$\because A\cos\theta+B\sin\theta$
$A=C\cos\theta_0$, $B=C\sin\theta_0$
$\therefore C\cos\theta_0\cos\theta+C\sin\theta_0\sin\theta$
$=C\cos(\theta-\theta_0)$

where C & θ_0 are constants of Integration
θ_0 can be made '0' by rotating base line $\theta=0$

$$\text{So } u=\frac{\mu}{h^2}+C\cos\theta \quad \text{——— (2)}$$

(1) is the most general eq of the orbit under the central force varying as inverse square of distance

from (2)
$$\frac{1}{r}=\frac{\mu}{h^2}\left[1+\frac{C}{\mu/h^2}\cos\theta\right]$$
$$\frac{h^2/\mu}{r}=\left[1+\frac{Ch^2}{\mu}\cos\theta\right] \quad \text{——— (3)}$$

(3) is of the form $\frac{l}{r}=1+e\cos\theta$ which is polar Eq of Conic

So the orbit is a conic with focus at the centre of the force and semi latus rectum $l=\frac{h^2}{\mu}$

eccentricity $e=\frac{h^2C}{\mu}$

$\because$ we get by comparing (3) with $\frac{l}{r}=1+e\cos\theta$

Eq. of the orbit of planet with sun at the focus (centre) in terms of total energy.

Sol Eq of the orbit of the planet with sun at the focus is

$$u = \frac{\mu}{h^2} + c\cos\theta \quad \text{---} (1)$$

(We find c in terms of total energy and put in eq (1))

Total energy = K.E + P.E

$$E = T + V \quad \text{---} (2)$$

where T = K.E per unit mass
V = P.E per unit mass

$$V = -\int f(r)\,dr$$

$$= -\int -\frac{\mu}{r^2}\,dr$$

$$V = -\frac{\mu}{r} \quad \text{Integrating}$$

where $f(r)$ is central force per unit mass
$\therefore f(r) = \frac{F}{m} = \frac{GMm}{r^2 m} = \frac{\mu}{r^2}$
$\because f(r)$ is attractive force $\therefore f(r) = -\frac{\mu}{r^2}$

$$T = \frac{1}{2}v^2$$

$$T = \frac{1}{2}\left((\dot{r})^2 + r^2(\dot{\theta})^2\right)$$

K.E per unit mass $T = \frac{1}{2}mv^2 = \frac{1}{2}\cdot 1\cdot v^2$
$\because \bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{s}$
$\therefore v^2 = \bar{v}\cdot\bar{v} = (\dot{r})^2 + r^2(\dot{\theta})^2$

So from (2)

$$E = \frac{1}{2}\left((\dot{r})^2 + r^2(\dot{\theta})^2\right) + \left(-\frac{\mu}{r}\right)$$

$$= \frac{1}{2}\left[\left(-h\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}(hu^2)^2\right] - \mu u$$

$$= \frac{h^2}{2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - \mu u$$

$\because$ We know $r = \frac{1}{u}$
$\dot{r} = \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$
$= -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta}$
$\dot{r} = -\frac{1}{\cancel{u^2}}\frac{du}{d\theta}(h\cancel{u^2})$
$\dot{r} = -h\frac{du}{d\theta}$

$\because r^2\dot{\theta} = h$
$\frac{1}{u^2}\dot{\theta} = h$
$\dot{\theta} = hu^2$

$$= \frac{h^2}{2}\left[(-c\sin\theta)^2 + \left(\frac{\mu}{h^2} + c\cos\theta\right)^2\right] - \mu\left(\frac{\mu}{h^2} + c\cos\theta\right)$$

$\because$ from (1) $u = \frac{\mu}{h^2} + c\cos\theta$
$\frac{du}{d\theta} = 0 - c\sin\theta$

$$= \frac{h^2}{2}\left[c^2\sin^2\theta + \frac{\mu^2}{h^4} + c^2\cos^2\theta + 2\frac{\mu c\cos\theta}{h^2}\right] - \frac{\mu^2}{h^2} - \mu c\cos\theta$$

$$= \frac{h^2c^2\sin^2\theta}{2} + \frac{h^2\mu^2}{2h^4} + \frac{h^2c^2\cos^2\theta}{2} + \frac{\cancel{h^2}}{\cancel{2}}\frac{\cancel{2}\mu c\cos\theta}{\cancel{h^2}} - \frac{\mu^2}{h^2} - \mu c\cos\theta$$

$$= \frac{h^2c^2}{2}(\sin^2\theta + \cos^2\theta) + \frac{\mu^2}{2h^2} - \frac{\mu^2}{h^2} + \cancel{\mu c\cos\theta} - \cancel{\mu c\cos\theta}$$

$$= \frac{h^2c^2}{2} + \frac{\mu^2 - 2\mu^2}{2h^2}$$

$$E = \frac{h^2c^2}{2} - \frac{\mu^2}{2h^2}$$

$$E + \frac{\mu^2}{2h^2} = \frac{h^2c^2}{2} \quad \Rightarrow c^2 = \frac{2E}{h^2} + \frac{\mu^2}{h^4}. \quad \Rightarrow c^2 = \frac{\mu^2}{h^4}\left(\frac{\frac{2E}{h^2}}{\mu^2/h^4} + 1\right)$$

So $c = \frac{\mu}{h^2}\sqrt{1+\frac{2Eh^2}{\mu^2}}$ ——— (3) Put in (1)

from (1) $u = \frac{\mu}{h^2} + \left(\frac{\mu}{h^2}\sqrt{1+\frac{2Eh^2}{\mu^2}}\right)\cos\theta$

$u = \frac{\mu}{h^2}\left[1+\sqrt{1+\frac{2Eh^2}{\mu^2}}\cos\theta\right]$

$\frac{1}{r} = \frac{\mu}{h^2}\left[1+\sqrt{1+\frac{2Eh^2}{\mu^2}}\cos\theta\right]$ ——— (4)

Polar Eq of Conic $\frac{l}{r} = 1+e\cos\theta$

compare (4) & (5) $\frac{1}{r} = \frac{1}{l}(1+e\cos\theta)$ ——— (5)

So $\frac{1}{l} = \frac{\mu}{h^2} \Rightarrow \boxed{l = \frac{h^2}{\mu}}$

$e = \sqrt{1+\frac{2Eh^2}{\mu^2}}$

Constants of Elliptic orbit

If a is semi Major axis of an ellipse then

$l = a(1-e^2)$

$\Rightarrow a = \frac{l}{1-e^2}$

$a = \frac{h^2/\mu}{1-\left(1+\frac{2Eh^2}{\mu^2}\right)}$

$a = \frac{\frac{h^2}{\mu}}{\cancel{1}-\cancel{1}-\frac{2Eh^2}{\mu^2}}$

$a = -\frac{\cancel{h^2}}{\cancel{\mu}}\cdot\frac{\mu^{\cancel{2}}}{2E\cancel{h^2}}$

$a = \boxed{-\frac{\mu}{2E}} \Rightarrow \boxed{E = -\frac{\mu}{2a}}$

and

$h = \sqrt{\mu l} = \sqrt{\mu a(1-e^2)}$

Th Show that velocity at any pt of the orbit (ellipse) is given by

$V^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right)$

Proof We know $E = T+V$

$E = \frac{1}{2}V^2 - \mu u$

$-\frac{\mu}{2a} = \frac{1}{2}V^2 - \mu u$ $\quad\because E = -\frac{\mu}{2a}$ for Ellipse, a = Semi Major Axis of Ellipse

$\mu u - \frac{\mu}{2a} = \frac{1}{2}V^2$

$\frac{2a\mu u-\mu}{\cancel{2}a} = \frac{1}{\cancel{2}}V^2$

$\frac{\mu(2au-1)}{a} = V^2$

$\frac{\mu}{a}\left(\frac{2a}{r}-1\right) = V^2$

$\mu\left(\frac{2}{r}-\frac{1}{a}\right) = V^2$

proved

x ——— x

Th Show that the time taken by a particle to describe the whole ellipse is

$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$

Proof Areal Speed $\frac{dA}{dt} = \frac{1}{2}r^2\dot\theta = \frac{1}{2}h$

$\frac{dA}{dt}$ i.e Area described in unit time $= \frac{h}{2}$

So Area described in T time $= T\frac{h}{2}$

$\therefore$ $T\frac{h}{2}$ = Area of ellipse

$T\frac{h}{2} = \pi ab$

$T = \frac{2\pi ab}{h}$

$= \frac{2\pi ab}{\sqrt{l\mu}}$ $\quad\left(\because \frac{h^2}{\mu} = l,\ h^2 = \mu l\right.$

$= \frac{2\pi ab}{\sqrt{l}\sqrt{\mu}}$

$= \frac{2\pi a\cancel{b}}{\frac{\cancel{b}}{\sqrt{a}}\sqrt{\mu}}$ $\quad\left(\because l = \frac{b^2}{a}\right.$

$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$

In Terms of energy $T = \frac{2\pi}{\sqrt{\mu}}\left(-\frac{\mu}{2E}\right)^{3/2}$ $\quad\left(\text{for an ellipse } \because a = -\frac{\mu}{2E}\right.$

Derivation of Newton's Law of Gravitation from Kepler's L

By the first law of Kepler "each planet describes an ellipse with the sun at one focus", if we take the sun at the focus (origin) the eq of orbit can be written as

$$\frac{l}{r} = 1 + e\cos\theta$$

$$lu = 1 + e\cos\theta \quad \text{———} (1)$$

Diff

$$l\frac{du}{d\theta} = -e\sin\theta$$

Diff

$$l.\frac{d^2u}{d\theta^2} = -e\cos\theta$$

$$l\frac{d^2u}{d\theta^2} = 1 - lu \quad \text{using (1)} \quad (2)$$

D.Eq of orbit is

$$f(u) = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= h^2u^2\left(\frac{1}{l} - \cancel{u} + \cancel{u}\right) \quad \text{using (2)}$$

$$= \frac{h^2u^2}{l}$$

$$= \frac{h^2}{l}\cdot\frac{1}{r^2}$$

$$f(u) = \frac{\mu}{r^2} \qquad \text{where } \mu = \frac{h^2}{l}$$

$f(u) \propto \frac{1}{r^2}$ Thus for a planet the force varies inversely as square of distance from sun, which is according to Newton's Law of Gravitation.

Note For an ellipse $\frac{h^2}{b^2} = \frac{\mu}{a}$

$$h^2 = \frac{b^2\mu}{a}$$

Also $h^2 = \mu l$

$$\therefore \boxed{l = \frac{b^2}{a}} \quad \left(\text{Semi Latus Rectum} = \frac{b^2}{a}\right)$$

In an ellipse $\because b^2 = a^2(1-e^2)$ $\therefore l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = \boxed{a(1-e^2)}$

Exercise

① A particle describes the following curves under force F to the pole, show that the force is as stated.

(i) $\frac{a}{r} = e^{n\theta}$ Eq of orbit.

$au = e^{n\theta}$ $\Rightarrow u = \frac{e^{n\theta}}{a}$ —— ①

$a\frac{du}{d\theta} = ne^{n\theta}$ —— ②

$a\frac{d^2u}{d\theta^2} = n^2 e^{n\theta}$ —— ③

$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$ D.Eq of orbit

$= h^2\left(\frac{e^{2n\theta}}{a^2}\right)\left[\frac{e^{n\theta}}{a} + \frac{n^2e^{n\theta}}{a}\right]$ using ①

$= h^2 \frac{e^{2n\theta}}{a^2}\frac{e^{n\theta}}{a}(1+n^2)$

$= \frac{h^2e^{3n\theta}}{a^3}(1+n^2)$

$= \frac{h^2}{a^3}\left(\frac{a^3}{r^3}\right)(1+n^2)$ $\because \frac{a}{r} = e^{n\theta}$

$= \frac{h^2}{r^3}(1+n^2)$

$f \propto \frac{1}{r^3}$

(ii) $\frac{a}{r} = n\theta$

$au = n\theta$ —— ①

$a\frac{du}{d\theta} = n$ —— ②

$a\frac{d^2u}{d\theta^2} = 0$ —— ③

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$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$ D.Eq of orbit

$= h^2u^2(u+0)$ using ③

$= h^2u^3$

$= \frac{h^2}{r^3}$

$f \propto \frac{1}{r^3}$

(iii) $\dfrac{a}{r} = \cosh n\theta$

$au = \cosh n\theta$ ——— (i)

$a\dfrac{du}{d\theta} = n\sinh n\theta$ (ii)

$a\dfrac{d^2u}{d\theta^2} = n^2\cosh n\theta$

$a\dfrac{d^2u}{d\theta^2} = n^2 au$ using (i)

$\dfrac{d^2u}{d\theta^2} = n^2u$ ——— (iii)

$f = h^2u^2\left(u+\dfrac{d^2u}{d\theta^2}\right)$

$= h^2u^2(u+n^2u)$ using (iii)

$= h^2u^3(1+n^2)$

$= \dfrac{h^2(1+n^2)}{r^3}$

$f \propto \dfrac{1}{r^3}$

(iv) $\dfrac{a}{r} = \sin n\theta$

$au = \sin n\theta$ ——— (i)

$a\dfrac{du}{d\theta} = n\cos n\theta$ ——— (ii)

$a\dfrac{d^2u}{d\theta^2} = -n^2\sin n\theta$

$a\dfrac{d^2u}{d\theta^2} = -n^2(au)$ using (i)

$\dfrac{d^2u}{d\theta^2} = -n^2u$ ——— (iii)

$f = h^2u^2\left(u+\dfrac{d^2u}{d\theta^2}\right)$

$= h^2u^2(u-un^2)$

$= h^2u^3(1-n^2)$

$= \dfrac{h^2}{r^3}(1-n^2)$

$f \propto \dfrac{1}{r^3}$

Q2 $\quad r^n \cos n\theta = a^n \qquad$ Show that $f \propto r^{2n-3}$

$$r^n = \frac{a^n}{\cos n\theta}$$

$$u^n = \frac{\cos n\theta}{a^n}$$

$$u^n a^n = \cos n\theta \quad \text{——} (1)$$

Diff

$$a^n \cancel{n} u^{n-1} \frac{du}{d\theta} = -\sin n\theta (\cancel{n})$$

$$a^n u^n \frac{du}{d\theta} = -u \sin n\theta \qquad \times \text{by } u$$

$$\cancel{a^n} \frac{\cos n\theta}{\cancel{a^n}} \frac{du}{d\theta} = -u \sin n\theta$$

$$\frac{du}{d\theta} = \frac{-u \sin n\theta}{\cos n\theta}$$

$$\frac{du}{d\theta} = -u \tan n\theta \quad \text{——} (2)$$

Diff

$$\frac{d^2u}{d\theta^2} = -\left(\frac{du}{d\theta} \tan n\theta + nu \sec^2 n\theta\right)$$

$$= -\left(-u \tan^2 n\theta + nu \sec^2 n\theta\right) \quad \text{using } (1)$$

$$= u \tan^2 n\theta - nu \sec^2 n\theta$$

$$= u(\sec^2 n\theta - 1) - nu \sec^2 n\theta$$

$$= u \sec^2 n\theta - u - nu \sec^2 n\theta$$

$$u + \frac{d^2u}{d\theta^2} = u \sec^2 n\theta (1-n) \quad \text{——} (3)$$

$$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) \quad \text{D. Eq of orbit}$$

$$= h^2u^2\left(u \sec^2 n\theta (1-n)\right) \quad \text{using } (3)$$

$$f = h^2 u^3 \sec^2 n\theta (1-n)$$

$$= h^2 u^3 \left(\frac{1}{a^n u^n}\right)^2 (1-n) \quad \text{using } (1)$$

$$= \frac{h^2}{a^{2n} u^{2n-3}} (1-n)$$

$$= \frac{h^2}{a^{2n}} (1-n) r^{2n-3}$$

$$f \propto r^{2n-3} \quad \text{proved}$$

Dr. Munawar Annual 2000

Find the Law of force for the following orbit, the pole being

$$r^2 = a^2\cos 2\theta$$

$$u^2 = \frac{1}{a^2}\sec 2\theta \quad \text{---}(\mathrm{I})$$

Diff

$$\cancel{2}u\frac{du}{d\theta} = \frac{1}{a^2}\sec 2\theta\tan 2\theta(\cancel{2})$$

$$u\frac{du}{d\theta} = u^2 \tan 2\theta$$

$$\frac{du}{d\theta} = u\tan 2\theta \quad \text{---}(\mathrm{II})$$

Diff

$$\frac{d^2u}{d\theta^2} = 2u\sec^2 2\theta + \tan 2\theta\frac{du}{d\theta}$$

$$= 2u\sec^2 2\theta + u\tan^2 2\theta \quad \text{using (II)}$$

$$= 2u\sec^2 2\theta + u(\sec^2 2\theta - 1)$$

$$\frac{d^2u}{d\theta^2} = 2u\sec^2 2\theta + u\sec^2 2\theta - u$$

$$u + \frac{d^2u}{d\theta^2} = 3u\sec^2 2\theta \quad \text{---}(\mathrm{III})$$

$$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$= h^2u^2(3u\sec^2 2\theta)$$

$$= h^2u^2(3u(a^2u^2)^2) \quad \text{using (I)}$$

$$f = h^2u^7 3a^4$$

$$f = \frac{3a^4h^2}{r^7} \quad \text{Ans.}$$

Example (1)

Q.

$$r^n = a^n\cos n\theta \qquad \text{Show that } f = \frac{(n+1)h^2a^{2n}}{r^{2n+3}}$$

$$u^n = \frac{1}{a^n}\sec n\theta$$

$$a^n u^n = \sec n\theta \quad \text{---}(1)$$

Diff

$$a^n \cancel{n}u^{n-1}\frac{du}{d\theta} = \sec n\theta\tan n\theta(\cancel{n})$$

$$a^n u^n\frac{du}{d\theta} = u\sec n\theta\tan n\theta \quad \times \text{ by } u$$

$$\cancel{\sec n\theta}\frac{du}{d\theta} = u\cancel{\sec n\theta}\tan n\theta \quad \text{using (1)}$$

$$\frac{du}{d\theta} = u\tan n\theta \quad \text{---}(2)$$

* see above

Remaining part of Example (1)

*
$$\frac{d^2u}{d\theta^2} = u\sec^2 n\theta(n) + \frac{du}{d\theta}\tan n\theta$$

$$= un\sec^2 n\theta + u\tan^2 n\theta \quad \text{using (2)}$$

$$= un\sec^2 n\theta + u(\sec^2 n\theta - 1)$$

$$= un\sec^2 n\theta + u\sec^2 n\theta - u$$

$$u + \frac{d^2u}{d\theta^2} = u\sec^2 n\theta(n+1) \quad \text{---}(3).$$

$$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$= h^2u^2\{u\sec^2 n\theta(n+1)\} \quad \text{using (3)}$$

$$= h^2u^3\sec^2 n\theta(n+1)$$

$$= h^2u^3(a^nu^n)^2(n+1) \quad \text{using (1)}$$

$$= h^2u^3a^{2n}u^{2n}(n+1)$$

$$f = h^2u^{2n+3}a^{2n}(n+1)$$

$$f = \frac{h^2}{r^{2n+3}}a^{2n}(n+1) \quad \text{Ans.}$$

Imp Example Dr Munawar

③ $r^n = A\cos n\theta + B\sin n\theta$: Show that $F \propto \frac{1}{r^{2n+3}}$

Put $A = R\cos\alpha$ R is const

$B = R\sin\alpha$

$$r^n = R\cos\alpha\cos n\theta + R\sin\alpha\sin n\theta$$

$$r^n = R\cos(n\theta-\alpha)$$

$$u^n = \frac{1}{R}\sec(n\theta-\alpha) \quad \text{——— ①}$$

Diff $nu^{n-1}\frac{du}{d\theta} = \frac{1}{R}\sec(n\theta-\alpha)\tan(n\theta-\alpha)\cancel{n}$

$$u^n\frac{du}{d\theta} = \frac{u}{R}\sec(n\theta-\alpha)\tan(n\theta-\alpha) \qquad \times \text{ by } u$$

$$\frac{du}{d\theta} = \frac{\frac{u}{\cancel{R}}\cancel{\sec(n\theta-\alpha)}\tan(n\theta-\alpha)}{\frac{1}{\cancel{R}}\cancel{\sec(n\theta-\alpha)}} \qquad \text{using ①}$$

$$\frac{du}{d\theta} = u\tan(n\theta-\alpha) \quad \text{——— ②}$$

Diff $\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan(n\theta-\alpha) + u\sec^2(n\theta-\alpha)n$

$$= u\tan^2(n\theta-\alpha) + u\sec^2(n\theta-\alpha)n \qquad \text{using ②}$$

$$= u[\sec^2(n\theta-\alpha)-1] + u\sec^2(n\theta-\alpha)n$$

$$= u\sec^2(n\theta-\alpha) - u + u\sec^2(n\theta-\alpha)n$$

$$u + \frac{d^2u}{d\theta^2} = u\sec^2(n\theta-\alpha)[1+n] \quad \text{——— ③}$$

$$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$= h^2u^2\left(u\sec^2(n\theta-\alpha)(1+n)\right)$$

$$= h^2u^3(Ru^n)^2(1+n) \qquad \text{using ①} \quad \because Ru^n = \sec(n\theta-\alpha)$$

$$= h^2u^{2n+3}R^2(1+n)$$

$$f = \frac{h^2R^2(1+n)}{r^{2n+3}}$$

$$\therefore f \propto \frac{1}{r^{2n+3}} \qquad \because h^2R^2(1+n) \text{ is const.}$$

×————×

④ A particle of unit mass describes an ellipse under the a central force Mr. Show that the normal component of acceleration at any instant is $\frac{abM^{\frac{3}{2}}}{v}$, where v is the velocity at that instant and a, b are the semi-axes of the ellipse.

Normal Comp of Acc $= \frac{V^2}{\rho}$ $\rho = ?$

$\rho = \frac{a^2b^2}{p^3}$ $p = ?$

$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$

D.Eq of orbit $f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$

$$f = Mr$$

$$\& \quad f = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = Mr$$

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{M}{u^3} \qquad \because r = \frac{1}{u}$$

$$h^2\left[2\frac{du}{d\theta}u + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right] = \frac{M}{u^3}\,2\frac{du}{d\theta}$$

$$h^2\frac{d}{d\theta}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2Mu^{-3}\frac{du}{d\theta}$$

Integrating

$$h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \frac{2Mu^{-2}}{-2} + c$$

$$h^2\left(\frac{1}{p^2}\right) = -\frac{M}{u^2} + c \qquad \because \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$\frac{h^2}{p^2} = -Mr^2 + c \quad \text{———} ①$$

Since $r = \frac{1}{u} \Rightarrow \frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$

$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$= u^2 + u^4\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2$

$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

We know pedal eq of Ellipse is

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

$$= \frac{b^2 + a^2 - r^2}{a^2b^2}$$

$$\frac{a^2b^2}{p^2} = b^2 + a^2 - r^2 \quad \text{———} ②$$

Comparing ① & ②

$$\frac{h^2}{a^2b^2} = \frac{M}{1} = \frac{c}{a^2+b^2}$$

$$h^2 = Ma^2b^2, \quad c = M(a^2+b^2)$$

Also $v^2p^2 = h^2$

$$v^2p^2 = Ma^2b^2 \qquad \because h^2 = Ma^2b^2$$

$$p^2 = \frac{Ma^2b^2}{v^2}$$

$$p = \frac{\sqrt{M}\,ab}{v} \quad \text{———} ③$$

Normal component of Acc $= \frac{v^2}{\rho}$ ——— ④

For Ellipse Radius of Curvature $\rho = \frac{a^2b^2}{p^3}$ (see Q10 Ex 7.7)

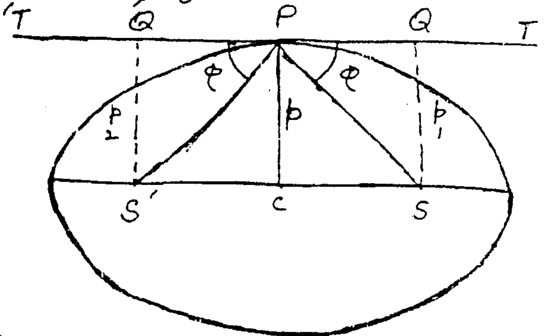
$$\rho = \frac{a^2b^2}{\left(\frac{\sqrt{M}\,ab}{v}\right)^3} \quad ☆ \text{ see above}$$

☆ Hence $\rho = \frac{a^2b^2v^3}{M^{3/2}a^3b^3} = \frac{v^3}{M^{3/2}ab}$

$\therefore$ Normal Comp of Acc $= \frac{v^2}{\frac{v^3}{M^{3/2}ab}}$

$$= \boxed{\frac{abM^{3/2}}{v}} \text{ Ans.}$$

Q5 If a particle is describing an ellipse about a centre of force in the centre, show that the sum of the reciprocal of its angular velocities about the foci is constant.



Let at any time 'T' position of particle be 'P' describing an ellipse under central force through the centre, and 'V' be the velocity of particle at 'P' along the tangent then

$$vp = h \qquad \text{———— (1)}$$

where p is length of ⊥ from centre 'C' to tangent PT

$\angle QPS = \angle Q'PS' = \varphi$ since we know that tangent at 'P' is equally inclined with focal distances.

If ω_1 & ω_2 are angular velocities of the particle at pt 'P' about foci S & S' then

$$\omega_1 = \frac{V\sin\varphi}{SP} \quad \& \quad \omega_2 = \frac{V\sin\varphi}{S'P}$$

$$\frac{1}{\omega_1} + \frac{1}{\omega_2} \qquad \text{(according to Q)}$$

$$= \frac{SP}{V\sin\varphi} + \frac{S'P}{V\sin\varphi} \qquad \star$$

$$= \frac{SP + S'P}{V\sin\varphi}$$

$$= \frac{2a}{V\sin\varphi} \qquad \left(\begin{array}{l}\because \text{In an ellipse sum of} \\ \text{focal distances is const} = 2a\end{array}\right)$$

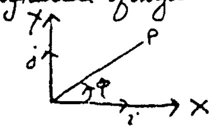
$$= \frac{2a}{V\frac{p}{a}}$$

$$= \frac{2a^2}{Vp} = \frac{2a^2}{h} \text{ (const)}$$

proved.

Angular Velocity 'ω'

It is rate of change of angular displacement φ w.r.t time 't'. At 't' if P is position of particle about 'O' then $\frac{d\varphi}{dt} = \omega$ gives the magnitude of angular velocity.

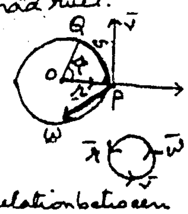


Angular Velocity vector $\bar{\omega}$

$$\bar{\omega} = \frac{d\varphi}{dt}\hat{k} \quad \text{where } \hat{k} = \hat{i}\times\hat{j}$$

i.e. unit vector ⊥ to plane XY.

If a particle is rotating with angular velocity $\bar{\omega}$ in a ⊙ with centre at origin its velocity is ⊥ to $\bar{\omega}$ & $\bar{r}$ and its direction is connected to the direction of $\bar{\omega}$ & $\bar{r}$ by right hand rule.



$S = r\varphi$

$\frac{ds}{dt} = r\frac{d\varphi}{dt}$

$|V| = |r||\omega|$

$$\boxed{\bar{V} = \bar{\omega}\times\bar{r}}$$

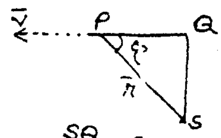
The relation between Linear Velocity & Angular Velocity

$\bar{\omega} = \bar{r}\times\bar{v}$

$\bar{r} = \bar{v}\times\bar{\omega}$

$$\omega = \frac{d\theta}{dt} = \frac{d\theta}{ds}\cdot\frac{ds}{dt} = \frac{1}{r}\left(r\frac{d\theta}{ds}\right)(v)$$

$$\omega = \frac{v}{r}(\sin\varphi)$$



$\frac{SQ}{PS} = \sin\varphi$

$SQ = PS\sin\varphi$

$SQ = r\sin\varphi$

From rt $\triangle PSQ$ & $\triangle PS'Q'$

$$\sin\varphi = \frac{p_1}{SP} = \frac{p_2}{PS'}$$

$p_1 = SP\sin\varphi$

$p_2 = PS'\sin\varphi$

$p_1 + p_2 = (SP + PS')\sin\varphi$

$p_1 + p_2 = 2a\sin\varphi$

$\frac{p_1+p_2}{2a} = \sin\varphi$

Imp MA, Munawar.

(6) A particle of mass 'm' moves under the central force $mM\{3au^4 - 2(a$ and is projected from an apse at a distance $a+b$ with velocity $\frac{\sqrt{M}}{a+b}$. Show the the orbit is $r = a + b\cos\theta$.

Sol If 'P' is ⊥ distance from centre of orbit to the apse.

then $P = r = a+b$ (given) —— (I)

and $v = \frac{\sqrt{M}}{a+b}$ (given) —— (II)

Since $h = vP$ —— (III)

The length of radius vector at an apse is known as apsidal distance. $a+b$ is apsidal distance. At an apse the radius vector is ⊥ to tangent. so $p = a+b$

Put (I) & (II) in (III) $h = \frac{\sqrt{M}}{\cancel{(a+b)}}\cdot\cancel{(a+b)} \Rightarrow h = \sqrt{M}$

$$F = mM(3au^4 - 2(a^2-b^2)u^5)$$

$$f = M(3au^4 - 2(a^2-b^2)u^5)$$

force per unit mass

$\because f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$ force per unit mass.

$\because h = \sqrt{M}$

$$h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = M\left[3au^4 - 2(a^2-b^2)u^5\right]$$

$$\cancel{Mu^2}\left(u + \frac{d^2u}{d\theta^2}\right) = \cancel{Mu^2}\left(3au^2 - 2(a^2-b^2)u^3\right)$$

$$\frac{d^2u}{d\theta^2} = 3au^2 - 2(a^2-b^2)u^3 - u$$

$\times$ by $2\frac{du}{d\theta}$

$$2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} = 3au^2\left(2\frac{du}{d\theta}\right) - 2(a^2-b^2)u^3\left(2\frac{du}{d\theta}\right) - u\left(2\frac{du}{d\theta}\right)$$

$$\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2 = \frac{d}{d\theta}(2au^3) - (a^2-b^2)\frac{d}{d\theta}(u^4) - \frac{d}{d\theta}(u^2)$$

Integrate

$$\left(\frac{du}{d\theta}\right)^2 = 2au^3 - (a^2-b^2)u^4 - u^2 + A \quad \text{—— (IV)}$$

Since apse $r = a+b$

& $u = \frac{1}{a+b} \Rightarrow \frac{du}{d\theta} = 0$

putting in (IV)

$$0 = 2a\left(\frac{1}{a+b}\right)^3 - (a^2-b^2)\left(\frac{1}{a+b}\right)^4 - \left(\frac{1}{a+b}\right)^2 + A$$

$$0 = \frac{2a(a+b) - (a^2-b^2) - (a+b)^2}{(a+b)^4} + A$$

$$0 = \frac{\cancel{2a^2} + \cancel{2ab} - \cancel{a^2} + \cancel{b^2} - \cancel{a^2} - \cancel{b^2} - \cancel{2ab}}{(a+b)^4} + A$$

$0 = A$ Put in (IV)

$$\left(\frac{du}{d\theta}\right)^2 = 2au^3 - (a^2-b^2)u^4 - u^2 + 0$$

$$\left(\frac{du}{d\theta}\right)^2 = u^2\left(2au - (a^2-b^2)u^2 - 1\right) \quad \text{———} \quad (v)$$

Since $r = \frac{1}{u}$

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

$$\left(-u^2\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2$$

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2$$

$$\left(\frac{-1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2}\left[2a\left(\frac{1}{r}\right) - (a^2-b^2)\left(\frac{1}{r^2}\right) - 1\right]$$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{r^2}\left(\frac{2a}{r} - \frac{(a^2-b^2)}{r^2} - 1\right)$$

$$= 2ar - (a^2-b^2) - r^2$$

$$= 2ar - a^2 + b^2 - r^2$$

$$= b^2 - (r^2 + a^2 - 2ar)$$

$$\left(\frac{dr}{d\theta}\right)^2 = b^2 - (r-a)^2$$

$$\frac{dr}{d\theta} = \pm\sqrt{b^2 - (r-a)^2}$$

Separating Variables.

$$\int \frac{dr}{\sqrt{b^2-(r-a)^2}} = \pm\int d\theta$$

$$\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}$$

$$-\cos^{-1}\left(\frac{r-a}{b}\right) = \pm\theta + B$$

At apse $r = a+b$, $\theta = 0$

$$\Rightarrow -\cos^{-1}\left(\frac{a+b-a}{b}\right) = 0 + B \qquad \because \cos^{-1}(0) = 0$$

$$\Rightarrow \boxed{B = 0}$$

$$\therefore -\cos^{-1}\left(\frac{r-a}{b}\right) = \pm\theta + 0$$

$$\cos^{-1}\left(\frac{r-a}{b}\right) = \mp\theta$$

$$\frac{r-a}{b} = \cos(\mp\theta)$$

$$\frac{r-a}{b} = \cos\theta \qquad \because \cos(-\theta) = \cos\theta$$

$$r - a = b\cos\theta$$

$$\boxed{r = a + b\cos\theta}$$

(7) The Law of force is Mu^5 and a particle is projected from an apse 'a'. Find the orbit when the velocity of projection is $\frac{\sqrt{M}}{a^2}$.

Sol We know that the Law of force is

$$f(u) = h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$Mu^5 = h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right) \quad \because f(u)=Mu^5 \text{ given}$$

$\because v = \frac{\sqrt{M}}{a^2}$ (given) $p = r = a$ given

$\therefore vp = h \Rightarrow \frac{\sqrt{M}}{a^2}a = h$

$\Rightarrow \frac{\sqrt{M}}{a} = h$

$$Mu^5 = \left(\frac{\sqrt{M}}{a}\right)^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$u^3 = \frac{1}{a^2}\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$a^2u^3 = u+\frac{d^2u}{d\theta^2}$$

$$a^2u^3 2\frac{du}{d\theta} = \left(2\frac{du}{d\theta}\right)u + \left(2\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} \quad \times \text{ by } 2\frac{du}{d\theta}$$

$$2a^2u^3\frac{du}{d\theta} = \frac{d}{d\theta}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]$$

Integrating

$$2a^2\frac{u^4}{4}+A = u^2+\left(\frac{du}{d\theta}\right)^2$$

To find const A.

At $r=a$, $u=\frac{1}{a}$ & $\frac{du}{d\theta}=0$

& $\frac{du}{d\theta}=0 \therefore \frac{a^2}{2}\left(\frac{1}{a^4}\right)+A = \frac{1}{a^2} \Rightarrow A = \frac{1}{a^2}-\frac{1}{2a^2} = \boxed{\frac{1}{2a^2}}$

$$\frac{a^2u^4}{2}+A = u^2+\left(\frac{du}{d\theta}\right)^2$$

$$\frac{a^2u^4}{2}+\frac{1}{2a^2} = u^2+\left(\frac{du}{d\theta}\right)^2$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{a^2u^4}{2}+\frac{1}{2a^2}-u^2$$

$$= \frac{a^4u^4+1-2a^2u^2}{2a^2}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{(a^2u^2-1)^2}{2a^2}$$

$$\left(\frac{du}{d\theta}\right) = \frac{a^2u^2-1}{\sqrt{2}a}$$

$$\frac{du}{a^2u^2-1} = \frac{d\theta}{\sqrt{2}\,a}$$

$$\frac{1}{a^2}\left(\frac{du}{u^2-\frac{1}{a^2}}\right) = \frac{d\theta}{\sqrt{2}\,a}$$

Integrating

$$\frac{1}{a^2}\left[\frac{1}{2\left(\frac{1}{a}\right)}\ln\left\{\frac{\left(u-\frac{1}{a}\right)}{\left(u+\frac{1}{a}\right)}\right\}\right] = \frac{\theta}{\sqrt{2}\,a}+B$$

$$\frac{1}{a^2}\left(\frac{a}{2}\right)\ln\left\{\frac{au-1}{au+1}\right\} = \frac{\theta}{\sqrt{2}\,a}+B$$

$$\frac{1}{2a}\ln\left(\frac{au-1}{au+1}\right) = \frac{\theta}{\sqrt{2}\,a}+B$$

$$\ln\left(\frac{au-1}{au+1}\right) = \frac{2\cancel{a}\theta}{\sqrt{2}\cancel{a}}+B$$

$$\frac{au-1}{au+1} = e^{\sqrt{2}\theta+B}$$

$$\frac{a\frac{1}{r}-1}{a\frac{1}{r}+1} = e^{\sqrt{2}\theta+B}$$

$$\frac{a-r}{a+r} = e^{\sqrt{2}\theta+B}$$

At $r=a$, $\theta=0 \Rightarrow 0 = e^{0+B} \Rightarrow \ln 0 = B$

$\boxed{-\infty = B}$

$\therefore \frac{a-r}{a+r} = e^{-\infty} = 0$

$a-r = 0 \Rightarrow \boxed{r=a}$ req. orbit

Ḡ Numan

(8) A particle moves under a central repulsive force $\frac{\mu}{r^3}$ and is projected from an apse at a distance 'a' with velocity V. Show that eq of path is $r\cos p\theta = a$ and angle described in time 't' is

$$\frac{1}{p}\tan^{-1}\left(\frac{pVt}{a}\right) \quad \text{where } p^2 = \frac{\mu + a^2V^2}{a^2V^2}$$

Sol Since force is repulsive so

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right) = -f(u) \quad \text{(negative sign is due to repulsive force)}$$

$$h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right) = -\frac{\mu}{r^3} \quad \because f = \frac{\mu}{r^3} \text{ given.}$$

$$h^2\cancel{u^2}\left(u+\frac{d^2u}{d\theta^2}\right) = -\mu u^{\cancel{3}}$$

$\because p = a,\ v = V$ (given) $\therefore h = pV$ $\boxed{h = aV}$

$$a^2V^2\left(u+\frac{d^2u}{d\theta^2}\right) = -\mu u$$

$$u+\frac{d^2u}{d\theta^2} = -\frac{\mu u}{a^2V^2}$$

$$\frac{d^2u}{d\theta^2} = -\frac{\mu u}{a^2V^2} - u$$

$$\frac{d^2u}{d\theta^2} = \frac{-\mu u - ua^2V^2}{a^2V^2}$$

$$\frac{d^2u}{d\theta^2} = -u\left(\frac{\mu + a^2V^2}{a^2V^2}\right)$$

$$\frac{d^2u}{d\theta^2} = -up^2 \quad \text{where } p^2 = \frac{\mu + a^2V^2}{a^2V^2} \text{ (given)}$$

$$\frac{d^2u}{d\theta^2} + up^2 = 0$$

$$\left(2\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} + \left(2\frac{du}{d\theta}\right)up^2 = 0 \quad \times \text{ by } 2\frac{du}{d\theta}$$

Integrating $\left(\frac{du}{d\theta}\right)^2 + p^2u^2 = A$ —— (1)

To find A
(At apse $r = a$
$\Rightarrow \frac{1}{u} = a$
$\Rightarrow u = \frac{1}{a}$
$\Rightarrow \frac{du}{d\theta} = 0$

$\therefore \left(\frac{du}{d\theta}\right)^2 + p^2u^2 = A$

$0 + p^2\cdot\frac{1}{a^2} = A$

$\boxed{\frac{p^2}{a^2} = A}$ Put in (1)

$$\therefore \left(\frac{du}{d\theta}\right)^2 + p^2u^2 = \frac{p^2}{a^2}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{p^2}{a^2} - p^2u^2$$

$$\Rightarrow \frac{du}{d\theta} = \pm p\sqrt{\frac{1}{a^2} - u^2}$$

$$\frac{du}{\sqrt{\left(\frac{1}{a}\right)^2 - u^2}} = \pm p\, d\theta$$

Integrating $-\cos^{-1}\left(\frac{u}{1/a}\right) = \pm p\theta + B$ —— (ii)

To find B

$r = a$ & $\theta = 0$

$\frac{1}{u} = a$

$u = \frac{1}{a}$

$-\cos^{-1}(a \cdot \frac{1}{a}) = \pm p\theta + B$

$-\cos^{-1}(1) = \pm p(0) + B$

$\boxed{0 = B}$ Put in (ii)

$\therefore -\cos^{-1}(au) = \pm p\theta + 0$

$\Rightarrow au = \mp \cos p\theta$

$\Rightarrow a = \frac{1}{u}\cos p\theta$

$\Rightarrow a = r\cos p\theta$ Eq of the path.

To find angle θ.

$r^2\dot\theta = h$

$\dot\theta = \frac{h}{r^2}$

$\dot\theta = hu^2$

$\dot\theta = aV \cdot \left(\frac{1}{a^2}\cos^2 p\theta\right)$ $\left(\because h = aV\right.$ & $\left. au = \cos p\theta\right)$

$\frac{d\theta}{dt} = \frac{V\cos^2 p\theta}{a}$

$\frac{d\theta}{\cos^2 p\theta} = \frac{V}{a}dt$

$\int \sec^2 p\theta \, d\theta = \frac{V}{a}\int dt$

$\frac{\tan p\theta}{p} = \frac{V}{a}t + C$ —— (iii)

Initially at apse i.e

At $t=0, \theta=0$

$\Rightarrow C = 0$ Put in (iii)

$\therefore \frac{\tan p\theta}{p} = \frac{Vt}{a} + 0$.

$\tan p\theta = \frac{Vtp}{a}$

$p\theta = \tan^{-1}\left(\frac{Vtp}{a}\right)$

$\theta = \frac{1}{p}\tan^{-1}\left(\frac{Vtp}{a}\right)$

where $p^2 = \frac{\mu + a^2V^2}{a^2V^2}$

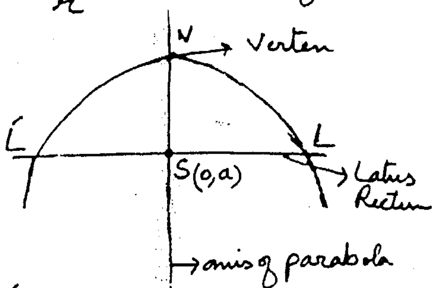
Two Particles describe in equal times, the arc of a Parabola bounded by the latus rectum, one under an attraction to the focus and the other with const acceleration 'g' parallel to the axis. Show that the acceleration of the first particle at the vertex of the parabola is $\frac{16g}{9}$.

Sol Eq of Parabola in Polar form is

$\frac{l}{r} = 1+e\cos\theta$ Eq of Conic

$$\frac{l}{r} = 1+1.\cos\theta \qquad \because e=1 \text{ for Parabola}$$

where 'l' is semi latus rectum $=2a$



LL' = length of Latus Rectum $=4a$
length of Semi latus Rectum $=2a$.

Time taken by a particle to go from L' to L
$= 2$(Time taken from V to L)

Let the force is attractive towards focus 'S'.

So $h = r^2\dot{\theta}$

$$= r^2\frac{d\theta}{dt}$$

$$\int h\,dt = \int_0^{\pi/2} r^2\,d\theta \qquad \because r = \frac{l}{1+\cos\theta}$$

$\because \theta=0$ to $\theta=\pi/2$

$$ht = \int_0^{\pi/2}\left(\frac{l}{1+\cos\theta}\right)^2 d\theta$$

$$= l^2\int_0^{\pi/2}\left(\frac{1}{2\cos^2\frac{\theta}{2}}\right)^2 d\theta$$

$$= \frac{l^2}{4}\int_0^{\pi/2}\sec^4\frac{\theta}{2}\,d\theta$$

$$= \frac{l^2}{4}\int_0^{\pi/2}\sec^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta$$

$$= \frac{l^2}{4}\int_0^{\pi/2}\left(\sec^2\frac{\theta}{2} + \sec^2\frac{\theta}{2}\tan^2\frac{\theta}{2}\right)d\theta$$

$$= \frac{l^2}{4}\left|\frac{\tan\frac{\theta}{2}}{\frac{1}{2}} + \frac{\tan^3\theta/2}{3\left(\frac{1}{2}\right)}\right|_0^{\pi/2}$$

$$= \frac{l^2}{4}\left|2\tan\frac{\theta}{2} + \frac{2}{3}\tan^3\theta/2\right|_0^{\pi/2}$$

$$= \frac{l^2}{4}\left[2\tan\frac{\pi}{4} + \frac{2}{3}\tan^3\frac{\pi}{4} - 0 - 0\right]$$

$$= \frac{l^2}{4}\left(2\cdot 1 + \frac{2}{3}\cdot 1\right)$$

$$ht = \frac{l^2}{4}\left(\frac{8}{3}\right) = \frac{2l^2}{3} \Rightarrow t = \frac{2l^2}{3h}$$

Thus time from V to L $= \frac{2\ell^2}{3h}$

So Total time from L' to L $= 2\left[\frac{2\ell^2}{3h}\right] = \frac{4\ell^2}{3h}$

$= \frac{4}{3h}(2a)^2 = \frac{16a^2}{3h}$ —— (1)

(ii) Now we find time from L' to L along the arc under constant acceleration 'g' // to the axis of parabola i,e y-axis (vertically downwards). Now time from V to L along the arc is same as time from V to S under gravity vertical downward.

$S = ut + \frac{1}{2}gt^2$ (∵ at pt V, initial velocity $u=0$ and distance $S = \overline{VS} = a$

$a = 0t + \frac{1}{2}gt^2$

$\Rightarrow \frac{2a}{g} = t^2$

$\Rightarrow t = \sqrt{\frac{2a}{g}}$

Total time required from L' to L is $2\left(\sqrt{\frac{2a}{g}}\right)$ —— (2)

from (1)&(2) $\frac{16a^2}{3h} = 2\sqrt{\frac{2a}{g}}$ $\Rightarrow h = \frac{\cancel{16}^{8}a^2}{3\cancel{2}}\sqrt{\frac{g}{2a}} \Rightarrow \boxed{h = \frac{8a^2}{3}\sqrt{\frac{g}{2a}}}$ —— (3)

(iii) Now if 'f' is the acceleration of 1st particle towards focus S,

then 'f' $= h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$

{∵ $\bar{F} = m\bar{a}$, $F = ma$, $\frac{F}{m} = a$, $f = a$, force/mass = acc. By Law of force we often mean Acc as f is force/unit mass.

$f = \left(\frac{8a^2}{3}\sqrt{\frac{g}{2a}}\right)^2 u^2\left(\frac{d^2u}{d\theta^2} + u\right)$ using (3)

$= \frac{64a^4}{9}\frac{g}{2a}u^2\left(\frac{d^2u}{d\theta^2} + u\right)$

$= \frac{32a^3g}{9}u^2\left(\frac{d^2u}{d\theta^2} + u\right)$ —— (4)

Now orbit of particle is $\frac{\ell}{r} = 1 + \cos\theta$

$\ell u = 1 + \cos\theta$

Diff $\ell\frac{du}{d\theta} = -\sin\theta$

Diff $\ell\frac{d^2u}{d\theta^2} = -\cos\theta$

Adding $\ell u + \ell\frac{d^2u}{d\theta^2} = 1 + \cancel{\cos\theta} - \cancel{\cos\theta}$

$\ell\left(u + \frac{d^2u}{d\theta^2}\right) = 1$

$u + \frac{d^2u}{d\theta^2} = \frac{1}{\ell} = \frac{1}{2a}$ —— (5) Put in (4)

$= \frac{32a^3g}{9}u^2\left(\frac{1}{2a}\right)$ using (5)

$= \frac{16a^2gu^2}{9}$

$f = \frac{16a^2g}{9}\left(\frac{1}{a^2}\right)$ ∵ at vertex $r = a$, $\frac{1}{u} = a$, $u = \frac{1}{a}$

$\boxed{f = \frac{16g}{9}}$ proved

decreasing (27) ellipse

(10) A planet is describing an ellipse about the sun as focus. Show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is $\dfrac{2\pi ae}{T(1-e^2)}$ where '2a' the major axis, 'e' the eccentricity and T is the periodic time.

OR

A particle of mass 'm' describes an elliptic orbit about an attracting force centre situated at one focus. The force is that of inverse square law. If 'e' is the eccentricity 'T' the time period, 2a the major axis, show that the greatest radial velocity of the particle is $\dfrac{2\pi ae}{T\sqrt{1-e^2}}$

Sol Since the orbit is an ellipse with force centre at one focus, its eq in polar form is

$$r = \frac{l}{1+e\cos\theta} = l(1+e\cos\theta)^{-1} \quad \text{—— (1)}$$

Diff w.r.t 't'

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -l(1+e\cos\theta)^{-2}(-e\sin\theta)\cdot\frac{d\theta}{dt}$$

$$= \frac{le\sin\theta}{(1+e\cos\theta)^2}\cdot\dot\theta$$

x by l ÷

$$= \frac{l^2 e\sin\theta}{l(1+e\cos\theta)^2}\cdot\dot\theta$$

using (1)

$$= \frac{r^2}{l}\, e\sin\theta\cdot\dot\theta$$

$\because r^2\dot\theta = h$

$$= h\,\frac{e\sin\theta}{l}$$

$\because h^2 = \mu l$

$$\frac{dr}{dt} = \sqrt{\mu l}\;\frac{e\sin\theta}{l} = \frac{\sqrt{\mu}}{\sqrt{l}}\,e\sin\theta$$

$\frac{dr}{dt}$ will be Max if $\sin\theta$ is Max i.e equal to '1' i.e $\theta = \frac{\pi}{2}$ since e, μ, & l are consts

Hence

$$\left(\frac{dr}{dt}\right)_{Max} = \frac{\sqrt{\mu}}{\sqrt{l}}\cdot e\cdot 1 \quad \text{—— (11)}$$

Now We know $T = \dfrac{2\pi a^{3/2}}{\sqrt{\mu}} \Rightarrow \sqrt{\mu} = \dfrac{2\pi a^{3/2}}{T}$

and $l = \dfrac{b^2}{a} = \dfrac{a^2(1-e^2)}{a} = a(1-e^2)$

Put values of $\sqrt{\mu}$ & l in (11)

$$= \frac{2\pi a^{3/2}}{T}\cdot\frac{1}{\sqrt{a(1-e^2)}}\cdot e$$

$$= \frac{2\pi a e}{T\sqrt{1-e^2}} \quad \text{proved}$$

Example 2 Prove that the speed at any pt of a central orbit is given by where h is the areal speed and p is the perpendicular distance from the of force of the tangent at that point.

Hence find the expression for V when a particle subject to inverse square Law of force, describes (i) Elliptic, (ii) Parabolic (iii) hyperbolic orbit.

Sol We know $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$ —(1) $\quad \left(\because v = \dot{r}\hat{r} + r\dot{\theta}\hat{s} \ \ \& \ \ \bar{v}.\bar{v} = v^2\right.$

since $r = \frac{1}{u}$

$\Rightarrow \frac{dr}{dt} = \dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$

$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\dot{\theta}$

and $h = r^2\dot{\theta}$

from (1) $\therefore v^2 = \left(-\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta}\right)^2 + r^2\dot{\theta}^2$

$= \left[\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2 + r^2\right]\dot{\theta}^2$

$= \left[r^4\left(\frac{du}{d\theta}\right)^2 + r^2\right]\dot{\theta}^2$

$= \left[\left(\frac{du}{d\theta}\right)^2 + \frac{1}{r^2}\right]r^4\dot{\theta}^2$

$\because h = r^2\dot{\theta}$ $\quad = \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]h^2$

$v^2 = \left(\frac{1}{p^2}\right)h^2 \qquad \left(\because \frac{1}{p^2} = \left(\frac{du}{d\theta}\right)^2 + u^2\right.$ pedal Eq of orbit see page 6

$v^2p^2 = h^2$

$\boxed{vp = h}$ —(2)

Newton's Law of gravitation from Keplers Law

$f = \frac{\mu}{r^2} = \mu u^2$

but $f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$ (eq of orbit)

$\therefore \mu \cancel{u^2} = h^2\cancel{u^2}\left(u + \frac{d^2u}{d\theta^2}\right)$

$\mu = h^2\left(u + \frac{d^2u}{d\theta^2}\right)$

$\mu\, 2\frac{du}{d\theta} = h^2\left(2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right)$ (x by $2\frac{du}{d\theta}$)

Integrating $\mu 2u + c = h^2\left(\cancel{2}\frac{u^2}{\cancel{2}} + \left(\frac{du}{d\theta}\right)^2\right)$

$\frac{h^2}{p^2} = 2\mu u + c \qquad \because \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

$\boxed{\frac{h^2}{p^2} = \frac{2\mu}{r} + c}$ —— ③ Pedal Eq of orbit

We know (r,p) Eq of an Ellipse, Parabola & Hyperbola referred to focus as pole are

$\frac{b^2}{p^2} = \frac{2a}{r} - 1$ (Ellipse) —— ④

$p^2 = ar$ (Parabola) —— ⑤

$\frac{b^2}{p^2} = \frac{2a}{r} + 1$ Hyperbola —— ⑥

where 'a' is semi major axis, 'b' is semi minor axis, 'l' is semi Latus Rectum of Conic.

Comparing ④ & ③ we get

$\frac{\frac{h^2}{\cancel{p^2}}}{\frac{b^2}{\cancel{p^2}}} = \frac{\cancel{2}\frac{\mu}{\cancel{r}}}{\cancel{2}\frac{a}{\cancel{r}}} = \frac{c}{-1} \implies \frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c}{-1}$

$\implies h^2 = \frac{b^2\mu}{a} = \mu l$, $\boxed{c = -\frac{\mu}{a}}$ (i.e. c is –ive in elliptic orbit)

Comparing ⑤ & ③ we get

$\frac{\frac{h^2}{p^2}}{p^2} = \frac{2\frac{\mu}{r}}{ar} = \frac{c}{0} \implies \frac{h^2}{p^4} = \frac{2\mu}{ar^2} = \frac{c}{0}$

$\overset{\times \div \text{ by } a}{\implies} h^2 = a\frac{2\mu p^4}{a^2r^2}$, $\frac{2\mu(0)}{ar^2} = c$

$\overset{\text{using ⑤}}{\implies} h^2 = \frac{a\,2\mu\,(\cancel{p^4})}{\cancel{p^4}}$, $\boxed{0 = c}$ (i.e. $c = 0$ in Parabolic orbit)

Similarly comparing ⑥ & ③ we get $h^2 = \mu l$ & $\boxed{c = \frac{\mu}{a}}$ (i.e. c is +ive in Hyperbolic orbit)

Hence the orbit will be an Ellipse, Parabola, Hyperbola according $c \lesseqgtr 0$

From eq ③ $v^2 = 2\frac{\mu}{r} + c$ —— ⑦ $\quad \because v^2p^2 = h^2$ from ②

$v^2 = 2\frac{\mu}{r} - \frac{\mu}{a}$ when the orbit is ellipse (by $c = -\frac{\mu}{a}$ in ⑦)

$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$ —— ⑧

$v^2 = 2\frac{\mu}{r} + 0$ —— ⑨ when the orbit is Parabola (by $c = 0$ in ⑦)

when the orbit is Hyperbola (by $c = \frac{\mu}{a}$ in ⑦)

$v^2 = 2\frac{\mu}{r} + \frac{\mu}{a}$

$v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right]$ —— ⑩

For circular orbit under inverse square ...