Orbital Motion

Central Force

If a particle is moving under the action of a force which is always directed towards or away from a fixed point such a force is called a central force. The fixed point is called Centre of Force and is usually taken as origin.
The central force may be Attractive or Repulsive as it is directed towards or away from fixed point.
The central force at a point is a function of distance of that point from the centre of force. This functional relationship is called Law of Force.
The path described by the particle moving under a central force is called the central Orbit.

Thus \( \vec{F} \) is a central force if \( \vec{F} \times \vec{r} = 0 \) where \( \vec{F} = F \hat{r} \)

\[
\begin{align*}
\vec{r} \times \vec{F} &= 0 \\
\implies \vec{r} \times \vec{F} &= r \hat{r} \times F \hat{r} \\
&= r F \hat{r} \times \hat{r} \\
&= 0
\end{align*}
\]

Examples

1. Motion of earth around the sun, takes place under a force which is attractive and is always directed towards the sun.
2. Motion of planet round the sun.
3. Motion of electron about the nucleus in atom.

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The orbit of a particle under a central force is necessarily a plane curve.

**Proof**

Let \( F \) be a central force acting on a particle of mass \( m \).

Let origin \( O \) be the centre of force, then

\[
\vec{r} \times \vec{F} = 0
\]

(\( \vec{F} \) acts along some direction)

\[
\Rightarrow \vec{r} \times \frac{m \vec{v}}{dt} = 0
\]

\[
\Rightarrow \vec{r} \times \vec{a} = 0
\]

\[
\Rightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0
\]

Hence \( \vec{r} \times \vec{v} \) is a constant vector.

\( \Rightarrow \) The normal to the plane formed by \( \vec{r} \times \vec{v} \) has a constant direction.

\( \Rightarrow \) It is only possible when particle moves along a plane curve.

**3rd Statement of this**

"Motion under a Central Force is always in a Plane."

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**Magnitude of Angular Momentum \( \vec{H} \)**

We know \( \vec{v} = \vec{r} \dot{\vec{r}} + \vec{r} \dot{\vec{\theta}} \hat{\vec{s}}\) (radial + tangential comp.)

Momentum = \( m\hat{\vec{v}} \)

Angular Momentum

\[
\vec{H} = \vec{r} \times m \vec{\dot{v}}
\]

\[
H = m \vec{r} \times \vec{v}
\]

\[
H = m \vec{r} \times (\vec{r} \dot{\vec{r}} + \vec{r} \dot{\vec{\theta}} \hat{\vec{s}})
\]

\[
= m \vec{r} \times \vec{r} \dot{\vec{r}} + m \vec{r} \times \vec{r} \dot{\vec{\theta}} \hat{\vec{s}}
\]

\[
= m \vec{r} \times \dot{\vec{r}} + m \vec{r} \dot{\vec{\theta}} \times \hat{\vec{s}}
\]

\[
\vec{H} = 0 + m \vec{r} \dot{\vec{\theta}} \times \hat{\vec{s}}
\]

\[
H = |\vec{H}| = m |\vec{r} \times \dot{\vec{\theta}} | = 1
\]

\[
\frac{|\vec{H}|}{m} = \vec{r} \dot{\vec{\theta}}
\]

\[
\vec{H} = \frac{\vec{r} \dot{\vec{\theta}}}{\vec{r} \dot{\vec{\theta}}}
\]

Where \( \vec{h} = \frac{|\vec{H}|}{m} \) is angular momentum of particle of unit mass.

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The orbit described under a central attractive force varying directly as the distance is an **Ellipse** having centre at the
centre of force.

**Prop.** Let the plane $g$ of the orbit be $xy$-plane,
then \[ \vec{F} \propto -\vec{r} \]
\[ \vec{F} = -k \vec{r} \quad \text{--- 1} \]

\[ m \ddot{x} = -k \vec{r} \]
\[ m \ddot{y} = -k \vec{r} \]

\[ m(x \ddot{x} + y \ddot{y}) = -k \left( x \dot{x} + y \dot{y} \right) \]

\[ \ddot{x} = -k \dot{x} \]
\[ \ddot{y} = -k \dot{y} \]

\[ (k \dot{x}^2 + k \dot{y}^2) = 0 \]
\[ \ddot{x} + k \dot{x} = 0 \]
\[ \ddot{y} + k \dot{y} = 0 \]

\[ m \dot{x}^2 + k \dot{x} = 0 \]
\[ m \dot{y}^2 + k \dot{y} = 0 \]

\[ m = \pm k \dot{x} \]
\[ m = \pm k \dot{y} \]

**The General Sol of these**

\[ x = e^{(A \cos kt + B \sin kt)} \quad \text{--- 2} \]

\[ y = e^{(C \cos kt + D \sin kt)} \quad \text{--- 3} \]

\[ x \Delta y \Rightarrow C x = A \cos kt + B \sin kt \]

\[ x \Delta y \Rightarrow A y = C \cos kt + A \sin kt \]

\[ x \Delta d \Rightarrow \frac{C x - A y}{B C - A D} = \sin kt \quad \text{--- 4} \]

\[ x \Delta d \Rightarrow \frac{x d - b y}{A D - B C} = \cos kt \quad \text{--- 5} \]

**Squaring and Adding:**

\[ \left( \frac{x d - b y}{A D - B C} \right)^2 + \left( \frac{C x - A y}{B C - A D} \right)^2 = 1 \]

\[ (x d - b y)^2 = (A D - B C)^2 \]

\[ (C x - A y)^2 = (B C - A D)^2 \]

which is the equation of a central conic and is an **Ellipse**, $AD$ and $BC$ being determined by

\[ \begin{align*}
\text{initial conditions.}
\end{align*} \]
Prove that \( \vec{r} \cdot \vec{\Omega} = h = \text{const} \)

Proof:

Let \((r, \theta)\) be the position of the particle on the orbit. Let \(F\) be the central attractive force, then

\[
\begin{align*}
-\vec{F} &= m \vec{a} \\
&= m \left( \vec{r} - \vec{r}_{\text{eq}} \right) \vec{\hat{r}} + \left( 2 \vec{r} \cdot \vec{\Omega} + \vec{r}_{\text{eq}} \cdot \vec{\omega} \right) \vec{\hat{\Omega}} \\
&= m \left( \vec{r} - \vec{r}_{\text{eq}} \right) \vec{\hat{r}} + \frac{1}{h} \left( 2 \vec{r} \cdot \vec{\Omega} + \vec{r}_{\text{eq}} \cdot \vec{\omega} \right) \vec{\hat{\Omega}} \\
-\vec{F} &= m \left( \vec{r} - \vec{r}_{\text{eq}} \right) \vec{\hat{r}} + \frac{1}{h} \frac{d}{dt} \left( \vec{r} \cdot \vec{\Omega} \right) \vec{\hat{\Omega}}
\end{align*}
\]

\(-F_r \vec{\hat{r}} - F_\theta \vec{\hat{\Omega}} = m \left( \vec{r} - \vec{r}_{\text{eq}} \right) \vec{\hat{r}} + \frac{m}{h} \frac{d}{dt} \left( \vec{r} \cdot \vec{\Omega} \right) \vec{\hat{\Omega}}
\]

where \(F_r + F_\theta\) are components of central forces \(\vec{F}\), along and perpendicular to radius vector \(\vec{r}\), respectively.

Since \(\vec{F}\) is central attractive force directed towards \(O\), so \(\vec{F} = -m \vec{a}
\]

and the central force is always directed along radius vector \(\vec{r}\).

So \(F_\theta = 0\).

\[
\begin{align*}
\vec{F}_r &= m \left( \vec{r} - \vec{r}_{\text{eq}} \right) \\
-\vec{F}_\theta &= \frac{m}{h} \frac{d}{dt} \left( \vec{r} \cdot \vec{\Omega} \right)
\end{align*}
\]

\[
\begin{align*}
\int \vec{F}_r \, dr &= m \int \left( \vec{r} - \vec{r}_{\text{eq}} \right) \, dr \\
\int -\vec{F}_\theta \, d\theta &= \frac{m}{h} \int \frac{d}{dt} \left( \vec{r} \cdot \vec{\Omega} \right) \, d\theta
\end{align*}
\]

Integrating \(\vec{F}_r = \text{const} = h\) (say)

The Areal Speed of a particle moving under a central force is constant.

To support the radius vector \(\vec{OP}\) sweeps an area \(\Delta A\) in time \(\Delta t\),

\[
\Delta A = \text{area of } \triangle OPE
\]

\[
= \frac{1}{2} (OE)(PE) \\
= \frac{1}{2} (r_0 + r)(r_0 \sin \theta) \\
= \frac{1}{2} (r_0^2 \sin \theta + r_0 \cdot r \cdot \sin \theta)
\]

Taking limit as \(\Delta t \to 0\), \(\Delta r \to 0\), \(\Delta \theta \to 0\),

\[
\frac{\Delta A}{\Delta t} = \frac{1}{2} \left( r_0^2 \sin \theta + r_0 \cdot r \cdot \sin \theta \right)
\]

Areal Speed = \(\frac{1}{2} r_0 \sin \theta \)

and

\[
\Delta A = \frac{1}{2} h
\]

\[
\operatorname{Areal\, Speed} = \frac{1}{2} h
\]

\[
\therefore \vec{v} \cdot \vec{\Omega} = h
\]

\[
\frac{d}{dt} (\text{Areal Speed}) = \frac{1}{2} h
\]

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Derive the Differential Eq of the motion of particle of orbit under central force per unit mass.

Proof: Let \( \vec{F} \) be the attractive central force on the particle, then

\[
-F = ma
\]

\[
-(Fr \hat{r} + Fo \hat{\theta}) = m (\ddot{\hat{r}} - \hat{r} \dot{\theta}^2) \hat{r} + m (\dot{r} \dot{\theta} + \dot{\theta} \hat{r}) \hat{\theta}
\]

\[
-F \hat{r} = m (\ddot{r} - \hat{r} \dot{\theta}^2) \hat{r}
\]

\[
-F \hat{\theta} = m (\dot{r} \dot{\theta} + \dot{\theta} \hat{r}) \hat{\theta}
\]

\[
\text{Since } F \hat{r} \text{ is } \perp \text{ to radius vector, i.e. transverse component of } \vec{F}, \text{ so } F \hat{\theta} = 0
\]

\[
\text{and } \vec{F} \text{ is along the radius vector directed towards } O, \text{ so } -F = -(Fr \hat{r} + Fo \hat{\theta})
\]

\[
\therefore -F = m (\ddot{r} - \hat{r} \dot{\theta}^2)
\]

\[
= m (\ddot{\hat{r}} - \frac{\dot{\theta}^2}{\hat{r}^2})
\]

\[
\therefore \ddot{r} = \frac{\dot{\theta}^2}{\hat{r}^2} \quad \tag{1}
\]

Using (1)

\[
-F = m (\ddot{r} - \hat{r} \dot{\theta}^2)
\]

\[
\therefore \ddot{r} = \frac{-F}{m} = \frac{-F}{m} \hat{r}
\]

\[
\therefore \ddot{\theta} = \frac{\dot{F}}{m \hat{r}} = \frac{-\hat{r}}{m \hat{r}}
\]

Let \( u = \frac{1}{r} \Rightarrow r = \frac{1}{u} \)

\[
\frac{du}{dt} = \frac{\dot{r}}{r} = -\frac{1}{u} \frac{du}{dt}
\]

\[
= \frac{-1}{u} \frac{du}{dt} \frac{de}{dt} = -\dot{u} \frac{du}{dt}
\]

\[
\Rightarrow \dot{u} = \frac{du}{dt}
\]

\[
\frac{d^2 \hat{r}}{dt^2} = \frac{d}{dt} (\frac{-F}{m} \hat{r})
\]

\[
= -\frac{F}{m} \hat{r} \frac{d}{dt} \frac{de}{dt} = -F \hat{r} \dot{\theta}^2
\]

\[
= -F \hat{r} (\frac{\dot{\theta}^2}{\hat{r}^2}) \dot{\theta}^2
\]

\[
= -F \hat{r} \dot{\theta}^2 \frac{\dot{\theta}^2}{\hat{r}^2}
\]

\[
\therefore \ddot{u} = \frac{-1}{u^3} \hat{r}
\]

\[
\therefore \ddot{\theta} = -\frac{\dot{u}^2}{u^2}
\]
Derive the D^3^2 \rho = m \textit{ orbit in Pedal form}.

Let \( P(r, \theta) \) be a point on the orbit described by a pair of mass \( m \) moving under the central attractive force.

We know
\[
\frac{1}{\rho^2} = \frac{1}{r^2} + \frac{1}{r^2} \left( \frac{d\rho}{d\theta} \right)^2
\]
\[
\frac{1}{\rho^2} = u^2 + u \left( \frac{du}{d\theta} \right)^2
\]
\[
\frac{1}{\rho^2} = u^2 + \left( \frac{du}{d\theta} \right)^2
\]

\( \rho \) are called Pedal Coordinates

\( r = \rho \), \( \frac{dr}{d\theta} = -\frac{1}{u} \frac{du}{d\theta} \)

Differentiate:
\[
-\frac{2}{\rho^3} \frac{d\rho}{d\theta} = \frac{d}{d\theta} \left[ u^2 + \left( \frac{du}{d\theta} \right)^2 \right]
\]
\[
= \frac{d}{d\theta} \left[ u^2 + \left( \frac{du}{d\theta} \right)^2 \right] + \frac{d}{d\theta} \left( \frac{du}{d\theta} \right) \frac{du}{d\theta}
\]
\[
= 2u \frac{du}{d\theta} \left( u + \frac{du}{d\theta} \right) \frac{du}{d\theta}
\]

\[
\frac{1}{\rho^3} \frac{d\rho}{d\theta} = \frac{f}{h^2}
\]

Pedal eqn of motion of orbit \( f = \frac{E}{m} \)

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An apse is a pt on a central orbit at which the radius vector drawn from the centre of the force is max or min.

At $AA'$ and $BB'$ on the elliptic orbit the radius vector from centre $C$ of the force is minor or min.

**Apsidal Distance**

The length (magnitude) of radius vector at an apse is known as apsidal distance $CA + CA'$.

**Apsis Line**

The line joining an apse to the centre of force is called an apsis line. $AA'$, $BB'$ are apsis lines.

**Apsidal Angle**

The angle between two consecutive apsis lines is called an apsidal angle. Angle between $AA'$ and $BB'$ is $\frac{\pi}{2}$.

**Theorem**

Analytical Condition for a pt $P(r, \theta)$ to be an Apsis is

$$\frac{dr}{d\theta} = 0 \quad \text{or} \quad \frac{d\theta}{dr} = 0$$

OR

The radius vector is perpendicular to the tangent at an Apsis.

**Proof**

Let $\theta$ be the angle between radius vector and tangent at $P$.

Then since $\tan \theta = \frac{r}{dr/d\theta}$

but $dr/d\theta = 0 \Rightarrow \tan \theta = 0$

$\Rightarrow \theta = 0$ or $\pi$

$\Rightarrow \theta = \frac{\pi}{2}$

*Note:* For Apsis $r$ is Max or Min.

From calculus we know the radius vector $r$ is Max or Min if $dr/d\theta = 0$.

Also $r = \frac{1}{u} = \frac{1}{u^2}$

$\frac{dr}{d\theta} = 0$

$-\frac{1}{u^2} \frac{du}{d\theta} = 0 \Rightarrow \frac{du}{d\theta} = 0$ at $\theta = 0$ or $\pi$.

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Theorem: If a particle describes an ellipse under central force towards its centre, the orbit has

i) four apses

ii) two apse lines

iii) two apsidal distances

Proof: The eq of orbit, referred to $O$, the force centre as origin is

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

In polar coordinates

\[ r^2 \cos^2 \theta + r^2 \sin^2 \theta = a^2 \]

\[ r^2 \left( \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) = 1 \]

\[ 2r \frac{d}{dr} \left( \frac{r \cos \theta \sin \theta}{ \frac{a^2}{r^2} + \frac{b^2}{r^2}} \right) + \frac{2r^2 \sin 2 \theta}{ \frac{a^2}{r^2} + \frac{b^2}{r^2}} = 0 \]

For an apse $dr = 0$

\[ \therefore \quad r^2 \sin 2 \theta \left( \frac{1}{a^2} + \frac{1}{b^2} \right) = 0 \]

\[ \Rightarrow \quad \sin 2 \theta = 0 \]

\[ \Rightarrow \quad 2 \theta = 0, \pi, 2\pi, 3\pi \]

\[ \Rightarrow \quad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \]

\[ \therefore \quad \text{So the apses are } A, A', B, B' \text{ which are extrema of major and minor axes} \]

\[ \text{ii) Apses lines are } OA, OA', OB \text{ and } OB' \text{ i.e } AA' \perp BB' \]

\[ \text{iii) Apsidal distance } |OA| = |OA'| = a \]

\[ |OB| = |OB'| = b \quad \text{two apsidal distances} \]

\[ \text{iv) Apsidal Angles are } \angle AOB = \angle BOA' = \angle A'OB = \angle BOA = \frac{\pi}{2} \]

So are apsidal angles, i.e \(90^\circ\).

Note: If a particle describes an ellipse under an attractive central force directed to one of its foci, then there are only two apses, \(AA'\)

\[ \text{two apsidal distances and only one apsee line } AA' \text{ and apsidal angle is } \pi. \]
Show that the orbit described by the planet around the sun is a Conic.

or

**Polar Eq of the Orbit**

**Proof** We consider the motion of the planet round the sun and the force is governed by Newton's Law of Gravitation.

If $M$ and $m$ are the mass of Sun and the planet then they attract each other with a force $\frac{GMM}{r^2}$, where $G$ is constant gravitation.

Take the sun as the pole, the D.Eq of the orbit is

$$\mu^2 (h^2 \dot{u}) \left( u + \frac{d^2 u}{d\theta^2} \right) = \frac{GMm}{r^2}$$

$$h^2 \dot{u}^2 (u + \frac{d^2 u}{d\theta^2}) = \mu u^2$$

$$u + \frac{d^2 u}{d\theta^2} = \frac{h}{h^2}$$

$$\frac{d^2 u}{d\theta^2} = 0$$

$$\frac{d^2 (u-H)}{d\theta^2} + u - \frac{H}{h^2} = 0$$

$$D^2 + 1 \left( u - \frac{H}{h^2} \right) = 0$$

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

**General Sol is**

$$u = \frac{H}{h^2} + C \cos(\theta - \theta_0)$$

where $C, \theta_0$ are constants of integration.

So $u = \frac{H}{h^2} + C \cos\theta$$

from (2) \( \frac{1}{h^2} = \frac{H}{h^2} \left( 1 + \frac{C}{h^2} \cos\theta \right) \)

(3) is of the form \( \frac{1}{h^2} = 1 + e \cos\theta \) which is polar Eq of Conic.

So the orbit is a conic with focus at the centre of the force and semi latus rectum \( \ell = \frac{h}{\mu} \)

eccentricity \( e = \frac{\ell}{h} \)

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Eq of the orbit of a planet with sun at the focus (central force)

terms of total energy.

So E = \frac{\mu}{2h^2} + c \cos \phi \quad (1)

Total energy = K.E + P.E
\[ E = T + V \quad (2) \]
where \( T = \text{K.E per unit mass} \)
\( V = \text{P.E per unit mass} \)

\[ V = -\int f(r) \, dr \]
\[ = -\int -\frac{\mu}{r^2} \, dr \]
\[ V = -\frac{\mu}{r} \quad \text{ending} \]

\[ T = \frac{1}{2} \dot{r}^2 \]
\[ T = \frac{1}{2} (\dot{r}^2 + \dot{\phi}^2) \]

So \[ E = \frac{1}{2} (\dot{r}^2 + \dot{\phi}^2) + \left( -\frac{\mu}{r} \right) \]

\[ = \frac{1}{2} \left( \frac{d}{dt} \left( \frac{\mu}{r} \right) + \frac{\mu}{r} \left( \frac{d}{dt} r \right) \right) - \mu \frac{d}{dt} r \]
\[ = \frac{1}{2} \left( \frac{d^2 \mu}{dt^2} + \frac{\mu}{r^3} \right) - \mu \frac{d}{dt} r \]
\[ = \frac{1}{2} \left( 2 \mu \sin \phi \left( -\sin \phi + \left( \frac{\mu}{h^2} + \mu \cos \phi \right) \right) - \mu \left( \frac{\mu}{h^2} + \mu \cos \phi \right) \right) \]
\[ = \frac{1}{2} \left( \frac{\mu}{h^2} \sin \phi + \mu \cos \phi + \mu \cos \phi - \mu \cos \phi \right) \]
\[ = \frac{1}{2} \left( \frac{\mu}{h^2} \sin \phi + \mu \cos \phi + \mu \cos \phi - \mu \cos \phi \right) \]
\[ = \frac{1}{2} \left( \frac{\mu}{h^2} \sin \phi + \mu \cos \phi + \mu \cos \phi - \mu \cos \phi \right) \]
\[ = \frac{1}{2} \left( \frac{\mu}{h^2} \sin \phi + \mu \cos \phi + \mu \cos \phi - \mu \cos \phi \right) \]
\[ = \frac{1}{2} \left( \frac{\mu}{h^2} \sin \phi + \mu \cos \phi + \mu \cos \phi - \mu \cos \phi \right) \]
\[ E = \frac{1}{2} \left( \frac{\mu}{h^2} \sin \phi + \mu \cos \phi + \mu \cos \phi - \mu \cos \phi \right) \]
\[ E + \frac{\mu}{2h^2} = \frac{1}{2} \left( \frac{\mu}{h^2} \sin \phi + \mu \cos \phi + \mu \cos \phi - \mu \cos \phi \right) \]
\[ \Rightarrow \frac{\mu}{h^2} \left( \frac{\mu}{h^2} + \frac{\mu}{h^2} \right) \cos \phi \]
\[ \Rightarrow \cos \phi = \frac{1}{\mu} \left( \frac{\mu}{h^2} \left( \frac{2E}{h^2} + \frac{\mu}{h^2} \right) \right) \]

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\[ C = \frac{\mu}{h^2} \sqrt{1 + \frac{2Eh^2}{\mu^2}} \]  
\[ U = \frac{\mu}{h^2} \left[ 1 + \frac{2Eh^2}{\mu^2} \right] \cos \theta \]
\[ \frac{1}{\ell} = \frac{\mu}{h^2} \left[ 1 + \frac{2Eh^2}{\mu^2} \right] \cos \theta \]

Polar Eqg Conic \[ \frac{1}{\ell} = 1 + e \cos \theta \]

\[ \text{Compare with } \frac{1}{\ell} = \frac{1}{\mu} (1 + e \cos \theta) \]

\[ \frac{\mu}{h^2} = \frac{1}{\mu} \Rightarrow \frac{1}{\mu} = \frac{\mu}{h^2} \]

\[ C = \sqrt{1 + \frac{2Eh^2}{\mu^2}} \]

\[ I = \frac{2\pi a}{h} \]

\[ \text{Show that the time taken by a particle to describe the whole ellipse is} \]
\[ I = \frac{2\pi a}{h} \]

\[ \text{Area of ellipse} \]
\[ A = \pi ab \]

\[ I = \frac{2\pi ab}{h} \]

\[ \text{Instantaneous} \]
\[ I = \frac{2\pi a}{\mu} \]

\[ \text{Instantaneous} \]
\[ I = \frac{2\pi a}{\mu} \left( \frac{\pi a^2}{2\mu} \right) \]

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Derivation of Newton's Law of Gravitation from Kepler's.

By the first law of Kepler, "each planet describes an ellipse with the sun at one focus," if we take the sun at the focus (origin), the eq of orbit can be written as

\[ \frac{r}{r_0} = 1 + e \cos \theta \]

\[ \frac{d\theta}{dE} = -e \sin \theta \]

\[ \frac{d\theta^2}{dE} = -e \cos \theta \]

\[ \frac{d^2\theta}{dE^2} = 1 - \frac{1}{r} \]

The Eq of orbit is

\[ f(u) = \frac{h^2 u^2}{2} + \left( \frac{h}{r} \right)^2 \]

\[ = \frac{h^2 u^2}{2} \left( \frac{1}{r} \right) \]

\[ = \frac{h^2}{r^2} \]

\[ f(u) = \frac{H^2}{r^2} \]

where \( H = \frac{h}{r} \)

For a planet, the force varies inversely as square of the distance from sun, which is according to Newton's Law of Gravitation.

\[ \frac{1}{r} \]

Note. For an ellipse \( \frac{h^2}{b^2} = \frac{H^2}{a^2} \)

Also \( H = \frac{h}{r} \)

\[ \frac{c^2}{a^2} = 1 - \frac{b^2}{a^2} \]

\( \frac{h}{r} = \frac{c^2}{a^2} \)

\[ \frac{e}{a} = \frac{c}{a} \]

\[ c^2 = a^2 (1 - e^2) \]

\[ \frac{c}{a} = \frac{1}{a} \]

\[ \frac{c}{\sqrt{a^2 - c^2}} = \frac{a}{\sqrt{a^2 - c^2}} \]

\[ \frac{c}{\sqrt{a^2 - c^2}} = \frac{a}{\sqrt{a^2 - c^2}} = \frac{a}{\sqrt{a^2 - c^2}} \]

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Exercise

(i) A particle describes the following curves under force $F$ to the pole, show that the force is as stated.

\[
\begin{align*}
\frac{a}{c} &= e^n \\
au &= e^n \\
\frac{d^2u}{dx^2} &= ne^n \\
\frac{d^3u}{dx^3} &= ne^n
\end{align*}
\]

\[f = h^n \frac{d^2u}{dx^2} \text{ Egg orbit}\]

\[= h^n \left( \frac{e^n}{a^n} \right) \text{ Egg orbit}\]

\[= h^n \frac{e^n}{a^n} \text{ Egg orbit}\]

\[= h^n \frac{e^n}{a^n} (1 + n^2)\]

\[= h^n \frac{e^n}{a^n} \left( \frac{a^2}{r^2} (1 + n^2) \right)\]

\[= h^n \frac{e^n}{a^n} \left( \frac{a^2}{r^3} (1 + n^2) \right)\]

\[f \propto \frac{1}{r^3}\]

(ii) $\frac{a}{c} = n^2$

\[au = n^2 \]

\[\frac{d^2u}{dx^2} = n \]

\[\frac{d^3u}{dx^3} = 0 \]

\[f = h^n \frac{d^2u}{dx^2} \text{ Egg orbit}\]

\[= h^n \frac{e^n}{a^n} \text{ Egg orbit}\]

\[= h^n \frac{e^n}{a^n} \text{ Egg orbit}\]

\[f \propto \frac{1}{r^3}\]
(iii) \[ \frac{a}{r} = \cosh n \theta \]

\[ a_u = \cosh n \theta \quad \text{(1)} \]

\[ \frac{a}{d \theta} = n \sinh n \theta \quad \text{(2)} \]

\[ \frac{a d u}{d \theta} = n^2 \cosh n \theta \]

\[ \frac{a d^2 u}{d \theta^2} = n^2 a \quad \text{(3)} \]

\[ \frac{d^2 u}{d \theta^2} = n^2 u \quad \text{(4)} \]

\[ f = \frac{h^2 u^2 (u + d_u)}{d \theta^2} \]

\[ = \frac{h^2 u^2 (u + n^2 u)}{d \theta^2} \quad \text{(5)} \]

\[ = \frac{h^2 u^2 (1 + n^2)}{d \theta^2} \]

\[ = \frac{h^2 (1 + n^2)}{n^2} \]

\[ f \propto \frac{1}{n^2} \]

(iv) \[ \frac{a}{r} = \sin n \theta \]

\[ a_u = \sin n \theta \quad \text{(6)} \]

\[ \frac{a}{d \theta} = n \cos n \theta \quad \text{(7)} \]

\[ \frac{a d u}{d \theta} = -n \sin n \theta \]

\[ \frac{a d^2 u}{d \theta^2} = -n^2 \sin n \theta \quad \text{(8)} \]

\[ \frac{a d^2 u}{d \theta^2} = -n^2 a \quad \text{(9)} \]

\[ \frac{d^2 u}{d \theta^2} = -n^2 u \quad \text{(10)} \]

\[ f = \frac{h^2 u^2 (u + d_u)}{d \theta^2} \]

\[ = \frac{h^2 u^2 (u - u_n)}{d \theta^2} \]

\[ = \frac{h^2 u^2 (1 - n^2)}{d \theta^2} \]

\[ = \frac{h^2 (1 - n^2)}{n^2} \]

\[ f \propto \frac{1}{n^2} \]

Available at www.mathcity.org
\[ x^2 \cos \theta = a^n \]

Show that for \( n \neq 3 \):

\[ n a^n = \frac{a^n}{\cos \theta} \]

\[ u^n = \cos \theta \]

\[ u a^n = \cos \theta \]

\[ \frac{du}{dx} \]

\[ \frac{u^n}{n-1} \frac{du}{dx} = -\sin \theta (n) \]

\[ a u^n \frac{du}{dx} = -u \sin \theta \times b u \]

\[ \frac{d(u^n)}{dx} = -u \sin \theta \]

\[ \frac{du}{dx} = -u \tan \theta \]

\[ \frac{d(u^n)}{dx} = \frac{u \tan \theta + U \sec \theta}{u \tan \theta + U \sec \theta} \]

\[ \frac{d(u^n)}{dx} = \frac{u \tan \theta - U \sec \theta}{u \tan \theta - U \sec \theta} \]

\[ \frac{d(u^n)}{dx} = \frac{U \sec \theta (1-n)}{U \sec \theta (1-n)} \]

\[ f = \frac{h^2 u^2 \left( u + \frac{du}{dx} \right)}{dx^2} \]

\[ f = \frac{h^2 u^2 \left( U \sec \theta (1-n) \right)}{dx^2} \]

\[ f = \frac{h^2 u^2 \sec \theta (1-n)}{dx^2} \]

\[ f = \frac{h^2 u^2 \left( \frac{1}{a \tan} \right)^2 (1-n)}{dx^2} \]

\[ f = \frac{h^2 u^2 \tan ^2 (1-n)}{dx^2} \]

\[ f \propto \frac{1}{a^n \tan ^2} \]

Available at

www.mathcity.org
Find the Law of force for the following orbit, the pole being

\[ r^2 = a \cos \lambda \]
\[ u = \frac{1}{a^2} \sec \lambda \]

\[ 2 \frac{du}{d\lambda} = \frac{1}{a^2} \sec \lambda \tan \lambda \tan \lambda (i) \]
\[ u \frac{du}{d\lambda} = U \sec \lambda \tan \lambda \]
\[ \frac{du}{d\lambda} = U \tan \lambda \]  

\[ \frac{d^2u}{d\lambda^2} = \frac{2U \sec \lambda + U \sec \lambda \tan \lambda}{d\lambda} 
\]
\[ = \frac{2U \sec \lambda + U \sec \lambda \tan \lambda}{d\lambda} \]
\[ = \frac{2U \sec \lambda + U (\sec \lambda - 1)}{d\lambda} \]
\[ \frac{d^2u}{d\lambda^2} = 2 \frac{U \sec \lambda + U \sec \lambda - U}{d\lambda} \]

\[ u + \frac{du}{d\lambda} = 3U \sec \lambda \]  

\[ f = h^2 \left( U + \frac{du}{d\lambda} \right) \]
\[ = h^2 \left( 3U \sec \lambda \right) \]
\[ = h^2 \left( 3U \left( \sec \lambda \right)^2 \right) \]  

\[ f = \frac{3h^2 \sec \lambda^2}{r^2} \]

---

**Example**

\[ \frac{d^2u}{d\lambda^2} = \frac{U \sec \lambda (n) + \frac{du}{d\lambda} \tan \lambda}{d\lambda} \]
\[ = U \sec \lambda + U \tan \lambda \]  

\[ = U \sec \lambda + U \sec \lambda \]  

\[ = U \sec \lambda + U \sec \lambda - U \]

\[ U + \frac{du}{d\lambda} = U \sec \lambda (n+1) \]

\[ f = h^2 \left( U + \frac{du}{d\lambda} \right) \]
\[ = h^2 \left( U \sec \lambda (n+1) \right) \]
\[ = h^2 \sec \lambda (n+1) \]
\[ = h^2 \sec \lambda (n+1) \]  

\[ f = \frac{h^2 \sec \lambda (n+1)}{r^2} \]

---

**Note:**

\[ r^2 = a \cos \theta \]
\[ u = \frac{1}{a^2} \sec \theta \]
\[ a \frac{du}{d\theta} = \sec \theta \tan \theta \]  

\[ a \frac{d^2u}{d\theta^2} = USec \theta \tan \theta \]

\[ \frac{du}{d\theta} = U \sec \theta \]

\[ \frac{d^2u}{d\theta^2} = U \sec \theta \tan \theta \]

\[ \frac{d\lambda}{d\theta} = U \sec \theta \]  

\[ \frac{d\lambda}{d\lambda} = U \sec \theta \]  

\[ \frac{d\lambda}{d\theta} = U \sec \theta \]
Example 12.14

\( r^n = A \cos \theta + B \sin \theta \)

Show that for \( r \geq 3 \)

Put

\( A = R \cos \alpha \)

\( B = R \sin \alpha \)

\[ r^n = R \cos \alpha \cos \theta + R \sin \alpha \sin \theta \]

\[ r^n = R \cos (\theta - \alpha) \]

\[ u^n = \frac{1}{R} \sec (\theta - \alpha) \quad \text{(1)} \]

\[ \frac{du}{d\theta} = \frac{1}{R} \sec (\theta - \alpha) \tan (\theta - \alpha) \gamma \]

\[ \frac{u^n}{du} = \frac{u \sec (\theta - \alpha) \tan (\theta - \alpha)}{R} \quad \text{by (1)} \]

\[ \frac{du}{d\alpha} = \frac{u \sec (\theta - \alpha) \tan (\theta - \alpha)}{R \sec (\theta - \alpha)} \quad \text{by (1)} \]

\[ \frac{du}{d\alpha} = \frac{u \tan (\theta - \alpha)}{R} \quad \text{(2)} \]

\[ \frac{d^2u}{d\alpha^2} = \frac{du}{d\alpha} \frac{\tan (\theta - \alpha)}{R} + \frac{u \sec (\theta - \alpha) \tan (\theta - \alpha)}{R} \]

\[ = \frac{u \tan (\theta - \alpha)}{R} + \frac{u \sec (\theta - \alpha) \tan (\theta - \alpha)}{R} \quad \text{by (2)} \]

\[ = u \left[ \sec (\theta - \alpha) - 1 + \sec (\theta - \alpha) \right] \]

\[ = u \sec (\theta - \alpha) - u + u \sec (\theta - \alpha) \]

\[ u + \frac{du}{d\alpha^2} = u \sec (\theta - \alpha) \left[ 1 + n \right] \quad \text{(3)} \]

\[ f = \frac{h^n}{n} \left( u + \frac{du}{d\alpha^2} \right) \]

\[ = \frac{h^n}{n} \left( u \sec (\theta - \alpha) \left[ 1 + n \right] \right) \]

\[ = \frac{h^n}{n} \left( RU^n \right) \left[ 1 + n \right] \quad \text{by (3)} \]

\[ = \frac{h^n}{n} U^{n+3} R \left[ 1 + n \right] \]

\[ f = \frac{h^n R \left[ 1 + n \right]}{n^m} \]

\[ \star \frac{1}{m^3} \]

\[ \star \frac{h^n R \left[ 1 + n \right]}{n^m} \quad \text{is constant.} \]

Available at

www.mathcity.org
A particle of unit mass describes an ellipse under the central force $F = \frac{2}{r^3}$. Show that the normal component of acceleration at any instant is $\frac{2bM}{v}$, where $v$ is the velocity at that instant, and $a, b$ are the semi-axes of the ellipse.

\[ f = \frac{M}{r} + \frac{1}{2}\frac{\text{d}u^2}{\text{d}t^2} \]
\[ \frac{\text{d}u}{\text{d}t} = \frac{M}{r} \]
\[ \frac{\text{d}u}{\text{d}t} = \frac{M}{U^3} \]
\[ \frac{\text{d}u}{\text{d}t} = \frac{M}{U^3} \]
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]

Integrating,
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]

We know the pedal eq of ellipse is
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]

Comparing (1) & (2)
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]

Also $\frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t}$
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]
\[ \frac{\text{d}u^2}{\text{d}t^2} = \frac{2M}{U^3} \frac{\text{d}u}{\text{d}t} \]

Normal component of Acc $= \frac{2bM}{v}$
\[ \text{Normal component of Acc} = \frac{2bM}{v} \]

For Ellipsee Radius of Curvature $\rho^* = \frac{ab}{v^2}$
\[ \rho^* = \frac{ab}{v^2} \]

Hence $\rho = \frac{ab}{v^2}$
\[ \rho = \frac{ab}{v^2} \]
\[ \rho = \frac{ab}{v^2} \]
\[ \rho = \frac{ab}{v^2} \]

Available at www.mathcity.org
If a particle is describing an ellipse about a centre of force in the centre, show that the sum of the reciprocal of its angular velocities about the foci is constant.

Let at any time 'T' position of particle be 'P' describing an ellipse under central force through the centre, and 'V' be the velocity of particle 'P' along the tangent then

\[ V_P = h \] where \( h \) is length \( OQ \) from centre to tangent PT

\[ \angle QPS = \angle QPS' = \phi \] since we know that tangent at 'P' is equally inclined with focal distance.

\[ \omega_1, \omega_2 \] are angular velocities of the particle at 'P' 'P' about goes S & S then

\[ \omega_1 = \frac{V \sin \phi}{SP} \quad \omega_2 = \frac{V \sin \phi}{SP} \]

\[ \frac{1}{\omega_1} + \frac{1}{\omega_2} = \frac{SP + SP}{V \sin \phi} \] (according to \( \theta \))

\[ \omega = \frac{2 a}{V \sin \phi} \quad \text{(Sum of ellipse sum \( \omega \))}

\[ \omega = \frac{2 a}{V \sin \phi} \quad \text{(From ellipses sum \( \omega \))}

\[ \omega = \frac{2 a^2}{VP} \quad \text{(constant)} \] provided.
A particle of mass $m$ moves under the central force $mM(3au - 2(a-b)u)$ and is projected from an apoapsis at a distance $a+b$ with velocity $\sqrt{\frac{\mu}{2a}}$. Show that the orbit is $r = a + b + \frac{\mu}{2a}$. 

Set $P'$ as the distance from centre $C$ to orbit to the apoapsis, then $P = r = a + b$ (given) — (i)

and $v = \frac{\sqrt{\mu}}{a+b}$ (given) — (ii)

since $h = \sqrt{\mu}$

Putting (ii) $h = \sqrt{\mu}$ \[ F = mM(3au - 2(a-b)u) \]

a) $\frac{r}{d\theta} = M(3au - 2(a-b)u)$

b) $\frac{2d}{d\theta} = 3au(2du) - (a-b)3u^2(2du) - U(2du) \times \text{by 2du}$

c) $\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}(2au^3) - (a-b)^2\frac{d}{d\theta}(u^4) - \frac{d}{d\theta}(u^4)$

This gives $\frac{d}{d\theta} = 2au^3 - (a-b)^2u^4 + A$ — (v)

Since apoapsis $r = a+b$

$u = \frac{r}{a+b} \Rightarrow \frac{du}{d\theta} = 0$

Putting in (iv)

$0 = 2a(a+b) - (a-b)^2\left(\frac{1}{a+b}\right)^4 - \left(\frac{1}{a+b}\right)^2 + A$

$0 = 2a(a+b) - (a-b)^2 - (a+b)^3 + A$

$0 = 2a^2 + 2ab - a^2 + b^2 - a^2 - 2ab + A$

$0 = A$ Put in (v)
\[
\begin{align*}
(d^2 u/\partial \theta^2) &= 2a u^3 - (a^2 - b^2) u^2 - u^1 + \sigma \\
(d u/\partial \theta) &= u^1 (2au - (a^2 - b^2) u^2) \\
(-d^2 r/\partial \theta^2) &= \frac{r^4}{n^4} \left[ 2a \left( \frac{r}{n} \right)^2 - (a^2 - b^2) \left( \frac{r}{n} \right)^2 \right] - 1 \\
(d r/\partial \theta) &= \frac{r^4}{n^4} \left( \frac{2a}{n} - \left( \frac{a^2 - b^2}{n^2} \right) \right) - 1 \\
&= 2ar - (a^2 - b^2) n^2 \\
&= 2ar - a^2 + b^2 - n^2 \\
(d s^2/\partial \theta^2) &= b^2 - (n^2 + a^2 - 2ar) \\
(d s/\partial \theta) &= b^2 - (n - a)^2 \\
\frac{dr}{d\theta} &= \pm \sqrt{b^2 - (n - a)^2} \\
\text{Separating variables,} \\
\int \frac{dr}{\sqrt{b^2 - (n - a)^2}} &= \pm \int d\theta \\
-\cos^{-1} \left( \frac{n-a}{b} \right) &= \pm \theta + \beta \\
\text{At points} \ r = a + b, \ \theta = 0 \\
\Rightarrow -\cos^{-1} \left( \frac{a+b-a}{b} \right) &= 0 + \beta \\
\Rightarrow \beta &= 0 \\
\therefore -\cos^{-1} \left( \frac{n-a}{b} \right) &= \pm \theta + \beta \\
\cos^{-1} \left( \frac{n-a}{b} \right) &= \pm \theta \\
\frac{n-a}{b} &= \cos \left( \pm \theta \right) \\
\frac{n-a}{b} &= \cos \theta \\
\frac{n-a}{b} &= b \cos \theta \\
(n-a) &= b \cos \theta \\
(r &= a + b \cos \theta)}
\end{align*}
\]
The Law of Force is \( M \ddot{u} \) and a particle is projected from a gun.

(a) Find the orbit when the velocity of projection is \( \sqrt{\frac{M}{a^2}} \).

We know that the Law of Force is

\[
\dot{f}(u) = \frac{h}{u^2} (u + \frac{d}{u^2})
\]

\[
\ddot{M} = \frac{h}{u^2} \left( u + \frac{d}{u^2} \right) \quad : f(u) = \ddot{M} \sin \frac{\pi}{2}
\]

\[
\ddot{M} = \left( \frac{M}{a^2} \right) \left( u + \frac{d}{u^2} \right)
\]

\[
\ddot{u} = \frac{1}{a^2} \left( u + \frac{d}{u^2} \right)
\]

\[
\frac{du}{dt} = \left( u + \frac{d}{u^2} \right)
\]

\[
\frac{du}{dt} = \left( \frac{1}{a^2} \right) \left( \frac{u}{u^2} \right)
\]

\[
\frac{du}{dt} = \frac{1}{a^2} \left( \frac{u}{u^2} \right)
\]

\[
\frac{du}{dt} = \frac{1}{a^2} \ln \left( \frac{au - 1}{au + 1} \right)
\]

\[
\frac{1}{a^2} \left[ 2 \left( \frac{1}{a} \right) \ln \left( \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \right) \right] = \frac{\theta}{a^2} + B
\]

Available at

www.mathcity.org
\[ \text{A particle moves under a central repulsive force } \frac{1}{r^3} \text{ and is projected from an apse at a distance } a \text{ with velocity } v. \text{ Show that eq. path is } r \cos \theta = a \text{ and angle described in time } t \text{ is} \]
\[ \frac{t}{p} \tan^{-1} \left( \frac{rV}{a^2} \right) \quad \text{where } p = \frac{\mu + a^2 V^2}{a^2 V^2} \]

Since force is repulsive so
\[ h^2 (u + u^2) = -f(u) \quad \text{(negative sign due to repulsive force)} \]
\[ h^2 u (u + u^2) = -\frac{\mu}{a^2} \]
\[ h^2 u^2 (u + u^2) = -\mu u \]
\[ \frac{dV}{dt} (u + u^2) = -\mu \]
\[ u + u^2 = -\mu \frac{dV}{dt} \]
\[ \frac{dV}{dt} = -\mu \frac{u - u^2}{a^2 V^2} \]
\[ \frac{dV}{dt} = -\mu \frac{u(1 + \frac{V}{a})}{a^2 V^2} \]
\[ \frac{dV}{dt} = -\mu \frac{u}{a^2 V^2} \quad \text{where } p = \mu \frac{V}{a^2 V^2} \]
\[ \frac{dV}{dt} + u^2 = 0 \]
\[ (\frac{du}{dt})^2 + u^2 = A \quad \text{by } 2 \frac{du}{dt} \]
\[ \frac{du}{dt} + u^2 = A \]
\[ \frac{du}{dt} = \frac{p^2 - u^2}{a^2} \]
\[ \frac{du}{dt} = \pm \sqrt{\frac{p^2 - u^2}{a^2}} - u^2 \]

\[ \frac{du}{dt} = \pm p \frac{\sqrt{\frac{p^2}{a^2}} - u^2}{a} \]

\[ \text{Available at www.mathcity.org} \]
Two particles describe in equal times, the arc of a Parabola bounded by the latus rectum, one under an attraction to the focus and the other with constant acceleration g, parallel to the axis. Show that the acceleration of the first particle at the vertex of the parabola is \(\frac{16g}{9}\).

Set

Eq of Parabola in Polar form is

\[
\frac{r}{c} = 1 + \cos \theta
\]

where c is semi latus rectum = 2a

Time taken by a particle to go from \(L\) to \(L\)

\[t = 2 \times \text{Time taken from } V \text{ to } L\]

Let the force is attractive towards focus \(S\).

\[
\frac{h}{r} = \frac{\pi}{2} \cos \theta
\]

\[
\int h \, dt = \int \frac{\pi}{2} \cos \theta \, dt
\]

\[
h = \int \frac{\pi}{2} \cos \theta \, dt
\]

\[
= \frac{\pi}{4} \left[ \sec \theta + \tan \theta \right] \left. \right|_0^\pi
\]

\[
= \frac{\pi}{4} \left[ \sec \theta + \tan \theta \right]_0^\pi
\]

\[
= \frac{\pi}{4} \left[ \sec \theta + \tan \theta \right]_0^\pi
\]

\[
= \frac{\pi}{4} \left[ \frac{1}{2} + \frac{3}{2} \right]
\]

\[
h = \frac{\pi}{4} (8)
\]

\[
th = \frac{\pi}{4} \left( \frac{8}{3} \right) = 2 \frac{\pi}{3}
\]

\[
\Rightarrow t = \frac{2 \pi}{3}\]
Thus time from $V$ to $L = \frac{2e^t}{3h}$

So total time from $V$ to $L = 2\left(\frac{2e^t}{3h}\right) = \frac{4e^t}{3h}$

$= \frac{4(2a)^2}{3h} = \frac{16a^2}{3h}$ ——(1)

Now we find time from $L$ to $L$ along the arc under constant acceleration $g$, to the axis of parabola i.e. $y$ -axis (vertically downwards). Now time from $V$ to $L$ along the arc is same as time from $V$ to $S$ under gravity, vertical downward.

$s = ut + \frac{1}{2}gt^2$ (at $V$, initial velocity $u=0$ and distance $s = \sqrt{3a}$)

$a = ct + \frac{1}{2}gt^2$

$\Rightarrow \frac{2a}{g} = t^2$

$\Rightarrow t = \sqrt{\frac{2a}{g}}$

Total time required from $L$ to $L$ is $2\left(\frac{2a}{g}\right)$ ——(2)

from(1)$\Rightarrow \frac{16a^2}{3h} = 2\left(\frac{2a}{g}\right) \Rightarrow h = \frac{8a^2}{3g} = \frac{8a^2}{3} \Rightarrow h = \frac{8a^2}{2a}$ ——(3)

Now $f$ is the acceleration $g$, 1st particle towards focus $S,$

then $f = h^2\left(\frac{d^2u}{dt^2} + u\right)$

$f = \frac{16a^2}{3g} u^2 \left(\frac{d^2u}{dt^2} + u\right)$ ——(3)

$= \frac{64a^2}{3g} \frac{u}{2a} u \left(\frac{d^2u}{dt^2} + u\right)$

$= \frac{32a^3}{g} \frac{u}{2a} u \left(\frac{d^2u}{dt^2} + u\right)$ ——(4)

Now orbiting particle is $l = 1 + \cos\theta$

$\Rightarrow \frac{l}{\sin\theta} \frac{dU}{dt} = 1 + \cos\theta$

$\Rightarrow \frac{dU}{dt} = -\sin\theta$ ——(3)

Using

$\frac{d^2U}{dt^2} = -\cos\theta - \sin\theta$

$\Rightarrow \frac{l}{(1+\frac{d^2u}{dt^2})} = 1 + \cos\theta - \sin\theta$

$\Rightarrow \frac{U + \frac{d^2u}{dt^2}}{l} = 1$

$\Rightarrow \frac{U}{l} + \frac{d^2u}{dt^2} = \frac{l}{2a}$ ——(5)

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A planet is deriving an ellipse about the sun as focus. Show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is \( \frac{2\pi a e}{T(1 - e^2)} \) where 2a is the major axis, e the eccentricity and T is the periodic time.

A particle of mass \( m \) describes an elliptic orbit about an attracting force centre situated at one focus. The force is that of inverse square law. If \( e \) is the eccentricity, \( T \) the time period, \( 2a \) the major axis, show that the greatest radial velocity of the particle is \( \frac{2\pi a e}{T(1 - e^2)} \).

Since the orbit is an ellipse with force centre at one focus, its eq in polar form is

\[ r = \frac{h}{1 + e \cos \theta} \]  

Differentiating, \( \frac{dr}{d\theta} = \frac{dr}{d\theta} \cdot \frac{d\theta}{d\theta} = \frac{h e \sin \theta}{(1 + e \cos \theta)^2} \)

\[ x = \frac{re \cos \theta}{h} = \frac{h e \sin \theta \cdot \theta}{h} = \frac{e \sin \theta}{1 + e \cos \theta} \]

\[ y = \frac{re \sin \theta}{h} = \frac{h e \sin \theta \cdot \theta}{h} = \frac{e \sin \theta}{1 + e \cos \theta} \]

\[ \frac{dr}{dt} = \frac{h e \sin \theta}{1 + e \cos \theta} \]

\[ \frac{d\theta}{dt} \] will be Max if \( \sin \theta \) is Maxi.e. equal to 1, i.e. \( \theta = \frac{\pi}{2} \)

Hence

\[ \frac{dr}{dt} \mid_{\max} = \frac{h e \sin \theta}{1 + e \cos \theta} \]  \[ \frac{d\theta}{dt} \mid_{\max} = \frac{h e \sin \theta}{1 + e \cos \theta} \]

Now we know

\[ \frac{\pi a}{T} = \frac{h}{a(1 - e)} \]

and

\[ \frac{2\pi a e}{T(1 - e^2)} \]

\[ \sqrt{h^2 + e^2} \]

Hence

\[ \frac{dr}{dt} \mid_{\max} = \frac{2\pi a e}{T(1 - e^2)} \]
Example: Prove that the speed at any point of a central orbit is given by

\[ v = \sqrt{\frac{1}{\mu} - \frac{1}{r} - r \theta^2} \]

where \( r \) is the radial speed and \( \theta \) is the perpendicular distance from the force of the tangent at that point.

Hence find the expression for \( v \) when a particle is subject to inverse square law of force, describing (i) Elliptic, (ii) Parabolic and (iii) Hyperbolic orbit.

\[ \frac{dr}{dt} = -\frac{1}{u} \frac{du}{dt} \frac{d\theta}{dt} \]

and \( h = r \theta \).

Then:

\[ v^2 = \left( \frac{1}{u} - \frac{1}{r} \right) \theta^2 + r \theta^2 \]

\[ = \left( \frac{1}{u} \frac{du}{dt} \right)^2 + r \theta^2 \]

\[ = \left( \frac{1}{u} \frac{du}{dt} + \frac{r}{u} \right) \theta^2 \]

\[ = \left( \frac{du}{dt} + \frac{r}{u} \right) \frac{1}{u} \theta^2 \]

\[ \therefore h = \theta \]

\[ v^2 = \left( \frac{1}{r} \right) \frac{1}{u} \theta^2 \]

\[ \sqrt{v^2} = \frac{\theta}{u} \]

\[ \sqrt{\theta} = \frac{h}{u} \]

Newton's law of gravitation from Kepler's law

\[ f = \frac{H}{r^2} = \mu \frac{u^2}{r} \]

but \( f = h \frac{u}{r} (u + \frac{du}{dt}) \) (eq. 3 orbit)

\[ \therefore \frac{u}{v} = h \frac{u}{r} (u + \frac{du}{dt}) \]

\[ v = h \frac{u}{r} (u + \frac{du}{dt}) \]

\[ H = h \frac{u}{r} (u + \frac{du}{dt}) \]

\[ \mu \frac{du}{dt} = h \left( \frac{2u du}{dt} + \frac{2u u}{dt} \right) \]

\[ \mu 2u = h \left( \frac{u^2}{v} + \frac{du}{dt} \right) \]
\[ \frac{p}{p_i} = 2\frac{\mu}{\alpha} + c \]

\[ \frac{b_i}{p_i} = \frac{2\alpha}{\pi} - 1 \]  \(\text{Elliptic} \)  \(\text{P1} \)

\[ \frac{p^2}{a^2} = c \]  \(\text{Parabolic} \)  \(\text{P2} \)

\[ \frac{b_i}{p_i} = \frac{2\alpha}{\pi} + 1 \]  \(\text{Hyperbolic} \)  \(\text{P3} \)

**Comparison (P1) to (P2):**

\[ \frac{b_i}{p_i} = \frac{2\alpha}{\pi} = \frac{c}{-1} \]

\[ \frac{h}{b_i} = \frac{h}{a} = \frac{c}{-1} \]

\[ h = \frac{2\alpha^2}{\pi} \]  \(\text{Ellipse in elliptic orbit} \)

**Comparison (P2) to (P3):**

\[ \frac{b_i}{p_i} = \frac{2\alpha}{ar^2} = \frac{c}{0} \]

\[ \frac{h}{p^2} = \frac{2\alpha}{ar^2} = \frac{c}{0} \]

\[ h = \frac{2\alpha^2}{\pi} \]  \(\text{Parabola in parabolic orbit} \)

**Comparison (P1) to (P3):**

\[ h = p\sqrt{\frac{2\mu}{\alpha}} \]

\[ \frac{c}{a} = \frac{2\mu}{\alpha} \]  \(\text{Hyperbola in hyperbolic orbit} \)

Since the orbit will be an Elliptic, Parabolic, Hyperbolic according to \( c \leq 0 \)