## Partial differentiation (ch – 9)

Partial derivative

let $z = f(x,y)$

i.e., $z$ is a function of two independent variables $x$ & $y$ then its partial derivative w.r.t. $x$ is calculated as

$$z = f(x,y)$$

$$\Rightarrow z + \delta z = f(x+\delta x, y) \qquad (y \text{ is kept constt.})$$

or $$\delta z = f(x+\delta x, y) - f(x,y)$$

Dividing both sides by $\delta x$

$$\frac{\delta z}{\delta x} = \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

taking limit as $\delta x \to 0$ on both sides

$$\lim_{\delta x \to 0} \frac{\delta z}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

If this limit exists as a unique & definite quantity, then it is called partial derivative of $z$ w.r.t. $x$ & is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x(x,y)$

Similarly the partial derivative of $z = f(x,y)$ w.r.t. $y$ keeping $x$ constt is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_y(x,y)$.

**Ex** If $z = x^3 + 7x^2y + 8xy^2 + y^3$

find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

**Sol:** Given function is

$$z = x^3 + 7x^2y + 8xy^2 + y^3$$

then $\frac{\partial z}{\partial x} = 3x^2 + 14xy + 8y^2$

$$\frac{\partial z}{\partial y} = 7x^2 + 16xy + 3y^2$$

**Homogeneous function:**

A function $f(x,y)$ is said to be a homogeneous function of degree $n$ if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

**Ex** Show that the function $\frac{\sqrt{x}+\sqrt{y}}{x+y}$ is a homogeneous function of degree $-\frac{1}{2}$.

**Sol.**

Let $f(x,y) = \frac{\sqrt{x}+\sqrt{y}}{x+y}$

Put $x = \lambda x$ & $y = \lambda y$

$$f(\lambda x, \lambda y) = \frac{\sqrt{\lambda x}+\sqrt{\lambda y}}{\lambda x + \lambda y}$$

$$= \frac{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})}{\lambda(x+y)}$$

$$= \lambda^{-1/2}\left(\frac{\sqrt{x}+\sqrt{y}}{x+y}\right)$$

$$f(\lambda x, \lambda y) = \lambda^{-1/2} f(x,y)$$

So $f(x,y) = \frac{\sqrt{x}+\sqrt{y}}{x+y}$ is a homogeneous function of degree $-\frac{1}{2}$.

Ex Show that the function $x\sin(y/x) + y\cos\left(\frac{x}{y}\right)$ is a homogeneous function of degree 1.

Sol.

Let $f(x,y) = x\sin(y/x) + y\cos(x/y)$

Put $x = \lambda x$, $y = \lambda y$

So $f(\lambda x,\lambda y) = \lambda x\sin\left(\frac{\lambda y}{\lambda x}\right) + \lambda y\cos\left(\frac{\lambda x}{\lambda y}\right)$

$= \lambda\left[x\sin(y/x) + y\cos(x/y)\right]$

$f(\lambda x,\lambda y) = \lambda f(x,y)$

So $f(x,y) = x\sin(y/x) + y\cos(x/y)$ is a homogeneous function of degree 1.

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Note If a function $f(x,y)$ is homogeneous of degree $n$ then we know by def.

$f(\lambda x,\lambda y) = \lambda^n f(x,y)$

Put $\lambda = \frac{1}{x}$

$f\left(1, \frac{1}{x}\cdot y\right) = \left(\frac{1}{x}\right)^n f(x,y)$

$f(1, y/x) = \frac{1}{x^n} f(x,y)$

$f(y/x) = \frac{1}{x^n} f(x,y)$

$\Rightarrow f(x,y) = x^n f(y/x)$

Hence if $f(x,y)$ is a homogeneous function of degree $n$ then it can be written as

$f(x,y) = x^n f(y/x)$

Euler's theorem:

Statement: If $U = f(x,y)$ is a homogeneous function of degree $n$ then

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = nu$$

Proof: Since $U = f(x,y)$ is a homogeneous function of degree $n$, so it can be written as

$$U = f(x,y) = x^n f(y/x)$$

$$\text{or} \quad U = x^n f(y/x) \quad \text{---} \quad (1)$$

Diff. (1) partially w.r.t. $x$

$$\text{Now} \quad \frac{\partial U}{\partial x} = x^n f'(y/x) \cdot -\frac{y}{x^2} + f(y/x) \cdot n x^{n-1}$$

$$\text{or} \quad \frac{\partial U}{\partial x} = -x^{n-2} y f'(y/x) + n x^{n-1} f(y/x)$$

Multiplying both sides by $x$.

$$x \frac{\partial U}{\partial x} = -x^{n-1} y f'(y/x) + n x^n f(y/x) \quad \text{---} \quad (2)$$

Now diff. (1) partially w.r.t. $y$

$$\frac{\partial U}{\partial y} = x^n f'(y/x) \cdot \frac{1}{x}$$

$$\text{or} \quad \frac{\partial U}{\partial y} = x^{n-1} f'(y/x) \quad \text{---}$$

Multiplying both sides by $y$,

$$y \frac{\partial U}{\partial y} = x^{n-1} y f'(y/x) \quad \text{---} \quad (3)$$

Adding (2) & (3)

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = n x^n f(y/x).$$

$$\text{or} \quad x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = nu$$

## Exercise No. 9.1

Q1 verify Euler's theorem for

(a) $U = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

(b) $U = x^n \ln\left(\frac{y}{x}\right)$

(c) $U = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Sol.

(a) Given function is

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

It is a homogeneous function of degree zero, then we have to verify Euler's theorem i.e., we have to prove

$$x\frac{\partial U}{\partial x} + y\cdot\frac{\partial U}{\partial y} = 0.$$

Now $U = \sin^{-1}(x/y) + \tan^{-1}(y/x)$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2}$$

$$= \frac{1}{\frac{\sqrt{y^2-x^2}}{y}}\cdot\frac{1}{y} + \frac{x^2}{x^2+y^2}\cdot\frac{-y}{x^2}$$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\Rightarrow x\frac{\partial U}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

Now $$\frac{\partial U}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot -\frac{x}{y^2} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}$$

$$= -\frac{1}{\frac{\sqrt{y^2-x^2}}{y}}\cdot\frac{x}{y^2} + \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

Now

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

which is req. verification.

(b) Sol. Given function is

$$u = x^n \ln\left(\frac{y}{x}\right)$$

It is a homogeneous function of degree $n$ then we have to verify Euler's theorem i.e.,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Now $u = x^n \ln(y/x)$

$$\frac{\partial u}{\partial x} = x^n \cdot \frac{1}{\frac{y}{x}} \cdot -\frac{y}{x^2} + \ln(y/x)\cdot nx^{n-1}$$

$$= -\frac{x^n \cdot x \cdot y}{y \cdot x^2} + \ln(y/x)\cdot nx^{n-1}$$

$$\frac{\partial u}{\partial x} = -x^{n-1} + nx^{n-1}\ln(y/x)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = -x^n + nx^n \ln(y/x)$$

Now $\frac{\partial u}{\partial y} = x^n \cdot \frac{1}{\frac{y}{x}} \cdot \frac{1}{x}$

$$\frac{\partial u}{\partial y} = \frac{x^n}{y}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = x^n$$

Now $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n x^n \ln(y/x)$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

viz req. verification of Euler's theorem.

(c) Sol. Given function is

$$u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

$$= \frac{x^{1/4}\left[1 + \frac{y^{1/4}}{x^{1/4}}\right]}{x^{1/5}\left[1 + \frac{y^{1/5}}{x^{1/5}}\right]}$$

$$= x^{\frac{1}{4}-\frac{1}{5}}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

$$u = x^{\frac{1}{20}}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

Which shows that $u$ is a homogeneous function of degree $\frac{1}{20}$ then we have to verify Euler's theorem

i.e., $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}u$

Now $u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$x\frac{\partial u}{\partial x} = \frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

Now $$\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}y^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{(\frac{1}{4}y^{1/4})(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

Now

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{\frac{1}{4}x^{1/4}(x^{\frac{1}{5}}+y^{\frac{1}{5}}) - \frac{1}{5}x^{\frac{1}{5}}(x^{\frac{1}{4}}+y^{1/4}) + \frac{1}{4}y^{1/4}(x^{\frac{1}{5}}+y^{\frac{1}{5}}) - \frac{1}{5}y^{\frac{1}{5}}(x^{\frac{1}{4}}+y^{\frac{1}{4}})}{(x^{\frac{1}{5}}+y^{\frac{1}{5}})^2}$$

$$= \frac{(x^{\frac{1}{5}}+y^{\frac{1}{5}})(\frac{1}{4}x^{\frac{1}{4}}+\frac{1}{4}y^{\frac{1}{4}}) - (x^{\frac{1}{4}}+y^{1/4})(\frac{1}{5}x^{\frac{1}{5}}+\frac{1}{5}y^{1/5})}{(x^{\frac{1}{5}}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}(x^{\frac{1}{5}}+y^{\frac{1}{5}})(x^{\frac{1}{4}}+y^{\frac{1}{4}}) - \frac{1}{5}(x^{\frac{1}{4}}+y^{1/4})(x^{\frac{1}{5}}+y^{1/5})}{(x^{\frac{1}{5}}+y^{1/5})^2}$$

$$= \frac{(x^{\frac{1}{4}}+y^{1/4})\cancel{(x^{1/5}+y^{1/5})}\left\{\frac{1}{4}-\frac{1}{5}\right\}}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\left(\frac{5-4}{20}\right)$$

$$= \frac{1}{20}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{20}u$$

viz req. verification of Euler's theorem.

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Q2. If $u = f(y/x)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$

Sol. Given function is

$$u = f(y/x)$$

Now $\frac{\partial u}{\partial x} = f'(y/x) \cdot -\frac{y}{x^2}$

$$\Rightarrow x\frac{\partial u}{\partial x} = -\frac{y}{x}f'(y/x)$$

$$\frac{\partial u}{\partial y} = f'(y/x)\cdot\frac{1}{x}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{x}f'(y/x)$$

Now $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y}{x}f'(y/x) + \frac{y}{x}f'(y/x)$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

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Q3 If $u = xyf(x/y)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$.

Sol. Given function is

$$u = xyf(x/y)$$

$$\frac{\partial u}{\partial x} = y\cdot\frac{\partial}{\partial x}\left(xf(x/y)\right)$$

$$\frac{\partial u}{\partial x} = y\left[xf'(x/y)\cdot\frac{1}{y} + f(x/y)\cdot 1\right]$$

$$\frac{\partial u}{\partial x} = xf'(x/y) + yf(x/y)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = x^2f'(x/y) + xyf(x/y)$$

Now $\frac{\partial u}{\partial y} = x\cdot\frac{\partial}{\partial y}\left(yf(x/y)\right)$

$$= x\left[yf'(x/y)\cdot -\frac{x}{y^2} + f(x/y)\cdot 1\right]$$

$$\frac{\partial u}{\partial y} = -\frac{x^2}{y}f'(x/y) + xf(x/y)$$

$$\Rightarrow y\frac{\partial u}{\partial y} = -x^2f'(x/y) + xyf(x/y)$$

Now $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xyf(x/y)$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

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**Q4** If $z = \tan^{-1}(y/x)$, verify that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

**Sol.** Given function is

$$z = \tan^{-1}(y/x)$$

Diff. partially w.r.t. $x$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot -\frac{y}{x^2}$$

$$= -\frac{x^2}{x^2+y^2} \cdot \frac{y}{x^2}$$

$$\frac{\partial z}{\partial x} = -\frac{y}{x^2+y^2}$$

Now

$$\frac{\partial^2 z}{\partial x^2} = -y \cdot \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= -y \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2x$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \quad \text{———} (1)$$

Now $$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$$

Now $$\frac{\partial^2 z}{\partial y^2} = x \cdot \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$= x \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2} \quad \text{———} (2)$$

Adding (1) & (2)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

<u>Q5</u> If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that

$$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

<u>Sol.</u> Given function is

$$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

This is not a homogeneous function but can be reduced to homogeneous by putting

$$z = \sin u = \frac{x^2+y^2}{x+y}$$

Here $z$ is a homogeneous function of degree 1, then by Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad \text{———} (1)$$

Here $z = \sin u$

$$\Rightarrow \frac{\partial z}{\partial x} = \cos u.\frac{\partial u}{\partial x}$$

$$\& \quad \frac{\partial z}{\partial y} = \cos u.\frac{\partial u}{\partial y}$$

Put in (1)

$$x.\cos u.\frac{\partial u}{\partial x} + y.\cos u.\frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

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<u>Q6</u> If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 0$$



<u>Sol.</u> Given function is

$$u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$$

This is not a homogeneous function but can be reduced to homogeneous by putting

$$z = \sin u = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

Here $z$ is a homogeneous function of degree 0 then by Euler's theorem

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0 \quad \text{———} \quad (1)$$

Here $z = \sin u$

$$\frac{\partial z}{\partial x} = \cos u \cdot \frac{\partial u}{\partial x}$$

& $$\frac{\partial z}{\partial y} = \cos u \cdot \frac{\partial u}{\partial y}$$

Put in (1)

$$x \cdot \cos u \cdot \frac{\partial u}{\partial x} + y \cdot \cos u \cdot \frac{\partial u}{\partial y} = 0$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

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Q7 If $u = \ln\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1$

Sol: Given function is

$$u = \ln\left(\frac{x^2+y^2}{x+y}\right)$$

This is not a homogeneous function but can be reduced to homogeneous by putting

$$z = e^u = \frac{x^2+y^2}{x+y}$$

Here $z$ is a homogeneous function of degree 1 then by Euler's theorem

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z \quad \text{———} \quad (1)$$

Here $z = e^u$

$$\frac{\partial z}{\partial x} = e^u \cdot \frac{\partial u}{\partial x}$$

⋮

& $\frac{\partial z}{\partial y} = e^{u} \cdot \frac{\partial u}{\partial y}$

Put in ①

$$x \cdot e^{u} \cdot \frac{\partial u}{\partial x} + y \cdot e^{u} \cdot \frac{\partial u}{\partial y} = e^{u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 1$$

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Q8 If $u = f(x, y)$ is a homogeneous function of degree $n$, prove that

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1) f$$

Sol:- Since $u = f(x, y)$ is a homogeneous function of degree $n$ then by Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

or $x f_x + y f_y = nf$ ——— ①

Diff. ① partially w.r.t. $x$ & $y$

$x f_{xx} + f_x + y f_{yx} = n f_x$ ——— ②

& $x f_{xy} + y f_{yy} + f_y = n f_y$ ——— ③

Multiplying ② by $x$ & ③ by $y$ & adding

$$x(x f_{xx} + f_x + y f_{yx}) + y(x f_{xy} + y f_{yy} + f_y) = nx f_x + ny f_y$$

$$x^2 f_{xx} + x f_x + xy f_{yx} + xy f_{xy} + y^2 f_{yy} + y f_y = nx f_x + ny f_y$$

Assume that $f_{xy} = f_{yx}$

So $x^2 f_{xx} + y^2 f_{yy} + 2xy f_{xy} = nx f_x - x f_x + ny f_y - y f_y$

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = (n-1) x f_x + (n-1) y f_y$$

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = (n-1)[x f_x + y f_y]$$

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$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = (n-1).nf$$

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f$$

**Q9** If $u = f(r)$ where $r = \sqrt{x^2+y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

**Sol.** Here $r = \sqrt{x^2+y^2}$

$$\Rightarrow r^2 = x^2+y^2$$

Diff. partially w.r.t. $x$

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$



Now as $u = f(r)$

Diff. partially w.r.t. $x$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$$

Now $$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{r.1 - x.\frac{\partial r}{\partial x}}{r^2}\right] + \frac{x}{r} \cdot f''(r) \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \cdot \left[\frac{r - \frac{x^2}{r}}{r^2}\right] + \frac{x}{r} \cdot f''(r) \cdot \frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{r^2 - x^2}{r^3}\right] + \frac{x^2}{r^2} f''(r) \quad \text{———} \quad (1)$$

Similarly by symmetry we have

$$\frac{\partial^2 u}{\partial y^2} = f'(r) \cdot \left[\frac{r^2 - y^2}{r^3}\right] + \frac{y^2}{r^2} f''(r) \quad \text{———} \quad (2)$$

Adding (1) & (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}\left[r^2 - x^2 + r^2 - y^2\right] + \frac{f''(r)}{r^2}(x^2+y^2)$$

$$= \frac{f'(r)}{r^3}\left[2r^2 - (x^2+y^2)\right] + \frac{f''(r)}{r^2} \cdot r^2$$

$$= \frac{f'(r)}{r^3}(2r^2 - r^2) + f''(r)$$

$$= \frac{f'(r)}{r^3} \cdot r^2 + f''(r)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

**Q10** If $V = \rho^m$, where $\rho^2 = x^2+y^2+z^2$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)\rho^{m-2}$$

**Sol.** As $\rho^2 = x^2+y^2+z^2$

Diff. partially w.r.t. $x$

$$2\rho \cdot \frac{\partial \rho}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial \rho}{\partial x} = \frac{x}{\rho}$$

Similarly $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$

& $\frac{\partial \rho}{\partial z} = \frac{z}{\rho}$

Now as $V = \rho^m$

Diff. partially w.r.t. $x$

$$\frac{\partial V}{\partial x} = m\rho^{m-1} \cdot \frac{\partial \rho}{\partial x}$$

$$= m\rho^{m-1} \cdot \frac{x}{\rho}$$

$$\frac{\partial V}{\partial x} = m x \rho^{m-2}$$

1520 (16)

$$\frac{\partial^2 v}{\partial x^2} = m\left[x \cdot (m-2)\rho^{m-3}\cdot\frac{\partial \rho}{\partial x} + \rho^{m-2}\cdot 1\right]$$

$$= m\left[x(m-2)\cdot\rho^{m-3}\cdot\frac{x}{\rho} + \rho^{m-2}\right]$$

$$\frac{\partial^2 v}{\partial x^2} = m(m-2)x^2\rho^{m-4} + m\rho^{m-2}$$

Similarly by symmetry we have

$$\frac{\partial^2 v}{\partial y^2} = m(m-2)y^2\rho^{m-4} + m\rho^{m-2}$$

$$\& \quad \frac{\partial^2 v}{\partial z^2} = m(m-2)z^2\rho^{m-4} + m\rho^{m-2}$$

Now

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} = m(m-2)\rho^{m-4}(x^2+y^2+z^2) + 3m\rho^{m-2}$$

$$= m(m-2)\rho^{m-4}\cdot\rho^2 + 3m\rho^{m-2}$$

$$= m(m-2)\rho^{m-2} + 3m\rho^{m-2}$$

$$= m\rho^{m-2}(m-2+3)$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} = m(m+1)\rho^{m-2}$$

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<u>Differential of a function</u>:

Let $u = f(x,y)$ be a differentiable function of two variables $x$ & $y$ then differential or total differential of $u$ is defined as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

where $dx = \Delta x$

& $dy = \Delta y$ are differentials of $x$ & $y$.

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Similarly if $w = f(x,y,z)$ then

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$