Analytic geometry of three dimensions

(Chapter No. 8)

Coordinate System:
We select a point O for the origin, \( x, y, z \) as the directed distances along \( x \)-axis, \( y \)-axis, and \( z \)-axis respectively. Then we define coordinates of \( P(x, y, z) \).

Distance b/w two points in space

Let \( P(x_1, y_1, z_1) \) and \( Q(x_2, y_2, z_2) \) be two points in space then the distance b/w the points \( P(x_1, y_1, z_1) \) and \( Q(x_2, y_2, z_2) \)

\[ |PQ| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2} \]

Point dividing a line seg. \( t \) in a given ratio

The coordinates of a point \( P(x, y, z) \) dividing a line segment in the ratio \( m:n \) are

\[ P\left(\frac{mx_1 + nx_2}{m+n}, \frac{my_1 + ny_2}{m+n}, \frac{mz_1 + nz_2}{m+n}\right) \]

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8.1.2

P and Q are the opposite vertices of a parallelepiped having its faces parallel to the coord. planes.
Find the co-ords. of the other vertices & sketch the parallelepiped.

Q1:  \[ P(-1,1,2) \rightarrow Q(2,3,5) \]

5.12 Complete the parallelepiped with faces parallel to the coord. planes & PA as a diagonal.
The co-ords. of the other vertices are as
\[ A(2,1,2) \rightarrow B(-1,3,2) \]
\[ C(-1,1,5) \rightarrow R(2,2,2) \]
\[ S(-1,3,5) \rightarrow T(2,1,5) \]

Q2:  \[ P(2,-1,3) \rightarrow Q(4,0,-1) \]

5.12 Complete the parallelepiped with faces parallel to the coord. planes & PA as a diagonal.
The co-ords. of the other vertices are
\[ A(4,-1,3) \rightarrow B(2,0,-3) \]
\[ C(2,1,3) \rightarrow R(4,0,-3) \]
\[ S(2,0,-1) \rightarrow T(4,-1,1) \]

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Q3. \( P(2, 5, -3), Q(-4, 2, 1) \)

Solve: Complete the parallelogram with face \( II \) to the
coordinate planes \( \triangle PA \) as a diagonal.
Then the coordinates of the remaining
vertices are:

\( A(-4, 5, -3), B(2, 2, -3), C(2, 5, 1) \)

\( R(-4, 2, -3), S(2, 2, 1), T(-4, 5, 1) \).  

Show that the three given points are either the
vertices of a triangle or the vertices of an
isosceles \( \triangle \) or both.

Q4. \( A(1, 5, 0), B(6, 6, 4), C(0, 9, 5) \)

Solve: Consider a \( \triangle ABC \) with vertices as given pts.

Now,

\[ |AB| = \sqrt{(1-6)^2 + (5-6)^2 + (0-4)^2} \]
\[ = \sqrt{25 + 1 + 16} \]
\[ = \sqrt{42} \]

\[ |BC| = \sqrt{(6-0)^2 + (9-6)^2 + (5-4)^2} \]
\[ = \sqrt{36 + 9 + 1} \]
\[ = \sqrt{46} \]

\[ |AC| = \sqrt{(1-0)^2 + (5-9)^2 + (0-5)^2} \]
(AC) = \sqrt{1^2 + 4^2 + 2^2} = \sqrt{1+16+4} = \sqrt{21}

Since |AB| = |AC|

Hence \( \triangle ABC \) is isosceles.

Q5 \( A(4, 9, 4), B(0, 11, 2), C(1, 0, 2) \)

Sub: Consider \( \triangle ABC \) with vertices as given pts.

Now

\[ |AB| = \sqrt{(0-4)^2+(11-9)^2+(2-4)^2} = \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6} \]

\[ |BC| = \sqrt{(1-0)^2+(0-11)^2+(1-2)^2} = \sqrt{1+121+1} = \sqrt{123} \]

\[ |AC| = \sqrt{(1-4)^2+(0-9)^2+(1-4)^2} = \sqrt{9+81+9} = \sqrt{99} = 3\sqrt{11} \]

Now \( |AB|^2 + |AC|^2 = 24 + 99 = 123 = (\sqrt{123})^2 = |BC|^2 \)

Thus \( \triangle ABC \) is a right triangle with right angle at A.
Consider a \( \triangle ABC \) with vertices as given pts.

Now

\[ |AB| = \sqrt{(4-1)^2 + (3-0)^2 + (2-2)^2} \]
\[ = \sqrt{9 + 9 + 0} \]
\[ = \sqrt{18} \]
\[ = 3\sqrt{2} \]

\[ |BC| = \sqrt{(0-4)^2 + (7-3)^2 + (6-2)^2} \]
\[ = \sqrt{16 + 16 + 16} \]
\[ = \sqrt{48} \]
\[ = 4\sqrt{3} \]

\[ |AC| = \sqrt{(1-0)^2 + (0-7)^2 + (2-6)^2} \]
\[ = \sqrt{1 + 49 + 16} \]
\[ = \sqrt{66} \]

Now \[ |AB|^2 + |BC|^2 = 18 + 48 \]
\[ = 66 \]
\[ = (\sqrt{66})^2 \]
\[ = |AC|^2 \]

Hence \( \triangle ABC \) is a right triangle with right angle at B.

\[ \overline{GT} \quad A(1,0,2), \; B(4,3,2), \; C(6,7,6) \]
\[ \overline{Sol.} \quad \text{A}(2,3,4), \; \text{B}(-3,-1,2), \; \text{C}(-4,1,0) \]
Consider a \( \triangle ABC \) with vertices as given pts. 

New:

\[
|AB| = \sqrt{(8-3)^2 + (-1-3)^2 + (2-1)^2} \\
= \sqrt{36 + 16 + 1} \\
= \sqrt{53}
\]

\[
|BC| = \sqrt{(-4-8)^2 + (1+1)^2 + (0-2)^2} \\
= \sqrt{144 + 4 + 4} \\
= \sqrt{152}
\]

\[
|AC| = \sqrt{(2+4)^2 + (3-1)^2 + (4-0)^2} \\
= \sqrt{36 + 4 + 16} \\
= \sqrt{56}
\]

Since \( |AB| = |AC| \)

Hence given \( \triangle ABC \) is an isosceles \( \triangle \).

Q8. Show that the pts. \((1,3,1)\), \((1,3,1)\), \((4,3,1)\) & \((0,2,0)\)
are the vertices of a regular tetrahedron.

Sol:-

Given pts. are \( A = (1,3,1) \), \( B = (1,3,1) \), \( C = (4,3,1) \) & \( D = (0,2,0) \)

To show that given pts. are vertices of a regular tetrahedron, we have
to show that \( |AB| = |AC| = |AD| = |BC| = |CD| = |BD| \).
\[ |AB| = \sqrt{(1-1)^2 + (3-6)^2 + (4-1)^2} \]
\[ = \sqrt{0 + 9 + 9} \]
\[ = \sqrt{18} \]
\[ = 3\sqrt{2} \]

\[ |AC| = \sqrt{(4-1)^2 + (3-6)^2 + (1-1)^2} \]
\[ = \sqrt{9 + 9 + 0} \]
\[ = \sqrt{18} \]
\[ = 3\sqrt{2} \]

\[ |AD| = \sqrt{(0-1)^2 + (2-6)^2 + (0-1)^2} \]
\[ = \sqrt{1 + 16 + 1} \]
\[ = \sqrt{19} \]
\[ = 3\sqrt{2} \]

\[ |BC| = \sqrt{(4-1)^2 + (3-3)^2 + (1-4)^2} \]
\[ = \sqrt{9 + 0 + 9} \]
\[ = \sqrt{18} \]
\[ = 3\sqrt{2} \]

\[ |CD| = \sqrt{(0-4)^2 + (2-3)^2 + (0-1)^2} \]
\[ = \sqrt{16 + 1 + 1} \]
\[ = \sqrt{18} \]
\[ = 3\sqrt{2} \]
\[ |BD| = \sqrt{(0-1)^2 + (2-3)^2 + (0-4)^2} \]
\[ = \sqrt{1 + 1 + 16} \]
\[ = \sqrt{18} \]
\[ = 3\sqrt{2} \]

Since \(|AB| = |AC| = |AD| = |BC| = |CD| = |BD|\)

Hence the given pts. are the vertices of a regular tetrahedron.

Q9 Show that the pts. \((3,-1,3)\), \((1,1,2)\), \((2,1,0)\) and \((4,1,1)\) are the vertices of a rectangle.

Sub.
Suppose given pts. are
\[ A = (3,-1,3) \], \[ B = (1,1,2) \], \[ C = (2,1,0) \], \[ D = (4,1,1) \]

The given pts. will form a rectangle if
\[ |AB| = |CD| \quad \text{and} \quad |AC| = |AD| \]
\[ \angle A = 90^\circ \]

Now
\[ |AB| = \sqrt{(1-3)^2 + (-1+1)^2 + (2-3)^2} \]
\[ = \sqrt{4+0+1} \]
\[ = \sqrt{5} \]
\[ |CD| = \sqrt{(4-2)^2 + (1-1)^2 + (1-0)^2} \]
\[ = \sqrt{4+0+1} = \sqrt{5} \]
Now
\[ |BC| = \sqrt{(2-1)^2 + (1+1)^2 + (1-2)^2} \]
\[ = \sqrt{1+4+4} \]
\[ = \sqrt{9} \]
\[ = 3 \]

\[ + |AD| = \sqrt{(4-3)^2 + (1+1)^2 + (1-3)^2} \]
\[ = \sqrt{1+4+4} \]
\[ = \sqrt{9} \]
\[ = 3 \]

Hence \[ |AB| = |CD| \] and \[ |BC| = |AD| \]

Now we place \[ \angle A = 90^\circ \]

Consider
\[ |AB|^2 + |AD|^2 = 5 + 9 \]
\[ = 14 \]

Since \[ |BD| = \sqrt{(4-1)^2 + (1+1)^2 + (1-2)^2} \]
\[ = \sqrt{9+4+1} \]
\[ = \sqrt{14} \]

So \[ |BD|^2 = 14 \]

Put in above eqn
\[ |AB|^2 + |AD|^2 = |BD|^2 \]

So \[ \angle A = 90^\circ \]. Hence given pts are vertices of a rectangle.
Q10 under what conditions on \( x, y, z \) is the pt. \( P(x, y, z) \) equidistant from the pts. \((-1, 3, 4)\) and \((-1, 5, 0)\) ?

Sol. Suppose the given pts. are \( A(-1, 3, 4) \) and \( B(-1, 5, 0) \)

Then \( P(x, y, z) \) is any pt.
According to given condition

\[ |PA| = |PB| \]

\[ \Rightarrow \sqrt{(x+1)^2 + (y-3)^2 + (z-4)^2} = \sqrt{(x+1)^2 + (y-5)^2 + (z-0)^2} \]

Squaring both sides

\[ (x+1)^2 + (y-3)^2 + (z-4)^2 = (x+1)^2 + (y-5)^2 + (z-0)^2 \]

\[ x^2 + 2x + 9 + y^2 - 6y + 9 + z^2 - 8z + 16 = x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 \]

\[ -6x + 2y - 8z + 26 = -2x - 10y + 26 \]

\[ -6x + 2y - 8z = 2x + 10y \]

\[ -8x + 12y - 8z = 0 \]

Dividing both sides by -4

\[ 2x - 3y + 2z = 0 \] in the req. condition.

All Find the Co-ords. of the pt. dividing the join of the pts. \((-3, 1, 4)\) and \((5, -1, 1)\) in the ratio 3:5.

Sol.
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Set: Given pts. are \( A(-3,1,4) \) & \( B(5,-1,6) \).

Let \( P(x,y,z) \) be the req. pt. dividing the line \( AB \) in ratio \( 3:5 \).

We know that the co-ords. of the pt. dividing the join of \((x_1,y_1,z_1) \) & \((x_2,y_2,z_2)\) in the ratio \( m_1:m_2 \) are

\[
\left( \frac{m_2x_1 + m_1x_2}{m_1+m_2}, \frac{m_2y_1 + m_1y_2}{m_1+m_2}, \frac{m_2z_1 + m_1z_2}{m_1+m_2} \right)
\]

Hence, co-ords. of pt. \( P \) are

\[
P\left( \frac{3(-3)+5 \cdot 5}{3+5}, \frac{3(1)+5 \cdot 1}{3+5}, \frac{3(4)+5 \cdot 6}{3+5} \right)
\]

\[
=P\left( \frac{-9+25}{8}, \frac{3+5}{8}, \frac{12+30}{8} \right)
\]

\[
=P\left( \frac{16}{8}, \frac{8}{8}, \frac{42}{8} \right)
\]

Thus, \( P(0,1,\frac{21}{4}) \).

Ans: Find the ratio in which \( yz\)-plane divides the segment joining the pts. \((-2,4,7) \) & \((3,-5,8)\).

Set: Given pts. are \( A(-2,4,7) \) & \( B(3,-5,8) \).

Let the \( yz\)-plane divides, the join of the given pts. in the ratio \( m_1:m_2 \).

Now the \( x\)-co-ord. of the pt. dividing the join of given pts. is

\[
x = \frac{3m_1 + (-2)m_2}{m_1 + m_2} = \frac{3m_1 - 2m_2}{m_1 + m_2}
\]
Since the pt. lies on yz-plane
So \( x = 0 \)

\[
\Rightarrow \frac{3m_1 - 2m_2}{m_1 + m_2} = 0
\]

\[
\Rightarrow 3m_1 - 2m_2 = 0
\]

\[
3m_1 = 2m_2 \quad \text{or} \quad m_1 : m_2 = 2 : 3 \quad \text{is req. ratio}
\]

Q13. Show that the Centroid of the triangle whose vertices are \((x_i, y_i, z_i), i = 1, 2, 3\) is

\[
\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)
\]

Solu.

Let the given vertices of the triangle are A\((x_1, y_1, z_1)\), B\((x_2, y_2, z_2)\), and C\((x_3, y_3, z_3)\).

Suppose D, E, F are the mid-
pnts. of the sides BC, AC, and
AB respectively.

Now Co-ords. of D are

\[
D\left( \frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2} \right)
\]

Suppose G is the Centroid of \(\triangle ABC\), then Co-ords.
of pt. G dividing AD in ratio 2:1 are

\[
G\left( \frac{1 \cdot x_1 + 2 \left( \frac{x_2 + x_3}{2} \right)}{1+2}, \frac{1 \cdot y_1 + 2 \left( \frac{y_2 + y_3}{2} \right)}{1+2}, \frac{1 \cdot z_1 + 2 \left( \frac{z_2 + z_3}{2} \right)}{1+2} \right)
\]
or: \( G \left( \frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3} \right) \)

Now the Co-ords. of pts. E & F are

\( E \left( \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right) \) & \( F \left( \frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2} \right) \)

Now the Co-ords. of the Centroid G, dividing BE in the ratio 2:1 are

\[ G \left( \frac{1 \cdot x_2 + 2 \left( \frac{x_1+x_2}{2} \right)}{1+2}, \frac{1 \cdot y_2 + 2 \left( \frac{y_1+y_2}{2} \right)}{1+2}, \frac{1 \cdot z_2 + 2 \left( \frac{z_1+z_2}{2} \right)}{1+2} \right) \]

\[ = G \left( \frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3} \right) \]

Similarly the Co-ords. of Centroid G, dividing CF in the ratio 2:1 are

\[ G \left( \frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3} \right) \]

Hence Co-ords. of Centroid G are

\[ G \left( \frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3} \right) \]

All Find the Centroid of tetrahedron whose vertices are \((x_i, y_i, z_i)\); \(i = 1, 2, 3, 4\).

Let the vertices of the tetrahedron are \( A = (x_1, y_1, z_1) \)
\( B = (x_2, y_2, z_2) \)
\( C = (x_3, y_3, z_3) \)
\( D = (x_4, y_4, z_4) \)

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let E, F, G, H are the centroids of
the triangles BCD, ACD, ABD & ABC resp.
Then their Co-ords are
\[ E = \left( \frac{x_1+x_2+x_4}{3}, \frac{y_1+y_2+y_4}{3}, \frac{z_1+z_2+2z_4}{3} \right) \]
\[ F = \left( \frac{x_1+x_3+x_4}{3}, \frac{y_1+y_3+y_4}{3}, \frac{z_1+z_3+2z_4}{3} \right) \]
\[ G = \left( \frac{x_1+x_3+x_4}{3}, \frac{y_1+y_3+y_4}{3}, \frac{z_1+z_3+2z_4}{3} \right) \]
\[ H = \left( \frac{x_1+x_4+x_3}{3}, \frac{y_1+y_4+y_3}{3}, \frac{z_1+z_4+2z_3}{3} \right) \]
Now Co-ords of Centroid dividing the line AE in
ratio 3:1 are
\[ \left( \frac{1 \cdot x_1 + 3 \left( \frac{x_1+x_3+x_4}{3} \right)}{1+3}, \frac{1 \cdot y_1 + 3 \left( \frac{y_1+y_3+y_4}{3} \right)}{1+3}, \frac{1 \cdot z_1 + 3 \left( \frac{z_1+z_3+2z_4}{3} \right)}{1+3} \right) \]
\[ = \left( \frac{x_1+x_3+x_4}{4}, \frac{y_1+y_3+y_4}{4}, \frac{z_1+z_3+2z_4}{4} \right) \]
Now Co-ords of Centroid dividing the line BF in ratio
3:1 are
\[ \left( \frac{1 \cdot x_2 + 3 \left( \frac{x_1+x_3+x_4}{3} \right)}{1+3}, \frac{1 \cdot y_2 + 3 \left( \frac{y_1+y_3+y_4}{3} \right)}{1+3}, \frac{1 \cdot z_2 + 3 \left( \frac{z_1+z_3+2z_4}{3} \right)}{1+3} \right) \]
\[ = \left( \frac{x_1+x_3+x_4}{4}, \frac{y_1+y_3+y_4}{4}, \frac{z_1+z_3+2z_4}{4} \right) \]
Similarly we can prove that Co-ords of Centroid
in case of CG and D H are
\[(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4})\]
So Co-ords of Centroid are
\[(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4})\]