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Notes of Fourier Series

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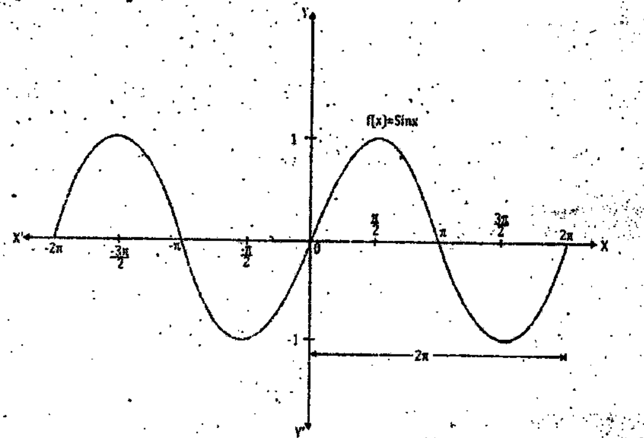
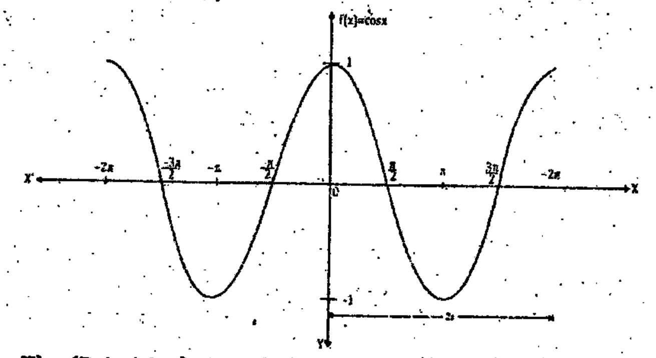
DEFINITION:
A function $f:\mathbb{R}\to\mathbb{R}$, is said to be periodic if there exists some real number $P>0$ such that $f(x+p)=f(x)\to(1)\ \forall x\in D_f$. The least value of $P>0$ satisfying Eq (1) is called the period (some-times called primitive period or fundamental period) of f.

EXAMPLES:
As $\sin x=\sin(x+2\pi)=\sin(\pi+4\pi)=\sin(x+6\pi)=\cdots\cdots$
So $2\pi, 4\pi, 6\pi,\cdots\cdots$, are periods of the function sin x. However, since 2π is the least of the periods $2\pi,4\pi,6\pi,\cdots\cdots$. So sin x is a periodic function with primitive period 2π.



As $\cos x=\cos(x+2\pi)=\cos(x+4\pi)=\cos(\pi+6\pi)=\cdots\cdots$
So the primitive period of Cos x is 2π.



The (Primitive) period of tan x and cot x is π.

NOTE:
If a function of is periodic with period p then $f(x+np)=f(x)\ \forall n\in Z^+=\{1,2,3,\cdots\cdots\}$
Thus $2P,3P,4P,\cdots\cdots$ are also its periods.

The period of sin nx and cos nx is $\frac{2\pi}{n}$, where $n\in Z^+=\{1,2,3,\cdots\cdots\}$.

If $f_1(x)$ and $f_2(x)$ have the some period p then the function
$f(x)=C_1f_1(x)+C_2f_2(x)\quad C_1,C_2\in\mathbb{R}$ also have the period p.
If $f_1(x)$ and $f_2(x)$ have periods p and q respectively then the period of the function $f(x)=C_1f_1(x)+C_2f_2(x)$ is the L.C.M of p and q.

i. A constant function is periodic and has any +ve real number as its period.

DEFINITION: Let a function f be defined in an interval $\left[\frac{-P}{2},\frac{P}{2}\right]$, where p is a +ve real number then the periodic extension 'φ' (of period P) of 'f' is defined as
$$\varphi(x)=\begin{cases} f(x) & \text{if } x\in\left[\frac{-P}{2},\frac{P}{2}\right] \\ f(x+P) & \text{if } x\notin\left[\frac{-P}{2},\frac{P}{2}\right]\end{cases}$$
Thus, to sketch the periodic extension $\varphi(x)$ of $f(x)$, simply sketch $f(x)$ for $\frac{-P}{2}\le x\le\frac{P}{2}$ and then go on repeating the same pattern of the graph with period P.

EXAMPLES:
let $f:[-\pi,\pi]\to\mathbb{R}$ be defined by $f(x)=x$.
Its graph is shown in fig-1

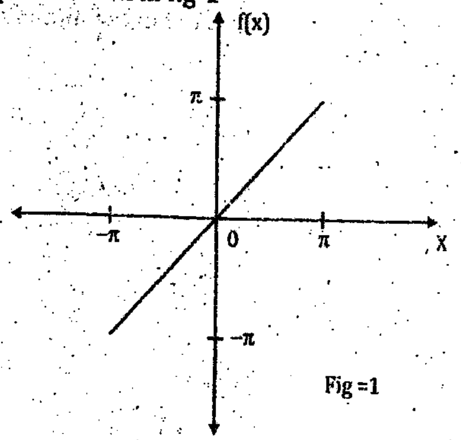


Fig =1

Clearly the function f is not periodic. Its periodic extension of period 2π is shown in Fig-2.
$\because[-\pi,\pi]=\left[\frac{-2\pi}{2},\frac{2\pi}{2}\right]\Rightarrow P=2\pi$

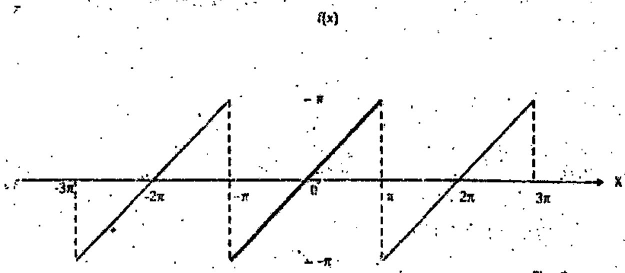


Fig-2

Let $f:[-1,1]\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}0 & \text{for } x<0\\ 1 & \text{for } x>0\end{cases}$$

Its graph is shown in Fig-3

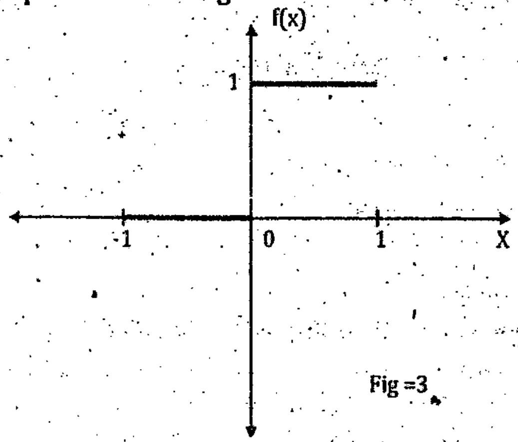


Fig =3

Clearly f is not periodic. Its periodic extension of period 2 is shown in fig-4.

$$\because [-1,1]=\left[\frac{-2}{2},\frac{2}{2}\right]\Rightarrow P=2$$

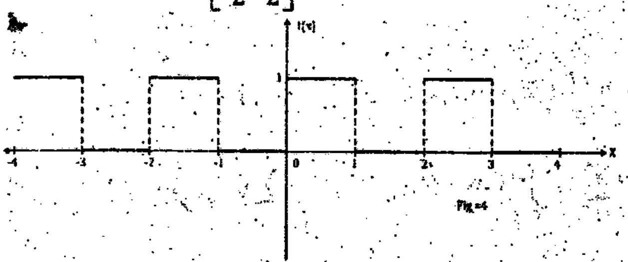


Fig-4

DEFINITION:

(Piecewise continuous function): A function f is said to be piecewise continuous on a finite interval $a\le x\le b$ if

f is defined on $[a,b]$

The internal $[a,b]$ can be divided into a finite number of subintervals in each of which f is continuous.

i. The one sided limits of f as x approaches the end points of each subinterval are finite but may or may not be equal see Fig-5.

OR A function f is said to be piecewise continuous on $a\le x\le b$ if there are at most a finite number of points $x_k\,(k=1,2,3,\cdots\cdots n)$ at which f has finite discontinuities and is continuous on each open interval $x_{k-1}<x<x_k$ of $[a,b]$. Thus a piecewise continuous function may have a finite number of jump discontinuities at the points of subdivisions of $[a,b]$.

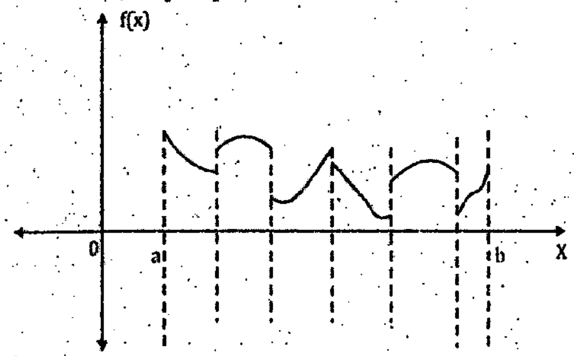


Fig-5

NOTE:

1. A continuous function is piecewise continuous.

If f is piecewise continuous on $[a,b]$ with the partition, $a=a_0<a_1<a_2<\cdots\cdots a_n=b$, then $\int_a^b f(x)dx$ exists and

$$\int_a^b f(x)dx=\int_0^{a_1} f_1(x)dx+\int_{a_1}^{a_2} f_2(x)dx+\cdots\cdots+\int_{a_{n-1}}^{a_n} f_n(x)dx$$

Where on each sub-interval $[a_{i-1},a_i]$ there is a continuous function f_i such that $f(x)=f_i(x)$ for $a_{i-1}<x<a_i$

DEFINITION:

A function f is said to be piecewise smooth on $[a,b]$ if f and its derivative f' are piecewise continuous on $[a,b]$.

DEFINITION:

A trigonometric infinite series of the form

$$\frac{a_0}{2}+a_1\text{Cos}\,x+a_2\text{Cos}\,2x+a_3\text{Cos}3x+\cdots\cdots$$

$$+b_1\text{Sin}\,x+b_2\text{Sin}\,2x+b_3\,\text{Sin}\,3x+\cdots\cdots$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\,\text{Cos}\,nx+b_n\,\text{Sin}\,nx)\longrightarrow(1)$$

Corresponding to a function $f(x)$ is called its four series and the constants a_n and b_n are known as four constants or coefficients.

In (1) above, we have chosen to write the constant term as $\frac{a_0}{2}$ instead of a_0. This is for convenience only, the Euler's formula for $a_n, n=1,2,3,\cdots$, which we are going to establish soon, will then reduce to a_0 for $n=0$.

THE COMPUTATION OF THE FOURIER COEFFICIENTS THEOREM

If the series $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx)$ is assumed integrable term-by-term in the interval $C \le x \le C+2\pi$ and has sum, then the coefficients a_n and b_n are given by, $a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\text{Cos } nx\, dx,\ n=0,1,2,\cdots$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\text{Sin } nx\, dx,\ n=1,2,3,\cdots$$

PROOF:

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx) \to (1)$

Where $f(x)$ is integrable term-by-term in the interval $C \le x \le C+2\pi$ and is a periodic function of period 2π.

To find a_0, we integrate both sides of Eq (1) term-by-term from $x=C$ to $x=C+2\pi$, and get

$$\int_C^{C+2\pi} f(x)dx = \frac{a_0}{2}\int_C^{C+2\pi} dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n \text{Cos } nx\right)dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n \text{Sin } nx\right)dx$$

$$\Rightarrow \int_C^{C+2\pi} f(x)dx = \frac{a_0}{2}\int_C^{C+2\pi} dx + \sum_{n=1}^{\infty} a_n\int_C^{C+2\pi} \text{Cos } nx\, dx + \sum_{n=1}^{\infty} b_n \int_C^{C+2\pi} \text{Sin } nx\, dx \to (2)$$

Now $\int_C^{C+2\pi} dx = [x]_C^{C+2\pi} = (C+2\pi)-C = 2\pi$

$$\Rightarrow \boxed{\int_C^{C+2\pi} dx = 2\pi}$$

$$\int_C^{C+2\pi} \text{Cos} nx dn = \left[\frac{\text{Sin } nx}{n}\right]_C^{C+2\pi} = \frac{1}{n}\left[\text{Sin } n(C+2\pi) - \text{Sin } nc\right]$$

$$\Rightarrow \int_C^{C+2\pi} \text{Cos} nx dn = \frac{1}{n}\left[\text{Sin}(nc+2n\pi) - \text{Sin } nc\right], n=1,2,3,\cdots$$

$$\Rightarrow \int_C^{C+2\pi} \text{Cos} nx dn = \frac{1}{n}[\text{Sin} nc - \text{Sin } nc]$$

$(\because \text{Sin}(nc+2n\pi) = \text{Sin } nc \text{ for } n=1,2,3,\cdots)$

$$\int_C^{C+2\pi} \text{Cos} nx dn = \frac{1}{n}(0) = 0$$

$$\boxed{\text{Thus } \int_C^{C+2\pi} \text{Cos } nx\, dx = 0}$$

$$\int_C^{C+2\pi} \text{Sin } nx\, dx = -\int_C^{C+2\pi} (-\text{Sin } nx)dx$$

$$= -\left[\frac{\text{Cos } nx}{n}\right]_C^{C+2\pi}$$

$$= \frac{-1}{n}\left[\text{Cos } n(C+2\pi) - \text{Cos } nc\right]_C^{C+2\pi}$$

$$\Rightarrow \int_C^{C+2\pi} \text{Sin } nx\, dx = \frac{-1}{n}[\text{Cos } nc - \text{Cos } nc] = 0$$

$(\because \cos n(c+2\pi) = \cos nc)$

$$\Rightarrow \boxed{\int_C^{C+2\pi} \text{Sin } nc\, dx = 0}$$

Putting the values in Eq (2), we have

$$\int_C^{C+2\pi} f(x)dx = \frac{a_0}{2}(2\pi) + \sum_{n=1}^{\infty} a_n(0) + \sum_{n=1}^{\infty} b_n(0) = a_0\pi$$

$$\Rightarrow \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx = a_0$$

$$\text{or } \boxed{a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx} \to (A_1)$$

To find a_n, we multiply both sides of Eq (1) by cos mx where m is fixed +ve integer and then integrate the resulting series term-by-term from $x=C$ to $x=C+2\pi$, and get

$$\int_C^{C+2\pi} f(x)\text{Cos} mx\, dx = \frac{a_0}{2}\int_C^{C+2\pi} \text{Cos} mx\, dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n \text{Cos } nx\right)\text{cos} mx dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n \text{Sin } nx\right)\text{Cos} mx\, dx$$

$$\Rightarrow \int_C^{C+2\pi} f(x)\text{Cos } mx\, dx = \frac{a_0}{2}\int_C^{C+2\pi} \text{Cos } mx\, dx + \sum_{n=1}^{\infty} a_n \int_C^{C+2\pi} (\text{Cos } nx \text{ Cos } mx)dx + \sum_{n=1}^{\infty} b_n \int_C^{C+2\pi} \text{Sin } nx \text{ Cos } mx\, dx \to$$

Now $\int_C^{C+2\pi} \text{Cos } mx\, dx = \left[\frac{\text{Sin } mx}{m}\right]_C^{C+2\pi} = 0$

$$\Rightarrow \boxed{\int_C^{C+2\pi} \text{Cos } mx\, dx = 0}$$

$$\int_C^{C+2\pi} \text{Cos } nx.\text{Cos} mx\, dx = \frac{1}{2}\int_C^{C+2\pi}\left[\text{Cos}(n+m)x + \text{Cos}(n-m)x\right]dx$$

$$\Rightarrow \int_C^{C+2\pi} \text{Cos} nx.\text{Cos } mx\, dx = \frac{1}{2}\int_C^{C+2\pi} (\text{Cos } 2nx + \text{Cos} o)dx \text{ for } m=n$$

$$\Rightarrow \int_C^{C+2\pi} \text{Cos } nx.\text{Cos } mx\, dx = \frac{1}{2}\int_C^{C+2\pi} \text{Cos} 2nx + \frac{1}{2}\int_C^{C+2\pi} 1 dx,\ m=n$$

$$\Rightarrow \boxed{\int_c^{c+2\pi} \text{Cos}\, nx . \text{Cos}\, mx\, dx = \pi \quad \text{for } m = n \\ = 0 \quad \text{for } m \neq n}$$

$$\int_c^{c+2\pi} \text{Sin}\, nx \,\text{Cos}\, mx\, dx = \frac{1}{2}\int_c^{c+2\pi}\left[\text{Sin}(n+m)x + \text{Sin}(n-m)x\right]dx$$

$$\int_c^{c+2\pi} \text{Sin}\, nx \,\text{Cos}\, mx\, dx = \frac{1}{2}\int_c^{c+2\pi}\text{Sin}(n+m)x dx + \frac{1}{2}\int_c^{c+2\pi}\text{Sin}(n-m)x dx$$

$$\int_c^{c+2\pi} \text{Sin}\, nx \,\text{Cos}\, mx\, dx = 0 + 0$$

$$\Rightarrow \boxed{\int_c^{c+2\pi} \text{Sin}\, nx \,\text{Cos}\, mx\, dx = 0}$$

Putting the values in (3) we have

$$\int_c^{x+2\pi} f(x)\text{Cos}\, mx\, dx = \frac{a_0}{2}(0) + a_n\pi + 0$$

$$\Rightarrow \int_c^{c+2\pi} f(x)\text{Cos}\, mx\, dx = a_n\pi$$

Putting $m = n$, we have

$$\int_c^{c+2\pi} f(x)\cos nx\, dx = a_n\pi$$

or $$\boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\text{Cos}\, nx\, dx \quad n = 1,2,3\cdots} \rightarrow (A_2)$$

Similarly, to find b_n, we multiply (1) by Sin mx and then integrating term-by-term form $x = c$ to $x = c + 2\pi$, we have

$$\int_c^{c+2\pi} f(x)\text{Sin}\, mx\, dx = \frac{a_0}{2}\int_c^{c+2\pi}\text{Sin}\, mx\, dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi}\text{Cos}\, nx . \text{Sin}\, mx\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi}\text{Sin}\, nx \,\text{Sin}\, mx\, dx$$

$$\Rightarrow \int_c^{c+2\pi} f(x)\text{Sin}\, mx\, dx = \frac{a_0}{2}(0) + 0 + b_n\pi$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\text{Sin}\, mx\, dx$$

or $$\boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\text{Sin}\, nx\, dx, \; n = 1,2,3,\cdots} \rightarrow (A_3)$$

The equations $(A_1), (A_2)$ and (A_3) are called Euler's formulas for the Fourier coefficients.

COROLLARY 1:

When $C = 0$, the formulae $(A_1), (A_2)$ and (A_3) reduce respectively to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{Cos}\, nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{Sin}\, nx\, dx$$

COROLLARY 2:

When $C = -\pi$, the formulae $(A_1), (A_2)$ and (A_3) reduce respectively to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Cos}\, nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Sin}\, nx\, dx$$

EXAMPLE 1: Find the four series representing the identity function $f(x) = x, 0 < x < 2\pi$ and sketchy its graph from $x = -4\pi$ to $x = 4\pi$.

SOLUTION:

Let $$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\text{Cos}\, nx + b_n\text{Sin}\, nx) \rightarrow (1)$$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left[\frac{4\pi^2}{2}\right] = 2\pi$$

$$\Rightarrow \boxed{a_0 = 2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{Cos}\, nx\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Cos}\, nx\, dx$$

Integrating by parts, we have

$$a_n = \frac{1}{\pi}\left[\frac{x\,\text{Sin}\, nx}{n}\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi}\frac{1.\,\text{Sin}\, nx}{n}dx$$

$$\Rightarrow a_n = \frac{1}{\pi}(0) + \frac{1}{n\pi}\int_0^{2\pi}(-\text{Sin}\, nx)dx$$

$$\Rightarrow a_n = 0 + \frac{1}{n\pi}\left[\frac{\text{Cos}\, nx}{n}\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[\text{Cos}\, nx\right]_0^{2\pi} = \frac{1}{n^2\pi}\left[\text{Cos}\, 2n\pi - \text{Cos}\, 0\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}[\text{Cos } nx]_0^{2\pi} = \frac{1}{n^2\pi}[1-1] = 0$$

$$\Rightarrow \boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{Sin } nx\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Sin } nx\, dx$$

Integrating by parts, we have

$$b_n = \frac{1}{\pi}\left[x\frac{(-\text{Cos } nx)}{n}\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi}\frac{(-\text{Cos } nx)dx}{n}$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[x\text{Cos } nx]_0^{2\pi} + \frac{1}{nx}\int_0^{2\pi}\text{Cos } nx\, dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[2\pi\,\text{Cos } 2n\pi - 0] + \frac{1}{nx}\left[\frac{\text{Sin } nx}{n}\right]_0^{2\pi}$$

$$\Rightarrow b_n = \frac{-2}{n}\text{Cos } 2n\pi + \frac{1}{n^2\pi}[\text{Sin } 2n\pi - \text{Sin } o]$$

$$\Rightarrow b_n = \frac{-2}{n}(1) + \frac{1}{n^2\pi}(0)$$

$$\Rightarrow b_n = \frac{-2}{n}$$

$$\Rightarrow \boxed{b_n = \frac{-2}{n}}$$

Putting $a_0 = 2\pi, a_n = 0, b_n = \frac{-2}{n}$ in (1) we have

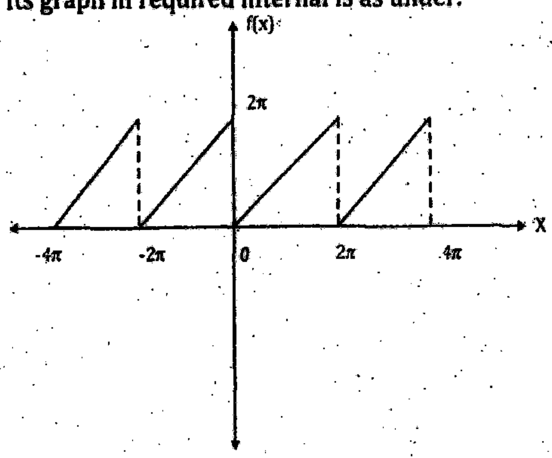
$$x = \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left(o\ \text{Cos } nx + \left(\frac{-2}{n}\right)\text{Sin } nx\right)$$

$$\Rightarrow x = \pi + (-2)\sum_{n=1}^{\infty}\frac{\text{Sin } nx}{n}$$

$$\Rightarrow x = \pi - 2\sum_{n=1}^{\infty}\frac{\text{Sin } nx}{n} = \pi - 2\left[\text{Sin } x + \frac{1}{2}\text{Sin } 2x + \frac{1}{3}\text{Sin } 3x + \cdots\cdots\right]$$

Which is the required four representation for the identity function in the given interval.

Its graph in required internal is as under.



EXAMPLE 2:

Obtain the Fourier series for e^x in the interval $-\pi < x < \pi$.

SOLUTION:

$$\text{Let } f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\text{Cos } nx + b_n\text{Sin } nx) \longrightarrow (1)$$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\, dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{2}{\pi}\left[\frac{e^{\pi} - e^{-\pi}}{2}\right] = \frac{2}{\pi}\text{Sin } h\pi$$

$$\Rightarrow \boxed{a_0 = \frac{2}{\pi}\text{Sin } h\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Cos } nx\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\text{Cos } nx\, dx \longrightarrow (2)$$

$$\text{Let } A = \int e^x\text{Cos } nx\, dx \longrightarrow (3)$$

Integrating (3) by parts, we have

$$\Rightarrow A = e^x\int\text{Cos } nx\, dx - \int\left(\left(\frac{de^x}{dx}\right)\int\text{Cos } nx\, dx\right)dx$$

$$\Rightarrow A = e^x\frac{\text{Sin } nx}{n} - \int e^x\frac{\text{Sin } nx}{n}dx$$

$$\Rightarrow A = \frac{e^x.\text{Sin } nx}{n} - \int e^x(-\text{Sin } nx)dx$$

$$\Rightarrow A = \frac{e^x\,\text{Sin } nx}{n} + \frac{1}{n}\left[e^x\int -\text{Sin } nx\, dx - \int\left(\left(\frac{de^x}{dx}\right)\int(-\text{Sin } nx\, dx)dx\right)\right]$$

$$\Rightarrow A = \frac{e^x\text{Sin } nx}{n} + \frac{1}{n}\left[\frac{e^x\,\text{Cos } nx}{n} - \int\frac{e^x\,\text{Cos} nx}{n}dx\right]$$

$$\Rightarrow A = \frac{e^x\text{Sin } nx}{n} + \frac{e^x\,\text{Cos } nx}{n^2} - \frac{1}{n^2}\int e^x\text{Cos } nx\, dx$$

$$\Rightarrow A = \frac{e^x\,\text{Sin} nx}{n} + \frac{e^x\,\text{Cos } nx}{n^2} - \frac{1}{n^2}A \text{ (By Eq (3))}$$

$$\Rightarrow A + \frac{1}{n^2}A = \frac{e^x}{n^2}(n\,\text{Sin } nx + \text{Cos } nx)$$

$$\Rightarrow \left(\frac{n^2+1}{n^2}\right)A = \frac{e^x}{n^2}(n.\text{Sin } nx + \text{Cos } nx)$$

$\Rightarrow A=\frac{e^x}{n^2+1}[n.\operatorname{Sin} nx+\operatorname{Cos} nx]$

$\Rightarrow A=\int e^x \operatorname{Cos} nx\,dx=\frac{e^x}{n^2+1}[n.\operatorname{Sin} nx+\operatorname{Cos} nx]$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\left[\frac{e^x}{n^2+1}(n.\operatorname{Sin} nx+\operatorname{Cos} nx)\right]_{-\pi}^{\pi}$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{1}{n^2+1}\times$

$\times\left[e^{\pi}(n\operatorname{Sin} n\pi+\operatorname{Cos} n\pi)-e^{-\pi}(n\operatorname{Sin}(-n\pi)+\operatorname{Cos}(-n\pi))\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{1}{n^2+1}\times$

$\times\left[e^{\pi}(n\operatorname{Sin} n\pi+\operatorname{Cos} n\pi)-e^{-\pi}(-n\operatorname{Sin} n\pi+\operatorname{Cos} n\pi)\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{1}{n^2+1}\times$

$\times\left[ne^{\pi}\operatorname{Sin} n\pi+e^{\pi}\operatorname{Cos} n\pi+ne^{-\pi}\operatorname{Sin} n\pi-e^{-\pi}\operatorname{Cos} n\pi\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{1}{n^2+1}\times$

$\times\left[(n\operatorname{Sin} n\pi)(e^{\pi}+e^{-\pi})+(e^{\pi}-e^{-\pi})\operatorname{Cos} n\pi\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{1}{n^2+1}\left[(n.0)(e^{\pi}+e^{-\pi})+(e^{\pi}-e^{-\pi})(-1)^n\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{1}{n^2+1}\left[0+(e^{\pi}-e^{-\pi})(-1)^n\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{(-1)^n}{n^2+1}(e^{\pi}-e^{-\pi})$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{2(-1)^n}{n^2+1}\frac{(e^{\pi}-e^{-\pi})}{2}$

$\Rightarrow \int_{-\pi}^{\pi} e^x \operatorname{Cos} nx\,dx=\frac{2(-1)^n}{n^2+1}\operatorname{Sin} h\pi$

Putting this value in Eq (2) we have

$$\boxed{a_n=\frac{2(-1)^n}{\pi(n^2+1)}\operatorname{Sin} h\pi}$$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{Sin} nx\,dx$

$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} e^x\operatorname{Sin} nx\,dx \longrightarrow (4)$

Let $B=\int e^x \operatorname{Sin} nx\,dx$

Integrating by parts, we have

$\Rightarrow B=e^x\int \operatorname{Sin} nx\,dx-\int\left(\left(\frac{de^x}{dx}\right)\int \operatorname{Sin} nx\,dx\right)dx$

$\Rightarrow B=e^x\left(\frac{-\operatorname{Cos} nx}{n}\right)-\int e^x\left(\frac{-\operatorname{Cos} nx}{n}\right)dx$

$\Rightarrow B=-\frac{e^x\operatorname{Cos} nx}{n}+\frac{1}{n}\int e^x\operatorname{Cos} nx\,dx$

$\Rightarrow B=-\frac{e^x\operatorname{Cos} nx}{n}+\frac{1}{n}\left[e^x\int \operatorname{Cos} nx\,dx-\int\left(\frac{de^x}{dx}\right)\int \operatorname{Cos} nx\,dx\right)dx$

$\Rightarrow B=-\frac{e^x\operatorname{Cos} nx}{n}+\frac{1}{n}\left[e^x\frac{\operatorname{Sin} nx}{n^2}-\int\frac{e^x\operatorname{Sin} nx}{n}dx\right]$

$\Rightarrow B=\frac{-e^x\operatorname{Cos} nx}{n}+\frac{e^x\operatorname{Sin} nx}{n^2}-\frac{1}{n^2}B$

$\Rightarrow B+\frac{1}{n^2}B=\frac{e^x}{n^2}(\operatorname{Sin} nx-n\operatorname{Cos} nx)$

$\Rightarrow\left(\frac{n^2+1}{n^2}\right)B=\frac{e^x}{n^2}(\operatorname{Sin} nx-n\operatorname{Cos} nx)$

$\Rightarrow B=\frac{e^x}{n^2+1}(\operatorname{Sin} nx-n\operatorname{Cos} nx)$

$\Rightarrow \int e^x\operatorname{Sin} nx\,dx=\frac{e^x}{n^2+1}(\operatorname{Sin} nx-n\operatorname{Cos} nx)$

$\Rightarrow \int_{-\pi}^{\pi} e^x\operatorname{Sin} nx\,dx=\frac{1}{n^2+1}\left[e^x(\operatorname{Sin} nx-n\operatorname{Cos} nx)\right]_{-\pi}^{\pi}$

$=\left[e^{\pi}(\operatorname{Sin} n\pi-n\operatorname{Cos} n\pi)-e^{-\pi}(\operatorname{Sin} n(-\pi)-n\operatorname{Cos} n(-\pi))\right]$

$=\left[e^{\pi}\operatorname{Sin} n\pi-ne^{\pi}\operatorname{Cos} n\pi+e^{-\pi}\operatorname{Sin} n\pi+e^{-\pi}n\operatorname{Cos} n\pi\right]$

$=\frac{1}{n^2+1}\left[(e^{\pi}+e^{-\pi})\sin n\pi-n(e^{\pi}-e^{-\pi})\cos n\pi\right]$

$=\frac{1}{n^2+1}\left[(e^{\pi}+e^{-\pi})(0)-2n\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)(-1)^n\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x\operatorname{Sin} nx\,dx=\frac{1}{n^2+1}\left[0-2n\operatorname{Sin} h\pi\,(-1)^n\right]$

$\Rightarrow \int_{-\pi}^{\pi} e^x\operatorname{Sin} nx\,dx=\frac{2n(-1)^{n+1}.\operatorname{Sin} h\pi}{n^2+1}$

Thus $\int_{-\pi}^{\pi} e^x\operatorname{Sin} nx\,dx=\frac{2n(-1)^{n+1}\operatorname{Sin} h\pi}{n^2+1}$

Putting it in eq (4) we have

$$\boxed{b_n=\frac{2n(-1)^{n+1}\operatorname{Sin} h\pi}{\pi(n^2+1)}}$$

Putting the values if a_0, a_n and b_n in (1) we have

$$e^x=\frac{\text{Sin }h\pi}{\pi}+\sum_{n=1}^{\infty}\left(\frac{2(-1)^n\text{Sin }h\pi}{\pi(n^2+1)}\text{Cos }nx+\frac{2n(-1)^{n+1}\text{Sin }h\pi}{\pi(n^2+1)}\text{Sin }nx\right)$$

$$e^x=\frac{\text{Sin }h\pi}{\pi}+\frac{2\text{Sin }h\pi}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n\text{Cos }nx}{n^2+1}+\frac{n(-1)^{n+1}.\text{Sin }nx}{n^2+1}\right)$$

Which is the required Fourier series for e^x in the given interval.

EXAMPLE 3:

Find the Fourier series corresponding to $f(x)=x+x^2$ in the interval $-\pi<x<\pi$ also deduce that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots$$

SOLUTION:

Let $f(x)=x+x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos }nx+b_n\text{Sin }nx)\rightarrow(1)$

By Euler's Formulae, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

Thus $\boxed{a_0=\frac{2\pi^2}{3}}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\text{Cos }nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\text{Cos }nx\,dx$$

In targeting by parts we have

$$a_n=\frac{1}{\pi}\left[(x+x^2)\frac{\text{Sin }nx}{n}\right]_{-\pi}^{\pi}-\frac{1}{\pi}\int_{-\pi}^{\pi}(1+2x)\frac{\text{Sin }nx}{n}dx$$

$$\Rightarrow a_n=0-\frac{1}{n\pi}\int_{-\pi}^{\pi}(1+2\pi)\text{Sin }nx\,dx$$

A gain integrating by parts, we have

$$a_n=\frac{-1}{n\pi}\left[\left[(1+2x)\frac{(-\text{Cos }nx)}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(0+2)\frac{(-\text{Cos }nx)}{n}dx\right]$$

$$\Rightarrow a_n=\frac{1}{n^2\pi}[(1+2\pi)\text{Cos }n\pi-(1-2\pi)\text{Cos}n\pi]-\frac{2}{n^2\pi}\int_{-\pi}^{\pi}\text{Cos }nx\,dx$$

$$\Rightarrow a_n=\frac{1}{n^2\pi}[(1+2\pi-1+2\pi)\text{Cos }n\pi]-\frac{2}{n^2\pi}\left[\frac{\text{Sin }nx}{n}\right]_{-\pi}^{\pi}$$

$$a_n=\frac{1}{n^2\pi}\left[4\pi(-1)^n\right]-0$$

$$\Rightarrow\boxed{a_n=\frac{4(-1)^n}{n^2}}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\text{ Sin }nx\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\text{Sin }nx\,dx$$

Integrating by parts, we have

$$b_n=\frac{1}{\pi}\left[(x+x^2)\left(\frac{-\text{Cos }nx}{n}\right)\right]_{-\pi}^{\pi}-\frac{1}{\pi}\int_{-\pi}^{\pi}(1+2x)\left(\frac{-\text{Cos }nx}{n}\right)dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[(\pi+\pi^2)\text{Cos }n\pi-(-\pi+\pi^2)\text{Cos }n\pi\right]$$

$$+\frac{1}{n\pi}\int_{-\pi}^{\pi}(1+2x)\text{Cos }nx\,dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[(\pi+\pi^2+\pi-\pi^2)\text{Cos }n\pi\right]+$$

$$+\frac{1}{n\pi}\left[(1+2x)\frac{\text{Sin }x}{n}\right]_{-\pi}^{\pi}-\frac{1}{n\pi}\int_{-\pi}^{\pi}(0+2)\frac{\text{Sin }nx}{n}dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[2\pi(-1)^n\right]+0-\frac{2}{n^2\pi}\int_{-\pi}^{\pi}\text{Sin }nx\,dx$$

$$\Rightarrow b_n=\frac{-2(-1)^n}{n}-\frac{2}{n^2\pi}\left[\frac{-\text{Cos }nx}{n}\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n=\frac{-2(-1)^n}{n}-0$$

$$\Rightarrow\boxed{b_n=\frac{-2(-1)^n}{n}}$$

Putting the values in Eq (1) we have

$$x+x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\text{Cos}nx-\frac{2(-1)^n}{n}\text{Sin }nx\right)$$

$$\Rightarrow x+x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\text{Cos }nx}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n\text{Sin }nx}{n}\rightarrow(2)$$

Which is the required Fourier representation of the given function in the given interval.

Now putting $x=\pi$ in Eq (2) we have

$$\pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\text{Cos}n\pi}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n\text{Sin }n\pi}{n}$$

$$\Rightarrow\pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}-2(0)$$

$$\Rightarrow \pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}\longrightarrow(3)$$

Next putting $x=-\pi$ in Eq (2), we have

$$-\pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\operatorname{Cos}n(-\pi)}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n\operatorname{Sin}n(-\pi)}{n}$$

$$\Rightarrow -\pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}-2(0)$$

$$\Rightarrow -\pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}\longrightarrow(4)$$

Adding Eq (3) and Eq (4) we have

$$2\pi^2=\frac{2\pi^2}{3}+8\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$2\pi^2-\frac{2\pi^2}{3}=8\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\frac{4\pi^2}{3}=8\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}\qquad \left(\because(-1)^{2n}=1\right)$$

$$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Which is the required result.

EXAMPLE 4:

Construct the Fourier series to represent

$$f(x)=\begin{cases}-x, & -\pi<x\le 0\\ x, & 0<x<\pi\end{cases}$$

and deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots$

SOLUTION:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\operatorname{Cos}nx+b_n\operatorname{Sin}nx)\to(1)$$

By Euler's formulae, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x)dx+\frac{1}{\pi}\int_{0}^{\pi}x\,dx$$

$$=\frac{-1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$\Rightarrow a_0=\frac{-1}{2\pi}\left[0^2-(-\pi)^2\right]+\frac{1}{2\pi}\left[\pi^2-0^2\right]$$

$$\Rightarrow a_0=\frac{-1}{2\pi}(-\pi^2)+\frac{1}{2\pi}(\pi^2)$$

$$\Rightarrow a_0=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

$$\Rightarrow \boxed{a_0=\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\operatorname{Cas}nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{0}(-x)\operatorname{Cos}nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}x\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_n=\frac{-1}{\pi}\int_{-\pi}^{0}x\operatorname{Cos}nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}x\operatorname{Cos}nx\,dx\longrightarrow(2)$$

In targeting by parts, we have

$$\int x\operatorname{Cos}nx\,dx=x\int\operatorname{Cos}nx\,dx-\int\left(\left(\frac{dx}{dx}\right)\int\operatorname{Cos}nx\,dx\right)dx$$

$$\Rightarrow \int x\operatorname{Cos}nx\,dx=\frac{x\operatorname{Sin}nx}{n}-\int\frac{1\operatorname{Sin}nx}{n}dx$$

$$\Rightarrow \int x\operatorname{Cos}nx\,dx=\frac{x\operatorname{Sin}nx}{n}+\frac{1}{n}\int-\operatorname{Sin}nx\,dx$$

$$\Rightarrow \int x\operatorname{Cos}nx\,dx=\frac{x\operatorname{Sin}nx}{n}+\frac{1}{n}\frac{\operatorname{Cos}nx}{n}$$

$$\Rightarrow \int x\operatorname{Cos}nx\,dx=\frac{x\operatorname{Sin}nx}{n}+\frac{\operatorname{Cos}nx}{n^2}$$

Now $\int_{-\pi}^{0}x\operatorname{Cos}nx\,dx=\left[\frac{x\operatorname{Sin}nx}{n}+\frac{\operatorname{Cos}nx}{n^2}\right]_{-\pi}^{0}$

$$\int_{-\pi}^{0}x\operatorname{Cos}nx\,dx=\left(0+\frac{1}{n^2}\right)-\left(0+\frac{(-1)^n}{n^2}\right)$$

$$\int_{-\pi}^{0}x\operatorname{Cos}nx\,dx=\frac{1}{n^2}-\frac{(-1)^n}{n^2}=\frac{1-(-1)^n}{n^2}=\frac{1+(-1)^{n+1}}{n^2}$$

$$\Rightarrow \boxed{\int_{-\pi}^{0}x\operatorname{Cos}nx\,dx=\frac{1+(-1)^{n+1}}{n^2}}$$

and $\int_{0}^{\pi}x\operatorname{Cos}nx\,dx=\left[\frac{x\operatorname{Sin}nx}{n}+\frac{\operatorname{Cos}nx}{n^2}\right]_{0}^{\pi}$

$$\Rightarrow \int_{0}^{\pi}x\operatorname{Cos}nx\,dx=\left(0+\frac{(-1)^n}{n^2}\right)-\left(0+\frac{1}{n^2}\right)$$

$$\Rightarrow \boxed{\int_{0}^{\pi}x\operatorname{Cos}nx\,dx=\frac{(-1)^n-1}{n^2}=\frac{-1+(-1)^n}{n^2}}$$

Then Eq (2) $\Rightarrow a_n = \frac{-1}{\pi}\left[\frac{1+(-1)^{n+1}}{n^2}\right]+\frac{1}{\pi}\left[\frac{-1+(-1)^n}{n^2}\right]$

$\Rightarrow a_n = \frac{(-1)}{n^2\pi}\left[1+(-1)^{n+1}+1-(-1)^n\right]$

$\Rightarrow a_n = \frac{(-1)}{n^2\pi}\left[2+(-1)^{n+1}+(-1)^{n+1}\right]$

$\Rightarrow a_n = \frac{(-1)}{n^2\pi}\left[2+2(-1)^{n+1}\right]$

$\Rightarrow \boxed{a_n = \frac{-2\left(1+(-1)^{n+1}\right)}{n^2\pi}}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\text{Sin } nx\, dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\text{ Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\text{ Sin } nx dx+\frac{1}{\pi}\int_{0}^{\pi} x\text{ Sin } nx\, dx$

$\Rightarrow b_n = \frac{-1}{\pi}\int_{-\pi}^{0} x\text{ Sin } nx\, dx+\frac{1}{\pi}\int_{0}^{\pi} x\text{ Sin } nx\, dx \longrightarrow (3)$

In targeting by parts, we have

$\int x\text{ Sin } nx\, dx = \left(\frac{-\text{Cos } nx}{n}\right)-\int 1\frac{(-\text{Cos } nx)}{n}dx$

$\Rightarrow \int x\text{ Sin } nx\, dx = \frac{-x\text{ Cos } nx}{n}+\frac{1}{n}\int \text{Cos } nx\, dx$

$\Rightarrow \int x\text{ Sin } nx\, dx = \frac{-x\text{ Cos } nx}{n}+\frac{1}{n}\frac{\text{Sin } nx}{n}$

$\Rightarrow \int x\text{ Sin } nx\, dx = \frac{1}{n^2}\text{Sin } nx - \frac{x\text{ Cos } nx}{n}$

Now $\int_{-\pi}^{0} x\text{ Sin } nx\, dx = \left[\frac{\text{Sin } nx}{n^2}-\frac{x\text{ Cos } nx}{n}\right]_{-\pi}^{0}$

$\Rightarrow \int_{-\pi}^{0} x\text{ Sin } nx\, dx = (0-0)-\left(0-\frac{(-\pi)(-1)^n}{n}\right)$

$\Rightarrow \boxed{\int_{-\pi}^{0} x\text{ Sin } nx\, dx = \frac{\pi(-1)^{n+1}}{n}}$

and $\int_{0}^{\pi} x\text{ Sin } nx\, dx = \left[\frac{\text{Sin } nx}{n^2}-\frac{x\text{ Cos } nx}{n}\right]_{0}^{\pi}$

$\Rightarrow \int_{0}^{\pi} x\text{ Sin } nx\, dx = \left(0-\frac{\pi(-1)^n}{n}\right)-(0-0)$

$\Rightarrow \boxed{\int_{0}^{\pi} x\text{ Sin } nx = \frac{\pi(-1)^n}{n}}$

Then Eq (3) $\Rightarrow b_n = \frac{-1}{\pi}\left[\frac{\pi(-1)^{n+1}}{n}\right]+\frac{1}{\pi}\left[\frac{\pi(-1)^{n+1}}{n}\right]$

$\Rightarrow \boxed{b_n = 0}$

Putting the values in (1), we have

$f(x) = \frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{(-2)\left(1+(-1)^{n+1}\right)}{n^2\pi}\text{Cos } nx$

$\Rightarrow f(x) = \frac{\pi}{2}-\frac{2}{\pi}\sum\left[\frac{1+(-1)^{n+1}}{n^2}\right]\text{Cos } nx$

$\Rightarrow f(x) = \frac{\pi}{2}-\frac{2}{\pi}\left[\frac{2}{1^2}\text{Cos}x+\frac{2}{3^2}\text{Cos }3x+\frac{2}{5^2}\text{Cos}5x+\cdots\cdots\right]$

$\Rightarrow f(x) = \frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\text{Cos } x}{1^2}+\frac{\text{Cos }3x}{3^2}+\frac{\text{Cos }5x}{5^2}+\cdots\cdots\right] \to (4)$

Which is the required representation of $f(x)$ in the given interval.

Putting $x=0$ in Eq (4) we have

$f(0) = \frac{\pi}{2}-\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right]$

$\Rightarrow 0 = \frac{\pi}{2}-\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right]$

$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right]$

$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots$

Which is the required result.

DIRCHLET CONDITIONS FOR CONVERGENCE

THEOREM: If a function $f(x)$

Is periodic, single valued and finite in $[-\pi, \pi]$

i. Has a finite number of finite discontinuities

ii. Has only a finite number of maxima and minima in the interval, then the representing Fourier series converges to $f(x)$ at every point at which it is continuous and converges to the average of the right hand and left hand limits of $f(x)$ at each point of its discontinuity. In other words, the series $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos } nx+b_n\text{Sin } nx)$ converges to

$f(x)$ if x is point of continuity

$\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity where

$f(x+0) = \lim_{h\to 0} f(x+h) =$ L.H.S Limit of $f(x)$ at x. h>0

$f(x-0) = \lim_{h\to 0} f(x-h) =$ R.H.S Limit of $f(x)$ at x, h>0.

EXAMPLE:

If $f(x)$ is defined by

$$f(x)=\begin{cases}-\pi & \text{for } -\pi<x<0\\ x & \text{for } 0<x<\pi.\end{cases}$$

Find the Fourier series expansion for $f(x)$. Also find the sum of the series for x=0 and $x=\pm\pi$ and deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$

SOLUTION:

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \operatorname{Cos} nx+b_n \operatorname{Sin} nx)\longrightarrow(1)$

By Euler's formulae, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{0}(-\pi)dx+\frac{1}{\pi}\int_{0}^{\pi}x\,dx$$

$$\Rightarrow a_0=-\int_{-\pi}^{0}dx+\frac{1}{\pi}\int_{0}^{\pi}x\,dx$$

$$\Rightarrow a_0=-[x]_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$\Rightarrow a_2=-[0-(-\pi)]+\frac{1}{2\pi}[\pi^2-0^2]$$

$$\Rightarrow a_0=-\pi+\frac{\pi}{2}=\frac{-\pi}{2}$$

$$\Rightarrow \boxed{a_0=\frac{-\pi}{2}}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\operatorname{Cos} nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\operatorname{Cos} nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\operatorname{Cos} nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\operatorname{Cos} nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}x\operatorname{Cos} nx\,dx$$

$$\Rightarrow a_n=-\int_{-\pi}^{0}\operatorname{Cos} nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}x\operatorname{Cos} nx\,dx$$

$$\Rightarrow a_n=-\left[\frac{\operatorname{Sin} nx}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{x\operatorname{Sin} nx}{n}\right]_0^{\pi}-\frac{1}{\pi}\int_0^{\pi}\frac{1\operatorname{Sin} nx}{n}dx$$

$$\Rightarrow a_n=\frac{-1}{n}(0)+\frac{1}{\pi n}(0)-\frac{1}{n\pi}\int_0^{\pi}\operatorname{Sin} nx\,dx$$

$$\Rightarrow a_n=\frac{-1}{n\pi}\left[\frac{-\operatorname{Cos} nx}{n}\right]_0^{\pi}$$

$$\Rightarrow a_n=\frac{1}{n^2\pi}[\operatorname{Cos} n\pi-\operatorname{Cos} 0]$$

$$\Rightarrow a_n=\frac{1}{n^2\pi}\left[(-1)^n-1\right]$$

$$\boxed{a_n=\frac{(-1)^n-1}{n^2\pi}}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\operatorname{Sin} nx\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\operatorname{Sin} nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\operatorname{Sin} nx\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\operatorname{Sin} nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}x\operatorname{Sin} nx\,dx$$

$$\Rightarrow b_n=\left[\frac{\operatorname{Cos} nx}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{x(-\operatorname{Cos} nx)}{n}\right]_0^{\pi}-\frac{1}{\pi}\int_0^{\pi}\frac{1(-\operatorname{Cos} nx)}{n}dx$$

$$\Rightarrow b_n=\frac{1}{n}[\operatorname{Cos} 0-\operatorname{Cos} n\pi]-\frac{1}{n\pi}[\pi\operatorname{Cos} n\pi-0]+\frac{1}{n\pi}\int_0^{\pi}\operatorname{Cos} nx\,dx$$

$$\Rightarrow b_n=\frac{1}{n}\left[1-(-1)^n\right]-\frac{1}{n\pi}\left[\pi(-1)^n\right]+\frac{1}{n\pi}\left[\frac{\operatorname{Sin} nx}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n=\frac{1-(-1)^n}{n}-\frac{(-1)^n}{n}+\frac{1}{n\pi}(0)$$

$$\boxed{b_n=\frac{1-2(-1)^n}{n}}$$

Putting the values in Eq (1) we have

$$f(x)=\frac{-\pi}{4}+\sum_{n=1}^{\infty}\left(\left(\frac{(-1)^n-1}{n^2\pi}\right)\operatorname{Cos} nx+\left(\frac{1-2(-1)^n}{n}\right)\operatorname{Sin} nx\right)$$

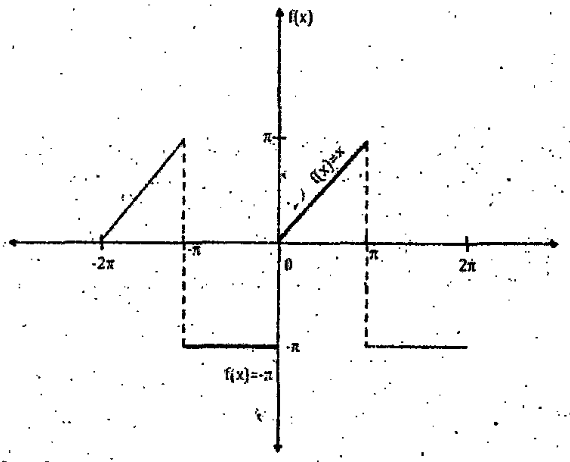
$$\Rightarrow f(x)=\frac{-\pi}{n}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi}\operatorname{Cos} nx+\sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\operatorname{Sin} nx$$

$$\Rightarrow f(x)=\frac{-\pi}{n}+\left[\frac{-2}{1^2\pi}\operatorname{Cos} x-\frac{2}{3^2\pi}\operatorname{Cos} 3x-\frac{2}{5^2\pi}\operatorname{Cos} 5x+\cdots\right]+$$

$$+\left[\frac{3}{1}\operatorname{Sin} x-\frac{1}{2}\operatorname{Sin} 2x+\frac{3}{3}\operatorname{Sin} 3x-\frac{1}{4}\operatorname{Sin} 4x+\cdots\cdots\right]$$

$$f(x)=\frac{-\pi}{4}-\frac{2}{\pi}\left[\frac{\operatorname{Cos} x}{1^2}+\frac{\operatorname{Cos} 3x}{3^2}+\frac{\operatorname{Cos} 5x}{5^2}+\cdots\cdots\right]+$$

$$+\left[3\operatorname{Sin} x-\frac{1}{2}\operatorname{Sin} 2x+\operatorname{Sin} 3x-\frac{1}{4}\operatorname{Sin} 4x+\cdots\cdots\right]\longrightarrow(2)$$



Clearly $x=-\pi,0,\pi$ are the points of discontinuities.

Putting $x=0$ in Eq (2) we have

$$f(0)=\frac{-\pi}{4}-\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right]\longrightarrow(3)$$

Since $f(x)$ is discontinuous at $x=0$,

So by Dirichlet conditions, we have

$$f(0)=\frac{\lim\limits_{x\to 0^-}f(x)+\lim\limits_{x\to 0^+}f(x)}{2}$$

$$\Rightarrow f(0)=\frac{\lim\limits_{x\to 0}(-\pi)+\lim\limits_{x\to 0}(x)}{2}$$

$$\Rightarrow f(0)=\frac{-\pi+0}{2}=\frac{-\pi}{2}$$

Putting $f(0)=\frac{-\pi}{2}$ in Eq (3) we have

$$\frac{-\pi}{2}=\frac{-\pi}{4}-\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right]$$

$$\Rightarrow\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots$$

Now when $x=\pi$ then

$$\text{Value of the series}=\frac{\lim\limits_{x\to\pi^-}f(x)+\lim\limits_{x\to\pi^+}\varphi(x)}{2}$$ where $\varphi(x)$ is the periodic extension of $f(x)$.

$$\text{Value of the series}=\frac{\lim\limits_{x\to\pi^-}f(x)+\lim\limits_{x\to\pi^+}f(x+2\pi)}{2}$$

$$\text{Value of the series}=\frac{\lim\limits_{x\to\pi}f(x)+\lim\limits_{x\to\pi}(-\pi)}{2}$$

$$\text{Value of the series}=\frac{\lim\limits_{x\to\pi^-}f(x)+\lim\limits_{x\to\pi^+}\varphi(x)}{2}$$

$$\text{Value of the series}=\frac{\pi-\pi}{2}$$

Value of the series $=0$

Finally, when $x=-\pi$ then

$$\text{Value of the series}=\frac{\lim\limits_{x\to-\pi^-}\varphi(x)+\lim\limits_{x\to-\pi^+}f(x)}{2}$$ where $\varphi(x)$ is the periodic extension of $f(x)$.

$$\text{Value of the series}=\frac{\lim\limits_{x\to-\pi^-}f(x+2\pi)+\lim\limits_{x\to-\pi^+}f(x)}{2}$$

$$\text{Value of the series}=\frac{\lim\limits_{x\to-\pi}(x+2\pi)+\lim\limits_{x\to-\pi}(-\pi)}{2}$$

$$\text{Value of the series}=\frac{-\pi+2\pi-\pi}{2}=0$$

EXERCISE 6.1

QUESTION 1:

Obtain the Fourier series representing $f(x)=x,\ -\pi<x<\pi$

SOLUTION:

Given that $f(x)=x,\ -\pi<x<\pi$

Let $f(x)=x=\frac{a_0}{2}+\sum\limits_{n=1}^{\infty}(a_n\operatorname{Cos}nx+b_n\operatorname{Sin}nx)\longrightarrow(1)$

By Euler's Formula, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x\,dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_0=\frac{1}{2\pi}\left[(\pi)^2-(-\pi)^2\right]$$

$$\Rightarrow a_0=\frac{1}{2\pi}\left[\pi^2-\pi^2\right]=0$$

Thus $\boxed{a_0=0}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\left[x.\left(\frac{\operatorname{Sin}nx}{n}\right)\right]_{-\pi}^{\pi}-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1.\operatorname{Sin}x}{n}dx$$

$$\Rightarrow a_n=\frac{1}{n\pi}(0)-\frac{1}{n\pi}\int_{-\pi}^{\pi}\operatorname{Sin}nx\,dx$$

$$\Rightarrow a_n=\frac{-1}{n\pi}\left[\frac{-\operatorname{Cos}nx}{n}\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_n=\frac{1}{n^2\pi}\left[\operatorname{Cos}n\pi-\operatorname{Cos}(-n\pi)\right]$$

$\Rightarrow a_n = \frac{1}{n^2\pi}(0)$

$\Rightarrow a_n = 0$

Thus $\boxed{a_n = 0}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Sin}\,nx\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\text{Sin}\,nx\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[x\left(\frac{-\text{Cos}\,nx}{n}\right)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} 1\left(\frac{-\text{Cos}\,nx}{n}\right)dx$

$\Rightarrow b_n = \frac{-1}{n\pi}[x\,\text{Cos}\,nx]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\text{Cos}\,nx\,dx$

$\Rightarrow b_n = \frac{-1}{n\pi}[\pi\,\text{Cos}\,n\pi - (-\pi)\,\text{Cos}(-n\pi)] + \frac{1}{n\pi}\left[\frac{\text{Sin}\,nx}{n}\right]_{-\pi}^{\pi}$

$\Rightarrow b_n = \frac{-1}{n\pi}[\pi\,\text{Cos}\,n\pi + \pi\,\text{Cos}\,n\pi] + \frac{1}{n\pi}(0)$

$\Rightarrow b_n = \frac{-1}{n\pi}(2\pi\,\text{Cos}\,n\pi)$

$\Rightarrow b_n = \frac{-2}{n}(-1)^n = \frac{2(-1)^{n+1}}{n}$

Thus $\boxed{b_n = \frac{2(-1)^{n+1}}{n}}$

Putting the values in Eq (1) we have

$x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\,\text{Cos}\,nx + b_n\,\text{Sin}\,nx)$

$\Rightarrow x = \frac{0}{2} + \sum_{n=1}^{\infty}\left((0)\text{Cos}\,nx + \frac{2(-1)^{n+1}}{n}\text{Sin}\,nx\right)$

$\Rightarrow x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\text{Sin}\,nx$

$\Rightarrow x = 2\left(\text{Sin}\,x - \frac{1}{2}\text{Sin}\,2x + \frac{1}{3}\text{Sin}\,3x - \frac{1}{4}\text{Sin}\,4x + \cdots\cdots\right)$

Which is the required Fourier series representation of the given function in the given interval.

QUESTION 2:

Obtain the Fourier series for $f(x) = e^{-x}$ in the internal $0 < x < 2\pi$

SOLUTION:

Let $f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\text{Cos}\,nx + b_n\,\text{Sin}\,nx) \rightarrow (1)$

By Euler's Formulae, we have

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$

$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$

$\Rightarrow a_0 = \frac{-1}{\pi}\left[e^{-2\pi} - e^0\right]$

$\Rightarrow a_0 = \frac{-1}{\pi}(e^{-2\pi} - 1)$

or $\boxed{a_0 = \frac{1}{\pi}(1 - e^{-2\pi})}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\text{Cos}\,nx\,dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,\text{Cos}\,nx\,dx \longrightarrow (2)$

Now let

$A = \int e^{-x}\,\text{Cos}\,nx\,dx = e^{-x}\frac{\text{Sin}\,nx}{n} - \int\left(\frac{e^{-x}}{-1}\right)\frac{\text{Sin}\,nx}{n}dx$

$\Rightarrow A = \frac{e^{-x}\,\text{Sin}\,nx}{n} + \frac{1}{n}\int e^{-x}\,\text{Sin}\,nx\,dx$

$\Rightarrow A = \frac{e^{-x}\,\text{Sin}\,nx}{n} + \frac{1}{n}\left[e^{-x}\left(\frac{-\text{Cos}\,nx}{n}\right) - \int\frac{e^{-x}}{-1}\frac{(-\text{cos}\,nx)}{n}dx\right]$

$\Rightarrow A = \frac{e^{-x}\,\text{Sin}\,nx}{n} - \frac{1}{n^2}e^{-x}\,\text{Cos}\,nx - \frac{1}{n^2}\int e^{-x}\,\text{Cos}\,nx\,dx$

$\Rightarrow A = \frac{e^{-x}\,\text{Sin}\,nx}{n} - \frac{e^{-x}\,\text{Cos}\,nx}{n^2} - \frac{1}{n^2}A$

$\Rightarrow A + \frac{1}{n^2}A = \frac{e^{-x}}{n^2}(n\,\text{Sin}\,nx - \text{Cos}\,nx)$

$\Rightarrow \frac{(n^2+1)}{n^2}A = \frac{e^{-x}}{n^2}(n\,\text{Sin}\,nx - \text{Cos}\,nx)$

$\Rightarrow A = \frac{e^{-x}}{n^2+1}(n\,\text{Sin}\,nx - \text{Cos}\,nx)$

$\Rightarrow \int e^{-x}\,\text{Cos}\,nx\,dx = \frac{e^{-x}}{n^2+1}(n\text{Sin}\,nx - \text{Cos}\,nx)$

$\Rightarrow \int_0^{2\pi} e^{-x}\,\text{Cos}\,nx\,dx = \frac{1}{n^2+1}\left[e^{-x}(n\,\text{Sin}\,nx - \text{Cos}\,nx)\right]_0^{2\pi}$

$\Rightarrow \int_0^{2\pi} e^{-x}\,\text{Cos}\,nx\,dx = \frac{1}{n^2+1}\left[\begin{array}{l} e^{-2\pi}(n\text{Sin}\,2\pi n - \text{Cos}\,2n\pi) \\ -e^0(n\text{Sin}0 - \text{Cos}0)\end{array}\right]$

$\Rightarrow \int_0^{2\pi} e^{-x}\,\text{Cos}\,nx\,dx = \frac{1}{n^2+1}\left[e^{-2\pi}(0-(+1)) - 1(0-1)\right]$

$\Rightarrow \int_0^{2\pi} e^{-x}\,\text{Cos}\,nx\,dx = \frac{1}{n^2+1}\left[1 - e^{-2\pi}\right]$

Then Eq (2) $\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx = \frac{1}{(n^2+1)\pi}(1-e^{-2\pi})$

$$\Rightarrow \boxed{a_n = \frac{1}{(n^2+1)\pi}(1-e^{-2\pi})}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\frac{n}{(n^2+1)}(1-e^{-2\pi})}$$

$$\left(\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right)$$

Putting the values in Eq (1), we have

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+1}+\frac{n\sin nx}{n^2+1}\right)\right]$$

$$\text{or } e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n^2+1}(\cos nx + n\sin nx)\right]$$

Which is the required Fourier series representation of the given function in the given interval.

QUESTION 3:

Find the Fourier series expansion of the step function $f(x)$ which is periodic with period 2π and is defined in $-\pi<x<\pi$ by

$$f(x)=\begin{cases}-1, & -\pi<x<0\\ 0, & x=0\\ +1, & 0<x<\pi\end{cases}$$

SOLUTION:

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \longrightarrow (1)$

By Euler's formulae we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_0^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-1)dx + \frac{1}{\pi}\int_0^{\pi}(1)dx$$

$$\Rightarrow a_0 = \frac{-1}{\pi}[x]_{-\pi}^{0} + \frac{1}{\pi}[x]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{-1}{\pi}[0-(-\pi)] + \frac{1}{\pi}[\pi-0]$$

$$\Rightarrow a_0 = -1+1=0$$

$$\boxed{\text{Thus } a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx$$

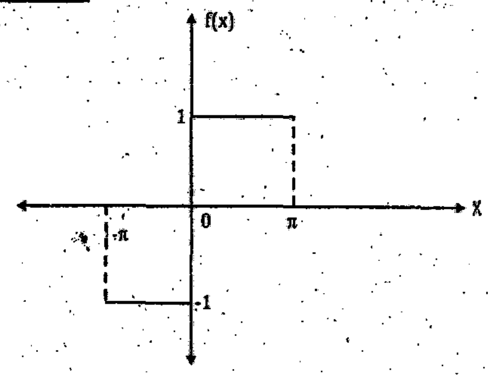
$$\Rightarrow a_n = \frac{-1}{\pi}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx$$

$$\Rightarrow a_n = \frac{-1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{-1}{\pi}(0) + \frac{1}{\pi}(0)$$

$$\Rightarrow a_n = 0+0=0$$

$$\boxed{\text{Thus } a_n = 0}$$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{-1}{\pi}\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$\Rightarrow b_n = \frac{-1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n = \frac{1}{n\pi}(\cos 0 - \cos(-n\pi)) - \frac{1}{n\pi}(\cos n\pi - \cos 0)$$

$$\Rightarrow b_n = \frac{1}{n\pi}(1-(-1)^n) - \frac{1}{n\pi}((-1)^n - 1)$$

$$\Rightarrow b_n = \frac{1}{n\pi}(1-(-1)^n-(-1)^n+1)$$

$$\Rightarrow b_n = \frac{1}{n\pi}(2-2(-1)^n)$$

$\Rightarrow b_n = \frac{2}{n\pi}\left(1-(-1)^n\right)$

Putting the values in Eq (1) we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx)$

$\Rightarrow f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left((0)\text{Cos } nx + \frac{2}{\pi}\left(\frac{1-(-1)^n}{n}\right)\text{Sin } nx\right)$

$\Rightarrow f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n}\text{Sin } nx$

$\Rightarrow f(x) = \frac{2}{\pi}\left(\frac{1}{2}\text{Sin } x + \frac{2}{3}\text{Sin } 3x + \frac{2}{5}\text{Sin } 5x + \cdots\cdots\right)$

$\Rightarrow f(x) = \frac{4}{\pi}\left(\text{Sin } x + \frac{\text{Sin } 3x}{3} + \frac{\text{Sin } 5x}{5} + \cdots\cdots\right)$

Which is the required Fourier series expansion of the given function in the given interval.

QUESTION 4:

Expand the function

$f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

in a Fourier series and examine the series at the points of discontinuity.

SOLUTION:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx) \longrightarrow (1)$

By Euler's formulae, we have

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi}(1)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2)\,dx$

$\Rightarrow a_0 = \frac{1}{\pi}[x]_0^{\pi} + \frac{1}{\pi}[2x]_{\pi}^{2\pi}$

$\Rightarrow a_0 = \frac{1}{\pi}[\pi - 0] + \frac{2}{\pi}[2\pi - \pi]$

$\Rightarrow a_0 = \frac{\pi}{\pi} + \frac{2\pi}{\pi}$

$\Rightarrow a_0 = 1 + 2 = 3$

$\Rightarrow \boxed{a_0 = 3}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\text{Cos } nx\, dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\text{Cos } nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\text{Cos } nx\, dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi}(1)\text{Cos } nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2)\text{Cos } nx\, dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi}\text{Cos } nx\, dx + \frac{2}{\pi}\int_{\pi}^{2\pi}\text{Cos } nx\, dx$

$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{\text{Sin } nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{\text{Sin } nx}{n}\right]_{\pi}^{2\pi}$

$\Rightarrow a_n = \frac{1}{n\pi}[\text{Sin } n\pi - \text{Sin } 0] + \frac{2}{n\pi}[\text{Sin } 2n\pi - \text{Sin } n\pi]$

$\Rightarrow a_n = \frac{1}{n\pi}(0) + \frac{2}{n\pi}(0)$

$\Rightarrow a_n = 0$

$\Rightarrow \boxed{a_n = 0}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\text{Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\,\text{Sin } nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,\text{Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi}(1)\text{Sin } nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2)\text{Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi}\text{Sin } nx\, dx + \frac{2}{\pi}\int_{\pi}^{2\pi}\text{Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{-\text{Cos } nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{-\text{Cos } nx}{n}\right]_{\pi}^{2\pi}$

$\Rightarrow b_n = \frac{-1}{n\pi}[\text{Cos } n\pi - \text{Cos } 0] - \frac{2}{n\pi}[\text{Cos } 2n\pi - \text{Cos } n\pi]$

$\Rightarrow b_n = \frac{-1}{n\pi}\left((-1)^n - 1\right) - \frac{2}{n\pi}\left(1 - (-1)^n\right)$

$\Rightarrow b_n = \frac{1}{n\pi}\left[1 - (-1)^n + 2(-1)^n - 2\right]$

$\boxed{b_n = \frac{1}{n\pi}\left[(-1)^n - 1\right]}$

Putting the values in (1), we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx)$

$\Rightarrow f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left((0)\text{Cos } nx + \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n}\right)\text{Sin } nx\right)$

$\Rightarrow f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n}\right)\text{Sin } nx$

$\Rightarrow f(x) = \frac{1}{\pi}\left(\frac{-2}{1}\text{Sin } x - \frac{2}{3}\text{Sin } 3x - \frac{2}{5}\text{Sin } 5x \cdots\cdots\right)$

$$\Rightarrow f(x)=\frac{-2}{\pi}\left(\operatorname{Sin}x+\frac{\operatorname{Sin}3x}{3}+\frac{\operatorname{Sin}5x}{5}+\cdots\cdots\right)$$

Which is the required Fourier series expansion of the given function in the given interval.

QUESTION 5:

Develop $f(x)$ in a Fourier series if defined as.

$$f(x)=\begin{cases}0, & -\pi<x\leq 0\\ 1, & 0<x\leq\pi\end{cases}$$

SOLUTION:

$$\text{Let } f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\operatorname{Cos}nx+b_n\operatorname{Sin}nx)\longrightarrow(1)$$

By Euler's Formulae, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{0}(0)dx+\frac{1}{\pi}\int_{0}^{\pi}(1)dx$$

$$\Rightarrow a_0=0+\frac{1}{\pi}[x]_0^{\pi}$$

$$\Rightarrow a_0=\frac{1}{\pi}[\pi-0]=1$$

$$\Rightarrow \boxed{a_0=1}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\operatorname{Cos}nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{0}(0)\operatorname{Cos}nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(1)\operatorname{Cos}nx\,dx$$

$$\Rightarrow a_n=0+\frac{1}{\pi}\left[\frac{\operatorname{Sin}nx}{n}\right]_0^{\pi}=0$$

$$\Rightarrow \boxed{a_n=0}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\operatorname{Sin}nx\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\operatorname{Sin}nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\operatorname{Sin}nx\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{0}(0)\operatorname{Sin}nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(1)\operatorname{Sin}nx\,dx$$

$$\Rightarrow b_n=0+\frac{1}{\pi}\int_{0}^{\pi}\operatorname{Sin}nx\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{-\operatorname{Cos}nx}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n=\frac{-1}{n\pi}[\operatorname{Cos}n\pi-\operatorname{Cos}0]$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left((-1)^n-1\right)$$

$$\boxed{b_n=\frac{1}{n\pi}\left(1-(-1)^n\right)}$$

Putting the values in Eq (1) we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\operatorname{Cos}nx+b_n\operatorname{Sin}nx)$$

$$\Rightarrow f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\left((0)\operatorname{Cos}nx+\frac{1}{\pi}\frac{\left(1-(-1)^n\right)}{n}\operatorname{Sin}nx\right)$$

$$\Rightarrow f(x)=\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\operatorname{Sin}nx$$

$$\Rightarrow f(x)=\frac{1}{2}+\frac{1}{\pi}\left[\frac{2}{1}\operatorname{Sin}x+\frac{2}{3}\operatorname{Sin}3x+\frac{2}{5}\operatorname{Sin}5x+\cdots\cdots\right]$$

$$\Rightarrow f(x)=\frac{1}{2}+\frac{2}{\pi}\left[\operatorname{Sin}x+\frac{1}{3}\operatorname{Sin}3x+\frac{1}{5}\operatorname{Sin}+\cdots\cdots\right]$$

Which is the required Fourier series representation of the given function in the given interval.

QUESTION 6:

Find the Fourier series for the function $f(x)$ defined as

$$f(x)=\begin{cases}x+\pi, & 0\leq x\leq\pi\\ -x-\pi, & -\pi\leq x<0\end{cases}$$

And $f(x+2\pi)=f(x)$

SOLUTION:

$$\text{Let } f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\operatorname{Cos}nx+b_n\operatorname{Sin}nx)\longrightarrow(1)$$

By Euler's Formulae, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{-x^2}{2}-\pi x\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{\pi^2}{2}+\pi x\right]_0^{\pi}$$

$$\Rightarrow a_0=\frac{-1}{\pi}\left(0-\left(\frac{(-\pi)^2}{2}+\pi(-\pi)\right)\right)+\frac{1}{\pi}\left(\frac{\pi^2}{2}+\pi(\pi)-0\right)$$

$$\Rightarrow a_0 = \frac{-1}{\pi}\left(0-\left(\frac{\pi^2}{2}-\pi^2\right)\right)+\frac{1}{\pi}\left[\frac{\pi^2}{2}+\pi^2\right]$$

$$\Rightarrow a_0 = \frac{+1}{\pi}\left(\frac{-\pi^2}{2}\right)+\frac{1}{\pi}\left(\frac{3\pi^2}{2}\right)$$

$$\Rightarrow a_0 = \frac{-\pi}{2}+\frac{3\pi}{2}=\frac{2\pi}{2}=\pi$$

$$\Rightarrow \boxed{a_0=\pi}$$

$bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Cos}\, nx\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\text{Cos}\, nx\, dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\text{Cos}\, nx\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\text{Cos}\, nx\, dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\text{Cos}\, nx\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[(-x-\pi)\frac{\text{Sin}\, nx}{n}\right]_{-\pi}^{0}\frac{-1}{\pi}\int_{-\pi}^{0}(-1-0)\frac{\text{Sin}\, nx}{n}dx$$

$$+\frac{1}{\pi}\left[(x+\pi)\frac{\text{Sin}\, nx}{n}\right]_{0}^{\pi}-\frac{1}{\pi}\int_{0}^{\pi}(1+0)\frac{\text{Sin}\, nx}{n}dx$$

$$\Rightarrow a_n = \frac{1}{\pi}(0)+\frac{1}{n\pi}\int_{-\pi}^{0}\text{Sin}\, nx\, dx+\frac{1}{\pi}(0)-\frac{1}{n\pi}\int_{0}^{\pi}\text{Sin}\, nx\, dx$$

$$\Rightarrow a_n = \frac{1}{n\pi}\left[\frac{-\text{Cos}\, nx}{n}\right]_{-\pi}^{0}-\frac{1}{n\pi}\left[\frac{-\text{Cos}\, nx}{n}\right]_{0}^{\pi}$$

$$\Rightarrow a_n = \frac{-1}{n^2\pi}[\text{Cos}0-\text{Cos}\, n\pi]+\frac{1}{n^2\pi}[\text{Cos}\, n\pi-\text{Cos}0]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}[-\text{Cos}0+\text{Cos}\, n\pi+\text{Cos}\, n\pi-\text{Cos}\, 0]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}[2\,\text{Cos}\, n\pi-2\text{Cos}\, 0]$$

$$\Rightarrow a_n = \frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$\Rightarrow \boxed{a_n = \frac{2}{n^2\pi}\left((-1)^n-1\right)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Sin}\, nx\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,\text{Sin}\, nx\, dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\,\text{Sin}\, nx\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\,\text{Sin}\, nx\, dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\,\text{Sin}\, nx\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[(-x-\pi)\frac{(-\text{Cos}\, nx)}{n}\right]_{-\pi}^{0}-\frac{1}{\pi}\int_{-\pi}^{0}(-1-0)\frac{(-\text{Cos}\, nx)}{n}dx+$$

$$+\frac{1}{\pi}\left[(x+\pi)\frac{(-\text{Cos}\, nx)}{n}\right]_{0}^{\pi}-\frac{1}{\pi}\int_{0}^{\pi}(1+0)\frac{(-\text{Cos}\, nx)}{n}dx$$

$$\Rightarrow b_n = \frac{1}{n\pi}\left[(x+\pi)\text{Cos}\, nx\right]_{-\pi}^{0}-\frac{1}{n\pi}\int_{-\pi}^{0}\text{Cos}\, nx\, dx+$$

$$-\frac{1}{n\pi}\left[(x+\pi)\text{Cos}\, nx\right]_{0}^{\pi}+\frac{1}{n\pi}\int_{0}^{\pi}\text{Cos}\, nx\, dx$$

$$\Rightarrow b_n = \frac{1}{n\pi}\left[(0+\pi)\text{Cos}\, 0-(-\pi+\pi)\text{Cos}(-n\pi)\right]-\frac{1}{n\pi}\left[\frac{\text{Sin}\, nx}{n}\right]_{-\pi}^{0}+$$

$$-\frac{1}{n\pi}\left[(\pi+\pi)\text{Cos}\, n\pi-(0+\pi)\text{Cos}\, 0\right]+\frac{1}{n\pi}\left[\frac{\text{Sin}\, nx}{n}\right]_{0}^{\pi}$$

$$\Rightarrow b_n = \frac{1}{n\pi}(\pi-0)-\frac{1}{n\pi}(0)-\frac{1}{n\pi}\left(2\pi(-n)^n-\pi\right)+\frac{1}{n\pi}(0)$$

$$\Rightarrow b_n = \frac{1}{n}-0-\frac{1}{n}\left(2(-1)^n-1\right)+0$$

$$\Rightarrow b_n = \frac{1}{n}\left[1-2(-1)^n+1\right]$$

$$\Rightarrow b_n = \frac{1}{n}\left(2-2(-1)^n\right)$$

$$\Rightarrow \boxed{b_n = \frac{2}{n}\left(1-(-1)^n\right)}$$

Putting the values in Eq (1) we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\,\text{Cos}\, nx+\sum_{n=1}^{\infty}b_n\,\text{Sin}\, nx$$

$$\Rightarrow f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left((-1)^n-1\right)\text{Cos}\, nx+\sum_{n=1}^{\infty}\frac{2}{n}\left(1-(-1)^n\right)\text{Sin}\, nx$$

$$\Rightarrow f(x)=\frac{\pi}{2}+\left[\frac{-4}{1^2\pi}\text{Cos}\, x-\frac{4}{3^2\pi}\text{Cos}\, 3\pi-\frac{4}{5^2\pi}\text{Cos}5x+\cdots\cdots\right]$$

$$+\left[\frac{4}{1}\text{Sin}\, x+\frac{4}{5}\text{Sin}\, 3x+\frac{4}{5}\text{Sin}\, 5x+\cdots\cdots\right]$$

$$\Rightarrow f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\text{Cos}x+\frac{1}{3^2}\text{Cos}3x+\frac{1}{5^2}\text{Cos}5x+\cdots\cdots\right]$$

$$+4\left[\text{Sin}\, x+\frac{1}{3}\text{Sin}\, 3x+\frac{1}{5}\text{Sin}5x+\cdots\cdots\right]$$

Which is the required Fourier series for the given function in the given interval.

QUESTION 7:

Obtain the Fourier series representing the function

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ \text{Sin}x, & 0\le x\le\pi\end{cases}$$

SOLUTION:

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos}\, nx+b_n\text{Sin}\, nx)\longrightarrow(1)$

By Euler's Formulae, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \text{Sin}x\,dx$$

$$\Rightarrow a_0 = 0 + \frac{1}{\pi}\left[(-\text{Cos}\,x)\right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{-1}{\pi}[\text{Cos}\,\pi - \text{Cos}0]$$

$$\Rightarrow a_0 = \frac{-1}{\pi}[-1-1] = \frac{2}{\pi}$$

$$\Rightarrow \boxed{a_0 = \frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Cos}\,nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\text{Cos}\,nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\text{Cos}\,nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0}(0)\text{Cos}\,nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\text{Sin}\,x\,\text{Cos}\,nx\,dx$$

$$\Rightarrow a_n = 0 + \frac{1}{2\pi}\int_{0}^{\pi} 2\,\text{Sin}x\,\text{Cos}\,nx\,dx$$

$$\Rightarrow a_n = \frac{1}{2\pi}\int_{0}^{\pi}\left[\text{Sin}\,(n+1)x - \text{Sin}(n-1)x\right]dx$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[\frac{-\text{Cos}\,(n+1)x}{n+1} + \frac{\text{Cos}\,(n-1)x}{n-1}\right]_0^{\pi} \quad \text{where } n \neq 1$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[\left(\frac{-\text{Cos}(n+1)\pi}{n+1} + \frac{\text{Cos}(n-1)\pi}{n-1}\right) - \left(\frac{-\text{Cos}0}{n+1} + \frac{\text{Cos}\,0}{n-1}\right)\right]$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[(\text{Cos}\,n\pi)\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[(\text{Cos}\,n\pi)\left(\frac{-2}{n^2-1}\right) + \frac{(-2)}{n^2-1}\right]$$

$$a_n = \frac{1}{2\pi}\left(\frac{-2}{n^2-1}\right)(\text{Cos}\,n\pi + 1)$$

$$\Rightarrow \boxed{a_n = \frac{-1}{(n^2-1)\pi}(\text{Cos}\,n\pi + 1) \quad \text{where } n \neq 1}$$

For $n = 1$ we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Cos}x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,\text{Cos}x\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\text{Cos}x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_{-\pi}^{0}(0)\text{Cos}\,x\,dx + \frac{1}{\pi}\int_{0}^{\pi}\text{Sin}x\,\text{Cos}x\,dx$$

$$\Rightarrow a_1 = 0 + \frac{1}{2\pi}\int_{0}^{\pi} 2\,\text{Sin}\,x\,\text{Cos}x\,dx$$

$$\Rightarrow a_1 = \frac{1}{2\pi}\int_{0}^{\pi}\text{Sin}\,2x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[\frac{-\text{Cos}\,2x}{2}\right]_0^{\pi}$$

$$\Rightarrow a_1 = \frac{-1}{2\pi}[\text{Cos}\,2\pi - \text{Cos}\,0]$$

$$\Rightarrow a_1 = \frac{-1}{2\pi}(0) = 0$$

$$\Rightarrow \boxed{a_1 = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Sin}\,nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\text{Sin}\,nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\text{Sin}\,nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0}(0)\text{Sin}\,nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\text{Sin}x.\text{Sin}\,nx\,dx$$

$$\Rightarrow b_n = 0 + \frac{1}{2\pi}\int_{0}^{\pi} 2\,\text{Sin}\,x.\text{Sin}nx\,dx$$

$$\Rightarrow b_n = \frac{1}{2\pi}\int_{0}^{\pi}\left[\text{Cos}\,(n-1)x - \text{Cos}(n+1)x\right]dx$$

$$\Rightarrow b_n = \frac{1}{2\pi}\left[\frac{\text{Sin}\,(n-1)x}{n-1} - \frac{\text{Sin}(n+1)x}{n+1}\right]_0^{\pi} \quad n \neq 1$$

$$\Rightarrow b_n = \frac{1}{2\pi}(0) = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

For $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Sin}x\,dx$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\text{Sin}x\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\text{Sin}x\,dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\int_{-\pi}^{0}(0)\text{Sin}x\,dx + \frac{1}{\pi}\int_{0}^{\pi}\text{Sin}x\,\text{sin}\,x\,dx$$

$$\Rightarrow b_1 = 0 + \frac{1}{\pi}\int_{0}^{\pi}\text{sin}^2 x\,dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{1-\text{Cos}\,2x}{2}\right)dx$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\left[\frac{x - \text{Sin}\,2x}{2}\right]_0^{\pi}$$

$$\Rightarrow b_1 = \frac{1}{2\pi}[\pi] = \frac{1}{2}$$

$\Rightarrow \boxed{b_1 = \frac{1}{2}}$

Now Eq (1) can be written as

$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + \sum_{n=2}^{\infty}(a_n \cos nx + b_n \sin nx)$

putting the values, we have

$f(x) = \frac{1}{\pi} + (0)\cos n + \frac{1}{2}\sin x +$

$+\sum_{n\neq 2}^{\infty} \frac{-1}{(n^2-1)\pi}([\cos(n\pi)+1]\cos nx + (0)\sin nx)$

or $f(x) = \frac{1}{\pi} - \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty}\left(\frac{\cos(n\pi)+1}{n^2-1}\cos nx\right)$

$\Rightarrow f(x) = \frac{1}{\pi} - \frac{1}{\pi}\frac{1}{2}\sin x + \left[\frac{2}{2^2-1}\cos 2x + \frac{2}{4^2-1}\cos 4\pi + \cdots\cdots\right]$

$\Rightarrow f(x) = \frac{1}{\pi} + \frac{2}{\pi}\frac{1}{2}\sin x + \left[\frac{1}{2^2-1}\cos 2x - \frac{1}{4^2-1}\cos 4x + \cdots\cdots\right]$

Which is the required Fourier series representation of the given function in the given interval.

QUESTION 8:

Find the Fourier series corresponding to the function $f(x)$ defined by

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

SOLUTION:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \longrightarrow (1)$

By Euler's Formulae, we have

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi}\left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}$

$\Rightarrow a_0 = \frac{1}{2\pi}\left[\pi^2 - o^2\right] + \frac{1}{\pi}\left[\left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]$

$\Rightarrow a_0 = \frac{1}{2\pi}(\pi^2) + \frac{1}{\pi}\left[2\pi^2 - \frac{3\pi^2}{2}\right]$

$\Rightarrow a_0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

$\Rightarrow \boxed{a_0 = \pi}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos nx\,dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx$

$a_n = \frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}\frac{(1)\sin nx\,dx}{n} + \frac{1}{\pi}\left[(2\pi - x)\frac{\sin nx}{n}\right]_{\pi}^{\pi} +$

$-\frac{1}{\pi}\int_{\pi}^{2\pi}(0-1)\frac{\sin nx}{n}dx$

$\Rightarrow a_n = \frac{1}{\pi}(0) - \frac{1}{n\pi}\int_0^{\pi}\sin nx\,dx + \frac{1}{\pi}[0] + \frac{1}{n\pi}\int_{\pi}^{2\pi}\sin nx\,dx$

$\Rightarrow a_n = \frac{-1}{n\pi}\left[\frac{(-\cos nx)}{n}\right]_0^{\pi} + \frac{1}{n\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}$

$\Rightarrow a_n = \frac{1}{n^2\pi}[\cos n\pi - \cos o] - \frac{1}{n^2\pi}[\cos 2n\pi - \cos n\pi]$

$\Rightarrow a_n = \frac{1}{n^2\pi}[\cos n\pi - \cos o - \cos 2n\pi + \cos n\pi]$

$\Rightarrow a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1 - 1 + (-1)^n\right]$

$\Rightarrow a_n = \frac{1}{n^2\pi}\left[2(1)^n - 2\right]$

$\Rightarrow a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$

$\Rightarrow \boxed{a_n = \frac{2((-1)^n - 1)}{\pi n^2}}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{x(-\cos nx)}{n}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}(1)\frac{(-\cos nx)}{n}dx +$

$+\frac{1}{\pi}\left[(2\pi - x)\frac{(-\cos nx)}{n}\right]_{\pi}^{2\pi} - \frac{1}{\pi}\int_{\pi}^{2\pi}(0-1)\frac{(-\cos nx)}{n}dx$

$\Rightarrow b_n = \frac{-1}{n\pi}[\pi\cos n\pi - 0] + \frac{1}{n\pi}\int_0^{\pi}\cos nx\,dx +$

$\frac{(-1)}{n\pi}[0 - \pi\cos n\pi] - \frac{1}{n\pi}\int_{\pi}^{2\pi}\cos nx\,dx$

$$\Rightarrow b_n = -\frac{1}{n}(-1)^n + \frac{1}{n\pi}\left[\frac{\text{Sin } nx}{n}\right]_0^{\pi} + \frac{1}{n}(-1)^n - \frac{1}{n\pi}\left[\frac{\text{Sin } nx}{n}\right]_{\pi}^{2\pi}$$

$$\Rightarrow b_n = \frac{-(-1)^n}{n} + \frac{1}{n\pi}[0] + \frac{(-1)^n}{n} - \frac{1}{n\pi}[0]$$

$$\Rightarrow b_n \frac{-(-1)^n}{n} + \frac{(-1)^n}{n} = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

Putting the values in Eq (1) we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx)$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2((-1)^n - 1)}{\pi n^2}\text{Cos } nx + (o)\text{Sin } nx\right)$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\text{Cos } nx$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2}{1^2}\text{Cos}x - \frac{2}{3^2}\text{Cos } 3x - \frac{2}{5^2}\text{Cos } 5x + \cdots\cdots\right]$$

$$\Rightarrow f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\text{Cos } x + \frac{1}{3^2}\text{Cos } 3x + \frac{1}{5^2}\text{Cos } 5x + \cdots\cdots\right]$$

Which is the required Fourier series corresponding to the given function.

QUESTION 9:

Obtain the Fourier series expansion for $f(x)$ defined as follows.

$$f(x) = \begin{cases} x + \frac{\pi}{2}, & -\pi < x \le 0 \\ \frac{\pi}{2} - x, & 0 \le x < \pi \end{cases}$$

SOLUTION:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx) \longrightarrow (1)$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0}\left(x + \frac{\pi}{2}\right)dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{\pi}{2}x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[0 - \left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right)\right] + \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2} - 0\right]$$

$$\Rightarrow a_0 = \frac{1}{\pi}(0) + \frac{1}{\pi}(0) = 0$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Cos } nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,\text{Cos } nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,\text{Cos } nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0}\left(x + \frac{\pi}{2}\right)\text{Cos } nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)\text{Cos } nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\left(x + \frac{\pi}{2}\right)\frac{\text{Sin } nx}{n}\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}(1+0)\frac{\text{Sin } nx}{n}dx$$

$$+\frac{1}{\pi}\left[\left(\frac{\pi}{2} - x\right)\frac{\text{Sin } nx}{n}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}(0-1)\frac{\text{Sin } nx}{n}dx$$

$$\Rightarrow a_n = \frac{1}{\pi}(0) - \frac{1}{n\pi}\int_{-\pi}^{0}\text{Sin } nx\,dx + \frac{1}{\pi}(0) + \frac{1}{n\pi}\int_0^{\pi}\text{Sin } nx\,dx$$

$$\Rightarrow a_n = \frac{-1}{n\pi}\left[\frac{-\text{Cos } nx}{n}\right]_{-\pi}^{0} + \frac{1}{n\pi}\left[\frac{-\text{Cos } nx}{n}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[\text{Cos } o - \text{Cos}(n\pi)\right] - \frac{1}{n\pi}\left[\text{Cos } n\pi - \text{Cos } o\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[1 - (-1)^n\right] - \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[2 - 2(-1)^n\right]$$

$$\Rightarrow \boxed{a_n = \frac{2(1 - (-1)^n)}{n^2\pi}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Sin } nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,\text{Sin } nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\,\text{Sin } nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0}\left(x + \frac{\pi}{2}\right)\text{Sin } nx\,dx + \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\text{Sin } nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\left(x + \frac{\pi}{2}\right)\left(\frac{-\text{Cos } nx}{n}\right)\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}(1+0)\left(\frac{-\text{Cos } nx}{n}\right)dx$$

$$+\frac{1}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{-\text{Cos } nx}{n}\right)\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}(0-1)\left(\frac{-\text{Cos } nx}{n}\right)dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[\frac{\pi}{2}\text{Cos } o + \frac{\pi}{2}\text{Cos}(-n\pi)\right] + \frac{1}{n\pi}\int_{-\pi}^{0}\text{Cos } nx\,dx$$

$$+\frac{(-1)}{n\pi}\left[\frac{-\pi}{2}\text{Cos } n\pi - \frac{\pi}{2}\text{Cos } o\right] - \frac{1}{n\pi}\int_0^{\pi}\text{Cos } nx\,dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}(-1)^n\right]+\frac{1}{n\pi}\left[\frac{\text{Sin } nx}{n}\right]_{-\pi}^{0}+$$

$$-\frac{1}{n\pi}\left[\frac{-\pi}{2}(-1)^n-\frac{\pi}{2}\right]-\frac{1}{n\pi}\left[\frac{\text{Sin } nx}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n = \frac{-\pi}{2n\pi}\left(1+(-1)^n\right)+\frac{1}{n\pi}(0)+\frac{\pi}{2n\pi}\left[(-1)^n+1\right]-\frac{1}{n\pi}(0)$$

$$\Rightarrow b_n = \frac{-1}{2n}\left(1+(-1)^n\right)+\frac{1}{2n}\left((-1)^n+1\right)$$

$$\Rightarrow b_n = \frac{1}{2n}\left[-1-(-1)^n+(-1)^n+1\right]$$

$$\Rightarrow \boxed{b_n = 0}$$

Putting the values in Eq (1), we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos } nx+b_n\text{Sin } nx)$$

$$\Rightarrow f(x)=\frac{0}{2}+\sum_{n=1}^{\infty}\left(\frac{2\left(1-(-1)^n\right)}{n^2\pi}\text{Cos } nx+(0)\text{Sin } nx\right)$$

$$\Rightarrow f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n^2}\text{Cos } nx$$

$$\Rightarrow f(x)=\frac{2}{\pi}\left[\frac{2}{1^2}\text{Cos } x+\frac{2}{3^2}\text{Cos } 3x+\frac{2}{5^2}\text{Cos } 5x+\cdots\cdots\right]$$

$$\Rightarrow f(x)=\frac{4}{\pi}\left[\text{Cos } x+\frac{1}{3^2}\text{Cos } 3x+\frac{1}{5^2}\text{Cos } 5x+\cdots\cdots\right]$$

Which is the required Fourier series of $f(x)$.

QUESTION 10:

Find the Fourier series to represent

$$f(x)=\frac{1}{4}(\pi-x)^2, 0<x<2\pi$$

SOLUTION:

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos } nx+b_n\text{Sin } nx)\longrightarrow(1)$

By Euler's Formulae, we have

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\,dx$$

$$\Rightarrow a_0=\frac{1}{4\pi}\left[\frac{-(\pi-x)^3}{3}\right]_0^{2\pi}$$

$$\Rightarrow a_0=\frac{-1}{12\pi}\left[(-\pi)^3-\pi^3\right]$$

$$\Rightarrow a_0=\frac{-1}{12\pi}\left[-\pi^3-\pi^3\right]$$

$$\Rightarrow a_0=\frac{2\pi^3}{12\pi}=\frac{\pi^2}{6}$$

$$\Rightarrow \boxed{a_0=\frac{1}{6}\pi^2}$$

$$a_n=\frac{1}{4}\int_0^{2\pi}f(x)\text{Cos } nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-\pi)^2\text{Cos } nx\,dx$$

$$\Rightarrow a_n=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\text{Cos } nx\,dx$$

$$\Rightarrow a_n=\frac{1}{4\pi}\left[(\pi-x)^2\frac{\text{Sin } nx}{n}\right]_0^{2\pi}-\frac{1}{4\pi}\int_0^{2\pi}-2(\pi-x)\frac{\text{Sin } nx}{n}dx$$

$$\Rightarrow a_n=\frac{1}{4\pi}[0]+\frac{2}{4n\pi}\int_0^{2\pi}(\pi-x)\text{ Sin } nx\,dx$$

$$\Rightarrow a_n=\frac{1}{2n\pi}\left[(\pi-x)\left(\frac{-\text{Cos } nx}{n}\right)\right]_0^{2\pi}-\frac{1}{2n\pi}\int_0^{2\pi}(0-1)\frac{(-\text{Cos } nx)}{n}dx$$

$$\Rightarrow a_n=\frac{-1}{2n^2\pi}\left[-\pi\text{Cos } 2n\pi-\pi\text{Cos } o\right]-\frac{1}{2n^2\pi}\int_0^{2\pi}\frac{\text{Cos } nx}{0}dx$$

$$\Rightarrow a_n=\frac{-1}{2n^2\pi}[-\pi-\pi]-\frac{1}{2n^2\pi}\left[\frac{\text{Sin } nx}{n}\right]_0^{2\pi}$$

$$\Rightarrow a_n=\frac{+2\pi}{2n^2\pi}-\frac{1}{2n^2\pi}[0]$$

$$\boxed{a_n=\frac{1}{n^2}}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\text{ Sin } nx\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\text{ Sin } nx\,dx$$

$$\Rightarrow b_n=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\text{ Sin } nx\,dx$$

$$\Rightarrow b_n=\frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\text{Cos } nx}{n}\right)\right]_0^{2\pi}-\frac{1}{4\pi}\int_0^{2\pi}-2(\pi-x)\frac{(-\text{Cos } nx)}{n}dx$$

$$\Rightarrow b_n=\frac{1}{4n\pi}\left[\pi^2\cos 2n\pi-\pi^2\cos 0\right]-\frac{2}{4n\pi}\int_0^{2\pi}(\pi-x)\cos nx\,dx$$

$$\Rightarrow b_n=\frac{-1}{4n\pi}\left[\pi^2-\pi^2\right]-\frac{1}{2n\pi}\left[(\pi-x)\frac{\text{Sin } nx}{n}\right]_0^{2\pi}+\frac{1}{2n\pi}\int_0^{2\pi}(0-1)\frac{\text{Sin } nx}{n}dx$$

$$\Rightarrow b_n=\frac{-1}{4n\pi}(0)-\frac{1}{2n^2\pi}(0)-\frac{1}{2n^2\pi}\int_0^{2\pi}\text{Sin } nx\,dx$$

$$\Rightarrow b_n=\frac{-1}{2n^2\pi}\left[\frac{-\text{Cos } nx}{n}\right]_0^{2\pi}$$

$\Rightarrow b_n = \frac{1}{2n^3\pi}[\text{Cos } 2n\pi - \text{Cos } 0] = 0$

$\Rightarrow \boxed{b_n = 0}$

Putting the values in Eq (1), we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx)$

$\Rightarrow f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\text{Cos } nx + (0)\text{Sin } nx\right)$

$\Rightarrow f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\text{Cos } nx}{n^2}$

$\Rightarrow \frac{1}{4}(\pi - x)^2 = \frac{\pi^2}{12} + \left[\text{Cos } x + \frac{\text{Cos } 2x}{2^2} + \frac{\text{Cos } 3x}{3^2} + \ldots\ldots\right]$

Which is the required Fourier series representation of the given function.

QUESTION 11:

Find the Fourier series expansion for $f(x)$, if $f(x) = x^2$, $0 < x < 2\pi$ and deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

SOLUTION:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx) \longrightarrow (1)$

By Euler's Formulae, we have

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx$

$a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} \quad a_n = \frac{1}{3\pi}\left[8\pi^3\right]$

$\Rightarrow \boxed{a_0 = \frac{8}{3}\pi^2}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{Cos } nx dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \text{Cos } nx\, dx$

$\Rightarrow a_n = \frac{1}{\pi}\left[x^2\frac{\text{Sin } nx}{n}\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi}\frac{2x \text{ Sin } nx}{n}dx$

$\Rightarrow a_n = 0 - \frac{2}{n\pi}\int_0^{2\pi} x \sin nx\, dx$

$\Rightarrow a_n = \frac{-2}{n\pi}\left[x\left(\frac{-\text{Cos } nx}{n}\right)\right]_0^{2\pi} + \frac{2}{n\pi}\int_0^{2\pi}(1)\left(\frac{-\text{Cos } nx}{n}\right)dx$

$\Rightarrow a_n = \frac{2}{n^2\pi}[x \text{ Cos } nx]_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi}\text{Cos } nx\, dx$

$\Rightarrow a_n = \frac{2}{n^2\pi}[2\pi \text{ Cos } 2n\pi - 0] - \frac{2}{n^2\pi}\left[\frac{\text{Sin } nx}{n}\right]_0^{2\pi}$

$\Rightarrow a_n = \frac{2}{n^2\pi}[2\pi] - \frac{2}{n^2\pi}(0)$

$\Rightarrow \boxed{a_n = \frac{4}{n^2}}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\text{ Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \text{ Sin } nx\, dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[x^2\left(\frac{-\text{Cos } nx}{n}\right)\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi} 2x\left(\frac{-\text{Cos } nx}{n}\right)dx$

$\Rightarrow b_n = \frac{-1}{n\pi}\left[4\pi^2 \text{Cos } 2n\pi - 0\right] + \frac{2}{n\pi}\int_0^{2\pi} x \text{ Cos } nx\, dx$

$\Rightarrow b_n = \frac{-4\pi}{n} + \frac{2}{n\pi}\left[\frac{x \text{ Sin } nx}{n}\right]_0^{2\pi} - \frac{2}{n\pi}\int_0^{2\pi}\frac{1 \text{ Sin } nx}{n}dx$

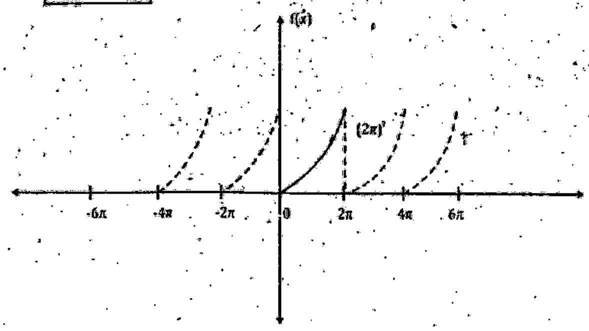
$\Rightarrow b_n = \frac{-4\pi}{n} + 0 - \frac{2}{n^2\pi}\int_0^{2\pi}\text{Sin } nx dx$

$\Rightarrow b_n = \frac{-4\pi}{n} - \frac{2}{n^2\pi}\left[\frac{-\text{Cos } nx}{n}\right]_0^{2\pi}$

$\Rightarrow b_n = \frac{-4\pi}{n} + \frac{2}{n^2\pi}[\text{Cos } 2n\pi - \text{Cos } o]$

$\Rightarrow b_n = \frac{-4\pi}{n} + 0$

$\Rightarrow \boxed{b_n = \frac{-4\pi}{n}}$



Putting the values in Eq (1), we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \text{Cos } nx + b_n \text{Sin } nx)$

$\Rightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\text{Cos } nx - \frac{4\pi}{n}\text{Sin } nx\right)$

$\Rightarrow x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\text{Cos } nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\text{Sin } nx}{n}$

$$\Rightarrow f(x)=x^2=\frac{4\pi^2}{3}+4\left[\cos x+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\cdots\right]$$
$$-4\pi\left[\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\cdots\cdots\right]\longrightarrow(2)$$

Which is the required Fourier series representation of the given function in the given interval.

Clearly $x=0,2\pi$ are the points of jump discontinuities of the function.

Putting $x=0$ in Eq (2), we have

$$f(0)=\frac{4\pi^2}{3}+4\left[\cos 0+\frac{\cos 0}{2^2}+\frac{\cos 0}{3^2}+\cdots\cdots\right]+$$
$$-4\pi\left[\sin 0+\frac{\sin 0}{2}+\frac{\sin 0}{3}+\cdots\cdots\right]$$
$$\Rightarrow f(0)=\frac{4\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots\right]\longrightarrow(3)$$

By Dirichlet conditions, we have

$$f(0)=\frac{\lim\limits_{x\to 0^-}f(x)+\lim\limits_{x\to 0^+}f(x)}{2}$$
$$\Rightarrow f(0)=\frac{(2\pi)^2+0}{2}$$
$$\Rightarrow f(0)=\frac{4\pi^2}{2}=2\pi^2$$
$$\Rightarrow f(0)=2\pi^2$$

Putting it in Eq (3), we have

$$2\pi^2=\frac{4\pi^3}{3}+\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots\right]$$
$$\Rightarrow 2\pi^2-\frac{4\pi^3}{3}=4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\Rightarrow \frac{2\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\Rightarrow \frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Which is the required result.

NOTE: We get same result by putting $x=2\pi$ in Eq (2) and then using Dirichlet conditions.

QUESTION 12:

Find a Fourier series to represent $x-x^2$ from $x=-\pi$ to $x=\pi$ and deduce that

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\cdots$$

SOLUTION:

Let $f(x)=x-x^2=\frac{a_0}{2}+\sum\limits_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\rightarrow(1)$

By Euler's Formulae, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$
$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx$$
$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-\pi}^{\pi}$$
$$\Rightarrow a_0=\frac{1}{\pi}\left[\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)-\left(\frac{(-\pi)^2}{2}-\frac{(-\pi)^3}{3}\right)\right]$$
$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{\pi^2}{2}-\frac{\pi^3}{3}-\frac{\pi^2}{2}-\frac{\pi^3}{3}\right]$$
$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{-2\pi^3}{3}\right]=\frac{-2\pi^2}{3}$$

Thus $\boxed{a_0=\frac{-2\pi^2}{3}}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$
$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$
$$\Rightarrow a_n=\frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n}\right]_{-\pi}^{\pi}-\frac{1}{\pi}\int_{-\pi}^{\pi}(1-2x)\frac{\sin nx}{n}dx$$
$$\Rightarrow a_n=\frac{1}{\pi}[0]-\frac{1}{n\pi}\left[(1-2x)\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{\pi}+$$
$$+\frac{1}{n\pi}\int_{-\pi}^{\pi}(0-2)\left(\frac{-\cos nx}{n}\right)dx$$
$$\Rightarrow a_n=0+\frac{1}{n^2\pi}\left[(1-2\pi)\cos n\pi-(1+2\pi)\cos(-n\pi)\right]+$$
$$+\frac{2}{n^2\pi}\int_{-\pi}^{\pi}\cos nx\,dx$$
$$\Rightarrow a_n=\frac{1}{n^2\pi}\left[\cos n\pi-2\pi\cos n\pi-\cos n\pi-2\pi\cos n\pi\right]+$$
$$+\frac{2}{n^2\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}$$
$$\Rightarrow a_n=\frac{1}{n^2\pi}\left[-4\pi\cos n\pi\right]+\frac{2}{n^3\pi}[0]$$
$$\Rightarrow a_n=\frac{-4\cos n\pi}{n^2}+0$$

Thus $\boxed{a_n=\frac{-4}{n^2}\cos n\pi}$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,\text{Sin}nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\text{Cos}\,nx}{n}\right)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}(1-2x)\left(\frac{-\text{Cos}\,nx}{n}\right)dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[(\pi-\pi^2)\text{Cos}\,n\pi - (-\pi-\pi^2)\text{Cos}n\pi\right] +$$

$$+\frac{1}{n\pi}\int_{-\pi}^{\pi}(1-2x)\,\text{Cos}nx\,dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[\pi\,\text{Cos}\,n\pi - \pi^2\,\text{Cos}n\pi + \pi\,\text{Cos}n\pi + \pi^2\,\text{Cos}n\pi\right] +$$

$$+\frac{1}{n\pi}\int_{-\pi}^{\pi}(1-2x)\text{Cos}nx\,dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[2\pi\,\text{Cos}n\pi] + \frac{1}{n\pi}\left[(1-2x)\frac{\text{Sin}\,x}{n}\right]_{-\pi}^{\pi}$$

$$-\frac{1}{n\pi}\int_{-\pi}^{\pi}(0-2)\text{Cos}nx\,dx$$

$$\Rightarrow b_n = \frac{-2}{n}\text{Cos}n\pi + \frac{1}{n^2\pi}[0] + \frac{2}{n^2\pi}\int_{-\pi}^{\pi}\text{Sin}nx\,dx$$

$$\Rightarrow b_n = \frac{-2}{n}\text{Cos}n\pi + \frac{2}{n^2\pi}\left[\frac{-\text{Cos}nx}{dx}\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n = \frac{-2}{n}\text{Cos}n\pi - \frac{2}{n^3\pi}\left[\text{Cos}n\pi - \text{Cos}n(-\pi)\right]$$

$$\Rightarrow b_n = \frac{-2}{n}\text{Cos}n\pi - \frac{2}{n^3\pi}[\text{Cos}n\pi - \text{Cos}n\pi]$$

$$\Rightarrow b_n = \frac{-2}{n}\text{Cos}n\pi - \frac{2}{n^2\pi}(0)$$

Thus $\boxed{b_n = \frac{-2}{n}\text{Cos}n\pi}$

Putting the values in Eq (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\text{Cos}nx + b_n\text{Sin}nx)$$

$$\Rightarrow x-x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\text{Cos}n\pi.\text{Cos}nx - \frac{2}{n}\text{Cos}n\pi.\,\text{Sin}nx\right)$$

$$\Rightarrow x-x^2 = -\frac{\pi^3}{3} - 4\sum_{n=1}^{\infty}\frac{\text{Cos}n\pi.\,\text{Cos}n\pi}{n^2} - 2\sum_{n=1}^{\infty}\frac{\text{Cos}n\pi\,\text{Sin}nx}{n}$$

$$\Rightarrow x-x^2 = \frac{-\pi^3}{3} - 4\left[\frac{\text{Cos}\pi.\,\text{Cos}x}{1^2} + \frac{\text{Cos}2\pi.\,\text{Sin}2\pi}{2^2} + \frac{\text{Cos}3\pi.\text{Cos}3x}{3^2} + \frac{\text{Cos}4\pi.\,\text{Cos}4x}{4^2} + \ldots\ldots\right]$$

$$-2\left[\frac{\text{Cos}\pi.\text{Sin}x}{1} + \frac{\text{Cos}2\pi.\,\text{Sin}2x}{2} + \frac{\text{Cos}3\pi.\,\text{Sin}3\pi}{3} + \frac{\text{Cos}4\pi.\,\text{Sin}4x}{4} + \ldots\right]$$

$$\Rightarrow x-x^2 = \frac{-\pi^3}{3} - 4\left[\frac{-\text{Cos}x}{1^2} + \frac{\text{Cos}2x}{2^2} - \frac{\text{Cos}3x}{3^2} + \frac{\text{Cos}4x}{4^2}\ldots\right] +$$

$$-2\left[-\text{Sin}x + \frac{\text{Sin}2x}{2} - \frac{\text{Sin}3x}{3} + \frac{\text{Sin}4x}{4} + \ldots\ldots\right]$$

or

$$x-x^2 = \frac{-\pi^3}{3} + 4\left[\frac{\text{Cos}x}{1^2} - \frac{\text{Cos}2x}{2^2} + \frac{\text{Cos}3x}{3^2} - \frac{\text{Cos}4x}{4^2} + \ldots\ldots\right] +$$

$$+2\left[\text{Sin}x - \frac{\text{Sin}2x}{2} + \frac{\text{Sin}3x}{3} - \frac{\text{Sin}4x}{4} + \ldots\ldots\right] \rightarrow (2)$$

which is the require Fourier series representation of the given function in the given interval.

Setting $x=0$ in Eq (2) we have

$$0-0^2 = \frac{-\pi^3}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right]$$

$$\Rightarrow \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right]$$

$$\Rightarrow \frac{\pi^3}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$$

Which is the required result.

QUESTION 13:

Expand $f(x) = x\text{Sin}x, 0 \le x \le 2\pi$ in a Fourier series.

SOLUTION:

Given that $f(x) = x\text{Sin}x, 0 \le x \le 2\pi$

By Euler's formula, we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi}x\,\text{Sin}x\,dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[x(-\text{Cos}x)\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi}1.(-\text{Cos}x)dx$$

$$\Rightarrow a_0 = \frac{-1}{\pi}[2\pi\,\text{Cos}\,2\pi - 0\,\text{Cos}\,0] + \frac{1}{\pi}\int_0^{2\pi}\text{Cos}\,x\,dx$$

$$\Rightarrow a_0 = \frac{-1}{\pi}[2\pi - 0] + \frac{1}{\pi}[\text{Sin}x]_0^{2\pi}$$

$$\Rightarrow a_0 = -2 + \frac{1}{\pi}[0] = -2$$

Thus $\boxed{a_0 = -2}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}f(x)\text{Cos}nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi}x\,\text{Sin}x.\,\text{Cos}nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}x.\,2\text{Sin}\,x.\,\text{Cos}nx\,dx$$

$$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\text{Sin}(n+1)x - \text{Sin}(n-1)x\right]dx$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[x\left[\left(\frac{-\text{Cos}(n+1)x}{n+1}\right) - \left(\frac{-\text{Cos}(n-1)x}{n-1}\right)\right]\right]_0^{2\pi} +$$

$$-\frac{1}{2\pi}\int_0^{2\pi} 1\left[\frac{-\text{Cos}(n+1)x}{n+1} - \frac{-\text{Cos}(n-1)x}{n-1}\right]dx$$

$$\Rightarrow a_n = \frac{-1}{2\pi}\left[x\left(\frac{\text{Cos}(n+1)x}{n+1} - \frac{\text{Cos}(n-1)x}{n-1}\right)\right]_0^{2\pi} +$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\text{Cos}(n+1)x}{n+1} - \frac{\text{Cos}(n-1)x}{n-1}\right]dx$$

$$\Rightarrow a_n = \frac{-1}{2\pi}\left[2\pi\left(\frac{\text{Cos}(n+1)2\pi}{n+1} - \frac{\text{Cos}(n-1)2\pi}{n-1}\right) - 0\right] +$$

$$+\frac{1}{2\pi(n+1)}\int_0^{2\pi}\text{Cos}(n+1)x\,dx - \frac{1}{2\pi(n-1)}\int_0^{2\pi}\text{Cos}(n-1)x\,dx$$

$$\Rightarrow a_n = \frac{-1}{2\pi}\left[2\pi\left(\frac{\text{Cos}(2n\pi+2\pi)}{n+1} - \frac{\text{Cos}(2n\pi-2\pi)}{n-1}\right)\right] +$$

$$+\frac{1}{2\pi(n+1)}\left[\frac{\text{Sin}(n+1)x}{(n+1)}\right]_0^{2\pi} - \frac{1}{2\pi(n-1)}\left[\frac{\text{Sin}(n-1)x}{n-1}\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{-1}{2\pi}\left[2\pi\left(\frac{\text{Cos}\,2\pi}{n+1} - \frac{\text{Cos}(-2\pi)}{n-1}\right)\right] + 0 - 0$$

$$\Rightarrow a_n = -\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = -\left[\frac{n-1-(n+1)}{(n+1)(n-1)}\right]$$

$$\Rightarrow a_n = -\left[\frac{n-1-n-1}{n^2-1}\right] = \frac{-(-2)}{n^2-1} = \frac{2}{n^2-1}$$

Thus $\boxed{a_n = \frac{2}{n^2-1}}$, $n \neq 1$

For $n=1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Sin}\,x\,\text{Cos}x\,dx$$

$$\Rightarrow a_1 = \frac{1}{2\pi}\int_0^{2\pi} x.\,2\text{Sin}x.\,\text{Cos}x\,dx$$

$$\Rightarrow a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\,\text{Sin}2x\,dx$$

$$\Rightarrow a_1 = \frac{1}{2\pi}\left[x\left(\frac{-\text{Cos}2x}{2}\right)\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi}(1)\left(\frac{-\text{Cos}\,2x}{2}\right)dx$$

$$\Rightarrow a_1 = \frac{-1}{4\pi}\left[x\,\text{Cos}2x\right]_0^{2\pi} + \frac{1}{4\pi}\int_0^{2\pi}\text{Cos}\,2x\,dx$$

$$\Rightarrow a_1 = \frac{-1}{4\pi}\left[2\pi\,\text{Cos}\,4\pi - 0\right] + \frac{1}{4\pi}\left[\frac{\text{Sin}\,2x}{2}\right]_0^{2\pi}$$

$$\Rightarrow a_1 = \frac{-1}{4\pi}[2\pi] + \frac{1}{4\pi}[0]$$

$$\boxed{a_1 = \frac{-1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\text{Sin}\,nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Sin}\,x.\,\text{Sinnx}\,dx$$

$$\Rightarrow b_n = \frac{1}{2\pi}\int_0^{2\pi} x2\,\text{Sinx}.\,\text{Sinx}\,dx$$

$$\Rightarrow b_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\text{Cos}(n-1)x - \text{Cos}(n+1)x\right]dx$$

$$\Rightarrow b_n = \frac{1}{2\pi}\left[x\left(\frac{\text{Sin}(n-1)x}{n-1} - \frac{\text{Sin}(n+1)x}{n+1}\right)\right]_0^{2\pi} +$$

$$-\frac{1}{2\pi}\int_0^{2\pi}(1)\left(\frac{\text{Sin}(n-1)x}{n-1} - \frac{\text{Sin}(n+1)}{n+1}\right)dx$$

$$\Rightarrow b_n = \frac{1}{2\pi}[0] - \frac{1}{2\pi(n-1)}\int_0^{2\pi}\text{Sin}(n-1)x\,dx$$

$$+\frac{1}{2\pi(n+1)}\int_0^{2\pi}\text{Sin}(n+1)x\,dx$$

$$\Rightarrow b_n = 0 - \frac{1}{2\pi(n-1)}\left[\frac{-\text{Cos}(n-1)x}{n-1}\right]_0^{2\pi}$$

$$+\frac{1}{2\pi(n+1)}\left[\frac{-\text{Cos}(n+1)x}{n+1}\right]_0^{2\pi}$$

$$\Rightarrow b_n = \frac{1}{2\pi(n-1)^2}\left[\text{Cos}(n-1)2\pi - \text{Cos}0\right] +$$

$$-\frac{1}{2\pi(n+1)^2}\left[\text{Cos}(n+1)2\pi - \text{Cos}0\right]$$

$$\Rightarrow b_n = \frac{1}{2\pi(n-1)^2}\left[\text{Cos}(-2\pi) - \text{Cos}\,0\right] +$$

$$-\frac{1}{2n(n+1)^2}\left[\text{Cos}2\pi - \text{Cos}\,0\right]$$

$$= \frac{1}{2\pi(n-1)^2}\left[\text{Cos}\,2\pi - \text{Cos}0\right] - \frac{1}{2n(n+1)^2}\left[\text{Cos}\,2\pi - \text{Cos}0\right]$$

$$\Rightarrow b_n = 0, \quad n \neq 1$$

Thus $\boxed{b_n = 0}$ for $n \neq 1$

For n=1, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Sin}x.\,\text{Sin}\,x\,dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Sin}^2x\,dx$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\,2\text{Sin}^2x\,dx$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\int_0^{2\pi} x(1-\text{Cos}\,2x)dx$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\left[x\left(x-\frac{\sin 2x}{2}\right)\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi}(1)\left(x-\frac{\sin 2x}{2}\right)dx$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\left[2\pi\left(\frac{2\pi - \text{Sin}\,4\pi}{2}\right)-0\right] \frac{-1}{2\pi}\int_0^{2\pi}\left(x-\frac{\text{Sin}\,2x}{2}\right)dx$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\left[4\pi^2\right] - \frac{1}{2\pi}\left[\frac{x^2}{2}+\frac{\text{Cos}\,2x}{4}\right]_0^{2\pi}$$

$$\Rightarrow b_1 = 2\pi - \frac{1}{2\pi}\left[\frac{4\pi^2}{2}+\frac{\text{Cos}\,4\pi}{4}-\frac{o^2}{2}-\frac{\text{Cos}\,o}{4}\right]$$

$$\Rightarrow b_1 = 2\pi - \frac{1}{2\pi}\left[2\pi^2 + \frac{1}{4} - 0 - \frac{1}{4}\right]$$

$$\Rightarrow b_1 = 2\pi - \pi = \pi$$

Thus $\boxed{b_1 = \pi}$

Hence the required Fourier is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{Cos}nx + \sum_{n=1}^{\infty} b_n \text{Sin}nx$$

or $f(x) = \frac{a_0}{2} + a_1\text{Cos}\,x + b_1\text{Sin}\,x + \sum_{n=2}^{\infty} a_n\text{Cos}\,nx + \sum_{n=2}^{\infty} b_n\text{Sin}\,nx$

$$\Rightarrow x\,\text{Sin}\,x = \frac{-2}{2} + \left(\frac{-1}{2}\right)\text{Cos}x + \pi\text{Sin}x + $$

$$+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\text{Cos}nx + \sum_{n=2}^{\infty}(o)\text{Sin}\,nx$$

$$\Rightarrow x\text{Sin}\,x = -1 - \frac{\text{Cos}\,x}{2} + \pi\,\text{Sin}x + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\text{Cos}nx$$

or $x\text{Sin}x = -1 + \pi\,\text{Sin}x - \frac{1}{2}\text{Cos}x +$

$$+2\left(\frac{1}{2^2-1}\text{Cos}\,2x + \frac{1}{3^2-1}\text{Cos}3x + \frac{1}{4^2-1}\text{Cos}\,4x + \cdots\cdots\right) \text{Ans}$$

QUESTION 14 (i):

Expand the following function in Fourier series

$f(x) = x, \quad \text{if } -\pi < x \le 0$

$\quad = 2x, \quad \text{if } 0 \le x \le \pi$

SOLUTION:

By Euler's Formulae, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_0^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_0^{\pi} 2x.dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{2x^2}{2}\right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[0-\pi^2\right] + \frac{1}{\pi}\left[\pi^2 - 0\right]$$

$$\Rightarrow a_0 = \frac{-\pi^2}{2\pi} + \frac{\pi^2}{\pi} = \frac{-\pi}{2} + \pi = \frac{\pi}{2}$$

Thus $\boxed{a_0 = \frac{\pi}{2}}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Cos}nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\text{Cos}nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\text{Cos}nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} x.\,\text{Cos}nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2x.\,\text{Cos}nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{x\,\text{Sin}nx}{n}\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}(1)\frac{\text{Sin}nx}{n}dx +$$

$$+\frac{2}{\pi}\left[\frac{x\,\text{Sin}nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}(1)\frac{\text{Sin}nx}{n}dx$$

$$\Rightarrow a_n = \frac{1}{\pi}[0] - \frac{1}{n\pi}\int_{-\pi}^{0}\text{Sin}nx\,dx + \frac{2}{\pi}[0] - \frac{2}{n\pi}\int_0^{\pi}\text{Sin}nx\,dx$$

$$\Rightarrow a_n = 0 - \frac{1}{n\pi}\left[\frac{-\text{Cos}nx}{n}\right]_{-\pi}^{0} + 0 - \frac{2}{n\pi}\left[\frac{-\text{Cos}nx}{n}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[\text{Cos}nx\right]_{-\pi}^{0} + \frac{2}{n^2\pi}\left[\text{Cos}nx\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[\text{Cos}0 - \text{Cos}n\pi\right] + \frac{2}{n^2\pi}\left[\text{Cos}n\pi - \text{Cos}o\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[1-(-1)^n\right] + \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[1-(-1)^n + 2(-1)^n - 2\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

Thus $\boxed{a_n = \frac{(-1)^n - 1}{n^2\pi}}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Sinnx}\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\text{Sinnx}\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,\text{Sinnx}\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} x.\text{Sinnx}\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x.\text{Sinnx}\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\,\text{Sinnx}\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\,\text{Sinnx}\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[x\left(\frac{-\text{Cosnx}}{n}\right)\right]_{-\pi}^{0} - \frac{1}{\pi}\int (1)\left(\frac{-\text{Cosnx}}{n}\right)dx$

$+\frac{2}{\pi}\left[x\left(\frac{-\text{Cosnx}}{n}\right)\right]_{0}^{\pi} - \frac{2}{\pi}\int_{0}^{\pi}(1)\left(\frac{-\text{Cosnx}}{n}\right)dx$

$\Rightarrow b_n = \frac{-1}{n\pi}[x.\text{Cosnx}]_{-\pi}^{0} + \frac{1}{n\pi}\int_{-\pi}^{0}\text{Cosnx}\,dx - \frac{2}{\pi}[x.\text{Cosnx}]_{0}^{\pi}$

$+\frac{2}{n\pi}\int_{0}^{\pi}\text{Cosnx}\,dx$

$\Rightarrow b_n = \frac{-1}{n\pi}[0+\pi\,\text{Cosnx}] + \frac{1}{n\pi}\left[\frac{\text{Sinnx}}{n}\right]_{-\pi}^{0} +$

$-\frac{2}{n\pi}[\pi\,\text{Cos}n\pi - 0] + \frac{2}{n\pi}\left[\frac{\text{Sinnx}}{n}\right]_{0}^{\pi}$

$\Rightarrow b_n = \frac{1}{n}\text{Cos}n\pi + 0 - \frac{2}{n}\text{Cos}n\pi + 0$

$\Rightarrow b_n = \frac{-1}{n}(-1)^n - \frac{2}{n}(-1)^n = \left(\frac{-1}{n} - \frac{2}{n}\right)(-1)^n = \frac{-3}{n}(-1)^n$

Thus $\boxed{b_n = \frac{3(-1)^{n+1}}{n}}$

Hence the required Fourier series is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\text{Cosnx} + b_n\text{Sinnx})$

$\Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\left[\frac{(-1)^n - 1}{n^2\pi}\right]\text{Cosnx} + \frac{3(-1)^{n+1}}{n}\text{Sinnx}\right)$

or $f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\text{Cosnx} + 3\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\text{Sinnx}$

QUESTION 14 (ii):

Expand the following function in a Fourier series

$f(x) = x^2, \qquad -\pi \le x \le \pi$

SOLUTION:

By Euler's Formulae, we have

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx$

$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{3\pi}\left[\pi^3 - (-\pi)^3\right]$

$\Rightarrow a_0 = \frac{1}{3\pi}\left[\pi^3 + \pi^3\right] = \frac{2\pi^3}{3\pi} = \frac{2}{3}\pi^2$

Thus $\boxed{a_0 = \frac{2}{3}\pi^2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\text{Cosnx}\,dx$

$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\text{Cosnx}\,dx$

$\Rightarrow a_n = \frac{1}{\pi}\left[x^2\frac{\text{Sinnx}}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x.\frac{\text{Sinnx}}{n}dx$

$\Rightarrow a_n = \frac{1}{n\pi}\left[x^2.\text{Sinnx}\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x.\text{Sinnx}\,dx$

$\Rightarrow a_n = \frac{1}{n\pi}[0] - \frac{2}{n\pi}\left[x\left(\frac{-\text{Cosnx}}{n}\right)\right]_{-\pi}^{\pi} + \frac{2}{n\pi}\int_{-\pi}^{\pi}(1)\frac{(-\text{Cosnx})}{n}dx$

$\Rightarrow a_n = 0 + \frac{2}{n^2\pi}[x\,\text{Cos}n\pi]_{-\pi}^{\pi} - \frac{2}{n^2\pi}\left[\frac{\text{Sinnx}}{n}\right]_{-\pi}^{\pi}$

$\Rightarrow a_n = \frac{2}{n^2\pi}\left[2\pi(-1)^n\right] - 0$

$\Rightarrow a_n = \frac{4\pi(-1)^n}{n^2\pi} = \frac{4(-1)^n}{n^2}$

Thus $\boxed{a_n = \frac{4(-1)^n}{n^2}}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\text{Sinnx}\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\text{Sinnx}\,dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[x^2\left(\frac{-\text{Cosnx}}{n}\right)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\left(\frac{-\text{Cosnx}}{n}\right)dx$

$\Rightarrow b_n = \frac{-1}{n\pi}\left[x^2.\text{Cosnx}\right]_{-\pi}^{\pi} + \frac{2}{n\pi}\int_{-\pi}^{\pi} x\,\text{Cosnx}\,dx$

$\Rightarrow b_n = \frac{-1}{n\pi}\left[x^2\,\text{Cosnx} - \pi^2\text{Cos}n\pi\right] + \frac{2}{n\pi}\left[\frac{x\,\text{Sinnx}}{n}\right]_{-\pi}^{\pi} +$

$-\frac{2}{n\pi}\int_{-\pi}^{\pi}(1)\frac{\text{Sinnx}\,dx}{n}$

$\Rightarrow b_n = \frac{-1}{n\pi}[0] + \frac{2}{n\pi}[0] + \frac{2}{n^2\pi}\int_{-\pi}^{\pi} -\text{Sinnx}\,dx$

$\Rightarrow b_n = 0+0+\frac{2}{n^2\pi}\left[\frac{\text{Cos}nx}{n}\right]_{-\pi}^{\pi}$

$\Rightarrow b_n = \frac{2}{n^3\pi}[\text{Cos}nx]_{-\pi}^{\pi} = \frac{2}{n^2\pi}[0] = 0$

Thus $\boxed{b_n = 0}$

Hence the required Fourier series is

$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos}nx + b_n\text{Sin}nx)$

$\Rightarrow x^2 = \frac{\pi^2}{3}+\sum\left(\frac{4(-1)^n}{n^2}\text{Cos}nx + 0.\text{Sin}nx\right)$

$\Rightarrow x^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\text{Cos}nx.$

FUNCTIONS HAVING ARBITRARY PERIODS

Some times we require Fourier series representation of a function over an interval of length different from 2π. In such cases, the interval of arbitrary length is converted into one of length 2π by suitable substitution.

Let a function $f(x)$ with period 2P be defined over the interval $-P<x<P$, where p is any +ve number we introduce a new variable "t" to change $f(x)$ to $g(t)$ with period 2π. For this purpose we let

$\frac{x}{P} = \frac{t}{\pi}$ or $x = \frac{P}{\pi}t$ then

$f(x) = f\left(\frac{P}{\pi}t\right) = g(t) \longrightarrow (1)$

Now $g(t+2\pi) = f\left(\frac{P}{\pi}(t+2\pi)\right)$

$g(t+2\pi) = f\left(\frac{Pt}{\pi}+2P\right)$

$g(t+2\pi) = f\left(\frac{Pt}{\pi}\right)$ ($\because$ f is a function of period 2P)

$g(t+2\pi) = g(t)$ (By(1))

Thus $g(t+2\pi) = g(t), -\pi<t<\pi$

Showing that $g(t)$ has period 2π.

Hence the Fourier series corresponding to $g(t)$ is

$g(t) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos}nt + b_n\text{Sin}nt) \rightarrow (2)$

Where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\text{Cos}nt\,dt, \quad n = 0,1,2,\cdots\cdots$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\text{Sin}\,nt\,dt$

To find the Fourier series for $f(x)$ in $-P<x<P$,

We replace t by $\frac{\pi}{P}x$ in (2) and get

$f(x) = g\left(\frac{\pi}{P}x\right) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+b_n\text{Sin}\left(\frac{n\pi x}{P}\right)\right) \rightarrow (3)$

where

$a_n = \frac{1}{\pi}\int_{-P}^{P}g\left(\frac{\pi x}{P}\right)\text{Cos}\left(\frac{n\pi x}{P}\right)\frac{\pi}{P}dx = \frac{1}{P}+$

$+\int_{-P}^{P}f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx, \quad n = 0,1,2,\cdots\cdots$

$\Rightarrow b_n = \frac{1}{\pi}\int_{-P}^{P}g\left(\frac{\pi x}{P}\right)\text{Sin}\left(\frac{n\pi x}{P}\right)\frac{\pi}{P}dx = \frac{1}{P}$

$+\int_{-P}^{P}f(x)\text{Sin}\left(\frac{n\pi x}{P}\right)dx, \quad n = 1,2,3,\cdots$

NOTE: If we use the limits 0 and 2P instead if –P and P, then the corresponding Euler's formulae are

$a_n = \frac{1}{P}\int_{0}^{2P}f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx, \quad n = 0,1,2,3,\cdots\cdots$

$b_n = \frac{1}{P}\int_{0}^{2P}f(x)\,\text{Sin}\left(\frac{n\pi x}{P}\right)dx, \quad n = 1,2,3,\cdots\cdots$

EXAMPLE 1:

Expand into Fourier series $f(x) = x^2, \quad -1<x<1$.

SOLUTION:

Given that $f(x) = x^2, \quad -1<x<1$

Here period $= 2P = 1-(-1) = 1+1 = 2 \Rightarrow \boxed{P=1}$

The Fourier series of the function $f(x)$ is

$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+b_n\text{Sin}\left(\frac{n\pi x}{P}\right)\right)$

Putting $p = 1$, we have

$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos}n\pi x + b_n\text{Sin}n\pi x) \longrightarrow (1)$

Where $a_0 = \frac{1}{P}\int_{-P}^{P}f(x)dx$

$\Rightarrow a_0 = \frac{1}{1}\int_{-1}^{1}x^2dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}\left[(1)^3-(-1)^3\right]$

$\Rightarrow a_0 = \frac{1}{3}[1+1] = \frac{2}{3}$

Thus $\boxed{a_0 = \frac{2}{3}}$

$a_n = \frac{1}{P}\int_{-P}^{P}f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx$

$$\Rightarrow a_n = \frac{1}{1}\int_{-1}^{1} x^2 \cos\left(\frac{n\pi x}{1}\right)dx$$

$$\Rightarrow a_n = \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

$$\Rightarrow a_n = \left[x^2 \frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1} 2x \frac{\sin n\pi x}{n\pi} dx$$

$$\Rightarrow a_n = \frac{1}{n\pi}\left[x^2 . \sin n\pi x\right]_{-1}^{1} - \frac{2}{n\pi}\int_{-1}^{1} x . \sin n\pi x\, dx$$

$$\Rightarrow a_n = \frac{1}{n\pi}[0] - \frac{2}{n\pi}\left[\frac{x(-\cos n\pi x)}{n\pi}\right]_{-1}^{1} + \frac{2}{n\pi}\int_{-1}^{1}\frac{1(-\cos n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n = 0 + \frac{2}{n^2\pi^2}\left[x \cos n\pi x\right]_{-1}^{1} - \frac{2}{n^2\pi^2}\int_{-1}^{1} \cos n\pi x\, dx$$

$$\Rightarrow a_n = \frac{2}{n^2\pi^2}[2\cos n\pi] - \frac{2}{n^2\pi^2}\left[\frac{-\sin n\pi x}{n\pi}\right]_{-1}^{1}$$

$$\Rightarrow a_n = \frac{2}{n^2\pi^2}\left[2(-1)^n\right] + \frac{2}{n^3\pi^3}\left[\sin n\pi x\right]_{-1}^{1}$$

$$\Rightarrow a_n = \frac{4(-n)^n}{n^2\pi^2} + \frac{2}{n^2\pi^2}[0]$$

Thus $\boxed{a_n = \frac{4(-1)^n}{n^2\pi^2}}$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x) . \sin\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n = \frac{1}{1}\int_{-1}^{1} x^2 \sin\left(\frac{n\pi x}{1}\right)dx$$

$$\Rightarrow b_n = \int_{-1}^{1} x^2 \sin(n\pi x)dx$$

$$\Rightarrow b_n = \left[x^2\left(\frac{-\cos(n\pi x)}{n\pi}\right)\right]_{-1}^{1} - \int_{-1}^{1} 2x\left(\frac{-\cos(n\pi x)}{n\pi}\right)dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[x^2 . \cos n\pi x\right]_{-1}^{1} + \frac{2}{n\pi}\int_{-1}^{1} x \cos n\pi x\, dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[o] + \frac{2}{n\pi}\left[\frac{x . \sin(n\pi x)}{n\pi}\right]_{-1}^{1} - \frac{2}{n\pi}\int_{-1}^{1}\frac{(1)\sin(n\pi x)dx}{n\pi}$$

$$\Rightarrow b_n = 0 + \frac{2}{n^2\pi^2}\left[x . \sin(n\pi x)\right]_{-1}^{1} - \frac{2}{n^2\pi^2}\int_{-1}^{1}\sin(n\pi x)dx$$

$$\Rightarrow b_n = 0 + \frac{2}{n^2\pi^2}[o] - \frac{2}{n^2\pi^2}\left[\frac{-\cos n\pi x}{n\pi}\right]_{-1}^{1}$$

$$\Rightarrow b_n = \frac{2}{n^3\pi^3}\left[\cos n\pi x\right]_{-1}^{1}$$

$$\Rightarrow b_n = \frac{2}{n^3\pi^3}[\cos n\pi - \cos n\pi] = \frac{2}{n^3\pi^3}(0) = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

Putting the values in (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$$

$$\Rightarrow x^2 = \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2\pi^2}\cos n\pi x + (o)\sin n\pi x\right)$$

or $x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$.

EXAMPLE 2:

Prove, by Fourier series method, that

$$\frac{\ell}{2} - x = \frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{\ell}\right), \qquad 0 < x < \ell$$

SOLUTION:

Let $f(x) = \frac{\ell}{2} - x, \quad 0 < x < \ell$

Here period $= 2P = \ell - 0 \Rightarrow 2P = \ell \Rightarrow p = \frac{\ell}{2}$

The Fourier series of the function $f(x)$ in $0 < x < \ell$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{P}\right) + b_n \sin\left(\frac{n\pi x}{P}\right)\right)$$

OR

$$\frac{\ell}{2} - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell/2}\right) + b_n \sin\left(\frac{n\pi x}{\ell/2}\right) \quad \left(\because P = \frac{\ell}{2}\right)$$

OR

$$\frac{\ell}{2} - x = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi x}{\ell}\right) + b_n \sin\left(\frac{2n\pi x}{\ell}\right)\right) \longrightarrow (1)$$

Where $a_0 = \frac{1}{P}\int_0^{2P} f(x)dx$

$$\Rightarrow a_0 = \frac{1}{\ell/2}\int_0^{\ell}\left(\frac{\ell}{2} - x\right)dx \quad \left(\because P = \frac{\ell}{2} \Rightarrow 2P = \ell\right)$$

$$a_0 = \frac{2}{\ell}\left[\frac{\ell}{2}x - \frac{x^2}{2}\right]_0^{\ell}$$

$$\Rightarrow a_0 = \frac{2}{\ell}[0] = 0$$

Thus $\boxed{a_0 = 0}$

$$a_n = \frac{1}{P}\int_0^{2P} f(x)\cos\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n = \frac{1}{\ell/2}\int_0^{\ell}\left(\frac{\ell}{2} - x\right)\cos\left(\frac{2n\pi x}{\ell}\right)dx$$

$$\Rightarrow a_n=\frac{2}{\ell}\left[\frac{\left(\frac{\ell}{2}-x\right)\sin\left(\frac{2n\pi x}{\ell}\right)}{\frac{2n\pi}{\ell}}\right]_0^{\ell}-\frac{2}{\ell}\int_0^{\ell}\frac{(0-1)\sin\left(\frac{2n\pi x}{\ell}\right)}{\left(\frac{2n\pi}{\ell}\right)}dx$$

$$\Rightarrow a_n=\frac{1}{n\pi}[0]+\frac{2}{\ell}\times\frac{\ell}{2n\pi}\int_0^{\ell}\sin\left(\frac{2n\pi x}{\ell}\right)dx$$

$$\Rightarrow a_n=\frac{1}{n\pi}\left[\frac{-\cos\left(\frac{2n\pi x}{\ell}\right)}{\frac{2n\pi}{\ell}}\right]_0^{\ell}$$

$$\Rightarrow a_n=\frac{-\ell}{2n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{\ell}\right)\right]_0^{\ell}=\frac{-\ell}{2n^2\pi^2}[0]=0$$

Thus $\boxed{a_n=0}$

$$b_n=\frac{1}{P}\int_0^{2P}f(x).\sin\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\frac{1}{\ell/2}\int_0^{\ell}\left(\frac{\ell}{2}-x\right)\sin\left(\frac{2n\pi x}{\ell}\right)dx\quad\left(\because P=\ell/2\ \Rightarrow 2P=\ell\right)$$

$$\Rightarrow b_n=\frac{2}{\ell}\left[\left(\frac{\ell}{2}-x\right)\frac{\left(-\cos\left(\frac{2n\pi x}{\ell}\right)\right)}{\left(\frac{2n\pi}{\ell}\right)}\right]_0^{\ell}-\frac{2}{\ell}\times\frac{\ell}{2n\pi}\int_0^{\ell}\frac{\cos\left(\frac{2n\pi x}{\ell}\right)}{\left(\frac{2n\pi}{\ell}\right)}dx$$

$$\Rightarrow b_n=\frac{-2}{\ell}\times\frac{\ell}{2n\pi}\left[\left(\frac{\ell}{2}-x\right)\cos\left(\frac{2n\pi x}{\ell}\right)\right]_0^{\ell}-\frac{2}{\ell}\times\frac{\ell}{2n\pi}\int_0^{\ell}\cos\left(\frac{2n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[0-\frac{\ell}{2}\cos 0\right]-\frac{1}{n\pi}\left[\frac{\sin\left(\frac{2n\pi x}{\ell}\right)}{\left(\frac{2n\pi}{\ell}\right)}\right]_0^{\ell}$$

$$\Rightarrow b_n=\frac{\ell}{2n\pi}-\frac{1}{n\pi}\times\frac{\ell}{2n\pi}\left[\sin\left(\frac{2n\pi x}{\ell}\right)\right]_0^{\ell}$$

$$\Rightarrow b_n=\frac{\ell}{2n\pi}-\frac{\ell}{2n^2\pi^2}[0]=\frac{\ell}{2n\pi}$$

Thus $\boxed{b_n=\frac{\ell}{2n\pi}}$

Putting the values in (1), we have

$$\frac{\ell}{2}-x=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2n\pi x}{\ell}\right)+b_n\sin\left(\frac{2n\pi x}{\ell}\right)\right)$$

$$\Rightarrow\frac{\ell}{2}-x=\frac{0}{2}+\sum_{n=1}^{\infty}\left((0)\cos\left(\frac{2n\pi x}{\ell}\right)+\frac{\ell}{2n\pi}\sin\left(\frac{2n\pi x}{\ell}\right)\right)$$

$$\Rightarrow\frac{\ell}{2}-x=\frac{\ell}{2\pi}\sum\frac{1}{n}\sin\left(\frac{2n\pi x}{\ell}\right)$$

Which is the required result.

EXAMPLE 3:

Obtain the Fourier series for

$$f(x)=\begin{cases}x, & \text{for } -1<x\le 0\\ x+2, & \text{for } 0<x\le 1\end{cases}$$

SOLUTION:

Given that

$$f(x)=\begin{cases}x, & \text{for } -1<x\le 0\\ x+2, & \text{for } 0<x\le 1\end{cases}$$

Here period $=2P=1-(-1)=1+1=2\Rightarrow\boxed{P=1}$

The required Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{P}\right)+b_n\sin\left(\frac{n\pi x}{P}\right)\right)$$

Putting $P=1$, we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\pi x+b_n\sin n\pi x)\longrightarrow(1)$$

Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$

$$\Rightarrow a_0=\frac{1}{1}\int_{-1}^{1}f(x)dx\qquad(\because P=1)$$

$$\Rightarrow a_0=\int_{-1}^{0}f(x)dx+\int_0^1 f(x)dx$$

$$\Rightarrow a_0=\int_{-1}^{0}x\,dx+\int_0^1(x+2)dx$$

$$\Rightarrow a_0=\left[\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}+2x\right]_0^1$$

$$\Rightarrow a_0=\frac{1}{2}\left[(0)^2-(-1)^2\right]+\left[\frac{(1)^2}{2}+2(1)-\frac{(0)^2}{2}-2(0)\right]$$

$$\Rightarrow a_0=\frac{-1}{2}+\frac{1}{2}+2=2$$

Thus $\boxed{a_0=2}$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x)\cos\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\frac{1}{1}\int_{-1}^{1}f(x)\cos(n\pi x)dx\qquad(\because P=1)$$

$$\Rightarrow a_n=\int_{-1}^{0}f(x)\cos(n\pi x)dx+\int_0^1 f(x)\cos(n\pi x)dx$$

$$\Rightarrow a_n=\int_{-1}^{0}x\cos(n\pi x)dx+\int_0^1(x+2)\cos(n\pi x)dx$$

$$\Rightarrow a_n = \left[\frac{x\,Sin(n\pi x)}{n\pi}\right]_{-1}^{0} - \int_{-1}^{0}\frac{(1)\,Sin(n\pi x)}{n\pi}dx + \left[\frac{(x+2)\,Sin(n\pi x)}{n\pi}\right]_{0}^{1} - \int_{0}^{1}\frac{(1+0)Sin(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n = 0 - \frac{1}{n\pi}\int_{-1}^{0}Sin(n\pi x) + 0 - \frac{1}{n\pi}\int_{0}^{1}Sin(n\pi x)dx$$

$$\Rightarrow a_n = \frac{-1}{n\pi}\left[\frac{-Cos(n\pi x)}{n\pi}\right]_{-1}^{0} - \frac{1}{n\pi}\left[\frac{-Cos(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$\Rightarrow a_n = 0 - 0 = 0$$

Thus $\boxed{a_n = 0}$

$$b_n = \frac{1}{P}\int_{-P}^{P}f(x)Sin\frac{(n\pi x)}{P}dx$$

$$\Rightarrow b_n = \frac{1}{1}\int_{-1}^{1}f(x)Sin(n\pi x)dx \qquad (\because P=1)$$

$$\Rightarrow b_n = \int_{-1}^{0}f(x)Sin(n\pi x)dx + \int_{0}^{1}f(x)Sin(n\pi x)dx$$

$$\Rightarrow b_n = \int_{-1}^{0}x.\,Sin(n\pi x)dx + \int_{0}^{1}(x+2)Sin(n\pi x)\,dx$$

$$\Rightarrow b_n = \left[\frac{x(-Cos(n\pi x))}{n\pi}\right]_{-1}^{0} - \int\frac{(1)(-Cos(n\pi x))}{n\pi}dx + \left[(x+2)\frac{(-Cos(n\pi x))}{n\pi}\right]_{0}^{1} - \int_{0}^{1}\frac{(1+0)(-Cos(n\pi x))}{n\pi}dx.$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[x\,Cos(n\pi x)\right]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0}Cos(n\pi x)dx + \frac{-1}{n\pi}\left[(x+2)Cos(n\pi x)\right]_{0}^{1} + \frac{1}{n\pi}\int_{0}^{1}Cos(n\pi x)dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[0+Cosn\pi] + \frac{1}{n\pi}\left[\frac{Sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \frac{-1}{n\pi}[3Cosn\pi - 2Cos0] + \frac{1}{n\pi}\left[\frac{Sin(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$\Rightarrow b_n = \frac{-1(-1)^n}{n\pi} + 0 - \frac{1}{n\pi}\left[3(-1)^n - 2\right] + 0$$

$$\Rightarrow b_n = \frac{1}{n\pi}\left[-(-1)^n - 3(-1)^n + 2\right]$$

$$\Rightarrow b_n = \frac{1}{n\pi}\left[-4(-1)^n + 2\right]$$

$$\Rightarrow \boxed{b_n = \frac{2}{n\pi}\left(1 - 2(-1)^n\right)}$$

Putting the values in (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_nCos(n\pi x) + b_nSin(n\pi x)\right)$$

$$\Rightarrow f(x) = 1 + \sum_{n=1}^{\infty}\left((0)Cos(n\pi x) + \frac{2}{n\pi}\left(1-2(-1)^n\right)Sin(n\pi x)\right)$$

$$\Rightarrow f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]Sin(n\pi x)$$

OR

$$f(x) = 1 + \frac{2}{\pi}\left[\frac{3}{1}Sin\pi x - \frac{1}{2}Sin2\pi x + \frac{3}{3}Sin3\pi x - \frac{1}{4}Sin4\pi x + \frac{3}{5}Sin5\pi x - \frac{1}{6}Sin6\pi x + \ldots\ldots\right]$$

which is the required Fourier series representation of $f(x)$ in the given interval.

EXERCISE 6.2

QUESTION 1(i):

Expand into Fourier series the function
$f(x) = x, \quad 0<x<1$

SOLUTION:

Given that $f(x) = x,\ 0<x<1$

Here period $= 2P = 1-0 \Rightarrow 2P = 1 \Rightarrow \boxed{P = 1/2}$

The required series in given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_nCos\left(\frac{n\pi x}{P}\right) + b_nSin\left(\frac{n\pi x}{P}\right)\right)$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_nCos(2n\pi x) + b_nSin(2n\pi x)\right) \rightarrow (1) \quad \left(\because P = \frac{1}{2}\right)$$

Where $a_0 = \frac{1}{P}\int_{0}^{2P}f(x)dx$

$$\Rightarrow a_0 = 2\int_{0}^{1}x\,dx \quad \left(\because P = \frac{1}{2}\right)$$

$$\Rightarrow a_0 = 2\left[\frac{x^2}{2}\right]_{0}^{1} = (1)^2 - (0)^2 = 1$$

Thus $\boxed{a_0 = 1}$

$$a_n = \frac{1}{P}\int_{0}^{2P}f(x)Cos\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n = 2\int_{0}^{1}xCos(2n\pi x)dx \qquad \left(\because P = \frac{1}{2}\right)$$

$$\Rightarrow = 2\left[\frac{x.\,Sin(2n\pi x)}{2n\pi}\right]_{0}^{1} - 2\int_{0}^{1}\frac{(1)\,Sin(2n\pi x)}{2n\pi}dx$$

$$\Rightarrow a_n = \frac{1}{n\pi}\left[x.\text{Sin}(2n\pi x)\right]_0^1 - \frac{1}{n\pi}\int_0^1 \text{Sin}(2n\pi x)dx$$

$$\Rightarrow a_n = \frac{1}{n\pi}[0] - \frac{1}{n\pi}\left[\frac{-\text{Cos}(2n\pi x)}{2n\pi}\right]_0^1$$

$$\Rightarrow a_n = 0 + \frac{1}{2n^2\pi^2}\left[\text{Cos}(2n\pi x)\right]_0^1$$

$$\Rightarrow a_n = \frac{1}{2n^2\pi^2}[\text{Cos}2n\pi - \text{Cos}0] = \frac{1}{2n^2\pi^2}[1-1] = 0$$

Thus $\boxed{a_n = 0}$

$$b_n = \frac{1}{P}\int_0^{2P} f(x)\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n = 2\int_0^1 x\,\text{Sin}(2n\pi x)dx \left(\because P = \frac{1}{2}\right)$$

$$\Rightarrow b_n = 2\left[x\left(\frac{-\cos(2n\pi)}{2n\pi}\right)\right]_0^1 - 2\int_0^1 (1)\left(\frac{-\cos(2n\pi)}{2n\pi}\right)dx$$

$$\Rightarrow b_n = \frac{-2}{2n\pi}\left[x\cos(2n\pi)\right]_0^1 + \frac{2}{2n\pi}\int_0^1 \cos(2n\pi)dx$$

$$\Rightarrow b_n = -\frac{1}{n\pi}[\cos 2n\pi - 0] + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1$$

$$\Rightarrow b_n = \frac{-1}{n\pi} + \frac{1}{n\pi}[0] = \frac{(-1)^{n+1}}{n\pi}$$

Thus $\boxed{b_n = \frac{-1}{n\pi}}$

Putting the values in (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\text{Cos}(2n\pi x) + b_n\text{Sin}(2n\pi x)\right)$$

$$\Rightarrow x = \frac{1}{2} + \sum_{n=1}^{\infty}\left((0)\text{Cos}(2n\pi x) + \frac{(-1)}{n\pi}\text{Sin}(2n\pi x)\right)$$

$$\Rightarrow x = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)}{n}\text{Sin}(2n\pi x)$$

$$\Rightarrow x = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\text{Sin}(2n\pi x).$$

QUESTION 1(ii):

Expand into Fourier series the function

$f(x) = 2x+1, \quad 0<x<2$

SOLUTION:

Given that $f(x) = 2x+1, \quad 0<x<2$

Here period $= 2P = 2-0 = 2 \Rightarrow \boxed{P=1}$

The required Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{P}\right) + b_n\text{Sin}\left(\frac{n\pi x}{P}\right)\right)$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\text{Cos}(n\pi x) + b_n\text{Sin}(n\pi x)\right) \rightarrow (1) \quad (\because P=1)$$

Where $a_0 = \frac{1}{P}\int_0^{2P} f(x)dx$

$$\Rightarrow a_0 = \frac{1}{1}\int_0^2 (2x+1)dx \quad (\because P=1)$$

$$\Rightarrow a_0 = \left[\frac{2x^2}{2} + x\right]_0^2$$

$$\Rightarrow a_0 = \left[x^2 + x\right]_0^2 = (2)^2 + 2 - 0^2 - 0$$

$$\Rightarrow \boxed{a_0 = 6}$$

$$a_n = \frac{1}{P}\int_0^{2P} f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n = \frac{1}{1}\int_0^2 (2x+1)\text{Cos}(n\pi x)dx \quad (\because P=1)$$

$$\Rightarrow a_n = \left[(2x+1)\frac{\text{Sin}(n\pi x)}{n\pi}\right]_0^2 - \int_0^2 \frac{(2+0)\text{Sin}(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n = 0 - \frac{2}{n\pi}\int_0^2 \text{Sin}(n\pi x)dx = \frac{-2}{n\pi}\left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^2$$

$$\Rightarrow a_n = \frac{2}{n\pi}[\text{Cos}2n\pi - \text{Cos}0] = 0$$

Thus $\boxed{a_n = 0}$

$$b_n = \frac{1}{P}\int_0^{2P} f(x)\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n = \frac{1}{1}\int_0^2 (2x+1)\text{Sin}(n\pi x)dx \quad (\because P=1)$$

$$\Rightarrow b_n = \left[(2x+1)\left(\frac{-\text{Cos}(n\pi x)}{n\pi}\right)\right]_0^2 - \int_0^2 \frac{(2+0)(-\text{Cos}(n\pi x))}{n\pi}dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[(2x+1)\text{Cos}(n\pi x)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \text{Cos}(n\pi x)dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[5\text{Cos}2n\pi - \text{Cos}\,0] + \frac{2}{n\pi}\left[\frac{\text{Sin}(n\pi x)}{n\pi}\right]_0^2$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[5-1] + 0 = \frac{-4}{n\pi}$$

Thus $\boxed{b_n = \frac{-4}{n\pi}}$

Putting the values in Eq (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\text{Cos}(n\pi x) + b_n\text{Sin}(n\pi x)$$

$$\Rightarrow 2x+1=3+\sum_{n=1}^{\infty}\left(0\,\text{Cos}(n\pi x)+\left(\frac{-4}{n\pi}\right)\text{Sin}(n\pi x)\right)$$

$$\Rightarrow 2x+1=3+\sum_{n=1}^{\infty}\frac{(-4)}{n\pi}\text{Sin}(n\pi x)$$

$$\Rightarrow 2x+1=3-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\text{Sin}(n\pi x) \text{ Ans.}$$

NOTE: The book answer is wrong.

QUESTION 2:

Expand $f(x)$ in Fourier series in the interval $(-2,2)$

when $f(x)=\begin{cases}0, & -2<x<0\\ 1, & 0<x<2\end{cases}$

SOLUTION: Given that

$$f(x)=\begin{cases}0, & -2<x<0\\ 1, & 0<x<2\end{cases}$$

Here period $=2P=2-(-2)\Rightarrow 2P=4\Rightarrow \boxed{P=2}$

The required Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+b_n\text{Sin}\left(\frac{n\pi x}{P}\right)\right)$$

$$\Rightarrow f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{2}\right)+b_n\text{Sin}\left(\frac{n\pi x}{2}\right)\right)\longrightarrow(1)$$

Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$

$$\Rightarrow a_0=\frac{1}{2}\int_{-2}^{2}f(x)dx \qquad (\because P=2)$$

$$\Rightarrow a_0=\frac{1}{2}\int_{-2}^{0}f(x)dx+\frac{1}{2}\int_{0}^{2}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{2}\int_{-2}^{0}(0)dx+\frac{1}{2}\int_{0}^{2}1\,dx$$

$$\Rightarrow a_0=0+\frac{1}{2}[x]_0^2=\frac{1}{2}[2-0]=1$$

Thus $\boxed{a_0=1}$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x)\,\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{2}f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx \qquad (\because P=2)$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{0}f(x)\text{Cos}\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_{0}^{2}f(x)\text{Cos}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{0}(0)\text{Cos}(dx)+\frac{1}{2}\int_{0}^{2}(1)\text{Cos}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow a_n=0+\frac{1}{2}\int_{0}^{2}\text{Cos}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow a_n=\frac{1}{2}\left[\frac{\text{Sin}\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2=\frac{1}{n\pi}\left[\text{Sin}\left(\frac{n\pi x}{2}\right)\right]_0^2=0$$

$$\Rightarrow \boxed{a_n=0}$$

$$b_n=\frac{1}{P}\int_{-P}^{P}f(x)\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{2}f(x).\,\text{Sin}\left(\frac{n\pi x}{2}\right)dx \qquad (\because P=2)$$

$$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{0}f(x).\text{Sin}\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_{0}^{2}f(x).\,\text{Sin}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{0}(0)\text{Sin}\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_{0}^{2}(1)\text{Sin}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n=0+\frac{1}{2}\int_{0}^{2}\text{Sin}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n=\frac{1}{2}\left[\frac{-\text{Cos}\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[\text{Cos}\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$\Rightarrow b_n=\frac{-1}{n\pi}[\text{Cos}\,n\pi-\text{Cos}0]$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[(-1)^n-1\right]$$

$$\Rightarrow b_n=\frac{1}{n\pi}\left[1-(-1)^n\right]$$

Thus $\boxed{b_n=\frac{1}{n\pi}\left[1-(-1)^n\right]}$

Putting the values in Eq (1) we have

$$f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\left((0)\text{Cos}\left(\frac{n\pi x}{2}\right)+\left[\frac{1-(-1)^n}{n\pi}\right]\text{Sin}\left(\frac{n\pi x}{2}\right)\right)$$

$$\Rightarrow f(x)=\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\text{Sin}\left(\frac{n\pi x}{2}\right)$$

$$\Rightarrow f(x)=\frac{1}{2}+\frac{1}{\pi}\left[\frac{2}{1}\text{Sin}\left(\frac{\pi x}{2}\right)+\frac{2}{3}\text{Sin}\left(\frac{2\pi x}{2}\right)+\frac{2}{5}\text{Sin}\left(\frac{5\pi x}{2}\right)+\cdots\right]$$

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\left[\text{Sin}\left(\frac{\pi x}{2}\right)+\frac{1}{3}\text{Sin}\left(\frac{2\pi x}{2}\right)+\frac{1}{5}\text{Sin}\left(\frac{5\pi x}{2}\right)+\cdots\right]$$

QUESTION 3:

Find a Fourier series for $f(x)=x^2, \quad 1<x<2$

SOLUTION:

Here $f(x)=x^2, \quad 1<x<2$

To make the period 2π, we set $x=\frac{t}{2\pi}$

Then $f(x)=f\left(\frac{t}{2\pi}\right)=\left(\frac{t}{2\pi}\right)^2=\frac{t^2}{4\pi^2}=g(t)$

Thus $g(t)=f\left(\frac{t}{2\pi}\right)=\frac{t^2}{4\pi^2}, \quad 2\pi<t<4\pi$

The Fourier series of $g(t)$ is

$$g(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos}nt+b_n\text{Sin}nt)\longrightarrow(1)$$

Where $a_0=\frac{1}{\pi}\int_{2\pi}^{4\pi}g(t)dt$

$$a_0=\frac{1}{\pi}\int_{2\pi}^{4\pi}\left(\frac{t^2}{4\pi^2}\right)\text{Cos nt dt}$$

$$\Rightarrow a_0=\frac{1}{4\pi^3}\left[\frac{t^3}{3}\right]_{2\pi}^{4\pi}=\frac{1}{12\pi^3}\left[t^3\right]_{2\pi}^{4\pi}$$

$$\Rightarrow a_0=\frac{1}{12\pi^3}\left[(4\pi)^3-(2\pi)^3\right]$$

$$\Rightarrow a_0=\frac{1}{12\pi^3}\left[64\pi^3-8\pi^3\right]$$

$$\Rightarrow a_0=\frac{1}{12\pi^3}\left[65\pi^3\right]=\frac{65}{12}=\frac{14}{3}$$

Thus $\boxed{a_0=\frac{14}{3}}$

$$a_n=\frac{1}{\pi}\int_{2\pi}^{4\pi}g(t)\text{Cosnt dt}$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{2\pi}^{4\pi}\frac{t^2}{4\pi^2}.\text{cosnt dt}$$

$$\Rightarrow a_n=\frac{1}{4\pi^3}\int_{2\pi}^{4\pi}t^2.\text{Cosnt dt}$$

$$\Rightarrow a_n=\frac{1}{4\pi^3}\left[\frac{t^2\text{ Sinnt}}{n}\right]_{2\pi}^{4\pi}-\frac{1}{4\pi^3}\int_{2\pi}^{4\pi}\frac{2t.\text{Sinnt}}{n}dt$$

$$\Rightarrow a_n=0-\frac{2}{4n\pi^2}\int_{2\pi}^{4\pi}t.\text{Sinnt dt}$$

$$\Rightarrow a_n=\frac{-1}{2n\pi^3}\left[t\left(\frac{-\text{Cos nt}}{n}\right)\right]_{2\pi}^{4\pi}+\frac{1}{2n\pi^3}\int_{2\pi}^{4\pi}(1)\left(\frac{-\text{Cosnt}}{n}\right)dt$$

$$\Rightarrow a_n=\frac{1}{2n^2\pi^3}[t\text{ Cosnt}]_{2\pi}^{4\pi}-\frac{1}{2n^2\pi^3}\int_{2\pi}^{4\pi}\text{Cosnt dt}$$

$$\Rightarrow a_n=\frac{1}{2n^2\pi^3}[4\pi\text{Cos }4n\pi-2\pi\text{Cos }2n\pi]-\frac{1}{2n^2\pi^3}\left[\frac{\text{Sinnt}}{n}\right]_{2\pi}^{4\pi}$$

$$\Rightarrow a_n=\frac{1}{2n^2\pi^3}[4\pi-2\pi]-0$$

$$\Rightarrow a_n=\frac{2\pi}{2n^2\pi^2}=\frac{1}{n^2\pi^2}$$

Thus $\boxed{a_n=\frac{1}{n^2\pi^2}}$

$$b_n=\frac{1}{\pi}\int_{2\pi}^{4\pi}g(t)\text{Sinnt dt}$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{2\pi}^{4\pi}\frac{t^2}{4\pi^2}\text{Sinnt dt}$$

$$\Rightarrow b_n=\frac{1}{4\pi^3}\int_{2\pi}^{4\pi}t^2.\text{Sinnt dt}$$

$$\Rightarrow b_n=\frac{1}{4\pi^3}\left[t^2\frac{(-\text{Cosnt})}{n}\right]_{2\pi}^{4\pi}-\frac{1}{4\pi^3}\int_{2\pi}^{4\pi}2t\frac{(-\text{Cosnt})}{n}dt$$

$$\Rightarrow b_n=\frac{-1}{4n\pi^3}\left[t^2.\text{Cosnt}\right]_{2\pi}^{4\pi}+\frac{2}{4n\pi^3}\int_{2\pi}^{4\pi}t.\text{Cosnt dt}$$

$$\Rightarrow b_n=\frac{-1}{4n\pi^3}\left[16\pi^2\text{Cos}4n\pi-4\pi^2\text{Cos}2n\pi\right]+\frac{1}{2n\pi^3}\int_{2\pi}^{4\pi}t.\text{Cosnt dt}$$

$$\Rightarrow b_n=\frac{-1}{4n\pi^3}\left[16\pi^2-4\pi^2\right]+\frac{1}{2n\pi^3}\left[\frac{t\text{ Sinnt}}{n}\right]_{2\pi}^{4\pi}+$$

$$-\frac{1}{2n\pi^3}\int_{2\pi}^{4\pi}\frac{(1)\text{Sinnt}}{n}dt$$

$$\Rightarrow b_n=\frac{-1}{4n\pi^3}(12\pi^2)+0-\frac{1}{2n^2\pi^3}\int_{2\pi}^{4\pi}\text{Sinnt dt}$$

$$\Rightarrow b_n=\frac{-3}{n\pi}-\frac{1}{2n^2\pi^3}\left[\frac{-\text{Connt}}{n}\right]_{2\pi}^{4\pi}$$

$$\Rightarrow b_n=\frac{-3}{n\pi}+\frac{1}{2n^3\pi^3}[\text{Cosnt}]_{2\pi}^{4\pi}$$

$$\Rightarrow b_n=\frac{-3}{n\pi}+0=\frac{-3}{n\pi}$$

Thus $\boxed{b_n=\frac{-3}{n\pi}}$

Putting the values in Eq (1), we have

$$g(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cosnt}+b_n\text{Sinnt})$$

$$\Rightarrow g(t)=\frac{7}{3}+\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}\text{Cosnt}-\frac{3}{n\pi}\text{Sinnt}\right)$$

$$\Rightarrow g(t)=\frac{7}{3}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\text{Connt}-\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\text{Sinnt}\longrightarrow(2)$$

Replacing t by $2\pi x$ in Eq (2), we have

$$g(2\pi x)=\frac{7}{3}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\text{Cos}(2n\pi x)-\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\text{Sin}(2n\pi x) \text{ OR}$$

$$x^2=\frac{7}{3}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\text{Cos}(2n\pi x)-\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\text{Sin}(2n\pi x).$$

QUESTION 4:

Expand $f(x)=x-x^2$ as a Fourier series in the interval $(-1,1)$.

SOLUTION: Given that

$f(x)=x-x^2, \quad -1<x<1$

Here period $=2P=1-(-1)\Rightarrow 2P=2\Rightarrow \boxed{P=1}$

The required Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+b_n\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+b_n\text{Sin}(n\pi x)\rightarrow(1)$$

Where $\quad (\because P=1)$

$$a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$$

$$\Rightarrow a_0=\int_{-1}^{1}(x-x^2)dx \quad (\because p=1)$$

$$\Rightarrow a_0=\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{1}$$

$$\Rightarrow a_0=\left[\frac{(1)^2}{2}-\frac{(1)^3}{3}-\frac{(-1)^2}{2}+\frac{(-1)^3}{3}\right]$$

$$\Rightarrow a_0=\left[\frac{1}{2}-\frac{1}{3}-\frac{1}{2}-\frac{1}{3}\right]=\frac{-2}{3}$$

Thus $\boxed{a_0=\frac{-2}{3}}$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\int_{-1}^{1}(x-x^2)\text{Cos}(n\pi x)dx \quad (\because P=1)$$

$$\Rightarrow a_n=\left[(x-x^2)\frac{\text{Sin}(n\pi x)}{n\pi}\right]_{-1}^{1}-\int_{-1}^{1}(1-2x)\frac{\text{Sin}(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n=0-\frac{1}{n\pi}\int_{-1}^{1}(1-2x)\text{Sin}(n\pi x)dx$$

$$\Rightarrow a_n=\frac{-1}{n\pi}\left[(1-2x)\frac{(-\text{Cos}(n\pi x))}{n\pi}\right]_{-1}^{1}+\frac{1}{n\pi}\int_{-1}^{1}(0-2)\frac{(-\text{Cos}(n\pi x))}{n\pi}dx$$

$$\Rightarrow a_n=\frac{1}{n^2\pi^2}[-\text{Cos}n\pi-3\text{Cos}n\pi]+\frac{2}{n^2\pi^2}\left[\frac{\text{Sin}(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$\Rightarrow a_n=\frac{1}{n^2\pi^2}[-4\text{Cos}n\pi]+0$$

$$\Rightarrow a_n=\frac{-4(-1)^n}{n^2\pi^2}=\frac{4(-1)^{n+1}}{n^2\pi^2}$$

Thus $\boxed{a_n=\frac{4(-1)^{n+1}}{n^2\pi^2}}$

$$b_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\int_{-1}^{1}(x-x^2)\text{Sin}(n\pi x)dx \quad (\because P=1)$$

$$\Rightarrow b_n=\left[(x-x^2)\left(\frac{-\text{Cos}(n\pi x)}{n\pi}\right)\right]_{-1}^{1}-\int_{-1}^{1}(1-2x)\left(\frac{-\text{Cos}(n\pi x)}{n\pi}\right)dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[(x-x^2)\text{Cos}(n\pi x)\right]_{-1}^{1}+\frac{1}{n\pi}\int_{-1}^{1}(1-2x)\text{Cos}(n\pi x)dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}[0+2\text{Cos}(n\pi)]+\frac{1}{n\pi}\left[(1-2x)\frac{\text{Sin}(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$-\frac{1}{n\pi}\int_{-1}^{1}(0-2)\frac{\text{Sin}(n\pi x)}{n\pi}dx$$

$$\Rightarrow b_n=\frac{-2(-1)^n}{n\pi}+0+\frac{2}{n^2\pi^2}\int_{-1}^{1}\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}}{n\pi}+\frac{2}{n^2\pi^2}\left[\frac{-\text{Cos}(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}}{n\pi}-\frac{2}{n^3\pi^3}[\text{Cos}(n\pi x)]_{-1}^{1}$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}}{n\pi}-0$$

Thus $\boxed{b_n=\frac{2(-1)^{n+1}}{n\pi}}$

Putting the values in Eq (1), we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos}(n\pi x)+b_n\text{Sin}(n\pi x))$$

$$\Rightarrow x-x^2=\frac{-1}{3}+\sum_{n=1}^{\infty}\left(\frac{4(-1)^{n+1}}{n^2\pi^2}\text{Cos}(n\pi x)+\frac{2(-1)^{n+1}}{n\pi}\text{Sin}(n\pi x)\right)$$

$$\Rightarrow x-x^2=\frac{-1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\text{Cos}(n\pi x)+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\text{Sin}(n\pi)$$

QUESTION 5:

Develop $f(x)$ in a Fourier series in the interval $(0,2)$

$$\text{if } f(x)=\begin{cases}x, & 0<x<1\\ 0, & 1<x<2\end{cases}$$

SOLUTION: Given that

$$f(x)=\begin{cases} x, & 0<x<1 \\ 0, & 1<x<2 \end{cases}$$

Here period $=2P=2-0=2 \Rightarrow \boxed{P=1}$

The required Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \text{Cos}\left(\frac{n\pi x}{P}\right)+b_n \text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \text{Cos}(n\pi x)+b_n \text{Sin}(n\pi x) \rightarrow (1)$$

Where $(\because P=1)$

$$a_0=\frac{1}{P}\int_0^{2P} f(x)dx$$

$$\Rightarrow a_0=\int_0^2 f(x)dx \qquad (\because P=1)$$

$$\Rightarrow a_0=\int_0^1 f(x)dx+\int_1^2 f(x)dx$$

$$\Rightarrow a_0=\int_0^1 x\,dx+\int_1^2 0\,dx$$

$$\Rightarrow a_0=\left[\frac{x^2}{2}\right]_0^1+0=\left(\frac{(1)^2}{2}-\frac{(0)^2}{2}\right)$$

$$\Rightarrow a_0=\frac{1}{2} \text{ Thus } \boxed{a_0=\frac{1}{2}}$$

$$a_n=\frac{1}{P}\int_0^{2P} f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\int_0^2 f(x)\text{Cos}(n\pi x)dx \qquad (\because P=1)$$

$$\Rightarrow a_n=\int_0^1 f(x)\text{Cos}(n\pi x)dx+\int_1^2 f(x)\text{Cos}(n\pi x)dx$$

$$\Rightarrow a_n=\int_0^1 x\,\text{Cos}(n\pi x)dx+\int_1^2 (0)\text{Cos}(n\pi x)\,dx$$

$$\Rightarrow a_n=\left[\frac{x\,\text{Sin}(n\pi x)}{n\pi}\right]_0^1-\int_0^1 \frac{(1)\,\text{Sin}(n\pi x)}{n\pi}+0$$

$$\Rightarrow a_n=\frac{1}{n\pi}\left[x.\text{Sin}(n\pi x)\right]_0^1-\frac{1}{n\pi}\int_0^1 \text{Sin}(n\pi x)dx$$

$$\Rightarrow a_n=\frac{1}{n\pi}[0]-\frac{1}{n\pi}\left[\frac{(-\text{Cos}(n\pi x))}{n\pi}\right]_0^1$$

$$\Rightarrow a_n=\frac{1}{n^2\pi^2}\left[\text{Cos}(n\pi x)\right]_0^1$$

$$\Rightarrow a_n=\frac{1}{n^2\pi^2}[\text{Cos}n\pi-\text{Cos}\,0]$$

$$\Rightarrow a_n=\frac{1}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$\text{Thus } \boxed{a_n=\frac{(-1)^n-1}{n^2\pi^2}}$$

$$b_n=\frac{1}{P}\int_0^{2P} f(x)\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\int_0^2 f(x)\text{Sin}(n\pi x)dx \qquad (\because P=1)$$

$$\Rightarrow b_n=\int_0^1 f(x)\text{Sin}(n\pi x)dx+\int_1^2 f(x).\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n=\int_0^1 x.\text{Sin}(n\pi x)dx+\int_1^2 (0)\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n=\left[\frac{x(-\text{Cos}(n\pi x))}{n\pi}\right]_0^1-\int_0^1 \frac{(1)(-\text{Cos}(n\pi x))}{n\pi}dx+0$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[x\,\text{Cos}(n\pi x)\right]_0^1+\frac{1}{n\pi}\int_0^1 \text{Cos}(n\pi x)dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}[\text{Cos}\,n\pi-0]+\frac{1}{n\pi}\left[\frac{\text{Sin}(n\pi x)}{n\pi}\right]_0^1$$

$$\Rightarrow b_n=\frac{-1}{n\pi}(-1)^n+0$$

$$\text{Thus } \boxed{b_n=\frac{(-1)^{n+1}}{n\pi}}$$

Putting the values in Eq (1), we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}(n\pi x)+b_n\text{Sin}(n\pi x)\right)$$

$$\Rightarrow f(x)=\frac{1}{4}+\sum_{n=1}^{\infty}\left(\frac{((-1)^n-1)}{n^2\pi^2}\text{Cos}(n\pi x)+\frac{(-1)^{n+1}}{n\pi}\text{Sin}(n\pi x)\right)$$

$$f(x)=\frac{1}{4}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n^2}\text{Cos}(n\pi x)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\text{Sin}(n\pi x)$$

$$\Rightarrow f(x)=\frac{1}{4}+\frac{1}{\pi^2}\left[\frac{-2}{1^2}\text{Cos}(\pi x)-\frac{2}{3^2}\text{Cos}(3\pi x)-\frac{2}{5}\text{Cos}(5\pi x)\cdots\right]$$

$$+\frac{1}{\pi}\left[\frac{1}{1}\text{Sin}(\pi x)-\frac{1}{2}\text{Sin}(2\pi x)+\frac{1}{3}\text{Sin}(3\pi x)-\frac{1}{4}\text{Sin}(4\pi x)+\cdots\right]$$

$$\Rightarrow f(x)=\frac{1}{4}-\frac{2}{\pi^2}\left[\frac{\text{Cos}(\pi x)}{1^2}+\frac{\text{Cos}(3\pi x)}{3^2}+\frac{\text{Cos}(5\pi x)}{5^2}+\cdots\cdots\right]$$

$$+\frac{1}{\pi}\left[\frac{\text{Sin}(\pi x)}{1}-\frac{\text{Sin}(2\pi x)}{2}+\frac{\text{Sin}(3\pi x)}{3}-\frac{\text{Sin}(4\pi x)}{4}+\cdots\cdots\right] \text{ Ans.}$$

QUESTION 6:

Expand $f(x)=e^{-x}$ as a Fourier series in the interval $(-\ell,\ell)$

SOLUTION: Given that

$f(x)=e^{-x};\quad -\ell<x<\ell$

Here period $=2P=\ell-(-\ell)=2\ell \Rightarrow P=\ell$

The required Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+b_n\text{Sin}\left(\frac{n\pi x}{P}\right)\right)$$

$$\Rightarrow e^{-x}=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{\ell}\right)+b_n\text{Sin}\left(\frac{n\pi x}{\ell}\right)\right)\longrightarrow(A)$$

Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$

$$\Rightarrow a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}e^{-x}dx$$

$$\Rightarrow a_0=\frac{1}{\ell}\left[\frac{e^{-x}}{-1}\right]_{-\ell}^{\ell}=\frac{-1}{\ell}\left[e^{-\ell}-e^{-(-\ell)}\right]$$

$$\Rightarrow a_0=\frac{-1}{\ell}\left[e^{-\ell}-e^{\ell}\right]$$

$$\Rightarrow a_0=\frac{2}{2\ell}\left[e^{\ell}-e^{-\ell}\right]$$

$$\Rightarrow a_0=\frac{2}{\ell}\left[\frac{e^{\ell}-e^{-\ell}}{2}\right]=\frac{2}{\ell}\text{Sinh}\ell$$

Thus $\boxed{a_0=\frac{2}{\ell}\text{Sinh}\ell}$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx\longrightarrow(1)$$

Let $I_1=\int e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx$

$$\Rightarrow I_1=e^{-x}\frac{\text{Sin}\left(\frac{n\pi x}{\ell}\right)}{\left(\frac{n\pi}{\ell}\right)}-\int\frac{-e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}dx$$

$$\Rightarrow I_1=\frac{\ell}{n\pi}e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)+\frac{\ell}{n\pi}\int e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow I_1=\frac{\ell}{n\pi}e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)+\frac{\ell}{n\pi}\left[\frac{e^{-x}\left(-\text{Cos}\left(\frac{n\pi x}{\ell}\right)\right)}{\frac{n\pi}{\ell}}\right.$$

$$\left.-\frac{\ell}{n\pi}\int\frac{(-e^{-x})\left(-\text{Cos}\frac{(n\pi x)}{\ell}\right)}{\frac{n\pi}{\ell}}dx.\right]$$

$$\Rightarrow I_1=\frac{\ell e^{-x}}{n\pi}\text{Sin}\left(\frac{n\pi x}{\ell}\right)-\frac{\ell^2}{n^2\pi^2}\left[e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right)\right]$$

$$-\frac{\ell^2}{n^2\pi^2}\int e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx.$$

$$\Rightarrow I_1=\frac{\ell}{n\pi}e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)-\frac{\ell^2}{n^2\pi^2}e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)-\frac{\ell^2}{n^2\pi^2}I_1$$

$$\Rightarrow I_1+\frac{\ell^2}{n^2\pi^2}I_1=\frac{\ell}{n\pi}e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)-\frac{\ell^2}{n^2\pi^2}.e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow\left[\frac{n^2\pi^2+\ell^2}{n^2\pi^2}\right]I_1=\frac{\ell}{n\pi}e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)-\frac{\ell^2}{n^2\pi^2}.e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow I_1=\frac{\ell n\pi}{n^2\pi^2+\ell^2}.e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)-\frac{\ell^2}{n^2\pi^2+\ell^2}.e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)$$

i.e. $\int e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx=\frac{\ell n\pi}{n^2\pi^2+\ell^2}.e^{-x}.\text{Sin}\left(\frac{n\pi x}{\ell}\right)+$

$$\frac{\ell^2}{n^2\pi^2+\ell^2}.e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow\int_{-\ell}^{\ell}e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx=\frac{\ell n\pi}{n^2\pi^2+\ell}\left[e^{-x}\text{Sin}\left(\frac{n\pi x}{\ell}\right)\right]_{-\ell}^{\ell}$$

$$-\frac{\ell^2}{n^2\pi^2+\ell^2}\left[e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)\right]_{-\ell}^{\ell}$$

$$\Rightarrow\int_{-\ell}^{\ell}e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx=\frac{\ell n\pi}{n^2\pi^2+\ell^2}\left[e^{-\ell}\text{Sin}(n\pi)+e^{\ell}\text{Sin}(n\pi)\right]$$

$$-\frac{\ell^2}{n^2\pi^2+\ell^2}\left[e^{-\ell}\text{Cos}(n\pi)-e^{\ell}\text{Cos}n\pi\right]$$

$$\Rightarrow\int_{-\ell}^{\ell}e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx=0-\frac{\ell^2}{n^2\pi^2+\ell^2}\left[e^{-\ell}(-1)^n-e^{\ell}(-1)^n\right]$$

$$\Rightarrow\int_{-\ell}^{\ell}e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx=\frac{2\ell^2(-1)^n}{n^2\pi^2+\ell^2}\left[\frac{e^{\ell}-e^{-\ell}}{2}\right]$$

$$\Rightarrow\int_{-\ell}^{\ell}e^{-x}\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx=\frac{2\ell^2(-1)^n}{n^2\pi^2+\ell^2}\text{Sinh}\ell$$

$$\Rightarrow\frac{1}{\ell}\int_{-\ell}^{\ell}e^{-x}.\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx=\frac{2\ell(-1)^n}{n^2\pi^2+\ell^2}\text{Sinh}\ell$$

$$\Rightarrow\boxed{a_n=\frac{2\ell(-1)^n}{n^2\pi^2+\ell^2}.\text{Sinh}\ell}\qquad\text{(By Eq (1))}$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)dx \longrightarrow (2) \qquad (\because P=\ell)$$

Let $I_2 = \int e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$\Rightarrow I_2 = \frac{-e^{-x}\cos\left(\frac{n\pi x}{\ell}\right)}{\left(\frac{n\pi}{\ell}\right)} - \int \frac{e^{-x}\cos\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}dx$$

$$\Rightarrow I_2 = \frac{-\ell}{n\pi}\cdot e^{-x}\cdot\cos\left(\frac{n\pi x}{e}\right) - \frac{\ell}{n\pi}\int e^{-x}\cos\left(\frac{n\pi x}{e}\right)dx$$

$$\Rightarrow I_2 = \frac{-\ell}{n\pi}.e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right) - \frac{\ell}{n\pi}\left[\frac{e^{-x}.\sin\left(\frac{n\pi x}{\ell}\right)}{\left(\frac{n\pi}{\ell}\right)}\right]$$

$$+\frac{\ell}{n\pi}\int \frac{-e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)dx}{\left(\frac{n\pi}{\ell}\right)}$$

$$\Rightarrow I_2 = \frac{-\ell}{n\pi}.e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right) - \frac{\ell^2}{n^2\pi^2}\left[e^{-x}.\sin\left(\frac{n\pi x}{\ell}\right)\right]$$

$$-\frac{\ell^2}{n^2\pi^2}\int e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)dx.$$

$$\Rightarrow I_2 = -\frac{\ell}{n\pi}.e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right) - \frac{\ell^2}{n^2\pi^2}.e^{-x}.\sin\left(\frac{n\pi x}{\ell}\right) - \frac{\ell^2}{n^2\pi^2}I_2$$

$$\Rightarrow I_2 + \frac{\ell^2}{n^2\pi^2}I_2 = -\frac{\ell}{n\pi}.e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right) - \frac{\ell^2}{n^2\pi^2}.e^{-x}.\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \left(\frac{n^2\pi^2+\ell^2}{n^2\pi^2}\right)I_2 = -\frac{\ell}{n\pi}.e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right) - \frac{\ell^2}{n^2\pi^2}.e^{-x}.\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow I_2 = -\frac{\ell n\pi}{n^2\pi^2+\ell^2}.e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right) - \frac{\ell^2}{n^2\pi^2+\ell^2}.e^{-x}.\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \int_{-\ell}^{\ell} e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{-\ell n\pi}{n^2\pi^2+\ell^2}\left[e^{-x}.\cos\left(\frac{n\pi x}{\ell}\right)\right]_{-\ell}^{\ell}$$

$$-\frac{\ell^2}{n^2\pi^2+\ell^2}\left[e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)\right]_{-\ell}^{\ell}$$

$$\Rightarrow \int_{-\ell}^{\ell} e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{-\ell n\pi}{n^2\pi^2+\ell^2}\left[e^{-\ell}.\cos n\pi - e^{\ell}\cos n\pi\right] - 0$$

$$\Rightarrow \frac{1}{\ell}\int_{-\ell}^{\ell} e^{-x}\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{-n\pi}{n^2\pi^2+\ell^2}\left[e^{-\ell}(-1)^n - e^{\ell}(-1)^n\right]$$

$$\Rightarrow b_n = \frac{2n\pi(-1)^n}{n^2\pi^2+\ell^2}\left[\frac{e^{\ell}-e^{-\ell}}{2}\right]$$

$$\boxed{\Rightarrow b_n = \frac{2n(-1)^n\pi}{n^2\pi^2+\ell^2}.\sinh\ell}$$

Putting the values in Eq (A), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow e^{-x} = \frac{\sinh\ell}{\ell} + \sum_{n=1}^{\infty}\frac{2\ell(-1)^n}{n^2\pi^2+\ell^2}.\sinh\ell.\cos\left(\frac{n\pi x}{\ell}\right)$$

$$+\sum_{n=1}^{\infty}\frac{2n(-1)^n\pi}{n^2\pi^2+\ell^2}.\sinh\ell.\sin\left(\frac{n\pi x}{\ell}\right)$$

or $$e^{-x} = \frac{\sinh\ell}{\ell} + 2\ell.\sinh\ell\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2+\ell^2}.\cos\left(\frac{n\pi x}{\ell}\right)$$

$$+2\pi.\sinh\ell\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2\pi^2+\ell^2}.\sin\left(\frac{n\pi x}{\ell}\right) \qquad \text{Ans.}$$

QUESTION 7:

Expand into Fourier series

$$f(x) = \begin{cases} 0, & -2<x<-1 \\ k, & -1<x<1 \\ 0, & 1<x<2 \end{cases}$$

SOLUTION: Given that

$$f(x) = \begin{cases} 0, & -2<x<-1 \\ k, & -1<x<1 \\ 0, & 1<x<2 \end{cases}$$

Here period $= 2P = 2-(-2) \Rightarrow 2P = 4 \Rightarrow P = 2$

The required Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{P}\right) + b_n\sin\left(\frac{n\pi x}{P}\right)\right)$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n.\sin\left(\frac{n\pi x}{2}\right) \longrightarrow (1)$$

Where $(\because P=2)$

$$a_0 = \frac{1}{P}\int_{-P}^{P} f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{2}\int_{-2}^{-1} f(x)\,dx + \frac{1}{2}\int_{-1}^{1} f(x)\,dx + \frac{1}{2}\int_{1}^{2} f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{2}\int_{-2}^{-1} 0\,dx + \frac{1}{2}\int_{-1}^{1} k\,dx + \frac{1}{2}\int_{1}^{2} 0\,dx$$

$$\Rightarrow a_0 = 0 + \frac{1}{2}[kx]_{-1}^{1} + 0$$

$\Rightarrow a_0=\frac{1}{2}[k+k]=\frac{2k}{2}=k$

$\Rightarrow \boxed{a_0=k}$

$a_n=\frac{1}{P}\int_{-P}^{P}f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx$

$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{2}f(x)\text{Cos}\left(\frac{n\pi x}{2}\right)dx \qquad (\because P=2)$

$\Rightarrow a_n=\frac{1}{2}\int_{-1}^{1}f(x)\text{Cos}\left(\frac{n\pi x}{2}\right)dx$

$\Rightarrow a_n=\frac{1}{2}\int_{-1}^{1}k.\text{Cos}\left(\frac{n\pi x}{2}\right)dx$

$\Rightarrow a_n=\frac{1}{2}k\left[\frac{\text{Sin}\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1}$

$\Rightarrow a_n=\frac{k}{n\pi}\left[\text{Sin}\left(\frac{n\pi x}{2}\right)\right]_{-1}^{1}$

$\Rightarrow a_n=\frac{k}{n\pi}\left[\text{Sin}\left(\frac{n\pi}{2}\right)+\text{Sin}\left(\frac{n\pi}{2}\right)\right]$

$\Rightarrow \boxed{a_n=\frac{2k}{n\pi}.\text{Sin}\left(\frac{n\pi}{2}\right)}$

$b_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$

$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{2}f(x).\text{Sin}\left(\frac{n\pi x}{2}\right)dx$

$\Rightarrow b_n=\frac{1}{2}\int_{-1}^{1}k.\text{Sin}\left(\frac{n\pi x}{2}\right)dx$

$\Rightarrow b_n=\frac{-K}{2}\left[\frac{\text{Cos}\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1}$

$\Rightarrow b_n=\frac{-K}{n\pi}\left[\text{Cos}\left(\frac{n\pi x}{2}\right)\right]_{-1}^{1}=0$

$\Rightarrow \boxed{b_n=0}$

Putting the values in Eq (1), we have

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{2}\right)+\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{2}\right)$

$\Rightarrow f(x)=\frac{k}{2}+\sum_{n=1}^{\infty}\frac{2k}{n\pi}\text{Sin}\left(\frac{n\pi}{2}\right)\text{Cos}\left(\frac{n\pi x}{2}\right)+0$

$\Rightarrow f(x)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}.\text{Sin}\left(\frac{n\pi}{2}\right).\text{Cos}\left(\frac{n\pi x}{2}\right)$

$\Rightarrow f(x)=\frac{k}{2}+\frac{2k}{\pi}\times$

$$\left[\begin{array}{l}\frac{1}{1}.\text{Sin}\left(\frac{\pi}{2}\right).\text{Cos}\left(\frac{\pi x}{2}\right)+\frac{1}{2}.\text{Sin}\pi.\text{Cos}(\pi x)+\\ +\frac{1}{3}\text{Sin}\left(\frac{3\pi}{2}\right).\text{Cos}\left(\frac{3\pi x}{2}\right)+\frac{1}{4}\text{Sin}2\pi.\text{Cos}(2\pi x)+\\ +\frac{1}{5}\text{Sin}\left(\frac{5\pi}{2}\right)\text{Cos}\left(\frac{5\pi x}{2}\right)+\frac{1}{6}\text{Sin}(3\pi).\text{Cos}(3\pi x)+\cdots\end{array}\right]$$

$\Rightarrow f(x)=\frac{k}{2}+\frac{2k}{\pi}\left[\text{Cos}\left(\frac{\pi x}{2}\right)-\frac{1}{3}\text{Cos}\left(\frac{3\pi x}{2}\right)+\frac{1}{5}\text{Cos}\left(\frac{5\pi x}{2}\right)-\cdots\right]$

QUESTION 8:

Find the Fourier series for the function defined by

$$f(x)=\begin{cases}x+1, & -1\le x<0\\ x-1, & 0\le x\le 1\end{cases}$$

SOLUTION: Given that

$$f(x)=\begin{cases}x+1, & -1\le x<0\\ x-1, & 0\le x\le 1\end{cases}$$

Here period $=2P=1-(-1)\Rightarrow 2P=2\Rightarrow \boxed{P=1}$

The required Fourier series of $f(x)$ is

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{P}\right)$

$\Rightarrow f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}(n\pi x)+\sum_{n=1}^{\infty}b_n.\text{Sin}(n\pi x)\longrightarrow(1)$

Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$

$\Rightarrow a_0=\int_{-1}^{1}f(x)dx \qquad (\because P=1)$

$\Rightarrow a_0=\int_{-1}^{0}f(x)dx+\int_{0}^{1}f(x)dx$

$\Rightarrow a_0=\int_{-1}^{0}(x+1)dx+\int_{0}^{1}(x-1)dx$

$\Rightarrow a_0=\left[\frac{x^2}{2}+x\right]_{-1}^{0}+\left[\frac{x^2}{2}-x\right]_{0}^{1}$

$\Rightarrow a_0=\left[0-\frac{1}{2}+1\right]+\left[\frac{1}{2}-1-0\right]$

$\Rightarrow a_0=\frac{1}{2}-\frac{1}{2}=0$

$\Rightarrow \boxed{a_0=0}$

$a_n=\frac{1}{P}\int_{-P}^{P}f(x)\text{Cos}\left(\frac{n\pi x}{P}\right)dx$

$\Rightarrow a_n=\int_{-1}^{1}f(x)\text{Cos}(n\pi x)dx \qquad (\because P=1)$

$$\Rightarrow a_n = \int_{-1}^{0} f(x)\text{Cos}(n\pi x)\,dx + \int_{0}^{1} f(x).\text{Cos}(n\pi x)\,dx$$

$$\Rightarrow a_n = \int_{-1}^{0}(x+1)\text{Cos}(n\pi x)\,dx + \int_{0}^{1}(x-1)\text{Cos}(n\pi x)dx$$

$$\Rightarrow a_n = \left[\frac{(x+1)\text{Sin}(n\pi x)}{n\pi}\right]_{-1}^{0} - \int_{-1}^{0}\frac{(1+0)\text{Sin}(n\pi x)}{n\pi}dx +$$

$$+\left[\frac{(x-1)\text{Sin}(n\pi x)}{n\pi}\right]_{0}^{1} - \int_{0}^{1}\frac{(1-0)\text{Sin}(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n = 0 + \frac{1}{n\pi}\int_{-1}^{0}(-\text{Sin}(n\pi x))dx + 0 + \frac{1}{n\pi}\int_{0}^{1}(-\text{Sin}(n\pi x))dx$$

$$\Rightarrow a_n = \frac{1}{n\pi}\left[\frac{\text{Cos}(n\pi x)}{n\pi}\right]_{-1}^{0} + \frac{1}{n\pi}\left[\frac{\text{Cos}(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$\Rightarrow a_n = \frac{1}{n^2\pi^2}[\text{Cos}\,0 - \text{Cos}\,n\pi] + \frac{1}{n\pi}[\text{Cos}n\pi - \text{Cos}\,0]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi^2}[\text{Cos}\,0 - \text{Cos}n\pi + \text{Cos}n\pi - \text{Cos}\,0] = 0$$

$$\Rightarrow \boxed{a_n = 0}$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n = \int_{-1}^{1} f(x)\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n = \int_{-1}^{0} f(x).\text{Sin}(n\pi x)dx + \int_{0}^{1} f(x).\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n = \int_{-1}^{0}(x+1).\text{Sin}(n\pi x)dx + \int_{0}^{1}(x-1).\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n = \left[\frac{-(x+1)\text{Cos}(n\pi x)}{n\pi}\right]_{-1}^{0} + \int_{-1}^{0}\frac{(1+0)\text{Cos}(n\pi x)}{n\pi}dx +$$

$$+\left[\frac{-(x-1)\text{Cos}(n\pi x)}{n\pi}\right]_{0}^{1} + \int_{0}^{1}\frac{(1-0)\text{Cos}(n\pi x)}{n\pi}dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[\text{Cos}\,0 - 0] + \frac{1}{n\pi}\int_{-1}^{0}\text{Cos}(n\pi x)dx +$$

$$-\frac{1}{n\pi}[0 + \text{Cos}\,0] + \frac{1}{n\pi}\int_{0}^{1}\text{Cos}(n\pi x)dx$$

$$\Rightarrow b_n = -\frac{1}{n\pi} + \frac{1}{n\pi}\left[\frac{\text{Sin}(n\pi x)}{n\pi}\right]_{-1}^{0} - \frac{1}{n\pi} + \frac{1}{n\pi}\left[\frac{\text{Sin}(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$\Rightarrow b_n = -\frac{2}{n\pi} + 0 + 0$$

$$\Rightarrow \boxed{b_n = \frac{-2}{n\pi}}$$

Putting the values in Eq (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\text{Cos}(n\pi x) + \sum_{n=1}^{\infty} b_n\,\text{Sin}(n\pi x)$$

$$\Rightarrow f(x) = 0 + \sum_{n=1}^{\infty}(0)\text{Cos}(n\pi x) + \sum_{n=1}^{\infty}\frac{(-2)}{n\pi}\text{Sin}(n\pi x)$$

$$\Rightarrow f(x) = \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{\text{Sin}(n\pi x)}{n} \text{ Ans.}$$

QUESTION 9:
Obtain Fourier series for the function defined by

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$$

SOLUTION: Given that

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$$

Here period $= 2P = 2 - 0 \Rightarrow 2P = 2 \Rightarrow \boxed{P = 1}$

The required Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\text{Cos}\left(\frac{n\pi x}{P}\right) + \sum_{n=1}^{\infty} b_n\,\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\text{Cos}(n\pi x) + \sum_{n=1}^{\infty} b_n.\text{Sin}(n\pi x) \longrightarrow (1)$$

Where $(\because P = 1)$

$$a_0 = \frac{1}{P}\int_{0}^{2P} f(x)dx$$

$$\Rightarrow a_0 = \int_{0}^{2} f(x)dx$$

$$\Rightarrow a_0 = \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx$$

$$\Rightarrow a_0 = \int_{0}^{1}\pi x\,dx + \int_{1}^{2}\pi(2-x)dx$$

$$\Rightarrow a_0 = \frac{\pi}{2}\left[x^2\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$\Rightarrow a_0 = \frac{\pi}{2} + \pi\left[4 - 2 - 2 + \frac{1}{2}\right]$$

$$\Rightarrow a_0 = \frac{\pi}{2} + \pi\left[\frac{1}{2}\right] = \pi$$

Thus $\boxed{a_0 = \pi}$

$$a_n = \frac{1}{P}\int_{0}^{2P} f(x)\,\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n = \int_0^2 f(x).\operatorname{Cos}(n\pi x)dx$$

$$\Rightarrow a_n = \int_0^1 f(x)\operatorname{Cos}(n\pi x)dx + \int_1^2 f(x).\operatorname{Cos}(n\pi x)dx$$

$$\Rightarrow a_n = \int_0^1 \pi x.\operatorname{Cos}(n\pi x)dx + \int_1^2 \pi(2-x).\operatorname{Cos}(n\pi x)dx$$

$$\Rightarrow a_n = \pi\left[\frac{x\operatorname{Sin}(n\pi x)}{n\pi}\right]_0^1 - \pi\int_0^1 \frac{(1)\operatorname{Sin}(n\pi x)}{n\pi}dx +$$

$$+\pi\left[\frac{(2-x)\operatorname{Sin}(n\pi x)}{n\pi}\right]_1^2 - \pi\int_1^2 \frac{(0-1)\operatorname{Sin}(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n = 0 + \frac{1}{n}\int_0^1 -\operatorname{Sin}(n\pi x)dx + 0 - \frac{1}{n}\int_1^2 -\operatorname{Sin}(n\pi x)dx$$

$$\Rightarrow a_n = \frac{1}{n}\left[\frac{\operatorname{Cos}(n\pi x)}{n\pi}\right]_0^1 - \frac{1}{n}\left[\frac{\operatorname{Cos}(n\pi x)}{n\pi}\right]_1^2$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}[\operatorname{Cos}n\pi - \operatorname{Cos}0] - \frac{1}{n^2\pi}[\operatorname{Cos}2n\pi - \operatorname{Cos}n\pi]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1\right] - \frac{1}{n^2\pi}\left[1-(-1)^n\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1 - 1 + (-1)^n\right]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left[2(-1)^n - 2\right] = \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

Thus $\boxed{a_n = \frac{2}{n^2\pi}\left((-1)^n - 1\right)}$

$$b_n = \frac{1}{P}\int_0^{2P} f(x).\operatorname{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n = \int_0^2 f(x).\operatorname{Sin}(n\pi x)dx \qquad (\because P=1)$$

$$\Rightarrow b_n = \int_0^1 f(x).\operatorname{Sin}(n\pi x)dx + \int_1^2 f(x).\operatorname{Sin}(n\pi x)dx$$

$$\Rightarrow b_n = \int_0^1 \pi x.\operatorname{Sin}(n\pi x)dx + \int_1^2 \pi(2-x).\operatorname{Sin}(n\pi x)dx$$

$$\Rightarrow b_n = \pi\left[\frac{x(-\operatorname{Cos}(n\pi x))}{n\pi}\right]_0^1 + \pi\int_0^1 \frac{1.\operatorname{Cos}(n\pi x)}{n\pi} +$$

$$+\pi\left[-(2-x)\frac{\operatorname{Cos}(n\pi x)}{n\pi}\right]_1^2 + \pi\int_1^2 (0-1)\frac{\operatorname{Cos}(n\pi x)}{n\pi}dx$$

$$\Rightarrow b_n = -\frac{1}{n}\left[x\operatorname{Cos}(n\pi x)\right]_0^1 + \frac{1}{n}\left[\frac{\operatorname{Sin}(n\pi x)}{n\pi}\right]_0^1 +$$

$$-\frac{1}{n}\left[(2-x)\operatorname{Cos}(n\pi x)\right]_1^2 - \frac{1}{n}\left[\frac{\operatorname{Sin}(n\pi x)}{n\pi}\right]_1^2$$

$$\Rightarrow b_n = \frac{-1}{n}[\operatorname{Cos}n\pi - 0] + 0 - \frac{1}{n}[0 - \operatorname{Cos}n\pi] - 0$$

$$\Rightarrow b_n = \frac{-1}{n}(-1)^n + \frac{1}{n}(-1)^n = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

Putting the values in Eq (1), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\operatorname{Cos}(n\pi x) + \sum_{n=1}^{\infty} b_n\operatorname{Sin}(n\pi x)$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left((-1)^n - 1\right)\operatorname{Cos}(n\pi x) + \sum_{n=1}^{\infty}(0)\operatorname{Sin}(n\pi x)$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left((-1)^n - 1\right)}{n^2}.\operatorname{Cos}(n\pi x)$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2}{1^2}\operatorname{Cos}\pi x - \frac{2}{3^2}\operatorname{Cos}(3\pi x) - \frac{2}{5^2}\operatorname{Cos}(5\pi x)\cdots\right]$$

$$\Rightarrow f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}.\operatorname{Cos}\pi x + \frac{1}{3^2}.\operatorname{Cos}(3\pi x) + \frac{1}{5^2}.\operatorname{Cos}(5\pi x) + \cdots\cdots\right] \text{Ans.}$$

QUESTION 10:

Expand into Fourier series $f(x) = 1 - |x| \qquad -1 \le x \le 1$

SOLUTION:

Given that $f(x) = 1 - |x| \qquad -1 \le x \le 1$

Here period $= 2P = 1 - (-1) \Rightarrow 2P = 2 \Rightarrow \boxed{P = 1}$

The required Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\operatorname{Cos}\left(\frac{n\pi x}{P}\right) + \sum_{n=1}^{\infty} b_n.\operatorname{Sin}\left(\frac{n\pi x}{P}\right)$$

OR

$$f(x) = 1 - |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\operatorname{Cos}(n\pi x) + \sum_{n=1}^{\infty} b_n.\operatorname{Sin}(n\pi x) \longrightarrow (1)$$

Where $\qquad (\because P = 1)$

$$a_0 = \frac{1}{P}\int_{-P}^{P} f(x)dx$$

$$\Rightarrow a_0 = \int_{-1}^{1}(1 - |x|)dx$$

$$\Rightarrow a_0 = \int_{-1}^{1} 1dx - \int_{-1}^{1}|x|dx$$

$$\Rightarrow a_0 = [x]_{-1}^{1} - \int_{-1}^{0}(-x)dx - \int_0^1 x\,dx \qquad \left(\because |x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}\right)$$

$$\Rightarrow a_0 = 2+\left[\frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^2}{2}\right]_{0}^{1}$$

$$\Rightarrow a_0 = 2+\left[0-\frac{1}{2}\right]-\left[\frac{1}{2}-0\right]$$

$$\Rightarrow a_0 = 2-\frac{1}{2}-\frac{1}{2}=1$$

Thus $\boxed{a_0 = 1}$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x)\cos\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n = \int_{-1}^{1} f(1-|x|)\cos(n\pi x)dx \quad (\because P=1)$$

$$\Rightarrow a_n = \int_{-1}^{1}\cos(n\pi x)dx - \int_{-1}^{1}|x|\cos(n\pi x)dx$$

$$\Rightarrow a_n = \left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{1} - \int_{-1}^{0}(-x)\cos(n\pi x)dx - \int_{0}^{1}x.\cos(n\pi x)dx$$

$$\Rightarrow a_n = 0+\int_{-1}^{0}x.\cos(n\pi x)dx - \int_{0}^{1}x.\cos(n\pi x)dx$$

$$\Rightarrow a_n = \left[\frac{x.\sin(n\pi x)}{n\pi}\right]_{-1}^{0} - \int_{-1}^{0}\frac{(1).\sin(n\pi x)}{n\pi}dx +$$

$$-\left[\frac{x\sin(n\pi x)}{n\pi}\right]_{0}^{1} + \int_{0}^{1}\frac{(1)\sin(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n = 0+\frac{1}{n\pi}\left[\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0} - \frac{1}{n\pi}\left[\frac{\cos(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$\Rightarrow a_n = \frac{1}{n^2\pi^2}[\cos 0-\cos n\pi] - \frac{1}{n^2\pi^2}[\cos n\pi - \cos 0]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi^2}[\cos 0-\cos n\pi-\cos n\pi+\cos 0]$$

$$\Rightarrow a_n = \frac{1}{n^2\pi^2}[2\cos 0-2\cos n\pi]$$

$$\Rightarrow a_n = \frac{2}{n^2\pi^2}\left[1-(-1)^n\right]$$

Thus $\boxed{a_n = \frac{2\left(1-(-1)^n\right)}{n^2\pi^2}}$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x).\sin\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n = \int_{-1}^{1}(1-|x|)\sin(n\pi x)dx$$

$$\Rightarrow b_n = \int_{-1}^{1}\sin(n\pi x)dx - \int_{-1}^{1}|x|\sin(n\pi x)dx$$

$$\Rightarrow b_n = \left[\frac{-\cos(n\pi x)}{n\pi}\right]_{-1}^{1} - \int_{-1}^{1}(-x)\sin(n\pi x)dx + \int_{0}^{1}x.\sin(n\pi x)dx$$

$$\Rightarrow b_n = 0+\left[x\left(\frac{-\cos(n\pi x)}{n\pi}\right)\right]_{-1}^{0} - \int_{-1}^{0}(1)\left(\frac{-\cos(n\pi x)}{n\pi}\right)dx +$$

$$+\left[x\left(\frac{-\cos(n\pi x)}{n\pi}\right)\right]_{0}^{1} - \int_{0}^{1}(1)\left(\frac{-\cos(n\pi x)}{n\pi}\right)dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[x.\cos(n\pi x)]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0}\cos(n\pi x)dx +$$

$$\frac{-1}{n\pi}[x.\cos(n\pi x)]_{0}^{1} + \frac{1}{n\pi}\int_{0}^{1}\cos(n\pi x)dx$$

$$\Rightarrow b_n = \frac{-1}{n\pi}[0+\cos(n\pi)] + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{0} +$$

$$\frac{-1}{n\pi}[\cos n\pi - 0] + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$\Rightarrow b_n = \frac{-1(-1)^n}{n\pi}+0-\frac{1}{n\pi}(-1)^n+0$$

$$\Rightarrow b_n = \frac{-2(-1)^n}{n\pi}$$

Thus $\boxed{b_n = \frac{-2(-1)^n}{n\pi}}$

Putting the values in Eq (1), we have

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\cos(n\pi x)+\sum_{n=1}^{\infty}b_n.\sin(n\pi x)$$

$$\Rightarrow 1-|x| = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{2\left(1-(-1)^n\right)}{n^2\pi^2}\cos(n\pi x)+\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n\pi}\sin(n\pi x)$$

$$\Rightarrow 1-|x| = \frac{1}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n^2}\cos(n\pi x)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x)$$

$$\Rightarrow 1-|x| = \frac{1}{2}+\frac{2}{\pi^2}\left[\frac{2}{1^2}\cos\pi x+\frac{2}{3^2}\cos(3\pi x)+\frac{2}{5^2}\cos(5\pi x)+\ldots\ldots\right]+$$

$$-\frac{2}{\pi}\left[\frac{-1}{1}\sin(\pi x)+\frac{1}{2}\sin(2\pi x)\frac{-1}{3}\sin(3\pi x)+\ldots\ldots\right]$$

or $$1-|x| = \frac{1}{2}+\frac{4}{\pi^2}\left[\frac{\cos(\pi x)}{1^2}+\frac{\cos(3\pi x)}{3^2}+\frac{\cos(5\pi x)}{5^2}+\cdots\right]$$

$$+\frac{2}{\pi}\left[\sin(\pi x)-\frac{1}{2}\sin(2\pi x)+\frac{1}{3}\sin(3\pi x)-\frac{1}{4}\sin(4\pi x)+\cdots\right]$$ Ans.

QUESTION 11:

Find the Fourier series corresponding to the function $f(x)$ defined in $(-2,2)$ as

$$f(x) = \begin{cases} 2, & -2 \le x \le 0 \\ x, & 0 < x < 2 \end{cases}$$

SOLUTION: Given that

$$f(x)=\begin{cases}2, & -2\le x\le 0\\ x, & 0<x<2\end{cases}$$

Here period $=2P=2-(-2)\Rightarrow 2P=4\Rightarrow \boxed{P=2}$

The required Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{P}\right)+\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{P}\right)$$

OR

$$f(x)=\frac{a_0}{2}+\sum_{n1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{2}\right)+\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{2}\right)\rightarrow(1)$$

Where $(\because P=2)$

$$a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{2}\int_{-2}^{2}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{2}\int_{-2}^{0}f(x)dx+\frac{1}{2}\int_{0}^{2}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{2}\int_{-2}^{0}2\,dx+\frac{1}{2}\int_{0}^{2}x\,dx$$

$$\Rightarrow a_0=[x]_{-2}^{0}+\frac{1}{2}\left[\frac{x^2}{2}\right]_0^2=2+\frac{1}{4}[4]=3$$

Thus $\boxed{a_0=3}$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{p}\right)dx$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{2}f(x).\text{Cos}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{0}f(x).\text{Cos}\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_{0}^{2}f(x).\text{Cos}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow a_n=\frac{1}{2}\int_{-2}^{0}2.\text{Cos}\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_{0}^{2}x.\text{Cos}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow a_n=\frac{2}{n\pi}\left[\text{Sin}\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0}+\frac{1}{2}\left[\frac{x.\text{Sin}\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_0^2-\frac{1}{2}\int_0^2\frac{(1).\text{Sin}\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}dx$$

$$\Rightarrow a_n=0+0+\frac{1}{n\pi}\int_0^2-\text{Sin}\left(\frac{n\pi x}{2}\right)dx$$

$$a_n=\frac{1}{n\pi}\left[\frac{\text{Cos}\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2=\frac{2}{n^2\pi^2}\left[\text{Cos}\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$\Rightarrow a_n=\frac{2}{n^2\pi^2}[\text{Cos}\,n\pi-\text{Cos}0]$$

$$\Rightarrow \boxed{a_n=\frac{2}{n^2\pi^2}\left[(-1)^n-1\right]}$$

$$b_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{2}f(x).\text{Sin}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{0}f(x)\,\text{Sin}\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_0^2 f(x).\text{Sin}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n=\frac{1}{2}\int_{-2}^{0}2.\text{Sin}\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_0^2 x.\text{Sin}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n=\left[\frac{-\text{Cos}\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-2}^{0}+\frac{1}{2}\left[\frac{x.\left(-\text{Cos}\left(\frac{n\pi x}{2}\right)\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$+\frac{1}{2}\int_0^2\frac{(1)\text{Cos}\left(\frac{n\pi x}{2}\right)dx}{\left(\frac{n\pi}{2}\right)}$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[\text{Cos}\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0}-\frac{1}{n\pi}\left[x.\text{Cos}\left(\frac{n\pi x}{2}\right)\right]_0^2+\frac{1}{n\pi}\int_0^2\text{Cos}\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow b_n=\frac{-2}{n\pi}[\text{Cos}\,0-\text{Cos}n\pi]-\frac{1}{n\pi}[2\,\text{Cos}n\pi-0]+\frac{1}{n\pi}\left[\frac{\text{Sin}\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[1-(-1)^n\right]-\frac{2}{n\pi}(-1)^n+0$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[1-(-1)^n+(-1)^n\right]$$

$$\boxed{b_n=\frac{-2}{n\pi}}$$

Putting the values in Eq (1), we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}\left(\frac{n\pi x}{2}\right)+\sum_{n=1}^{\infty}b_n\text{Sin}\left(\frac{n\pi x}{2}\right)$$

$$f(x)=\frac{3}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left((-1)^n-1\right)\text{Cos}\left(\frac{n\pi x}{2}\right)+\sum_{n=1}^{\infty}\frac{(-2)}{n\pi}\text{Sin}\left(\frac{n\pi x}{2}\right)$$

$$f(x)=\frac{3}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\left((-1)^n-1\right)}{n^2}\text{Cos}\left(\frac{n\pi x}{2}\right)-\frac{2}{\pi}\sum\frac{1}{n}\text{Sin}\left(\frac{n\pi x}{2}\right)$$

$$f(x)=\frac{3}{2}+\frac{2}{\pi^2}\left[\frac{-2}{1^2}\text{Cos}\left(\frac{\pi}{2}x\right)-\frac{2}{3^2}\text{Cos}\left(\frac{3\pi}{2}x\right)-\frac{2}{5^2}\text{Cos}\left(\frac{5\pi}{2}x\right)\cdots\right]$$

$$-\frac{2}{\pi}\left[\frac{1}{1}\text{Sin}\left(\frac{\pi}{2}x\right)+\frac{1}{2}\text{Sin}\left(\frac{2\pi x}{2}\right)+\frac{1}{3}\text{Sin}\left(\frac{3\pi x}{2}\right)+\frac{1}{4}\text{Sin}\left(\frac{5\pi x}{2}\right)+\frac{1}{2}\text{Sin}\left(\frac{5\pi x}{2}\right)+\frac{1}{6}\text{Sin}\left(\frac{5\pi}{2}x\right)+\cdots\cdots\right]$$

$$f(x)=\frac{3}{2}-\frac{4}{\pi^2}\left[\frac{1}{1^2}\text{Cos}\left(\frac{\pi}{2}x\right)+\frac{1}{3^2}\text{Cos}\left(\frac{3\pi x}{2}\right)+\frac{1}{5^2}\text{Cos}\left(\frac{5\pi x}{2}\right)+\cdots\cdots\right]$$

$$-\frac{2}{\pi}\left[\text{Sin}\left(\frac{\pi}{2}x\right)+\frac{1}{2}\text{Sin}(\pi x)+\frac{1}{3}\text{Sin}\left(\frac{3\pi}{2}x\right)+\frac{1}{4}\text{Sin}(2\pi x)+\cdots\cdots\right] \text{Ans.}$$

QUESTION 12:

Obtain Fourier series for $f(x)$ of period 2ℓ defined as follows

$$f(x)=\begin{cases}\ell-x, & 0<x\le\ell\\ 0, & \ell\le x<2\ell\end{cases}$$

SOLUTION: Given that

$$f(x)=\begin{cases}\ell-x, & 0<x\le\ell\\ 0, & \ell\le x<2\ell\end{cases}$$

Here period $=2P=2\ell\Rightarrow P=\ell$

The required Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{P}\right)+\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{\ell}\right)+\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{\ell}\right)\rightarrow(1)$$

Where $(\because P=\ell)$

$$a_0=\frac{1}{P}\int_0^{2P}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\ell}\int_0^{2\ell}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\ell}\int_0^{\ell}f(x)dx+\frac{1}{\ell}\int_{\ell}^{2\ell}f(x)dx$$

$$\Rightarrow a_0=\frac{1}{\ell}\int_0^{\ell}(\ell-x)dx+\frac{1}{\ell}\int_{\ell}^{2\ell}(0)dx$$

$$\Rightarrow a_0=\frac{1}{\ell}\left[\ell x-\frac{x^2}{2}\right]_0^{\ell}+0$$

$$\Rightarrow a_0=\frac{1}{\ell}\left[\ell^2-\frac{\ell^2}{2}\right]=\frac{1}{\ell}\left[\frac{\ell^2}{2}\right]=\frac{\ell}{2}$$

Thus $\boxed{a_0=\frac{\ell}{2}}$

$$\Rightarrow a_n=\frac{1}{P}\int_0^{2P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\frac{1}{\ell}\int_0^{2\ell}f(x)\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow a_n=\frac{1}{\ell}\left[\frac{(\ell-x)\text{Sin}\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}\right]_0^{\ell}-\frac{1}{\ell}\int_0^{\ell}\frac{(0-1)\text{Sin}\left(\frac{n\pi x}{\ell}\right)}{\left(\frac{n\pi}{\ell}\right)}dx$$

$$\Rightarrow a_n=\frac{1}{n\pi}\left[(\ell-x)\text{Sin}\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell}-\frac{1}{n\pi}\int_0^{\ell}(-1)\text{Sin}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow a_n=\frac{1}{n\pi}[0]-\frac{1}{n\pi}\left[\frac{\text{Cos}\left(\frac{n\pi x}{\ell}\right)}{\frac{n\pi}{\ell}}\right]_0^{\ell}$$

$$\Rightarrow a_n=0-\frac{\ell}{n^2\pi^2}\left[\text{Cos}\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell}$$

$$\Rightarrow a_n=\frac{-\ell}{n^2\pi^2}\left[\text{Cos}(n\pi)-\text{Cos}\,0\right]$$

$$\Rightarrow a_n=\frac{-\ell}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$\Rightarrow\boxed{a_n=\frac{\ell\left(1-(-1)^n\right)}{n^2\pi^2}}$$

$$\Rightarrow b_n=\frac{1}{P}\int_0^{2P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\frac{1}{\ell}\int_0^{2\ell}f(x).\text{Sin}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n=\frac{1}{\ell}\int_0^{\ell}(\ell-x).\text{Sin}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n=\frac{1}{\ell}\left[\frac{(\ell-x)\left(\frac{-\text{Cos}(n\pi x)}{\ell}\right)}{\frac{n\pi}{\ell}}\right]_0^{\ell}-\frac{1}{\ell}\int_0^{\ell}(0-1)\frac{-\text{Cos}\left(n\pi x/\ell\right)}{\frac{n\pi}{\ell}}dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[(\ell-x)\text{Cos}\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell}-\frac{1}{n\pi}\int_0^{\ell}\text{Cos}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}[0-\ell\text{Cos}0]-\frac{1}{n\pi}\left[\frac{\text{Sin}\left(\frac{n\pi x}{\ell}\right)}{\left(\frac{n\pi}{\ell}\right)}\right]_0^{\ell}$$

$$\Rightarrow b_n=\frac{\ell}{n\pi}-0$$

$$\Rightarrow\boxed{b_n=\frac{\ell}{n\pi}}$$

Putting the values in the Eq (1), we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}\left(\frac{n\pi x}{\ell}\right)+\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow f(x)=\frac{\ell}{4}+\sum_{n=1}^{\infty}a_n\frac{\ell\left(1-(-1)^n\right)}{n^2\pi^2}\cos\left(\frac{n\pi x}{\ell}\right)+\sum_{n=1}^{\infty}\frac{\ell}{n\pi}.\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow f(x)=\frac{\ell}{4}+\frac{\ell}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n^2}.\cos\left(\frac{n\pi x}{\ell}\right)+\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow f(x)=\frac{\ell}{4}+\frac{\ell}{\pi^2}\left[\frac{2}{1^2}\cos\left(\frac{\pi x}{\ell}\right)+\frac{2}{3^2}\cos\left(\frac{3\pi x}{\ell}\right)+\frac{2}{5^2}\cos\left(\frac{5\pi x}{\ell}\right)+\cdots\right]$$

$$+\frac{\ell}{\pi}\left[\frac{1}{1}\sin\left(\frac{\pi x}{\ell}\right)+\frac{1}{2}\sin\left(\frac{2\pi x}{\ell}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{\ell}\right)+\cdots\cdots\right]$$

OR

$$f(x)=\frac{\ell}{4}+\frac{2\ell}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{\ell}\right)+\frac{1}{3^2}\cos\left(\frac{3\pi x}{\ell}\right)+\frac{1}{5^2}\cos\left(\frac{5\pi x}{\ell}\right)+\cdots\cdots\right]$$

$$+\frac{\ell}{\pi}\left[\sin\left(\frac{\pi x}{\ell}\right)+\frac{1}{2}.\sin\left(\frac{2\pi x}{\ell}\right)+\frac{1}{3}.\sin\left(\frac{3\pi x}{\ell}\right)+\cdots\cdots\right]$$ Ans.

EVEN AND ODD FUNCTIONS

The knowledge about even and odd functions helps us in reducing the amount of work during the calculations of Fourier coefficients while expanding a given function in Fourier series.

DEFINITION:

A function f is said to be even if

$f(-x)=f(x) \quad \forall x\in D_f$

e.g. $\cos x$, $|x|$, $x^{2n}(n=1,2,3,\cdots\cdots)$ are even functions. The graph of even functions is symmetric about y-axis. Moreover, the area under such a curve from $-C$ to C is double the area from 0 to C.

i.e. $\int_{-C}^{C}f(x)dx=2\int_{0}^{C}f(x)dx$

PROOF:

$$\int_{-C}^{C}f(x)dx=\int_{-C}^{0}f(x)dx+\int_{0}^{C}f(x)dx\rightarrow(1)$$

Let $I=\int_{-C}^{0}f(x)dx\rightarrow(2)$

Suppose $x=-t$ then $dx=-dt$

When $x=-C$ then $t=C$

When $x=0$ then $t=0$

So $I=\int_{-C}^{0}f(x)dx=\int_{C}^{0}-f(-t)dt$

$$\Rightarrow\int_{-C}^{0}f(x)dx=-\int_{C}^{0}f(t)dt \quad (\because f \text{ is even function})$$

$$\Rightarrow\int_{-C}^{0}f(x)dx=-\int_{C}^{0}f(t)dt=\int_{0}^{C}f(x)dx$$

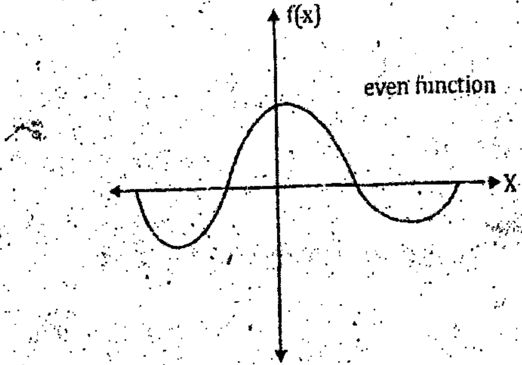
Thus $\int_{-C}^{0}f(x)dx=\int_{0}^{C}f(x)dx$

Then $(1)\Rightarrow\int_{-C}^{C}f(x)dx=\int_{-C}^{0}f(x)dx+\int_{0}^{C}f(x)dx$

$$\int_{-C}^{C}f(x)dx=\int_{0}^{C}f(x)dx+\int_{0}^{C}f(x)dx$$

$$\int_{-C}^{C}f(x)dx=2\int_{0}^{C}f(x)dx$$

Thus $\int_{-C}^{C}f(x)dx=2\int_{0}^{C}f(x)dx$



DEFINITION:

A function f is said to be odd if

$f(-x)=-f(x) \quad \forall x\in D_f$

e.g. $\sin x$, $x^{2n-1}(n=1,2,3,\cdots\cdots)$ are odd functions

The graph of odd functions are symmetric about the origin.

QUESTION:

Show that the area under an odd function from $-C$ to C is zero i.e. $\int_{-C}^{C}f(x)dx=0$ for f an odd function

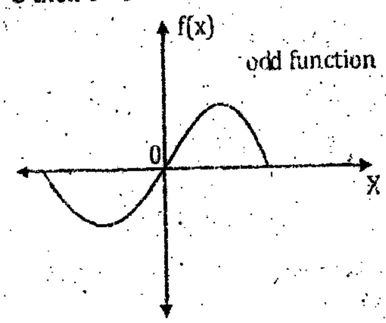
SOLUTION:

$$\int_{-C}^{C}f(x)dx=\int_{-C}^{0}f(x)dx+\int_{0}^{C}f(x)dx\rightarrow(1)$$

Let $x=-t$ then $dx=-dt$

When $x=-C$ then $t=C$

When $x=0$ then $t=0$



So $\int_{-c}^{0} f(x)dx = \int_{c}^{0} -f(-t)dt = -\int_{c}^{0} f(-t)dt$

$\int_{-c}^{0} f(x)dx = \int_{0}^{c} -f(t)dt = -\int_{0}^{c} f(t)dt$

$\Rightarrow \int_{-c}^{0} f(x)dx = -\int_{0}^{c} f(t)dt$

$\Rightarrow \int_{-c}^{0} f(x)dx = -\int_{0}^{c} f(x)dx$

Then $(1) \Rightarrow \int_{-c}^{c} f(x)dx = \int_{0}^{c} f(x)dx + \int_{0}^{c} f(x)dx$

$\int_{-c}^{c} f(x)dx = -\int_{0}^{c} f(x)dx + \int_{0}^{c} f(x)dx = 0$

Thus $\int_{-c}^{c} f(x)dx = 0$

QUESTION:

Show that the sum of even (odd) functions is even (odd)

SOLUTION:

Let f and g are even functions then

$f(-x) = f(x)\ \forall x \in D_f$

and $g(-x) = g(x)\ \forall x \in D_g$

Let $h(x) = f(x) + g(x)$

Then $h(-x) = f(-x) + g(-x)$

Then $h(-x) = f(x) + g(x) = h(x)$

Thus $h(-x) = h(x)$

i.e. the sum of even functions is even.

Similarly we can show that the sum of odd functions is odd.

QUESTION:

Show that the product of two even or odd functions is even.

SOLUTION:

Let f and g be two even function then

$f(-x) = f(x) \quad \forall x \in D_f$

and $g(-x) = g(x) \quad \forall x \in D_f$

Let $h(x) = f(x).g(x)$

Then $h(-x) = f(-x).g(-x)$

$\Rightarrow h(-x) = f(x).g(x) = h(x)$

$\Rightarrow h(-x) = h(x)\ \forall x \in D_h$

Showing that $h(x)$ i.e. $f(x).g(x)$ is even function.

Similarly we can show that the product of two odd functions is even.

QUESTION:

Show that the product of an even and an odd functions is odd.

SOLUTION:

Let f is an even and g is an odd function

Then $f(x) = f(x) \quad \forall x \in D_f$

and $g(-x) = -f(x) \quad \forall x \in D_g$

Let $\varphi(x) = f(x).g(x)$

Then $\varphi(-x) = f(-x).g(-x)$

$\Rightarrow \varphi(-x) = f(x)(-1)g(x)$

$\Rightarrow \varphi(-x) = -f(x).g(x) = -\varphi(x)$

Showing that $\varphi(x)$ i.e. $f(x).g(x)$ is an odd function.

QUESTION:

Show that any function can be expressed as a sum of an even and odd function.

SOLUTION:

Let $f(x)$ be any function.

Suppose $\varphi(x) = \dfrac{f(x)+f(-x)}{2}$ and $\psi(x) = \dfrac{f(x)-f(-x)}{2}$

then $\varphi(-x) = \dfrac{f(-x)+f(x)}{2}$

$\varphi(-x) = \dfrac{f(x)+f(-x)}{2}$

$\varphi(-x) = \varphi(x)$

$\psi(-x) = \dfrac{f(-x)-f(x)}{2}$

$\Rightarrow \psi(-x) = -\left[\dfrac{f(x)-f(-x)}{2}\right]$

$\Rightarrow \psi(-x) = -\psi(x)$

Showing that $\dfrac{f(x)+f(-x)}{2}$ is even while $\dfrac{f(x)-f(-x)}{2}$ is odd function.

Now

$\left[\dfrac{f(x)+f(-x)}{2}\right] + \left[\dfrac{f(x)-f(-x)}{2}\right] = \dfrac{1}{2}\left[f(x)+f(-x)+f(x)-f(-x)\right]$

$\Rightarrow \left[\dfrac{f(x)+f(-x)}{2}\right] + \left[\dfrac{f(x)-f(-x)}{2}\right] = \dfrac{1}{2}(2f(x))$

$\Rightarrow \left[\dfrac{f(x)+f(-x)}{2}\right] + \left[\dfrac{f(x)-f(-x)}{2}\right] = f(x)$

Thus $f(x) = \dfrac{f(x)+f(-x)}{2} + \dfrac{f(x)-f(-x)}{2}$

$\Rightarrow f(x) = \text{even} + \text{odd}$

e.g. $e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})$

$e^x = \text{Cosh}x + \text{Sinh}x$

$e^x = \text{even} + \text{odd}$

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

We know that when $f(x)$ is defined in $(-P,P)$
Where P is a +ve number, then the Fourier series of $f(x)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{P}\right)+b_n\text{Sin}\left(\frac{n\pi x}{P}\right)\right)\rightarrow(A)$$

Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx\longrightarrow(1)$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx\longrightarrow(2)$$

$$b_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx\longrightarrow(3)$$

Case 1: When $f(x)$ is an even function then

$$a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx=\frac{2}{P}\int_{0}^{P}f(x)dx$$

$$\Rightarrow\boxed{a_0=\frac{2}{P}\int_{0}^{P}f(x)dx}$$

Since $f(x)$ and $\text{Cos}\left(\frac{n\pi x}{P}\right)$ are both even function so is their product.

$$\because a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow\boxed{a_n=\frac{2}{P}\int_{0}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx}$$

Since $f(x)$ is even and $\text{Sin}\left(\frac{n\pi x}{P}\right)$ is an odd function

So $f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)$ is odd function.

$$\because b_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx=0$$

$\Rightarrow\boxed{b_n=0}$. Thus for an even function $f(x)$, we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}\left(\frac{n\pi x}{P}\right)$$

Case 2: When $f(x)$ is an odd function then

$$a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx=0$$

$$\Rightarrow\boxed{a_0=0}$$

Since $f(x)$ is an odd function and $\text{Cos}\left(\frac{n\pi x}{p}\right)$ is an even function, so $f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)$ is an odd function

$$\because a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx=0$$

$$\Rightarrow\boxed{a_n=0}$$

Since $f(x)$ and $\text{Sin}\left(\frac{n\pi x}{P}\right)$ are both odd function

So $f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)$ is an even function

$$\because b_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow\boxed{b_n=\frac{2}{P}\int_{0}^{P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx}$$

Thus for odd function $f(x)$, we have

$$f(x)=\sum_{n=1}^{\infty}b_n\text{Sin}\left(\frac{n\pi x}{P}\right)$$

EXAMPLE 1:
Find the Fourier expansion for the function
$f(x)=\text{Sin}kx,\quad -\ell<x<\ell$

SOLUTION:
Given that $f(x)=\text{Sin}(kx),\ -\ell<x<\ell$

Here period $=2P=2\ell\Rightarrow\boxed{P=\ell}$

The required Fourier expansion is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n.\text{Cos}\left(\frac{n\pi x}{\ell}\right)+b_n.\text{Sin}\left(\frac{n\pi x}{\ell}\right)\right)\rightarrow(1)$$

Since $f(x)$ is an odd function, so $a_0=0=a_n$

$$\text{So }(1)\Rightarrow\ f(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{\ell}\right)\longrightarrow(2)$$

Where $b_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$

$$\Rightarrow b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}\text{Sin}kx.\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\frac{2}{\ell}\int_{0}^{\ell}\text{Sin }kx.\text{Sin}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n=\frac{1}{\ell}\int_{0}^{\ell}2.\text{Sin}kx.\text{Sin}\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow b_n=\frac{1}{\ell}\int_{0}^{1}\left[\text{Cos}\left(\frac{n\pi}{\ell}-k\right)x-\text{Cos}\left(\frac{n\pi}{\ell}+k\right)x\right]dx$$

$$\Rightarrow b_n=\frac{1}{\ell}\left[\frac{\text{Sin}\left(\frac{n\pi}{\ell}-k\right)x}{\left(\frac{n\pi}{\ell}-k\right)}-\frac{\text{Sin}\left(\frac{n\pi}{\ell}+k\right)x}{\left(\frac{n\pi}{\ell}+k\right)}\right]_0^{\ell}$$

$$\Rightarrow b_n=\frac{1}{\ell}\left[\frac{\ell}{n\pi-\ell k}\text{Sin}\left(\frac{n\pi}{\ell}-k\right)x-\frac{\ell}{n\pi+\ell k}\text{Sin}\left(\frac{n\pi}{\ell}+k\right)x\right]_0^{\ell}$$

$$\Rightarrow b_n=\frac{1}{\ell}\left[\frac{\ell}{n\pi-\ell k}\text{Sin}\left(\frac{n\pi}{\ell}-k\right)\ell-\frac{\ell}{n\pi+\ell k}.\text{Sin}\left(\frac{n\pi}{\ell}+k\right)\ell\right]$$

$$\Rightarrow b_n=\frac{1}{n\pi-\ell k}\text{Sin}(n\pi-\ell k)-\frac{1}{n\pi+\ell k}\text{Sin}(n\pi+\ell k)$$

$$b_n=\frac{(n\pi+\ell k)\text{Sin}(n\pi-\ell k)-(n\pi-\ell k)\text{Sin}(n\pi+\ell x)}{n^2\pi^2-\ell^2k^2}$$

Simplifying, we have

$$b_n=\frac{2[\ell k.\text{Sin}(n\pi).\text{Cos}(\ell k)-n\pi.\text{Cos}(n\pi).\text{Sin}(\ell k)]}{n^2\pi^2-\ell^2k^2}\longrightarrow(3)$$

When $n=1$, then Eq (3) $\Rightarrow$

$$b_1=\frac{2[\ell k.\text{Sin}\pi.\text{Cos}(\ell k)-\pi\text{Cos}\pi.\text{Sin}(\ell k)]}{\pi^2-\ell^2k^2}$$

$$\Rightarrow \boxed{b_1=\frac{2\pi\,\text{Sin}(\ell k)}{\pi^2-\ell^2k^2}}$$

When $n=2$, then (3) $\Rightarrow$

$$b_2=\frac{2[\ell k.\text{Sin}(2\pi).\text{Cos}(\ell k)-2\pi\,\text{Cos}(2\pi).\text{Sin}(\ell k)]}{4\pi^2-\ell^2.k^2}$$

$$\Rightarrow \boxed{b_2=\frac{-4\pi\text{Sin}(\ell k)}{4\pi^2-\ell^2k^2}}$$

When $n=3$, then (3) $\Rightarrow$

$$b_3=\frac{2[\ell k\text{Sin}(3\pi).\text{Cos}(\ell k)-3\pi.\text{Cos}(3\pi).\text{Sin}(\ell k)]}{9\pi^2-\ell^2k^2}$$

$$\Rightarrow \boxed{b_3=\frac{6\pi.\text{Sin}(\ell k)}{9\pi^2-\ell^2k^2}}$$

Eq (2) can be written as

$$f(x)=\text{Sin}(kx)=b_1\text{Sin}\left(\frac{\pi x}{\ell}\right)+b_2\text{Sin}\left(\frac{2\pi x}{\ell}\right)+b_3.\text{Sin}\left(\frac{3\pi x}{\ell}\right)+\ldots$$

$$\Rightarrow \text{Sin}(kx)=\frac{2\pi\text{Sin}(\ell k)}{\pi^2-\ell^2.k^2}\text{Sin}\left(\frac{\pi x}{\ell}\right)-\frac{4\pi.\text{Sin}(\ell k)}{4\pi^2-\ell^2.k^2}\text{Sin}\left(\frac{2\pi x}{\ell}\right)+$$

$$+\frac{6\pi\text{Sin}(\ell k)}{9\pi^2-\ell^2.k^2}\text{Sin}\left(\frac{3\pi x}{\ell}\right)+\ldots\ldots$$

or $\text{Sin}(kx)=2\pi.\ \text{Sin}(\ell k)$

$$\times\left[\frac{1}{\pi^2-\ell^2.k^2}\text{Sin}\left(\frac{\pi x}{\ell}\right)-\frac{2}{4\pi^2-\ell^2.k^2}\text{Sin}\left(\frac{2\pi x}{\ell}\right)+ +\frac{3}{9\pi^2\ell^2.k^2}\text{Sin}\left(\frac{2\pi x}{\ell}\right)+\ldots\ldots\right]\text{ Ans.}$$

EXAMPLE 2:

Expand $f(x)=|\text{Cos}x|$ in a Fourier series in the interval $(-\pi,\pi)$

SOLUTION:

Given that $f(x)=|\text{Cos}\,x|$, $\quad -\pi<x<\pi$.

Here period $=2P=2\pi\Rightarrow\boxed{P=\pi}$

The required Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\text{Cos}\left(\frac{n\pi x}{P}\right)b_n\,\text{Sin}\left(\frac{n\pi x}{P}\right)\right)$$

or $$|\text{Cos}\,x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n.\text{Cos}\left(\frac{n\pi x}{\pi}\right).b_n\text{Sin}\left(\frac{n\pi x}{\pi}\right)\right)$$

or $$|\text{Cos}x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cos}(n\pi)+b_n.\text{Sin}(n\pi))$$

Since $f(x)=|\text{Cos}\,x|$ is an even function.

So $b_n=0$

$$\because|\text{Cos}\,x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}(n\pi)\longrightarrow(1)$$

Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|\text{Cos}\,x|dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi}|\text{Cos}\,x|dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi/2}|\text{Cos}\,x|dx+\frac{2}{\pi}\int_{\pi/2}^{\pi}|\text{Cos}\,x|dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi/2}\text{Cos}\,x\,dx-\frac{2}{\pi}\int_{\pi/2}^{\pi}\text{Cos}x\,dx$$

$$\Rightarrow a_0=\frac{2}{\pi}[\text{Sin}\,x]_0^{\pi/2}-\frac{2}{\pi}[\text{Sin}\,x]_{\pi/2}^{\pi}$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[\text{Sin}\frac{\pi}{2}-\text{Sin}\,0\right]-\frac{2}{\pi}\left[\text{Sin}\pi-\text{Sin}\frac{\pi}{2}\right]$$

$$\Rightarrow a_0=\frac{2}{\pi}[1-0]-\frac{2}{\pi}[0-1]$$

$$\Rightarrow a_0=\frac{2}{\pi}+\frac{2}{\pi}=\frac{4}{\pi}$$

Thus $\boxed{a_0=\frac{4}{\pi}}$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}(nx)dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|\text{Cos}\,x|.\text{Cos}(nx)dx$$

$$\Rightarrow a_n=\frac{2}{\pi}\int_0^{\pi}|\text{Cos}x|.\text{Cos}(nx)dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi/2}|\text{Cos }x|.\text{Cos}(n\pi)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}|\text{Cos }x|.\text{Cos}(nx)dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi/2}\text{Cos }x.\text{Cos}nx\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi}\text{Cos}x.\text{Cos }nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi/2}2.\text{Cos }x.\text{Cos }nx\,dx - \frac{1}{\pi}\int_{\pi/2}^{\pi}2.\text{Cos }x.\text{Cos }nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi/2}\left[\text{Cos}(n+1)x + \text{Cos}(n-1)x\right]dx +$$

$$\frac{-1}{\pi}\int_{\pi/2}^{\pi}\left[\text{Cos}(n+1)x + \text{Cos}(n-1)x\right]dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{\text{Sin}(n+1)x}{n+1} + \frac{\text{Sin}(n-1)x}{n-1}\right]_0^{\pi/2} +$$

$$-\frac{1}{\pi}\left[\frac{\text{Sin}(n+)x}{n+1} + \frac{\text{Sin}(n-1)x}{n-1}\right]_{\pi/2}^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{\text{Sin}(n+1)\frac{\pi}{2}}{n+1} + \frac{\text{Sin}(n-1)\frac{\pi}{2}}{n-1}\right] +$$

$$-\frac{1}{\pi}\left[\frac{\text{Sin}(n+1)\pi}{n+1} + \frac{\text{Sin}(n-1)\pi}{n-1} - \frac{\text{Sin}(n+1)\pi/2}{n+1} - \frac{\text{Sin}(n-1)\pi/2}{n-1}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\begin{array}{l}\frac{\text{Sin}(n+1)\frac{\pi}{2}}{n+1} + \frac{\text{Sin}(n-1)\frac{\pi}{2}}{n-1} - \frac{\text{Sin}(n+1)\pi}{n+1} \\ -\frac{\text{Sin}(n-1)\pi}{n-1} + \frac{\text{Sin}(n+1)\frac{\pi}{2}}{n+1} + \frac{\text{Sin}(n-1)\frac{\pi}{2}}{n-1}\end{array}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\begin{array}{l}\frac{2}{n+1}\text{Sin}(n+1)\frac{\pi}{2} + \frac{2}{n-1}\text{Sin}(n-1)\frac{\pi}{2} + \\ -\frac{\text{Sin}(n+1)\pi}{n+1} - \frac{\text{Sin}(n-1)\pi}{n-1}\end{array}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\begin{array}{l}\frac{2}{n+1}\text{Cos}\left(\frac{n\pi}{2}\right) - \frac{2}{n-1}\text{Cos}\left(\frac{n\pi}{2}\right) - \frac{1}{n+1}\times \\ (-\text{Sin}(n\pi)) - \frac{1}{n-1}(-\text{Sin}(n\pi))\end{array}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right)2\text{Cos}\left(\frac{n\pi}{2}\right) + \left(\frac{1}{n+1} + \frac{1}{n-1}\right)\text{Sin}(n\pi)\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\left(\frac{-2}{n^2-1}\right)2\text{Cos}\left(\frac{n\pi}{2}\right) + \frac{2n}{n^2+1}(0)\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{-4}{n^2-1}\text{Cos}\left(\frac{n\pi}{2}\right) + 0\right]$$

$$\boxed{a_n = \frac{-4}{\pi(n^2-1)}\text{Cos}\left(\frac{n\pi}{2}\right),\ n \neq 1}$$

For n=1, we have

$$a_1 = \frac{1}{p}\int_{-p}^{p}f(x)\cos x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\cos x|\cos x\,dx$$

$$\Rightarrow a_1 = \frac{2}{\pi}\int_0^{\pi}|\cos x|\cos x\,dx$$

$$\Rightarrow a_1 = \frac{2}{\pi}\int_0^{\pi/2}|\cos x|x\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}|\cos x|\cos x\,dx$$

$$\Rightarrow a_1 = \frac{2}{\pi}\int_0^{\pi/2}\cos x.\cos x\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(-\cos)\cos x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_0^{\pi/2}2\cos^2 x\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi}\cos^2 x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_0^{\pi/2}(\cos 2x+1)dx - \frac{1}{\pi}\int_{\pi/2}^{\pi}(\cos 2x+1)dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[\frac{\text{Sin}2x}{2} + x\right]_0^{\pi/2} - \frac{1}{\pi}\left[\frac{\text{Sin}2x}{2} + x\right]_{\pi/2}^{\pi}$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[0 + \frac{\pi}{2} - 0 - 0\right] - \frac{1}{\pi}\left[0 + \pi - 0 - \frac{\pi}{2}\right]$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[\frac{\pi}{2}\right] - \frac{1}{\pi}\left[\frac{\pi}{2}\right] = 0$$

$$\Rightarrow \boxed{a_1 = 0}$$

Eq (1) can be written as

$$|\cos x| = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty}a_n\cos nx$$

Putting the values, we have

$$|\cos x| = \frac{2}{\pi} + (0)\cos x + \sum_{n=2}^{\infty}\frac{-4}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right)\cos(nx)$$

$$\Rightarrow |\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\left(\frac{n\pi}{2}\right)\cos nx$$

$$\text{or } |\text{Cos}x| = \frac{2}{\pi} - \frac{4}{\pi}\left[\begin{array}{l}\frac{1}{2^2-1}(-1)\text{Cos }2x + \frac{1}{4^2-1}(1)\text{Cos}4x + \\ \frac{1}{6^2-1}(-1)\text{Cos }6x + \cdots\cdots\end{array}\right]$$

$$\text{or } |\text{Cos }x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{1}{3}\text{Cos }2x - \frac{1}{15}\text{Cos}4x + \frac{1}{35}\text{Cos}6x - \cdots\cdots\right]$$

EXERCISE 6.3

QUESTION 1:
Express the function $f(x)=x, -\pi<x<\pi$ in a Fourier series.

SOLUTION:
Here $f(x)=x, -\pi<x<\pi$ is an odd function.

and period $=2P=2\pi \Rightarrow \boxed{P=\pi}$

So the Fourier series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty} b_n . \text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\text{or } f(x)=\sum_{n=1}^{\infty} b_n . \text{Sin}\left(\frac{n\pi x}{\pi}\right) \quad (\because P=\pi)$$

$$\text{or } f(x)=\sum_{n=1}^{\infty} b_n . \text{Sin}(nx) \longrightarrow (1)$$

Where $b_n=\frac{1}{P}\int_{-P}^{P} f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x.\text{Sin}(nx)dx$$

$$\Rightarrow b_n=\frac{2}{\pi}\int_{0}^{\pi} x.\text{Sin}(nx)dx \quad (\because x.\text{Sin}(nx) \text{ is even function})$$

$$b_n=\frac{2}{\pi}\left[\frac{-x.\text{Cos}(n\pi)}{n}\right]_0^{\pi}+\frac{2}{\pi}\int_0^{\pi}\frac{(1).\text{Cos}(nx)dx}{n}$$

$$b_n=\frac{-2}{\pi n}\left[x.\text{Cos}(nx)\right]_0^{\pi}+\frac{2}{\pi n^2}\left[\text{Sin}(nx)\right]_0^{\pi}$$

$$b_n=\frac{-2}{\pi n}\left[\pi.\text{Cos}(n\pi)-0\right]+\frac{2}{\pi n^2}(0)$$

$$b_n=\frac{-2}{n}(-1)^n=\frac{2}{n}(-1)^{n+1}$$

Thus $\boxed{b_n=\frac{2}{n}(-1)^{n+1}}$

Then $(1) \Rightarrow$

$$f(x)=\sum_{n=1}^{\infty} b_n \text{Sin}(nx)$$

$$\text{or } x=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}.\text{Sin}(nx)$$

$$\text{or } x=2\left[\frac{1}{1}\text{Sin}x-\frac{1}{2}\text{Sin}2x+\frac{1}{3}\text{Sin }3x-\frac{1}{4}.\text{Sin}4x+\cdots\cdots\right]$$

$$\text{or } x=2\left[\text{Sin}x-\frac{1}{2}.\text{Sin}2x+\frac{1}{3}.\text{Sin}3x-\frac{1}{4}\text{Sin}4x+\cdots\cdots\right]$$

QUESTION 2:
Expand into Fourier series

$f(x)=\text{Cos}x, \quad \frac{-\pi}{2}\le x\le\frac{\pi}{2}$

SOLUTION:
Given that $f(x)=\text{Cos}x, \quad \frac{-\pi}{2}\le x\le\frac{\pi}{2}$

To make the interval of length 2π, we set $x=\frac{t}{2}$

Then $f(x)=f\left(\frac{t}{2}\right)=\text{Cos}\left(\frac{t}{2}\right)=g(t)$

$$\Rightarrow g(t)=\text{Cos}\left(\frac{t}{2}\right), -\pi\le t\le\pi$$

Since $g(t)$ is an even function, so it Fourier series is

$$g(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}nt \longrightarrow (1)$$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t)dt$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\text{Cos}\left(\frac{t}{2}\right)dt$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_{0}^{\pi}\text{Cos}\left(\frac{t}{2}\right)dt \quad \left(\because \text{Cos}\left(\frac{t}{2}\right) \text{ is even function}\right)$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[\frac{\text{Sin}\left(t/2\right)}{1/2}\right]_0^{\pi}=\frac{4}{\pi}$$

Thus $\boxed{a_0=4/\pi}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(t).\text{Cos }nt\ dt$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\text{Cos}\left(\frac{t}{2}\right).\text{Cos}(nt)dt$$

$$\Rightarrow a_n=\frac{2}{\pi}\int_{0}^{\pi}\text{Cos}(nt).\text{Cos}\left(\frac{t}{2}\right)dt$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{0}^{\pi}2.\text{Cos}(nt).\text{Cos}\left(t/2\right)dt$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{0}^{\pi}\left[\text{Cos}\left(n+\frac{1}{2}\right)t+\text{Cos}\left(n-\frac{1}{2}\right)t\right]dt$$

$$\Rightarrow a_n=\frac{1}{\pi}\left[\frac{\text{Sin}\left(n+\frac{1}{2}\right)t}{n+\frac{1}{2}}+\frac{\text{Sin}\left(n-\frac{1}{2}\right)t}{n-\frac{1}{2}}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{2}{2n+1}\text{Sin}\left(nt+\frac{t}{2}\right)+\frac{2}{2n-1}\text{Sin}\left(nt-\frac{t}{2}\right)\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{1}{2n+1}\text{Sin}\left(n\pi+\frac{\pi}{2}\right)+\frac{1}{2n-1}\text{Sin}\left(n\pi-\frac{\pi}{2}\right)-0\right]$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\frac{1}{2n+1}\text{Cos}(n\pi)+\frac{1}{2n-1}(-\text{Cos}(n\pi))\right]$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1}-\frac{(-1)^n}{2n-1}\right]$$

$$\Rightarrow a_n = \frac{2(-1)^n}{\pi}\left[\frac{2n-1-(2n+1)}{4n^2-1}\right]$$

$$\Rightarrow a_n = \frac{2(-1)^n}{\pi}\left[\frac{-2}{4n^2-1}\right]=\frac{4(-1)^{n+1}}{\pi(4n^2-1)}$$

Thus $\boxed{a_n = \frac{4(-1)^{n+1}}{\pi(4n^2-1)}}$

Putting the values in (1), we have

$$g(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cosnt}$$

$$\Rightarrow \text{Cos}\left(\frac{t}{2}\right)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{\pi(4n^2-1)}\text{Cos}(nt)$$

$$\Rightarrow \text{Cos}\left(\frac{t}{2}\right)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\text{Cos}(nt)$$

Putting $t=2x$, we have

$$\text{Cosx}=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\text{Cos}(2nx) \qquad \text{Ans.}$$

QUESTION 3:

Obtain a Fourier series for $f(x)=x^3,\ -\pi<x<\pi$

SOLUTION:

Given that $f(x)=x^3,\ -\pi<x<\pi$

Here $f(x)$ is an odd function and

Period $=2P=2\pi \Rightarrow \boxed{P=\pi}$

So the Fourier series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty}b_n\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow x^3=\sum_{n=1}^{\infty}b_n\text{Sin}\left(\frac{n\pi x}{\pi}\right) \quad (\because P=\pi)$$

$$\Rightarrow x^3=\sum_{n=1}^{\infty}b_n\text{Sin}(nx)\longrightarrow(1)$$

Where $b_n=\frac{1}{P}\int_{-P}^{P}f(x)\text{Sin}\left(\frac{n\pi x}{P}\right)$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^3\text{Sin}(nx)dx \because P=\pi$$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi}x^3.\text{Sin}(nx)dx \quad (\because x^3\text{Sin}(nx)\text{ is even function})$$

$$\Rightarrow b_n=\frac{2}{\pi}\left[x^3\frac{(-\text{Cos}(nx))}{n}\right]_0^{\pi}+\frac{2}{\pi}\int_0^{\pi}3x^2\frac{\text{Cos}(nx)}{n}dx$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[x^3\text{Cos}(nx)\right]_0^{\pi}+\frac{6}{n\pi}\int_0^{\pi}x^2.\text{Cosnx }dx$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[\pi^3\text{Cos }n\pi-0\right]+\frac{6}{n\pi}\left[\frac{x^2.\text{Sin}(nx)}{n}\right]_0^{\pi}-\frac{6}{n\pi}\int_0^{\pi}\frac{2x.\text{Sin}(nx)dx}{n}$$

$$\Rightarrow b_n=\frac{-2\pi^2}{n}(-1)^n+0-\frac{12}{n^2\pi}\int_0^{\pi}x.\text{Sin}(nx)dx$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}\pi^2}{n}-\frac{12}{n^2\pi}\left[\frac{-x.\text{Cos}(nx)}{n}\right]_0^{\pi}-\frac{12}{n^2\pi}\int_0^{\pi}\frac{(1)\text{Cos}(nx)dx}{n}$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}\pi^2}{n}+\frac{12}{n^3\pi}\left[\pi\text{Cos}(n\pi)-0\right]-\frac{12}{n^3\pi}\int_0^{\pi}\text{Cos}(nx)dx$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}\pi^2}{n}+\frac{12(-1)^n}{n^3}-\frac{12}{n^3\pi}\left[\frac{\text{Sin}(nx)}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}\pi^2}{n}+\frac{12(-1)^n}{n^3}-0$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}\pi^2}{n}+\frac{12(-1)^n}{n^3}$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}\pi^2}{n}-\frac{12(-1)^{n+1}}{n^3}$$

$$\text{or } b_n=2(-1)^{n+1}\left[\frac{\pi^2}{n}-\frac{6}{n^3}\right]$$

Then Eq (1) $\Rightarrow$

$$x^3=\sum_{n=1}^{\infty}2(-1)^{n+1}\left[\frac{\pi^2}{n}-\frac{6}{x^3}\right].\text{Sin}(nx)$$

$$\Rightarrow x^3=2\sum_{n=1}^{\infty}(-1)^{n+1}\left[\frac{\pi^2}{n}-\frac{6}{x^3}\right]\text{Sin}(nx)$$

$$\Rightarrow x^3=2\left[\begin{array}{l}\left(\frac{\pi^2}{1}-\frac{6}{1}\right)\text{Sinx}-\left(\frac{\pi^2}{2}-\frac{6}{2^3}\right)\text{Sin2x}+\left(\frac{\pi^2}{3}-\frac{6}{3^3}\right)\text{Sin3x}+\\-\left(\frac{\pi^2}{4}-\frac{6}{4^3}\right).\text{Sin}+\cdots\cdots\end{array}\right]$$

QUESTION 4:

Expand in a Fourier series $f(x)=|x|,\quad -1<x<1$

SOLUTION:

Given that $f(x)=|x|,\quad -1<x<1$

Here $f(x)$ is an even function and

Period $=2P=2\Rightarrow P=1$

So the Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{P}\right)$$

or $$|x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}(n\pi x)\longrightarrow(1)$$

Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$

$$\Rightarrow a_0=\frac{2}{P}\int_0^P f(x)dx \quad (\because f(x) \text{ is even function})$$

$$\Rightarrow a_0=2\int_0^1|x|dx$$

$$\Rightarrow a_0=2\int_0^1 x\,dx=2\left[\frac{x^2}{2}\right]_0^1 \quad (\because |x|=x, x\geq 0)$$

$$\Rightarrow \boxed{a_0=1}$$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\int_{-1}^{1}|x|.\text{Cos}(n\pi x)dx$$

$$\Rightarrow a_n=2\int_0^1|x|\text{Cos}(n\pi x)dx \quad (\because |x|.\text{Cos}(n\pi x) \text{ is even function})$$

$$\Rightarrow a_n=2\int_0^1 x.\text{Cos}(n\pi x)dx$$

$$\Rightarrow a_n=2\left[\frac{x.\text{Sin}(n\pi x)}{n\pi}\right]_0^1-2\int_0^1\frac{(1).\text{Sin}(n\pi x)dx}{n\pi}$$

$$\Rightarrow a_n=\frac{2}{n\pi}\left[x.\text{Sin}(n\pi x)\right]_0^1-\frac{2}{n\pi}\left[\frac{-\text{Cos}(n\pi x)}{n\pi}\right]_0^1$$

$$\Rightarrow a_n=\frac{2}{n\pi}(0)+\frac{2}{n^2\pi^2}\left[\text{Cos}n\pi-\text{Cos}\,0\right]$$

$$\Rightarrow a_n=0+\frac{2}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$\boxed{a_n=\frac{2}{n^2\pi^2}\left[(-1)^n-1\right]}$$

Putting the values in Eq (1), we have

$$|x|=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[(-1)^n-1\right].\text{Cos}(n\pi x)$$

$$\Rightarrow |x|=\frac{1}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n-1\right].\text{Cos}(n\pi x)$$

$$\Rightarrow |x|=\frac{1}{2}+\frac{2}{\pi^2}\left[\frac{1}{1^2}(-2).\text{Cos}\pi x+\frac{1}{3^2}(-2).\text{Cos}(3\pi x)+\frac{1}{5^2}(-2)\text{Cos}(5\pi x)+\cdots\cdots\right]$$

$$\Rightarrow |x|=\frac{1}{2}-\frac{4}{\pi^2}\left[\text{Cos}(\pi x)+\frac{\text{Cos}(3\pi x)}{3^2}+\frac{\text{Cos}(5\pi x)}{5^2}+\cdots\right] \text{ Ans.}$$

QUESTION 5:

Find a Fourier series for $f(x)=1-x^2, \quad -1\leq x\leq 1$

SOLUTION:

Given that $f(x)=1-x^2, \quad -1\leq x\leq 1$

Here $f(x)$ is an even function and

Period $=2P=2\Rightarrow \boxed{P=1}$

So that Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\text{Cos}(n\pi x)\longrightarrow(1)$$

Where $a_0=\frac{1}{P}\int_0^P f(x)dx$

$$\Rightarrow a_0=2\int_0^P f(x)dx \quad (\because f(x) \text{is even function})$$

$$\Rightarrow a_0=2\int_0^1(1-x^2)dx$$

$$\Rightarrow a_0=2\left[x-\frac{x^3}{3}\right]_0^1=2\left[1-\frac{1}{3}\right]=\frac{4}{3}$$

Thus $\boxed{a_0=\frac{4}{3}}$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\int_{-1}^{1}(1-x^2).\text{Cos}(n\pi x)dx$$

$$\Rightarrow a_n=2\int_0^1(1-x^2)\text{Cos}(n\pi x)\,dx \quad \left(\because (1-x^2)\text{Cos}(n\pi x) \text{ is even function}\right)$$

$$\Rightarrow a_n=2\left[\frac{(1-x^2)\text{Sin}(n\pi x)}{n\pi}\right]_0^1-2\int_0^1\frac{(0-2x)\text{Sin}(n\pi x)dx}{n\pi}$$

$$\Rightarrow a_n=\frac{2}{n\pi}\left[(1-x^2).\text{Sin}(n\pi x)\right]_0^1+\frac{4}{n\pi}\int_0^1 x.\text{Sin}(n\pi x)dx$$

$$\Rightarrow a_n=\frac{2}{n\pi}(0)+\frac{4}{n\pi}\left[\frac{-x.\text{Cos}(n\pi x)}{n\pi}\right]_0^1+\frac{4}{n\pi}\int_0^1\frac{(1)\text{Cos}(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n=0-\frac{4}{n^2\pi^2}\left[x.\text{Cos}(n\pi x)\right]_0^1+\frac{4}{n^2\pi^2}\left[\frac{\text{Sin}(n\pi x)}{n\pi}\right]_0^1$$

$$\Rightarrow a_n = \frac{-4}{n^2\pi^2}\left[\cos(n\pi)-0\right]+\frac{4}{n^3\pi^3}(0)$$

$$\Rightarrow a_n = \frac{-4}{n^2\pi^2}(-1)^n$$

$$\boxed{a_n = \frac{4(-1)^{n+1}}{n^2\pi^2}}$$

Putting the values is Eq (1), we have

$$1-x^2 = \frac{2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2\pi^2}\cos(n\pi x)$$

$$\Rightarrow 1-x^2 = \frac{2}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

$$\Rightarrow 1-x^2 = \frac{2}{3}+\frac{4}{\pi^2}\left[\cos\pi x-\frac{1}{2^2}\cos(2\pi x)+\frac{1}{3^2}\cos(3\pi x)+ -\frac{1}{4^2}\cos(4\pi x)+\cdots\right]$$

QUESTION 6:

Express $f(x)=\cos Px$, $-\pi<x<\pi$, where P is a fraction, as a Fourier series.

SOLUTION:

Given that $f(x)=\cos Px$, $-\pi<x<\pi$,

Here $f(x)$ is an even function and

Period $=2\ell=2\pi \Rightarrow \boxed{\ell=\pi}$

So the required Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\cos\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \cos(Px)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{\pi}\right) \qquad (\because \ell=\pi)$$

$$\Rightarrow \cos(Px)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\cos(nx) \longrightarrow (1)$$

Where

$$a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)dx$$

$$\Rightarrow a_0=\frac{2}{\ell}\int_0^{\ell}f(x)dx \quad (\because f(x) \text{ is even function})$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi}\cos(Px)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[\frac{\sin(Px)}{P}\right]_0^{\pi}=\frac{2}{P\pi}\left[\sin(P\pi)-\sin 0\right]$$

$$\Rightarrow \boxed{a_0=\frac{2}{P\pi}\sin(P\pi)}$$

$(\because$ P is fraction and not an integer so $\sin(p\pi)\neq 0)$

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x).\cos\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Rightarrow a_n=\frac{2}{\ell}\int_0^{\ell}f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad \left(\because f(x).\cos\left(\frac{n\pi x}{\ell}\right) \text{ even function}\right)$$

$$\Rightarrow a_n=\frac{2}{\pi}\int_0^{\pi}\cos(Px).\cos(nx)dx \quad (\because \ell=\pi)$$

$$a_n=\frac{1}{\pi}\int_0^{\pi}2\cos(nx).\cos(Px)dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_0^{\pi}\left[\cos(n+P)x+\cos(n-P)x\right]dx$$

$$\Rightarrow a_n=\frac{1}{P}\left[\frac{\sin(n+P)x}{n+P}+\frac{\sin(n-P)x}{n-P}\right]_0^{\pi}$$

$$\Rightarrow a_n=\frac{1}{P}\left[\frac{\sin(n+P)\pi}{n+P}+\frac{\sin(n-P)\pi}{n-P}\right]$$

$$\Rightarrow a_n=\frac{1}{P}\left[\frac{(-1)^n\sin(P\pi)}{n+P}+\frac{(-1)^n\sin(P\pi)}{n-P}\right]$$

$$\Rightarrow a_n=\frac{(-1)^n}{P}\left[\frac{1}{n+P}-\frac{1}{n-P}\right]\sin(P\pi)$$

$$\Rightarrow a_n=\frac{(-1)^n}{P}\left[\frac{n-P-n-P}{n^2-P^2}\right]\sin(P\pi)$$

$$\Rightarrow a_n=\frac{2(-1)^{n+1}P}{P(n^2-P^2)}\sin(P\pi)$$

$$\Rightarrow a_n=\frac{2(-1)^{n+1}}{n^2-P^2}.\sin(Px)$$

$$\boxed{a_n=\frac{2(-1)^{n+1}}{n^2-P^2}.\sin(P\pi)}$$

Putting the values in (1), we have

$$\cos(Px)=\frac{\sin(Px)}{P\pi}+\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^2-P^2}\sin(P\pi)\cos(nx)$$

$$\Rightarrow \cos(Px)=\frac{\sin(P\pi)}{P\pi}+2\sin(px)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-P^2}\cos(nx)$$

$$\Rightarrow \cos(Px)=\frac{\sin(P\pi)}{P\pi}+2.(\sin(Px))$$

$$\times\left[\frac{1}{1^2-P^2}\cos x-\frac{1}{2^2-P^2}\cos 2x+ +\frac{1}{3^2-P^2}\cos 3x-\frac{1}{4^2-P^2}\cos 4x+\cdots\right]$$

$$\Rightarrow \text{Cos}(Px) = \frac{\text{Sin}(P\pi)}{P\pi} + 2(\text{Sin}(P\pi))$$
$$\times\left[\frac{\text{Cos}x}{1-P^2} - \frac{\text{Cos}\,2x}{4-P^2} + \frac{\text{Cos}3x}{9-P^2} - \frac{\text{Cos}\,4x}{16-P^2} + \cdots\cdots\right]$$

QUESTION 7:

Find the Fourier series to represent the function $f(x)=|\text{Sin}\,x|, \quad -\pi<x<\pi$.

SOLUTION:

Given that $f(x)=|\text{Sin}\,x|, \quad -\pi<x<\pi$

Here $f(x)$ is an even function and

Period $= 2P = 2\pi \Rightarrow \boxed{P=\pi}$

So, the Fourier series of $f(x)$ is given by

$$f(x)\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\text{Cos}\left(\frac{n\pi x}{P}\right)$$

$$|\text{Sin}x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\text{Cos}(nx) \longrightarrow (1) \qquad (\because P=\pi)$$

Where $a_0 = \frac{1}{P}\int_{-P}^{P} f(x)dx$

$$\Rightarrow a_0 = \frac{2}{P}\int_0^P f(x)dx \quad (\because f(x) \text{ is even function})$$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^\pi |\text{Sin}x|dx$$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^\pi \text{Sin}x\,dx \quad (\because |\sin x| = \sin x, \text{ for } 0\le x\le\pi)$$

$$\Rightarrow a_0 = \frac{-2}{\pi}[\text{Cos}\,x]_0^\pi$$

$$\Rightarrow a_0 = \frac{-2}{\pi}[\text{Cos}\,\pi - \text{Cos}\,o]$$

$$\Rightarrow a = \frac{-2}{\pi}[-1-1]$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x).\text{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\text{Sin}x|.\text{Cos}(nx)dx \qquad (\because P=\pi)$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi |\sin x|.\text{Cos}(nx)dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi \text{Sin}x.\,\text{Cos}(nx)dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^\pi 2.\,\text{Sin}x.\text{Cos}(nx)dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^\pi [\text{Sin}(n+1)x - \text{Sin}(n-1)x]dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{-\text{Cos}(n+1)x}{n+1} + \frac{\text{Cos}(n-1)x}{n-1}\right]_0^\pi$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{-\text{Cos}(n+1)\pi}{n+1} + \frac{\text{Cos}(n-1)\pi}{n-1} + \frac{\text{Cos}\,0}{n+1} - \frac{\text{Cos}\,0}{n-1}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{-\text{Cos}\,n\pi}{n+1} + \frac{\text{Cos}\,n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[(-1)^n + 1\right]\left(\frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[(-1)^n + 1\right]\left[\frac{-2}{n^2-1}\right]$$

$$\Rightarrow a_n = \frac{-2\left((-1)^n + 1\right)}{\pi(n^2-1)}$$

Thus $\Rightarrow \boxed{a_n = \frac{2\left((-1)^{n+1} - 1\right)}{\pi(n^2-1)}, \qquad n\neq 1}$

For n=1, we have

$$a_1 = \frac{1}{P}\int_{-P}^{P} f(x).\text{Cos}\left(\frac{\pi x}{P}\right)dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\text{Sin}x|.\,\text{Cos}x\,dx \qquad (\because P=\pi)$$

$$\Rightarrow a_1 = \frac{2}{\pi}\int_0^\pi |\text{Sin}x|.\,\text{Cos}x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_0^\pi 2.\,\text{Sin}\,x.\,\text{Cos}x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_0^\pi \text{Sin}(2x)dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[\frac{-\cos(2x)}{2}\right]_0^\pi$$

$$\Rightarrow a_1 = \frac{-1}{2\pi}\left[\text{Cos}(2x)\right]_0^\pi$$

$$\Rightarrow a_1 = \frac{-1}{2\pi}[\text{Cos}2\pi - \text{Cos}\,0]$$

$$\Rightarrow a_1 = \frac{-1}{2\pi}[1-1] = 0$$

$\Rightarrow \boxed{a_1 = 0}$

Eq (1) can be written as

$$|\text{Sin}x| = \frac{a_0}{2} + a_1.\text{Cos}\,x + \sum_{n=2}^{\infty} a_n.\text{Cos}(nx)$$

$$\Rightarrow |\text{Sin}x| = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{2\left((-1)^{n+1} - 1\right)}{\pi(n^2-1)}\text{Cos}(nx)$$

$$\Rightarrow |\text{Sin}x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n - 1}{n^2-1}\right)\text{Cos}(nx)$$

$$\Rightarrow |\text{Sin}x| = \frac{2}{\pi} + \frac{2}{\pi}\left[\frac{-2}{3^2-1}\text{Cos}3x - \frac{2}{5^2-1}\text{Cos}5x - \frac{2}{7^2-1}\text{Cos}7x \cdots\cdots\right]$$

$$\Rightarrow |\text{Sin}x| = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{8}\text{Cos}3x + \frac{1}{24}\text{Cos}5x + \frac{1}{48}\text{Cos}7x + \cdots\cdots\right] \text{ Ans.}$$

QUESTION 8:

Find the Fourier series for $f(x)$ if

$$f(x) = \begin{cases} \frac{-1}{2}, & -\pi < x < 0 \\ \frac{1}{2}, & 0 < x < \pi \end{cases}$$

And $f(x+2\pi) = f(x)$

SOLUTION: Given that

$$f(x) = \begin{cases} \frac{-1}{2}, & -\pi < x < 0 \\ \frac{1}{2}, & 0 < x < \pi \end{cases}$$

and $f(x+2\pi) = f(x)$

Here period $= 2P = 2\pi \Rightarrow \boxed{P = \pi}$

The graph of the given function is symmetric about the origin, there for; $f(x)$ is an odd function, so its Fouries series is

$$f(x) = \sum_{n=1}^{\infty} b_n.\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n.\text{Sin}(nx) \longrightarrow (1) \quad (\because P = \pi)$$

Where $b_n = \frac{1}{P}\int_{n=1}^{\infty} f(x).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\text{Sin}(nx)dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\text{Sin}(nx)dx \quad (\because f(x)\sin(nx) \text{ is even function})$$

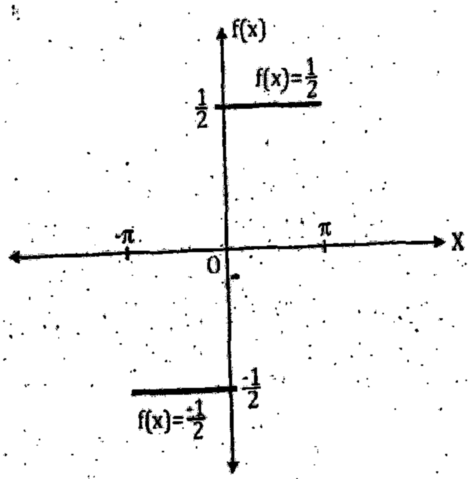
$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\text{Sin}(nx)dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{-\text{Cos}(nx)}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[\text{Cos}(nx)\right]_0^{\pi} = \frac{-1}{n\pi}\left[\text{Cos}(n\pi) - \text{Cos}0\right]$$

$$\Rightarrow b_n = \frac{-1}{n\pi}\left[(-1)^n - 1\right]$$

Thus $\boxed{b_n = \frac{1}{n\pi}\left(1 - (-1)^n\right)}$



Putting the values in (1), we have

$$f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\text{Sin}(nx)$$

$$f(x) = \frac{1}{\pi}\left[\frac{2}{1}\text{Sin}x + \frac{2}{3}\text{Sin}3x + \frac{2}{5}\text{Sin}5x + \cdots\cdots\right]$$

$$f(x) = \frac{2}{\pi}\left[\text{Sin}x + \frac{\text{Sin}(3x)}{3} + \frac{\text{Sin}(5x)}{5} + \cdots\cdots\right]$$

QUESTION 9:

Obtain Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1 + \frac{2}{\pi}x, & -\pi \le x \le 0 \\ 1 - \frac{2}{\pi}x, & 0 \le x \le \pi \end{cases}$$

And deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi^2}{8}$$

SOLUTION: Given that

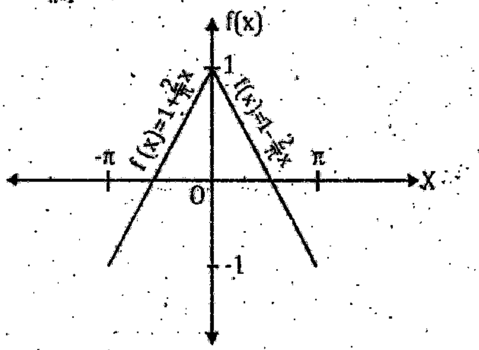
$$f(x) = \begin{cases} 1 + \frac{2}{\pi}x, & -\pi \le x \le 0 \\ 1 - \frac{2}{\pi}x, & 0 \le x \le \pi \end{cases}$$

Here period $= 2P = 2\pi \Rightarrow \boxed{P = \pi}$

Since the graph of the given function is symmetric about y-axis, so $f(x)$ is an even function, so the Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\operatorname{Cos}\left(\frac{n\pi x}{P}\right)$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\operatorname{Cos}(nx)\rightarrow(1)$$



Where $a_0=\frac{1}{P}\int_{-P}^{P}f(x)dx$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_{0}^{\pi}f(x)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{2}{\pi}x\right)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi}$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[\pi-\frac{\pi^2}{\pi}\right]=0$$

$$\Rightarrow \boxed{a_0=0}$$

$$a_n=\frac{1}{P}\int_{-P}^{P}f(x)\operatorname{Cos}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x).\operatorname{Cos}(nx)dx \qquad (\because P=\pi)$$

$$\Rightarrow a_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\operatorname{Cos}(nx)dx \qquad \left(\because f(x)\cos(nx)\text{ is even function}\right)$$

$$\Rightarrow a_n=\frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{2}{\pi}x\right)\operatorname{Cos}(nx)dx$$

$$\Rightarrow a_n=\frac{2}{\pi}\left[\left(1-\frac{2}{\pi}x\right)\frac{\operatorname{Sin}(nx)}{n}\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}\left(0-\frac{2}{\pi}\right)\frac{\operatorname{Sin}(nx)}{n}dx$$

$$\Rightarrow a_n=\frac{2}{n\pi}\left[\left(1-\frac{2}{\pi}x\right)\operatorname{Sin}(nx)\right]_0^{\pi}+\frac{4}{n\pi^2}\int_0^{\pi}\operatorname{Sin}(nx)dx$$

$$\Rightarrow a_n=\frac{2}{n\pi}[0]+\frac{4}{n\pi^2}\left[\frac{-\operatorname{Cos}(nx)}{n}\right]_0^{\pi}$$

$$\Rightarrow a_n=\frac{-4}{n^2\pi^2}[\operatorname{Cos}n\pi-\operatorname{Cos}0]$$

$$\Rightarrow a_n=\frac{-4}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$\boxed{a_n=\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]}$$

Putting the values in Eq (1), we have

$$f(x)=0+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\operatorname{Cos}(nx)$$

$$f(x)=\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\operatorname{Cos}(nx)$$

$$f(x)=\frac{4}{\pi^2}\left[\frac{2}{1^2}\operatorname{Cos}x+\frac{2}{3^2}\operatorname{Cos}3x+\frac{2}{5^2}\operatorname{Cos}5x+\cdots\right]$$

$$f(x)=\frac{8}{\pi^2}\left[\frac{\operatorname{Cos}x}{1^2}+\frac{\operatorname{Cos}3x}{3^2}+\frac{\operatorname{Cos}5x}{5^2}+\ldots\ldots\right]\longrightarrow(2)$$

Which is the required Fourier series of $f(x)$ putting $x=0$, in (2), we have

$$f(0)=1=\frac{8}{\pi^2}\left[\frac{\operatorname{Cos}0}{1^2}+\frac{\operatorname{Cos}0}{3^2}+\frac{\operatorname{Cos}0}{5^2}+\ldots\ldots\right]$$

$$\Rightarrow 1=\frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\right]$$

$$\Rightarrow \frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$$

Which is the required result.

QUESTION 10:

By Fourier series for $-\pi<x<\pi$ show that

$$x.\operatorname{Cos}x=\frac{-1}{2}\operatorname{Sin}x+2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}x.\operatorname{Sin}(nx)$$

SOLUTION:

Let $f(x)=x.\operatorname{Cos}x, \quad -\pi<x<\pi$

Here period $=2P=2\pi\Rightarrow\boxed{P=\pi}$

Since $f(x)=-x.\operatorname{Cos}(-x)=-x\operatorname{Cos}x=-f(x)$

So f(x) is odd function.

Hence its Fourier series is given by

$$f(x)=\sum_{n=1}^{\infty}b_n.\operatorname{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\text{or } x.\operatorname{Cos}x=\sum_{n=1}^{\infty}b_n.\operatorname{Sin}(nx)\longrightarrow(1) \quad (\because P=\pi)$$

Where $b_n=\frac{1}{P}\int_{-P}^{P}f(x).\operatorname{Sin}\left(\frac{n\pi x}{P}\right)dx$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x.\text{Cos}x.\text{Sin}(nx)dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_{0}^{\pi} x.\text{Cos}x.\text{Sin}(nx)dx \quad \left(\because x.\cos x.\sin(nx)\text{ is even function}\right)$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{0}^{\pi} x(2\,\text{Sin}nx.\text{Cos}\,x)dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{0}^{\pi} x[\text{Sin}(n+1)x + \text{Sin}(n-1)x]dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[x\left(\frac{-\text{Cos}(n+1)x}{n+1} - \frac{\text{Cos}(n-1)x}{n-1}\right)\right]_0^{\pi} + $$

$$-\frac{1}{\pi}\int_0^{\pi}\left[\frac{-\text{Cos}(n+1)x}{n+1} - \frac{\text{Cos}(n-1)x}{n-1}\right]dx$$

$$\Rightarrow b_n = -\frac{1}{\pi}\left[x\left(\frac{\text{Cos}(n+1)x}{n+1} + \frac{\text{Cos}(n-1)x}{n-1}\right)\right]_0^{\pi}$$

$$+\frac{1}{\pi}\int_0^{\pi}\left(\frac{\text{Cos}(n+1)x}{n+1} + \frac{\text{Cos}(n-1)x}{n-1}\right)dx$$

$$\Rightarrow b_n = \frac{-1}{\pi}\left[\pi\left(\frac{\text{Cos}(n+1)\pi}{n+1} + \frac{\text{Cos}(n-1)\pi}{n-1}\right) - 0\right] + $$

$$+\frac{1}{\pi}\left[\frac{\text{Sin}(n+1)x}{(n+1)^2} + \frac{\text{Sin}(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$\Rightarrow b_n = \frac{-1}{\pi}\left[\pi\left(\frac{\text{Cos}(n\pi)\text{Cos}\pi}{n+1} + \frac{\text{Cos}(n\pi).\text{Cos}\pi}{n-1}\right)\right]$$

$$+\frac{1}{\pi}\left[\frac{\text{Sin}(n+1)\pi}{(n+1)^2} + \frac{\text{Sin}(n-1)\pi}{(n-1)^2} - 0\right]$$

$$\Rightarrow b_n = \frac{-1}{\pi}\left[\pi\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right)\right] + \frac{1}{\pi}\left[\frac{\text{Sin}(n\pi).\text{Cos}\pi}{(n+1)^2} + \frac{\text{Sin}\,n\pi.\text{Cos}\pi}{(n-1)^2}\right]$$

$$\Rightarrow b_n = \frac{\pi}{\pi}\left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right] + 0$$

$$\Rightarrow b_n = (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$\Rightarrow b_n = (-1)^n\left[\frac{n-1+n+1}{n^2-1}\right]$$

$$\boxed{b_n = \frac{(-1)^n(2n)}{n^2-1}}, \qquad n \neq 1$$

For n=1, we have

$$b_1 = \frac{1}{P}\int_{-P}^{P} f(x)\text{Sin}\left(\frac{\pi x}{P}\right)dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x.\text{Cos}x.\text{Sin}x\,dx$$

$$\Rightarrow b_1 = \frac{2}{\pi}\int_0^{\pi} x.\text{Cos}x.\text{Sin}x\,dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\int_0^{\pi} x(2\,\text{Sin}x.\text{Cos}x)dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\int_0^{\pi} x\sin(2x)\,dx$$

$$\Rightarrow b_1 = \frac{1}{\pi}\left[x\left(\frac{-\text{Cos}(2x)}{2}\right)\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}(1)\left(\frac{-\text{Cos}(2x)}{2}\right)dx$$

$$\Rightarrow b_1 = \frac{-1}{2\pi}[x\,\text{Cos}(2x)]_0^{\pi} + \frac{1}{2\pi}\int_0^{\pi}\text{Cos}(2x)dx$$

$$\Rightarrow b_1 = \frac{-1}{2\pi}[\pi\,\text{Cos}\,2\pi - 0] + \frac{1}{2\pi}\left[\frac{\text{Sin}(2x)}{2}\right]_0^{\pi}$$

$$\Rightarrow b_1 = \frac{-1}{2\pi}[\pi(1)] + 0$$

$$\Rightarrow b_1 = \frac{-1}{2}$$

Thus $\boxed{b_1 = \frac{-1}{2}}$

Eq (1) can be wri0tten as

$$x.\text{Cos}x = b_1.\text{Sin}x + \sum_{n=2}^{\infty} b_n.\text{Sin}(nx)$$

$$\Rightarrow x.\text{Cos}x = \frac{-1}{2}.\text{Sin}x + \sum_{n=2}^{\infty}\frac{2(-1)^n n}{n^2-1}\text{Sin}(nx)$$

$$\Rightarrow x.\text{Cos}x = \frac{-1}{2}.\text{Sin}x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}x.\text{Sin}(nx)$$

Which is the required result.

HALF RANGE FOURIER SERIES

If a function is defined over half the range (say) o to L, instead of the full range from –L to L, it may be expended in a series of sine terms only or of cosine terms only. The series produced is then called a half-range Fourier series.

FOURIER SINE SERIES:

If we are required to expend a given function $f(x)$ as a series of sine in half-rang $0<x<L$, then we extent the function reflecting it in the origin so that $f(-x)=-f(x)$, that is we construct a function $\varphi(x)$ which is identical with $f(x)$ for $0<x<L$ but equal to $-f(-x)$ for $-L<x<0$

i.e. $\varphi(x)=\begin{cases} f(x) & 0<x<L \\ -f(x) & -L<x<0 \end{cases}$

Then the extended function $\varphi(x)$ is an odd function of x defined in the interval $-L<x<L$ and consequently will have a Fourier series involving sine terms alone given by

$$\varphi(x)=\sum_{n=1}^{\infty} b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$$

Where $b_n=\frac{1}{L}\int_{-L}^{L}\varphi(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$

$$\Rightarrow b_n=\frac{2}{L}\int_0^L\varphi(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$$

But $\varphi(x)=f(x)$ for $0<x<L$

$$\therefore f(x)=\sum_{n=1}^{\infty} b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)\rightarrow(1)$$

Where $b_n=\frac{2}{L}\int_0^L f(x)\,\text{Sin}\frac{(n\pi x)}{L}dx,\qquad n=1,2,3,\cdots\cdots$

The series (1) is known as Fourier sere series of $f(x)$ on the interval $0<x<L$.

FOURIER COSINE SERIES

If we are interested in expressing a function $f(x)$, defined in the interval $0<x<L$, as a Fourier series containing cosine terms only, then we create a new function $\varphi(x)$ such that

$$\varphi(x)=\begin{cases} f(x), & 0<x<L \\ f(-x), & -L<x<0 \end{cases}$$

Thus the extended function $\varphi(x)$ is an even function of x in $-L<x<L$ and its graph will be symmetrical about y-axis. Its Fourier series is given by

$$\varphi(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{L}\right)$$

Where $a_0=\frac{1}{L}\int_{-L}^{L}\varphi(x)dx$

$$\Rightarrow a_0=\frac{2}{L}\int_0^L\varphi(x)dx$$

and $a_n=\frac{1}{L}\int_{-L}^{L}\varphi(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$

$$a_n=\frac{2}{L}\int_0^L\varphi(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$$

But $\varphi(x)=f(x)$ for $0<x<L$

There fore, in $0<x<L$, $f(x)$ will be represented by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{L}\right)\rightarrow(2)$$

Where $a_0=\frac{2}{L}\int_0^L f(x)dx$

and $a_n=\frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$

The series (2) is known is known as Fourier cosine series of $f(x)$ on the interval $0<x<L$.

EXAMPLE 1:

Expand o $f(x)=x+1,\quad 0<x<\pi$ in a Fourier sine series.

SOLUTION:

Given that $f(x)=x+1,\quad 0<x<\pi$

Since we have to expand $f(x)$ in a Fourier sine series and we know that a function $f(x)$ has Fourier sine series when $f(x)$ is an odd function.

Here $f(x)$ is not and odd function since $f(-x)\neq -f(x)$

So we define a new function $\varphi(x)$ such that

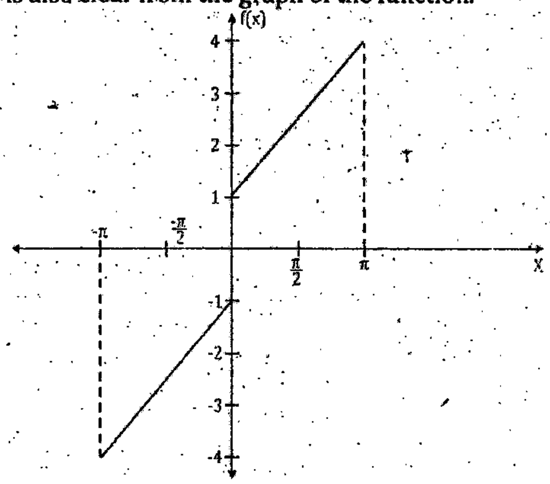
$$\varphi(x)=\begin{cases} f(x), & 0<x<\pi \\ -f(-x), & -\pi<x<0 \end{cases}$$

i.e. $\varphi(x)=\begin{cases} x+1, & 0<x<\pi \\ x-1, & -\pi<x<0 \end{cases}$

Now $\varphi(-x)=(-x)-1=-(x+1)=-\varphi(x)$

So $\varphi(x)$ is an odd function on $-\pi<x<\pi$

It is also clear from the graph of the function.



Here period $=2L=2\pi\Rightarrow\boxed{L=\pi}$

The Fourier series of $\varphi(x)$ is

$$\varphi(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$$

or $\varphi(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}(nx)\longrightarrow(1)\qquad(\because L=\pi)$

Where $b_n = \frac{1}{L}\int_{-L}^{L}\varphi(x).\text{Sin}(nx)dx$

$\Rightarrow b_n = \frac{2}{L}\int_0^L \varphi(x).\text{Sin}(nx)dx \quad \left(\because \varphi(x).\text{Sin}(nx) \text{ is even function}\right)$

$\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi (x+1).\text{Sin}(nx)dx$

$\Rightarrow b_n = \frac{2}{\pi}\left[\frac{-(x+1)\text{Cos}(nx)}{n}\right]_0^\pi + \frac{2}{\pi}\int_0^\pi \frac{(1+0)\text{Cos}(nx)}{n}dx$

$\Rightarrow b_n = \frac{-2}{n\pi}\left[(x+1).\text{Cos}(nx)\right]_0^\pi + \frac{2}{n\pi}\left[\frac{\text{Sin}(nx)}{n}\right]_0^\pi$

$\Rightarrow b_n = \frac{-2}{n\pi}\left[(\pi+1)\text{Cos}n\pi - \text{Cos }0\right] + 0$

$\Rightarrow \boxed{b_n = \frac{-2}{n\pi}\left[(\pi+1)(-1)^n - 1\right]}$

Then Eq (1) $\Rightarrow$

$\varphi(x) = \sum_{n=1}^{\infty}\frac{(-2)}{n\pi}\left[(\pi+1)(-1)^n - 1\right]\text{Sin}(nx)$

But $\varphi(x)$ is identical with $f(x)$ in $0<x<\pi$

So $f(x) = \sum_{n=1}^{\infty}\frac{(-2)}{n\pi}\left[(\pi+1)(-1)^n - 1\right]\text{Sin}(nx)$

$\Rightarrow f(x) = \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(\pi+1)(-1)^n - 1\right]}{n}.\text{Sin}(nx)$

$\Rightarrow f(x) = \frac{-2}{\pi}\left[\frac{(-\pi-2)}{1}\text{Sinx} + \frac{\pi}{2}\text{Sin}(2x) + \frac{(-\pi-2)}{3}\text{Sin3x} + \frac{\pi}{2}\text{Sin4x} + \cdots\right]$

$\Rightarrow f(x) = \frac{-2}{\pi}\left[-(\pi+2)\text{Sinx} + \frac{\pi}{2}\text{Sin2x} - \frac{(\pi+2)}{3}\text{Sin3x} + \frac{\pi}{2}\text{Sin4x} - \cdots\right]$

EXAMPLE 2:
Find the Fourier half-range
- **i.** Sine series
- **ii.** Cosine series

For the function defined by $f(x) = x^2,\ 0<x<\pi$

SOLUTION: Given that $f(x) = x^2,\ 0<x<\pi$

i. Since Fourier sine series involves only sine terms, so

Let $f(x) = x^2 = \sum_{n=1}^{\infty} b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$

or $x^2 = \sum_{n=1}^{\infty} b_n.\text{Sin}(nx) \longrightarrow (1) \quad (\because L=\pi)$

Where $b_n = \frac{2}{L}\int_0^L f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$

$\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi x^2.\text{Sin}(nx)dx$

$\Rightarrow b_n = \frac{2}{\pi}\left[\frac{-x^2.\text{Cos}(nx)}{n}\right]_0^\pi + \frac{2}{\pi}\int_0^\pi \frac{2x.\text{Cos}(nx)}{n}dx$

$\Rightarrow b_n = \frac{-2}{n\pi}\left[x^2.\text{Cos}(nx)\right]_0^\pi + \frac{4}{n\pi}\int_0^\pi x.\text{Cos}(nx)dx$

$\Rightarrow b_n = \frac{-2}{n\pi}\left[\pi^2.\text{Cos}n\pi - 0\right] + \frac{4}{n\pi}\left[\frac{x.\text{Sin}(nx)}{n}\right]_0^\pi - \frac{4}{n\pi}\int_0^\pi \frac{(1)\text{Sin}(nx)dx}{n}$

$\Rightarrow b_n = \frac{-2}{n}(-1)^n\pi + 0 + \frac{4}{n^2\pi}\left[\frac{\text{Cos}(nx)}{n}\right]_0^\pi$

$\Rightarrow b_n = \frac{2(-1)^{n+1}\pi}{n} + \frac{4}{n^3\pi}\left[\text{Cos}(n\pi) - \text{Cos }0\right]$

$\Rightarrow b_n = \frac{2(-1)^{n+1}\pi}{n} + \frac{4}{n^3\pi}\left[(-1)^n - 1\right] \longrightarrow (2)$

Putting $n = 1,2,3,4,\cdots\cdots$, in Eq (2) we get

$b_1 = 2\pi - \frac{8}{\pi}, b_2 = -\pi, b_3 = \frac{2\pi}{3} - \frac{8}{27\pi}, b_4 = \frac{-\pi}{2}, \cdots\cdots$

Eq (1) can be written as

$x^2 = b_1.\text{Sinx} + b_2.\text{Sin2x} + b_3.\text{Sin3x} + b_4.\text{Sin4x} + \cdots\cdots$

$\Rightarrow x^2 = \left(2\pi - \frac{8}{\pi}\right)\text{Sinx} - \pi.\text{Sin2x} + \left(\frac{2\pi}{3} - \frac{8}{27\pi}\right)\text{Sin3x} - \frac{\pi}{2}\text{Sin4x} + \cdots$

ii. Since Fourier cosine series does not involve sine terms, so

$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\text{Cos}\frac{(n\pi x)}{L}$

or $x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\text{Cos}(nx) \rightarrow (A) \quad (\because L=\pi)$

Where $a_0 = \frac{2}{L}\int_0^L f(x)dx$

$a_0 = \frac{2}{\pi}\int_0^\pi x^2 dx$

$\Rightarrow a_0 = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\pi^3\right] = 2\pi^2$

Thus $\boxed{a_0 = 2\pi^2}$

and $a_n = \frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi x^2.\text{Cos}(nx)dx$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\frac{x^2.\text{Sin}(nx)}{n}\right]_0^\pi - \frac{2}{\pi}\int_0^\pi \frac{2x.\text{Sin}(nx)}{n}dx$$

$$\Rightarrow a_n = 0 + \frac{4}{n\pi}\int_0^\pi x(-\text{Sin } nx)dx$$

$$a_n = \frac{4}{n\pi}\left[\frac{x.\text{Cos}(nx)}{n}\right]_0^\pi - \int_0^\pi (1)\frac{\text{Cos}nx}{n}dx$$

$$a_n = \frac{4}{n^2\pi}\left[x.\text{Cos}(nx)\right]_0^\pi - \frac{1}{n}\left[\frac{\text{Sin}(nx)}{n}\right]_0^\pi$$

$$\Rightarrow a_n = \frac{4}{n^2\pi}\left[\pi\text{Cos}n\pi - 0\right] - 0$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

Putting the values in Eq (A), we have

$$x^2 = \pi^2 + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\text{Cos}(nx)$$

$$\Rightarrow x^2 = \pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\text{Cos}(nx) \quad \text{Ans.}$$

EXAMPLE 3:

Represent as Fourier sine series

$$f(x) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

SOLUTION: Given that

$$f(x) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

Since Fourier sine series involve only sine terms so

$$f(x) = \sum_{n=1}^{\infty} b_n.\text{Sin}\left(\frac{n\pi x}{L}\right) \longrightarrow (1)$$

Where $b_n = \frac{2}{L}\int_0^L f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$

$$\Rightarrow b_n = \frac{2}{L}\int_0^{L/2} f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^{L} f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow b_n = \frac{2}{L}\int_0^{L/2} 1.\text{Sin}\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^{L} f(0)\text{sin}\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow b_n = \frac{2}{L}\int_0^{L/2}\text{Sin}\left(\frac{n\pi x}{L}\right)dx + 0$$

$$\Rightarrow b_n = \frac{2}{L}\left[\frac{-\text{Cos}\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)}\right]_0^{L/2}$$

$$\Rightarrow b_n = \frac{-2}{n\pi}\left[\text{Cos}\left(\frac{n\pi x}{L}\right)\right]_0^{L/2}$$

$$\Rightarrow b_n = \frac{-2}{n\pi}\left[\text{Cos}\left(\frac{n\pi}{2}\right) - \text{Cos } 0\right]$$

$$\Rightarrow b_n = \frac{-2}{n\pi}\left[\text{Cos}\left(\frac{n\pi}{2}\right) - 1\right]$$

$$\text{or } b_n = \frac{2}{n\pi}\left[1 - \text{Cos}\left(\frac{n\pi}{2}\right)\right] \longrightarrow (2)$$

Putting $n = 1,2,3,\cdots\cdots$, in Eq (2), we have

$$b_1 = \frac{2}{\pi}\left[1 - \text{Cos}\frac{\pi}{2}\right] = \frac{2}{\pi},$$

$$b_2 = \frac{2}{2\pi}\left[1 - \text{Cos}\pi\right] = \frac{2}{\pi},$$

$$b_3 = \frac{2}{3\pi}\left[1 - \text{Cos}\left(\frac{3\pi}{2}\right)\right] = \frac{2}{3\pi}$$

$$b_4 = \frac{2}{4\pi}\left[1 - \text{Cos}2\pi\right] = 0 \text{ and so on.}$$

Eq (1) can be written as

$$f(x) = b_1.\text{Sin}\left(\frac{\pi x}{L}\right) + b_2.\text{Sin}\left(\frac{2\pi x}{L}\right) + b_3.\text{Sin}\left(\frac{3\pi x}{L}\right) +$$

$$+ b_4.\text{Sin}\left(\frac{4\pi x}{L}\right) + \cdots\cdots$$

$$\Rightarrow f(x) = \frac{2}{\pi}.\text{Sin}\left(\frac{\pi x}{L}\right) + \frac{2}{\pi}.\text{Sin}\left(\frac{2\pi x}{L}\right) + \frac{2}{3\pi}.\text{Sin}\left(\frac{3\pi x}{L}\right) +$$

$$+ 0\text{Sin}\left(\frac{n\pi x}{L}\right) + \cdots\cdots$$

$$\Rightarrow f(x) = \frac{2}{\pi}\left[\text{Sin}\left(\frac{\pi x}{L}\right) + \text{Sin}\left(\frac{2\pi x}{L}\right) + \frac{1}{3}.\text{Sin}\left(\frac{3\pi x}{L}\right) + \cdots\cdots\right]$$

EXAMPLE 4:

Prove that for all values of x between $\frac{-\pi}{2}$ and $+\frac{\pi}{2}$

$$x = \frac{4}{\pi}\left(\text{Sin}x - \frac{\text{Sin}(3x)}{9} + \frac{\text{Sin}(5x)}{25}\cdots\cdots\right)$$

SOLUTION:

Here $\frac{-\pi}{2} < x < \frac{\pi}{2}$

$$\Rightarrow 0 < x + \frac{\pi}{2} < \pi$$

$$\Rightarrow 0 < t < \pi \qquad \text{where } t = x + \frac{\pi}{2}$$

Let $f(t) = t,\ 0 < t < \pi$

To get the required result we expand $f(t)$ in a cosine series.

Let $f(t)=t=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi t}{L}\right)$

or $t=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}(nt)$ $(\because L=\pi)$

Where $a_0=\frac{2}{L}\int_0^L f(t)dt$

$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi} t\,dt$

$\Rightarrow a_0=\frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi}=\pi$

$\Rightarrow \boxed{a_0=\pi}$

$a_n=\frac{2}{L}\int_0^L f(t).\text{Cos}\left(\frac{n\pi t}{L}\right)dt$

$\Rightarrow a_n=\frac{2}{\pi}\int_0^{\pi} t.\text{Cos}(nt)\,dt$

$\Rightarrow a_n=\frac{2}{\pi}\left[\frac{t.\text{Sin}(nt)}{n}\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}\frac{1.\text{Sin}(nt)dt}{n}$

$\Rightarrow a_n=\frac{2}{n\pi}\left[t.\text{Sin}(nt)\right]_0^{\pi}-\frac{2}{n\pi}\left[\frac{-\text{Cos}(nt)}{n}\right]_0^{\pi}$

$\Rightarrow a_n=0+\frac{2}{n^2\pi}\left[\text{Cos}(nt)\right]_0^{\pi}$

$\Rightarrow a_n=\frac{2}{n^2\pi}\left[\text{Con}(n\pi)-\text{Cos}0\right]$

$\boxed{a_n=\frac{2}{n^2\pi}\left[(-1)^n-1\right]}$

Putting the values in Eq (1), we have

$f(t)=t=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n-1\right]\text{Cos}(nt)$

$\Rightarrow t=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left((-1)^n-1\right)}{n^2}\text{Cos}(nt)$

$\Rightarrow t=\frac{\pi}{2}+\frac{2}{\pi}\left[\frac{-2}{1^2}\text{Cost}-\frac{2}{3^2}\text{Cos}3t-\frac{2}{5^2}\text{Cos}5t-\cdots\cdots\right]$

$\Rightarrow t=\frac{\pi}{2}-\frac{4}{\pi}\left[\text{Cost}+\frac{1}{9}\text{Cos}3t+\frac{1}{5}\text{Cos}5t+\cdots\cdots\right]$

Putting $t=x+\frac{\pi}{2}$, we have

$x+\frac{\pi}{2}=\frac{\pi}{2}-\frac{4}{\pi}\left[\text{Cos}\left(\frac{x+\pi}{2}\right)+\frac{1}{9}\text{Cos}3\left(\frac{x+\pi}{2}\right)+\frac{1}{5}\text{Cos}5\left(\frac{x+\pi}{2}\right)+\cdots\right]$

$\Rightarrow x=\frac{-4}{\pi}\left[-\text{Sin}x+\frac{\text{Sin}3x}{9}-\frac{\text{Sin}(5x)}{25}+\cdots\cdots\right]$

$\Rightarrow x=\frac{4}{\pi}\left[\text{Sin}x-\frac{\text{Sin}3x}{9}+\frac{\text{Sin}(5x)}{25}-\frac{\text{Sin}(7x)}{7}+\cdots\cdots\right]$

Which is the required result.

EXERCISE 6.4

QUESTION 1:

Given a constan 'K' show that, in the half-range

$0<x<\pi, K=\frac{4}{\pi}\left[\text{Sin}x+\frac{\text{Sin}(3x)}{3}+\frac{\text{Sin}(5x)}{5}+\cdots\cdots\right]K$

SOLUTION:

Let $f(x)=K;$ $0<x<\pi$

Since we need a Fourier sine series of $f(x)$ in the half-range $0<x<\pi$, so let

$f(x)=K=\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$

or $K=\sum_{n=1}^{\infty}b_n.\text{Sin}(nx)\rightarrow(1)\ \because L=\pi$

Where $b_n=\frac{2}{L}\int_0^L f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$

$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi}K.\text{Sin}(nx)dx$

$\Rightarrow b_n=\frac{2K}{\pi}\left[\frac{-\text{Cos}(nx)}{n}\right]_0^{\pi}$

$\Rightarrow b_n=\frac{-2K}{n\pi}\left[\text{Cos}(nx)\right]_0^{\pi}=\frac{-2K}{n\pi}\left[\text{Cos}n\pi-\text{Cos}\,0\right]$

$\Rightarrow b_n=\frac{-2K}{n\pi}\left[(-1)^n-1\right]$

or $\boxed{b_n=\frac{2K}{n\pi}\left(1-(-1)^n\right)}$

Then Eq (1) $\Rightarrow$

$K=\sum_{n=1}^{\infty}\frac{2K}{n\pi}\left(1-(-1)^n\right)\text{Sin}(nx)$

$\Rightarrow K=\frac{2K}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\text{Sin}(nx)$

$\Rightarrow K=\frac{2K}{\pi}\left[\frac{2}{1}\text{Sin}x+\frac{2}{3}\text{Sin}3x+\frac{2}{5}\text{Sin}5x+\cdots\cdots\right]$

or $K=\frac{4}{\pi}\left[\text{Sin}x+\frac{\text{Sin}3x}{3}+\frac{\text{Sin}5x}{5}+\cdots\cdots\right]K$

Which is the required result.

QUESTION 2:

Find the half-range Fourier cosine series for the function, $f(x)=x.$ $0<x<\pi$

SOLUTION:

Given that $f(x)=x, \quad 0<x<\pi$

Since the half-range Fourier cosine series does not involve sine terms, so let

$$f(x)=x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{K}\right)$$

$$\text{or } x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}(nx)\rightarrow(1) \qquad \because L=\pi$$

Where $a_0=\frac{2}{L}\int_0^L f(x)dx$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^\pi x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi=\pi$$

$\boxed{a_0=\pi}$

$$a_n=\frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow a_n=\frac{2}{\pi}\int_0^\pi x.\text{Cos}(nx)dx$$

$$\Rightarrow a_n=\frac{2}{\pi}\left[\frac{x.\text{Sin}(nx)}{n}\right]_0^\pi-\frac{2}{\pi}\int_0^\pi\frac{(1).\text{Sin}(nx)}{n}dx$$

$$\Rightarrow a_n=0+\frac{2}{n\pi}\left[\frac{\text{Cos}(nx)}{n}\right]_0^\pi$$

$$\Rightarrow a_n=\frac{2}{n^2\pi}\left[\text{Cos}(n\pi)-\text{Cos}\,0\right]$$

$\boxed{a_n=\frac{2}{n^2\pi}\left((-1)^n-1\right)}$

Putting the values in Eq (1), we have

$$x=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left((-1)^n-1\right)\text{Cos}(nx)$$

$$\Rightarrow x=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\text{Cos}(nx)$$

$$\Rightarrow x=\frac{\pi}{2}+\frac{2}{\pi}\left[\frac{-2}{1^2}\text{Cos}x-\frac{2}{3^2}\text{Cos}3x-\frac{2}{5^2}\text{Cos}5x\cdots\cdots\right]$$

$$\Rightarrow x=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\text{Cos}x}{1^2}+\frac{\text{Cos}3x}{3^2}+\frac{\text{Cos}5x}{5^2}+\cdots\cdots\right] \text{ Ans.}$$

QUESTION 3:

Obtain a half-range cosine series for

$f(x)=2x-1, \quad 0<x<1$

And show that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots$

SOLUTION:

Given that $f(x)=2x-1, \quad 0<x<1$

Since half-range cosine series does not involve sine terms, so we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{L}\right)$$

$$\text{or } 2x-1=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}(n\pi x)\rightarrow(1) \qquad (\because L=1)$$

Where $a_0=\frac{2}{L}\int_0^L f(x)dx$

$$a_0=2\int_0^1(2x-1)dx$$

$$\Rightarrow a_0=2\left[x^2-x\right]_0^1=0$$

$\Rightarrow\boxed{a_0=0}$

$$a_n=\frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow a_n=2\int_0^1(2x-1)\text{Cos}(n\pi x)dx$$

$$\Rightarrow a_n=2\left[(2x-1)\frac{\text{Sin}(n\pi x)}{n\pi}\right]_0^1-2\int_0^1(2-0)\frac{\text{Sin}(n\pi x)}{n\pi}dx$$

$$\Rightarrow a_n=\frac{2}{n\pi}\left[(2x-1)\text{Sin}(n\pi x)\right]_0^1+\frac{4}{n\pi}\left[\frac{\text{Cos}(n\pi x)}{n\pi}\right]_0^1$$

$$\Rightarrow a_n=0+\frac{4}{n^2\pi^2}\left[\text{Cos}(n\pi)-\text{Cos}\,0\right]$$

Thus $\boxed{a_n=\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]}$

Putting the values in Eq (1), we have

$$2x-1=\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left((-1)^n-1\right)\text{Cos}(n\pi x)$$

$$\Rightarrow 2x-1=\frac{4}{\pi^2}\left[\frac{-2}{1^2}\text{Cos}(\pi x)-\frac{2}{3^2}\text{Cos}(3\pi x)-\frac{2}{5^2}\text{Cos}(5\pi x)-\cdots\cdots\right]$$

$$\Rightarrow 2x-1=-\frac{8}{\pi^2}\left[\frac{\text{Cos}(\pi x)}{1^2}+\frac{\text{Cos}(3\pi x)}{3^2}+\frac{\text{Cos}(5\pi x)}{5^2}+\cdots\right]\rightarrow(2)$$

Which is the required half-range cosine series for $f(x)=2x-1$

To get the required result, we set $x=0$ or 1 in Eq (2).

Putting $x=0$ in Eq (2), we have

$$-1=\frac{-8}{\pi^2}\left[\frac{\text{Cos}0}{1^2}+\frac{\text{Cos}\,0}{3^2}+\frac{\text{Cos}\,0}{5^2}+\cdots\cdots\right]$$

$$\Rightarrow\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots$$

Which is the required result.

QUESTION 4:

Find the half-range sine series for the function defined as $f(x)=e^x, \quad 0<x<1$

SOLUTION:

Given that $f(x)=e^x,\ 0<x<1$

Since the half-range sine series involves only sine terns, so we have

$$f(x)=e^x=\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$$

or $e^x=\sum_{n=1}^{\infty}b_n.\text{Sin}(n\pi x)\rightarrow(1)\qquad(\because L=1)$

Where $b_n=\frac{2}{L}\int_0^1 f(x).\text{Sin}(n\pi x)dx$

$$\Rightarrow b_n=2\int_0^1 e^x\,\text{Sin}(n\pi x)dx\longrightarrow(2)$$

Integrating by parts, we have

$$\int e^x.\text{Sin}(n\pi x)dx=e^x\left(\frac{-\text{Cos}(n\pi x)}{n\pi}\right)-\int e^x\left(\frac{-\text{Cos}(n\pi x)}{n\pi}\right)dx$$

$$\Rightarrow\int e^x\,\text{Sin}(n\pi x)dx=-\frac{e^x.\text{Cos}(n\pi x)}{n\pi}+\frac{1}{n\pi}\int e^x\,\text{Cos}(n\pi x)dx$$

$$\Rightarrow\int e^x.\text{Sin}(n\pi x)dx=\frac{-e^x\,\text{Cos}(n\pi x)}{n\pi}+$$

$$+\frac{1}{n\pi}\left[\frac{e^x.\text{Sin}(n\pi x)}{n\pi}\right]-\frac{1}{n\pi}\int\frac{e^x\,\text{Sin}(n\pi x)dx}{n\pi}$$

$$\Rightarrow\int e^x.\text{Sin}(n\pi x)dx=\frac{-e^x.\text{Cos}(n\pi x)}{n\pi}+$$

$$+\frac{e^x\,\text{Sin}(n\pi x)}{n^2\pi^2}-\frac{1}{n^2\pi^2}\int e^x.\text{Sin}(n\pi x)dx$$

$$\Rightarrow\int e^x.\text{Sin}(n\pi x)dx+\frac{1}{n^2\pi^2}\int e^x\,\text{Sin}(n\pi x)dx$$

$$=\frac{e^x.\text{Sin}(n\pi x)}{n^2\pi^2}-\frac{e^x.\text{Cos}(n\pi x)}{n\pi}$$

$$\Rightarrow\left(\frac{n^2\pi^2+1}{n^2\pi^2}\right)\int e^x\,\text{Sin}(n\pi x)dx=\frac{e^x.\text{Sin}(n\pi x)}{n^2\pi^2}-\frac{e^x.\text{Cos}(n\pi x)}{n\pi}$$

$$\Rightarrow\int e^x.\text{Sin}\,n\pi x\,dx=\frac{e^x.\text{Sin}(n\pi x)}{n^2\pi^2+1}-\frac{n\pi\,e^x\text{Cos}(n\pi x)}{n^2\pi^2+1}$$

$$\Rightarrow\int_0^1 e^x\,\text{Sin}(n\pi x)dx=\left[\frac{e^x\,\text{Sin}(n\pi x)}{n^2\pi^2+1}-\frac{n\pi.e^x.\text{Cos}(n\pi x)}{n^2\pi^2+1}\right]_0^1$$

$$\Rightarrow\int_0^1 e^x.\text{Sin}(n\pi x)dx=0-\frac{n\pi e\,\text{Cos}n\pi}{n^2\pi^2+1}-0+\frac{n\pi\,e^0\text{Cos}0}{n^2\pi^2+1}$$

$$\Rightarrow\int_0^1 e^x.\text{Sin}(n\pi x)dx=\frac{n\pi}{n^2\pi^2+1}\left[e(-1)^{n+1}+1\right]$$

Then (2)$\Rightarrow$

$$\boxed{b_n=\frac{2n\pi}{n^2\pi^2+1}\left((-1)^{n+1}e+1\right)}$$

Putting the values of b_n in Eq (1), we have

$$e^x=\sum_{n=1}^{\infty}\frac{2n\pi}{n^2\pi^2+1}\left((-1)^{n+1}e+1\right)\text{Sin}(n\pi x)$$

$$\Rightarrow e^x=2\pi\sum_{n=1}^{\infty}\frac{n}{n^2\pi^2+1}\left((-1)^{n+1}e+1\right)\text{Sin}(n\pi x)$$

$$\Rightarrow e^x=2\pi\left[\frac{1}{1^2.\pi^2+1}(e+1)\text{Sin}(\pi x)+\frac{2}{2^2.\pi^2+1}(-e+1)\times\text{Sin}(2\pi x)+\frac{3}{3^2\pi^2+1}(e+1)\text{Sin}(3\pi x)+\cdots\cdots\right]$$

$$\text{or } e^x=2\pi\left[\frac{(1+e)}{\pi^2+1}\text{Sin}(\pi x)+\frac{2(1-e)}{4\pi^2+1}\text{Sin}(2\pi x)+\frac{3(1+e)}{9\pi^2+1}\text{Sin}(3\pi x)+\cdots\cdots\right]$$

QUESTION 5:

Expand $f(x)=e^x$ as Fourier cosine series for $0<x<1$

SOLUTION:

Given that $f(x)=e^x,\qquad 0<x<1$

Since the half-range Fourier cosine series does not involve sine terms, so we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{L}\right)$$

Since $L=1$, so

$$f(x)=e^x=\frac{a_0}{2}\sum_{n=1}^{\infty}a_n.\text{Cos}(n\pi x)\longrightarrow(1)$$

Where $a_0=\frac{2}{L}\int_0^L f(x)\,dx$

$$\Rightarrow a_0=2\int_0^1 e^x dx=2\left[e^x\right]_0^1\qquad(\because L=1)$$

$$\Rightarrow a_0=2\left[e^1-e^0\right]$$

$$\Rightarrow\boxed{a_0=2(e-1)}$$

$$a_n=\frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$$

$$a_n=2\int_0^1 e^x.\text{Cos}(n\pi x)dx\longrightarrow(2)$$

Integrating by parts, we have

$$\int e^x \cos(n\pi x)dx = \frac{e^x \sin(n\pi x)}{n\pi} - \int \frac{e^x.\sin(n\pi x)dx}{n\pi}$$

$$\Rightarrow \int e^x \cos(n\pi x)dx = \frac{e^x \sin(n\pi x)}{n\pi} + \frac{1}{n\pi}\int e^x(-\sin(n\pi x))dx$$

$$\Rightarrow \int e^x \cos(n\pi x)dx = \frac{e^x \sin(n\pi x)}{n\pi} + \frac{1}{n\pi}\left[\frac{e^x.\cos(n\pi x)}{n\pi}\right] + \frac{1}{n\pi}\int \frac{e^x.(\cos(n\pi x))dx}{n\pi}$$

$$\Rightarrow \int e^x \cos(n\pi x)dx = \frac{e^x \sin(n\pi x)}{n\pi} + \frac{e^x.\cos(n\pi x)}{n\pi} - \frac{1}{n\pi}\int \frac{e^x.\sin(n\pi x)dx}{n\pi}$$

$$\Rightarrow \int e^x \cos(n\pi x)dx + \frac{1}{n^2\pi^2}\int e^x.\cos(n\pi x)dx = \frac{e^x.\sin(n\pi x)}{n\pi} + \frac{e^x.\cos(n\pi x)}{n^2\pi^2}$$

$$\Rightarrow \left(\frac{n^2\pi^2+1}{n^2\pi^2}\right)\int e^x.\cos(n\pi x)dx = \frac{e^x.\sin(n\pi x)}{n\pi} + \frac{e^x.\cos(n\pi x)}{n^2\pi^2}$$

$$\Rightarrow \int e^x.\cos(n\pi x)dx = \frac{n\pi\, e^x.\sin(n\pi x)}{n^2\pi^2+1} + \frac{e^x \cos(n\pi x)}{n^2\pi^2+1}$$

$$\Rightarrow 2\int_0^1 e^x \cos(n\pi x)\,dx = \frac{2}{n^2\pi^2+1}\left[n\pi e^x \sin(n\pi x) + e^x \cos(n\pi x)\right]_0^1$$

$$\Rightarrow a_n = \frac{2}{n^2\pi^2+1}\left[0 + e^1 \cos n\pi - 0 - e^0 \cos 0\right] \qquad (\text{By }(2))$$

$$\Rightarrow \boxed{a_n = \frac{2}{n^2\pi^2+1}\left(e(-1)^n - 1\right)}$$

Putting the values in Eq (1), we have

$$e^x = (e-1) + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2+1}\left(e(-1)^n - 1\right)\cos(n\pi x)$$

$$\Rightarrow e^x = e - 1 + 2\sum_{n=1}^{\infty}\frac{\left((-1)^n e - 1\right)}{n^2\pi^2+1}\cos(n\pi x)$$

$$\Rightarrow e^x = e - 1 + 2\left[\frac{(-e-1)}{1^2.\pi^2+1}\cos(\pi x) + \frac{(e-1)}{2^2\pi^2+1}\cos(2\pi x) + \frac{(-e-1)}{3^2.\pi^2+1}\cos(3\pi x) + \cdots\cdots\right]$$

$$\Rightarrow e^x = e - 1 + 2\left[\frac{-(e+1)}{\pi^2+1}\cos\pi x + \frac{(e-1)\cos(2\pi x)}{4\pi^2+1} - \frac{(e-1)}{9\pi^2+1}\cos(3\pi x) + \cdots\cdots\right]$$

$$\Rightarrow e^x = e - 1 - 2\left[\left(\frac{1+e}{\pi^2+1}\right)\cos\pi x + \left(\frac{1-e}{4\pi^2+1}\right) + \cos 2\pi x\left(\frac{1+e}{9\pi^2+1}\right)\cos 3\pi x + \cdots\cdots\right]$$

QUESTION 6:

Represent by Fourier sine and cosine series the function defined by

$$f(x) = \begin{cases} 0, & 0 \le x < \pi/2 \\ 1, & \pi/2 \le x \le \pi \end{cases}$$

SOLUTION: Given that

$$f(x) = \begin{cases} 0, & 0 \le x < \pi/2 \\ 1, & \pi/2 \le x \le \pi \end{cases}$$

i. Fourier sine series:

Since the half-range Fourier sine series involves only the sine terms, so we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{or } f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \to (1) \qquad (\because L = \pi)$$

$$\text{Where } b_n = \frac{2}{L}\int_0^L f(x).\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi f(x).\sin(nx)dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi/2} f(x).\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)\sin(nx)dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi/2} (0)\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (1)\sin(nx)dx$$

$$\Rightarrow b_n = 0 + \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_{\pi/2}^{\pi}$$

$$b_n = \frac{-2}{n\pi}\left[\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{-2}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$\text{or } b_n = \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right) \longrightarrow (2)$$

Putting $n = 1, 2, 3, \cdots\cdots$, in Eq (2), we have

$$b_1 = \frac{2}{\pi}\left(\cos\left(\frac{\pi}{2}\right) + 1\right) = \frac{2}{\pi}$$

$b_2 = \frac{1}{\pi}(\text{Cos}\pi - 1) = \frac{-2}{\pi}$

$b_3 = \frac{2}{3\pi}\left(\text{Cos}\left(\frac{3\pi}{2}\right)+1\right) = \frac{2}{3\pi}$

$b_4 = \frac{1}{4\pi}(\text{Cos}(2\pi)-1) = 0$ and so on.

Eq (1) can be written as:

$f(x) = b_1.\text{Sin}x + b_2.\text{Sin}2x + b_3\text{Sin}3x + b_4.\text{Sin}4x + \cdots\cdots$

$\Rightarrow f(x) = \frac{2}{\pi}.\text{Sin}x - \frac{2}{\pi}.\text{Sin}2x + \frac{2}{3\pi}\text{Sin}3x + 0 + \cdots\cdots$

$f(x) = \frac{2}{\pi}\left[\text{Sin}x - \text{Sin}2x + \frac{1}{3}\text{Sin}3x + \cdots\cdots\right]$

ii. Fourier cosine series:

Since the half-range Fourier cosine series does not involve the sine terms, so we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{Cos}\left(\frac{n\pi x}{L}\right)$

or $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{Cos}(nx) \longrightarrow (3)$

Where $a_0 = \frac{2}{L}\int_0^L f(x)dx$

$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi/2} f(x)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)dx$

$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi/2}(0)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(1)dx$

$\Rightarrow a_0 = 0 + \frac{2}{\pi}[x]_{\pi/2}^{\pi}$

$\Rightarrow a_0 = \frac{2}{\pi}\left[\pi - \frac{\pi}{2}\right] = 1$

$\Rightarrow \boxed{a_0 = 1}$

$a_n = \frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\text{Cos}(nx)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi/2} f(x).\text{Cos}(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x).\text{Cos}(nx)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi/2}(0)\text{Cos}(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(1)\text{Cos}(nx)dx$

$\Rightarrow a_n = 0 + \frac{2}{\pi}\int_{\pi/2}^{\pi}\text{Cos}(nx)dx$

$\Rightarrow a_n = \frac{2}{\pi}\left[\frac{\text{Sin}(nx)}{n}\right]_{\pi/2}^{\pi}$

$\Rightarrow a_n = \frac{2}{n\pi}\left[\text{Sin } n\pi - \text{Sin}\left(\frac{n\pi}{2}\right)\right]$

$a_n = \frac{2}{n\pi}\left(0 - \text{Sin}\left(\frac{n\pi}{2}\right)\right)$

$\Rightarrow a_n = \frac{-2}{n\pi}\text{Sin}\left(\frac{n\pi}{2}\right) \longrightarrow (4)$

Putting $n = 1,2,3,4,\cdots\cdots$, in Eq (4), we have

$a_1 = \frac{-2}{\pi}\text{Sin}\left(\frac{\pi}{2}\right) = \frac{-2}{\pi}$

$a_2 = \frac{-1}{\pi}\text{Sin}\pi = 0$

$a_3 = \frac{-2}{3\pi}\text{Sin}\left(\frac{3\pi}{2}\right) = \frac{2}{3\pi}$

$a_4 = \frac{-1}{2\pi}\text{Sin}(2\pi) = 0$

$a_5 = \frac{-2}{5\pi}\text{Sin}\left(\frac{5\pi}{2}\right) = \frac{-2}{5\pi}$ and so on.

Eq (3) can be written as

$f(x) = \frac{a_0}{2} + a_1\text{Cos}x + a_2\text{Cos}2x + a_3\text{Cos}3x + a_4\text{Cos}4x + \cdots\cdots$

$\Rightarrow f(x) = \frac{1}{2} - \frac{2}{\pi}\text{Cos}x + 0.\text{Cos}2x + \frac{2}{3\pi}\text{Cos}3x + 0 - \frac{2}{5\pi}\text{Cos}5x + \cdots\cdots$

$\Rightarrow f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\text{Cos}x - \frac{\text{Cos}3x}{3} + \frac{\text{Cos}5x}{5} - \frac{\text{Cos}7x}{7} + \cdots\cdots\right]$

QUESTION 7:

Find the half-range sine and cosine series for the function

$$f(x) = \begin{cases} x, & 0 \le x < \pi/2 \\ \pi - x, & \frac{\pi}{2} \le x < \pi \end{cases}$$

SOLUTION: Given that

$$f(x) = \begin{cases} x, & 0 \le x < \pi/2 \\ \pi - x, & \frac{\pi}{2} \le x < \pi \end{cases}$$

i. Half-range sine series:

Since the half-range Fourier sine series involves only sine terms, so we have

$$f(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow f(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}(nx)\longrightarrow(1)\quad(\because L=\pi)$$

Where $b_n=\frac{2}{L}\int_0^L f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi} f(x).\text{Sin}(nx)dx$$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi/2} f(x).\text{Sin}(nx)dx+\frac{2}{\pi}\int_{\pi/2}^{\pi} f(x).\text{Sin}(nx)dx$$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi/2} x\text{Sin}(nx)dx+\frac{2}{\pi}\int_0^{\pi/2}(\pi-x)\text{Sin}(nx)dx$$

$$\Rightarrow b_n=\frac{2}{\pi}\left[\frac{-x.\text{Cos}(nx)}{n}\right]_0^{\pi/2}-\frac{2}{\pi}\int_0^{\pi/2}\frac{(1)(-\text{Cos}(nx))}{n}dx+$$

$$+\frac{2}{\pi}\left[\frac{(\pi-x)(-\text{Cos}(nx))}{n}\right]_{\pi/2}^{\pi}-\frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{(0-1)(-\text{Cos}nx)}{n}dx$$

$$\Rightarrow b_n=\frac{-2}{n\pi}[x.\text{Cos}nx]_0^{\pi/2}+\frac{2}{n\pi}\int_0^{\pi/2}\text{Cos}nx\,dx+$$

$$-\frac{2}{n\pi}[(\pi-x)\text{Cos}(nx)]_{\pi/2}^{\pi}-\frac{2}{\pi}\int_{\pi/2}^{\pi}\text{Cos}nx\,dx$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[\frac{\pi}{2}.\text{Cos}\left(\frac{n\pi}{2}\right)-0\right]+\frac{2}{n\pi}\left[\frac{\text{Sin}\,nx}{n}\right]_0^{\pi/2}+$$

$$-\frac{2}{n\pi}\left[0-\frac{\pi}{2}\text{Cos}\left(\frac{n\pi}{2}\right)\right]-\frac{2}{\pi}\left[\frac{\text{Sin}(nx)}{n}\right]_{\pi/2}^{\pi}$$

$$\Rightarrow b_n=\frac{-1}{n}\text{Cos}\left(\frac{n\pi}{2}\right)+\frac{2}{n^2\pi}\left[\text{Sin}\left(\frac{n\pi}{2}\right)-0\right]+\frac{1}{n}\text{Cos}\left(\frac{n\pi}{2}\right)+$$

$$-\frac{2}{n\pi}\left[0-\text{Sin}\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow b_n=\frac{2}{n^2\pi}\text{Sin}\left(\frac{n\pi}{2}\right)+\frac{2}{n\pi}\text{Sin}\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow b_n=\frac{2}{\pi}\left[\frac{1}{n^2}+\frac{1}{n}\right]\text{Sin}\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow b_n=\frac{2}{\pi}\left(\frac{1}{n^2}+\frac{1}{n}\right)\text{Sin}\left(\frac{n\pi}{2}\right)\longrightarrow(2)$$

Putting $n=1,2,3,4,\cdots\cdots$, in Eq (2), we have

$$b_1=\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{1}\right)\text{Sin}\left(\frac{\pi}{2}\right)=\frac{4}{\pi}$$

$$b_2=\frac{2}{\pi}\left(\frac{1}{2^2}+\frac{1}{2}\right)\text{Sin}\pi=0$$

$$b_3=\frac{2}{\pi}\left(\frac{1}{3^2}+\frac{1}{3}\right)\text{Sin}\left(\frac{3\pi}{3}\right)=\frac{-8}{9\pi}$$

$$b_4=\frac{2}{\pi}\left(\frac{1}{4^2}+\frac{1}{4}\right)\text{Sin}2\pi=0$$

$$b_5=\frac{2}{\pi}\left(\frac{1}{5^2}+\frac{1}{5}\right)\text{Sin}\left(\frac{5\pi}{2}\right)=\frac{12}{25\pi}\text{ and so on.}$$

Eq (1) can be written as

$$f(x)=b_1.\text{Sin}x+b_2.\text{Sin}2x+b_3\,\text{Sin}3x+b_4.\text{Sin}5x+\cdots\cdots$$

$$\Rightarrow f(x)=\frac{4}{\pi}\text{Sin}x+0.\text{Sin}2x-\frac{8}{9\pi}.\text{Sin}3x+0.\text{Sin}4x+\frac{12}{25\pi}.\text{Sin}5x+\cdots$$

$$\Rightarrow f(x)=\frac{4}{\pi}\text{Sin}x-\frac{8}{9\pi}.\text{Sin}3x+\frac{12}{25\pi}.\text{Sin}5x-\cdots\cdots$$

$$\text{or } f(x)=\frac{4}{\pi}\left[\text{Sin}x-\frac{2}{9}\text{Sin}3x+\frac{3}{25}.\text{Sin}5x-\cdots\cdots\right]$$

ii. Half-range cosine series:

Since the half-range Fourier cosine series does not involve sine terms, so we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{L}\right)$$

$$\text{or } f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}(nx)\longrightarrow(3)\quad(\because L=\pi)$$

Where $a_0=\frac{2}{L}\int_0^L f(x)dx$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi/2} f(x)dx+\frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\int_0^{\pi/2} x\,dx+\frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi-x)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2}+\frac{2}{\pi}\left[\pi x-\frac{x^2}{2}\right]_{\pi/2}^{\pi}$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{\pi^2}{4}-0\right]+\frac{2}{\pi}\left[\pi^2-\frac{\pi^2}{2}-\frac{\pi^2}{2}+\frac{\pi^2}{8}\right]$$

$$\Rightarrow a_0=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\pi^2}{8}\right]$$

$\Rightarrow a_0 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

Thus $\boxed{a_0 = \pi/2}$

$a_n = \frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{2}\right)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\,\text{Cos}(nx)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi/2} f(x)\text{Cos}(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x).\,\text{Cos}(nx)dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi/2} x.\,\text{Cos}nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\text{Cos}(nx)dx$

$\Rightarrow a_n = \frac{2}{\pi}\left[\frac{x.\,\text{Sin}(nx)}{n}\right]_0^{\pi/2} - \frac{2}{\pi}\int_0^{\pi/2}\frac{(1).\,\text{Sin}(nx)}{n}dx +$

$+\frac{2}{\pi}\left[\frac{(\pi - x).\,\text{Sin}(nx)}{n}\right]_0^{\pi/2} - \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{(0-1)\text{Sin}(nx)dx}{n}$

$\Rightarrow a_n = \frac{2}{n\pi}\left[x.\,\text{Sin}(nx)\right]_0^{\pi/2} + \frac{2}{n\pi}\left[\frac{\text{Cos}nx}{n}\right]_0^{\pi/2} +$

$+\frac{2}{n\pi}\left[(\pi - x)\text{Sin}(nx)\right]_{\pi/2}^{\pi} - \frac{2}{n\pi}\left[\frac{\text{Cos}\,nx}{n}\right]_{\pi/2}^{\pi}$

$\Rightarrow a_n = \frac{2}{n\pi}\left[\frac{\pi}{2}\text{Sin}\left(\frac{n\pi}{2}\right) - 0\right] + \frac{2}{n^2\pi}\left[\text{Cos}\left(\frac{n\pi}{2}\right) - \text{Cos}\,0\right] +$

$+\frac{2}{n\pi}\left[0 - \left(\pi - \frac{\pi}{2}\right)\text{Sin}\left(\frac{n\pi}{2}\right)\right] - \frac{2}{n^2\pi}\left[\text{Cos}n\pi - \text{Cos}\left(\frac{n\pi}{2}\right)\right]$

$\Rightarrow a_n = \frac{1}{n}\text{Sin}\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi}\left(\text{Cos}\left(\frac{n\pi}{2}\right) - 1\right) - \frac{1}{n}\text{Sin}\left(\frac{n\pi}{2}\right) +$

$-\frac{2}{n^2\pi}\left((-1)^n - \text{Cos}\left(\frac{n\pi}{2}\right)\right)$

$\Rightarrow a_n = \frac{2}{n^2\pi}\left[\text{Cos}\left(\frac{n\pi}{2}\right) - 1 - (-1)^n + \text{Cos}\left(\frac{n\pi}{2}\right)\right]$

$\Rightarrow a_n = \frac{2}{n^2\pi}\left[2\text{Cos}\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right] \longrightarrow (4)$

Putting $n = 1,2,3,4,\ldots\ldots$ in (4), we have

$a_1 = \frac{2}{1^2.\pi}\left[2\text{Cos}\left(\frac{\pi}{2}\right) - (-1)^1 - 1\right] = 0$

$a_2 = \frac{2}{2^2.\pi}\left[2\text{Cos}\,\pi - (-1)^2 - 1\right] = \frac{-2}{\pi}$

$a_3 = \frac{2}{3^2.\pi}\left[2\text{Cos}\left(\frac{3\pi}{2}\right) - (-1)^3 - 1\right] = 0$

$a_4 = \frac{2}{4^2.\pi}\left[2.\text{Cos}2\pi - (-1)^4 - 1\right] = 0$

$a_5 = \frac{2}{5^2.\pi}\left[2\text{Cos}\left(\frac{5\pi}{2}\right) - (-1)^5 - 1\right] = 0$

$a_6 = \frac{2}{6^2.\pi}\left[2.\text{Cos}3\pi - (-1)^6 - 1\right] = \frac{-2}{9\pi}$ and so on.

Eq^n(3) can be written as

$f(x) = \frac{a_0}{2} + a_1.\,\text{Cos}x + a_2.\text{Cos}2x + a_3.\text{Cos}3x + a_n\text{Cos}4x$

$+a_5\text{Cos}5x + a_6\text{Cos}6x + \cdots\cdots$

$\Rightarrow f(x) = \frac{\pi}{4} + 0.\,\text{Cos}x - \frac{2}{\pi}\text{Cos}2x + 0.\text{Cos}3x + 0.\text{Cos}4x +$

$+0.\text{Cos}5x - \frac{2}{9x}\text{Cos}6x + \cdots\cdots$

$\Rightarrow f(x) = \frac{\pi}{4} - \frac{2}{\pi}\text{Cos}2x - \frac{2}{9x}\text{Cos}6x - \cdots\cdots$

$\Rightarrow f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\text{Cos}2x + \frac{1}{9}\text{Cos}6x + \cdots\cdots\right]$

QUESTION 8:

Express sin x as a half-range cosine series in $0 < x < \pi$ and deduce that

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots = \frac{\pi}{4}$

SOLUTION:

Here $f(x) = \text{Sin}x, \quad 0 < x < \pi.$

Since the half-range Fourier cosine series does not involve sine terms, so we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\text{Cos}\left(\frac{n\pi x}{L}\right)$

$\Rightarrow \text{Sin}x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\,\text{Cos}(nx) \rightarrow (1) \quad \because L = \pi$

Where $a_0 = \frac{2}{L}\int_0^L f(x)dx$

$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi}\text{Sin}x\,dx$

$\Rightarrow a_0 = \frac{2}{\pi}\left[-\text{Cos}x\right]_0^{\pi}$

$\Rightarrow a_0 = \frac{-2}{\pi}\left[\text{Cos}\pi - \text{Cos}\,0\right]$

$\Rightarrow a_0 = \frac{-2}{\pi}\left[-1 - 1\right] = \frac{4}{\pi}$

$$\Rightarrow \boxed{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi \text{Sinx}.\text{Cos}(nx)dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^\pi 2.\text{Cos}(nx).\text{Sinx}\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^\pi [\text{Sin}(n+1)x - \text{Sin}(n-1)x]dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{-\text{Cos}(n+1)x}{n+1} + \frac{\text{Cos}(n-1)x}{n-1}\right]_0^\pi$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{-\text{Cos}(n+1)\pi}{n+1} + \frac{\text{Cos}(n-1)\pi}{n-1} + \frac{\text{Cos}\,0}{n+1} - \frac{\text{Cos}0}{n+1}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{\text{Cos}n\pi}{n+1} - \frac{\text{Cos}\,n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left((-1)^n + 1\right)\left(\frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$\Rightarrow a_n = \left(\frac{1+(-1)^n}{\pi}\right)\left(\frac{n-1-n-1}{n^2-1}\right)$$

$$\Rightarrow a_n = \left(\frac{1+(-1)^n}{\pi}\right)\frac{(-2)}{n^2-1}$$

$$\Rightarrow a_n = \frac{-2\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)}$$

$$\text{or } a_n = \frac{2\left((-1)^{n+1}-1\right)}{\pi\left(n^2-1\right)} \longrightarrow (2) \qquad n \neq 1$$

For $n = 1$, we have

$$a_1 = \frac{2}{L}\int_0^L f(x).\text{Cos}\left(\frac{\pi x}{L}\right)dx$$

$$\Rightarrow a_1 = \frac{2}{\pi}\int_0^\pi \text{Sinx}.\text{Cosx}\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_0^\pi 2\,\text{Sinx}.\text{Cosx}\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_0^\pi \text{Sin}(2x)dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[\frac{-\text{Cos}\,2x}{2}\right]_0^\pi = \frac{-1}{2\pi}[\text{Cos}\,2\pi - \text{Cos}0] = 0$$

$$\Rightarrow \boxed{a_1 = 0}$$

Eq (1) can be written as

$$\text{Sinx} = \frac{a_0}{2} + a_1.\text{Cosx} + \sum_{n=2}^\infty a_n.\text{Cos}(nx)$$

$$\Rightarrow \text{Sinx} = \frac{2}{\pi} + 0.\text{Cosx} + \sum_{n=2}^\infty \frac{2\left((-1)^{n+1}-1\right)}{\pi\left(n^2-1\right)}\text{Cos}(nx)$$

$$\Rightarrow \text{Sinx} = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^\infty \frac{(-1)^{n+1}-1}{n^2-1}.\text{Cos}(nx)$$

$$\Rightarrow \text{Sinx} = \frac{2}{\pi} + \frac{2}{\pi}\left[\frac{-2}{2^2-1}\text{Cos}2x - \frac{2}{4^2-1}\text{Cos}4x - \frac{2}{6^2-1}\text{Cos}6x - \cdots\cdots\right]$$

$$\Rightarrow \text{Sinx} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\text{Cos}2x + \frac{1}{15}\text{Cos}4x + \frac{1}{35}\text{Cos}6x\cdots\right] \longrightarrow (3)$$

Putting $x = \frac{\pi}{2}$ in Eq (3), we have

$$\text{Sin}\frac{\pi}{2} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}.\text{Cos}\pi + \frac{1}{15}\text{Cos}2\pi + \frac{1}{35}\text{Cos}3\pi + \cdots\cdots\right]$$

$$\Rightarrow 1 = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{-1}{3} + \frac{1}{15} - \frac{1}{35} + \cdots\cdots\right]$$

Multiplying by $\frac{\pi}{4}$, we have

$$\frac{\pi}{4} = \frac{1}{2} - \left[\frac{-1}{3} + \frac{1}{15} - \frac{1}{35} + \cdots\cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{2}{2}\left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35}\cdots\cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2}\left[\frac{2}{3} - \frac{2}{15} + \frac{2}{35}\cdots\cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2}\left(\frac{2}{3}\right) - \frac{1}{2}\left(\frac{2}{15}\right) + \frac{1}{2}\left(\frac{2}{35}\right) - \cdots\cdots$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) - \cdots\cdots$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2.3} - \frac{1}{2.3} + \frac{1}{2.5} + \frac{1}{2.5} - \frac{1}{2.7} - \frac{1}{2.7} + \cdots\cdots$$

Thus $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$

QUESTION 9:

Expand $f(x)=\begin{cases}\frac{1}{4}-x, & 0<x<\frac{1}{2}\\ x-\frac{3}{4}, & \frac{1}{2}<x<1\end{cases}$

As a Fourier sine series.

SOLUTION: Given that

$$f(x)=\begin{cases}\frac{1}{4}-x, & 0<x<\frac{1}{2}\\ x-\frac{3}{4}, & \frac{1}{2}<x<1\end{cases}$$

Since half-range Fourier sine series involves only sine terms, so we have

$$f(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$$

$$\text{or } f(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}(n\pi x)\rightarrow(1)\quad \because L=1$$

$$\text{Where } b_n=\frac{2}{L}\int_0^L f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow b_n=2\int_0^1 f(x).\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n=2\int_0^{1/2} f(x).\text{Sin}(n\pi x)dx+2\int_{1/2}^1 f(x).\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n=2\int_0^{1/2}\left(\frac{1}{4}-x\right)\text{Sin}(n\pi x)dx+2\int_{1/2}^1\left(x-\frac{3}{4}\right)\text{Sin}(n\pi x)dx$$

$$\Rightarrow b_n=2\left[-\left(\frac{1}{4}-x\right)\frac{\text{Cos}(n\pi x)}{n\pi}\right]_0^{1/2}-2\int_0^{1/2}-(0-1)\frac{\text{Cos}(n\pi x)}{n\pi}dx+$$

$$+2\left[-\left(x-\frac{3}{4}\right)\frac{\text{Cos}(n\pi x)}{n\pi}\right]_{1/2}^1-2\int_{1/2}^1-(1-0)\frac{\text{Cos}(n\pi x)}{n\pi}dx$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[\left(\frac{1}{4}-x\right)\text{Cos}(n\pi x)\right]_0^{1/2}-\frac{2}{n\pi}\left[\frac{\text{Sin } n\pi x}{n\pi}\right]_0^{1/2}+$$

$$-\frac{2}{n\pi}\left[\left(x-\frac{3}{4}\right)\text{Cos}(n\pi x)\right]_{1/2}^1+\frac{2}{n\pi}\left[\frac{\text{Sin}(n\pi x)}{n\pi}\right]_{1/2}^1$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[\frac{-1}{4}\text{Cos}\left(\frac{n\pi}{2}\right)-\frac{1}{4}\right]-\frac{2}{n^2\pi^2}\left[\text{Sin}\left(\frac{n\pi}{2}\right)-0\right]+$$

$$-\frac{2}{n\pi}\left[\frac{(-1)^n}{4}+\frac{1}{4}\text{Cos}\left(\frac{n\pi}{2}\right)\right]+\frac{2}{n^2\pi^2}\left[0-\text{Sin}\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow b_n=\frac{1}{2n\pi}\left[\text{Cos}\left(\frac{n\pi}{2}\right)+1\right]-\frac{2}{n^2\pi^2}\text{Sin}\left(\frac{n\pi}{2}\right)+$$

$$-\frac{1}{2n\pi}\left[(-1)^n+\text{Cos}\left(\frac{n\pi}{2}\right)\right]-\frac{2}{n^2\pi}\text{Sin}\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow b_n=\frac{1}{2n\pi}\text{Cos}\left(\frac{n\pi}{2}\right)+\frac{1}{2n\pi}-\frac{2}{n^2\pi^2}\text{Sin}\left(\frac{n\pi}{2}\right)-\frac{(-1)^n}{2x\pi}+$$

$$-\frac{1}{2n\pi}\text{Cos}\left(\frac{n\pi}{2}\right)-\frac{2}{n^2\pi^2}\text{Sin}\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow b_n=\frac{-4}{n^2\pi^2}\text{Sin}\left(\frac{n\pi}{2}\right)+\frac{1}{2n\pi}-\frac{(-1)^n}{2n\pi}$$

$$\Rightarrow b_n=\frac{-4}{n^2\pi^2}\text{Sin}\left(\frac{n\pi}{2}\right)+\frac{\left(1-(-1)^n\right)}{2n\pi}\rightarrow(2)$$

Putting $n=1,2,3,\cdots\cdots$, in Eq (2), we have

$$b_1=\frac{-4}{1^2.\pi^2}\text{Sin}\left(\frac{\pi}{2}\right)+\frac{\left(1-(-1)^1\right)}{2\pi}=\frac{-4}{\pi^2}+\frac{1}{\pi}$$

$$b_2=\frac{-4}{2^2.\pi^2}\sin\pi+\frac{1-(-1)^2}{2.2\pi}=0$$

$$b_3=\frac{-4}{3^2.\pi^2}\text{Sin}\left(\frac{3\pi}{2}\right)+\frac{1-(-1)^3}{2.4\pi}=\frac{4}{9\pi^2}+\frac{2}{6\pi}=\frac{4}{9\pi^2}+\frac{1}{3\pi}$$

$$b_4=\frac{-4}{4^2\pi^2}.\text{Sin}2\pi+\frac{1-(-1)^4}{2.4\pi}=0$$

$$b_5=\frac{-4}{5^2.\pi^2}\text{Sin}\left(\frac{5\pi}{2}\right)+\frac{1-(-1)^5}{2.5\pi}=\frac{-4}{25\pi^2}+\frac{2}{10\pi}=\frac{-4}{25\pi^2}+\frac{1}{5\pi}$$

and so on.

Eq (1) can be written as

$$\text{Sin}x=b_1\text{Sin}x+b_2\text{Sin}2x+b_3\text{Sin}3x+b_4\text{Sin}nx+b_5\text{Sin}5x+\cdots\cdots$$

$$\Rightarrow \text{Sin}x=\left(\frac{-4}{\pi^2}+\frac{1}{\pi}\right)\text{Sin}x+0.\text{Sin}2x+\left(\frac{4}{9\pi^2}+\frac{1}{3\pi}\right)\text{Sin}3x+$$

$$+0.\text{Sin}4x+\left(\frac{-4}{25\pi^2}+\frac{1}{5\pi}\right)\text{Sin}5x+\cdots\cdots$$

$$\text{Sin}x=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\text{Sin}x+\left(\frac{1}{3\pi}+\frac{4}{9\pi^2}\right)\text{Sin}3x+\left(\frac{1}{5\pi}-\frac{4}{25\pi^2}\right)\text{Sin}5x+\cdots\cdots$$

QUESTION 10:

Expand $x(\pi-x)$ in half-range sine series when $0<x<\pi$

SOLUTION:

Here $f(x)=x(\pi-x)=\pi x-x^2,\ 0<x<\pi$

Since the half-range sine series involve only since terms, so we have

$$f(x)=\sum_{n=1}^{\infty}b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$$

$$x(\pi-x)=\sum_{n=1}^{\infty} b_n.\text{Sin } nx \rightarrow (1) \quad \because L=\pi$$

Where $b_n=\frac{2}{L}\int_0^L f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\,\text{Sin}(nx)dx$$

$$\Rightarrow b_n=\frac{2}{\pi}\left[-(\pi n-x^2)\frac{\text{Cos}(nx)}{n}\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}-(\pi-2x)\frac{\text{Cos}(nx)}{n}dx$$

$$\Rightarrow b_n=\frac{-2}{\pi n}\left[(\pi x-x^2)\text{Cos}nx\right]_0^{\pi}+\frac{2}{\pi n}\int_0^{\pi}(\pi-2x)\text{Cos } nx\,dx$$

$$\Rightarrow b_n=\frac{-2}{\pi n}[0-0]+\frac{2}{\pi n}\left[(\pi-2x)\frac{\text{Sin } nx}{n}\right]_0^{\pi}+$$

$$-\frac{2}{\pi n}\int_0^{\pi}(0-2)\frac{\text{Sin } nx}{n}dx.$$

$$\Rightarrow b_n=\frac{2}{\pi n^2}\left[(\pi-2x)\text{Sin } nx\right]_0^{\pi}-\frac{4}{\pi n^2}\left[\frac{\text{Cos } nx}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n=\frac{2}{\pi n^2}[0-0]-\frac{4}{\pi n^3}[\text{Cos } n\pi-\text{Cos}0]$$

$$\Rightarrow b_n=\frac{-4}{\pi n^3}\left[(-1)^n-1\right]$$

or $\boxed{b_n=\frac{4\left(1-(-1)^n\right)}{n^3\pi}}$

Then Eq(1) $\Rightarrow$

$$x(\pi-x)=\sum_{n=1}^{\infty}\frac{4\left(1-(-1)^n\right)}{n^3\pi}\text{Sin } nx$$

$$\Rightarrow x(\pi-x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\text{Sin } nx$$

$$\Rightarrow x(\pi-x)=\frac{4}{\pi}\left[\frac{2}{1^3}\text{Sinx}+\frac{2}{3^2}\text{Sin3x}+\frac{2}{5^3}\text{Sin5x}+\cdots\cdots\right]$$

$$\Rightarrow x(\pi-x)=\frac{8}{\pi}\left[\frac{\text{Sin x}}{1^3}+\frac{\text{Sin 3x}}{3^2}+\frac{\text{Sin5x}}{5^3}+\cdots\cdots\right]$$

QUESTION 11:

Expand $f(x)=\pi-x$ into

i. Fourier series for $-\pi<x<\pi$

ii. Fourier sine series for $0<x<\pi$

iii. Fourier cosine series for $0\le x\le\pi$

SOLUTION: Given that $f(x)=\pi-x$

i. Fourier series for $-\pi<x<\pi$:

The Fourier series of $f(x)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\text{Cosnx}+b_n\text{Sin } nx)\rightarrow(1)$$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi-x)dx$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\pi x-\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\pi^2-\frac{\pi^2}{2}+\pi^2+\frac{\pi^2}{2}\right]$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[2\pi^2\right]=2\pi$$

$$\Rightarrow \boxed{a_0=2\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x).\text{Cosnx}\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi-x)\text{Cos } nx\,dx$$

$$\Rightarrow a_n=\frac{1}{\pi}\left[(\pi-x)\frac{\text{Sin } nx}{n}\right]_{-\pi}^{\pi}-\frac{1}{\pi}\int_{-\pi}^{\pi}(0-1)\frac{\text{Sin } nx}{n}dx$$

$$\Rightarrow a_n=\frac{1}{n\pi}[0-0]-\frac{1}{n\pi}\left[\frac{\text{Cosnx}}{n}\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_n=0-\frac{1}{n^2\pi}\left[\text{Cos } n\pi-\text{Cos}(-n\pi)\right]=0.$$

$$\Rightarrow \boxed{a_n=0}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x).\text{Sinnx}\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi-x)\text{Sinnx}\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{-(\pi-x)\text{Cosnx}}{n}\right]_{-\pi}^{\pi}-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{-(0-1)\text{Cos } nx}{n}dx$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[(\pi-x)\text{Cosnx}\right]_{-\pi}^{\pi}-\frac{1}{n\pi}\left[\frac{\text{Sin } nx}{n}\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n=\frac{-1}{n\pi}\left[0-2\pi\text{Cos}(-n\pi)\right]-0$$

$$\boxed{b_n=\frac{2}{n}(-1)^n}$$

Putting the values in Eq (1), we have

$$f(x)=\pi+\sum_{n=1}^{\infty}\left(0.\cos nx+\frac{2(-1)^n}{n}\sin nx\right)$$

$$\Rightarrow f(x)=\pi+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

Which is the required Fourier series of $f(x)$.

ii. Fourier sine series for $0<x<\pi$:

Since half-range Fourier sine series involve only sine terms, so we have

$$f(x)=\sum_{n=1}^{\infty}b_n.\sin\left(\frac{n\pi x}{L}\right)$$

or $\pi-x=\sum_{n=1}^{\infty}b_n.\sin nx\rightarrow(2)$ $\because L=\pi$

Where $b_n=\frac{2}{L}\int_0^L f(x).\sin nx\,dx$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\sin nx\,dx$$

$$\Rightarrow b_n=\frac{2}{\pi}\left[\frac{-(\pi-x)\cos nx}{n}\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}\frac{-(0-1)\cos nx\,dx}{n}$$

$$\Rightarrow b_n=\frac{-2}{n\pi}\left[(\pi-x)\cos nx\right]_0^{\pi}-\frac{2}{n\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$\Rightarrow b_n=\frac{-2}{n\pi}[0-\pi\cos 0]-\frac{2}{n^2\pi}[0-0]$$

$$\Rightarrow b_n=\frac{2}{n}$$

$$\Rightarrow \boxed{b_n=\frac{2}{n}}$$

Then Eq (2) $\Rightarrow$

$$\pi-x=\sum_{n=1}^{\infty}\frac{2}{n}\sin nx=2\sum\frac{\sin nx}{n}$$

$$\Rightarrow\pi-x=2\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\cdots\cdots\right]$$

iii. Fourier cosine series for $0\le x\le\pi$:

Since the half-range Fourier cosine series does not involve sine terms, so we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\cos\left(\frac{n\pi x}{L}\right)$$

or $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\cos(nx)\rightarrow(3)$ $\because L=\pi$

Where $a_0=\frac{2}{L}\int_0^L f(x)dx$

$$a_0=\frac{2}{\pi}\int_0^{\pi}(\pi-x)dx$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}$$

$$\Rightarrow a_0=\frac{2}{\pi}\left[\pi^2-\frac{\pi^2}{2}-0\right]$$

$$\Rightarrow a_0=\frac{2}{\pi}\left(\frac{\pi^2}{2}\right)=\pi$$

$$\Rightarrow\boxed{a_0=\pi}$$

$$a_n=\frac{2}{L}\int_0^L f(x).\cos\left(\frac{n\pi x}{n}\right)dx$$

$$\Rightarrow a_n=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\cos nx\,dx$$

$$\Rightarrow a_n=\frac{2}{\pi}\left[(\pi-x)\frac{\sin nx}{n}\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}(0-1)\frac{\sin nx}{n}dx$$

$$\Rightarrow a_n=\frac{2}{n\pi}\left[(\pi-x)\sin nx\right]_0^{\pi}-\frac{2}{n\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$\Rightarrow a_n=\frac{2}{n\pi}[0-0]-\frac{2}{n^2\pi}[\cos n\pi-\cos 0]$$

$$\Rightarrow a_n=\frac{-2}{n^2\pi}\left((-1)^n-1\right)$$

$$\boxed{a_n=\frac{2\left(1-(-1)^n\right)}{n^2\pi}}$$

Putting the values in (3), we have

$$f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2\left(1-(-1)^n\right)}{n^2\pi}\cos nx$$

$$\Rightarrow f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}.\cos nx$$

$$\Rightarrow f(x)=\frac{\pi}{2}+\frac{2}{\pi}\left[\frac{2}{1^2}\cos x+\frac{2}{3^2}\cos 3x+\frac{2}{5^2}\cos 5x+\cdots\cdots\right]$$

$$\Rightarrow f(x)=\frac{\pi}{2}+\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\cdots\right]$$

QUESTION 12:

Obtain Fourier sine series if $f(x)=\cos\left(\frac{\pi x}{P}\right), 0\le x\le P$

SOLUTION: Given that

$$f(x)=\cos\left(\frac{\pi x}{P}\right), 0\le x\le P$$

Since Fourier sine series involve only sine terms, so we have

$$f(x)=\sum_{n=1}^{\infty} b_n.\text{Sin}\left(\frac{n\pi x}{L}\right)$$

$$\text{or } \text{Cos}\left(\frac{\pi x}{P}\right)=\sum_{n=1}^{\infty} b_n.\text{Sin}\left(\frac{n\pi x}{P}\right)\to(1) \qquad \because L=P$$

Where $b_n=\frac{2}{L}\int_0^L f(x).\text{Sin}\left(\frac{n\pi x}{L}\right)dx$

$$b_n=\frac{2}{P}\int_0^P \text{Cos}\left(\frac{\pi x}{P}\right).\text{Sin}\left(\frac{n\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\frac{1}{P}\int_0^P 2.\text{Sin}\left(\frac{n\pi x}{P}\right).\text{Cos}\left(\frac{\pi x}{P}\right)dx$$

$$\Rightarrow b_n=\frac{1}{P}\int_0^P\left[\text{Sin}\left(\frac{(n+1)\pi x}{P}\right)+\text{Sin}\left(\frac{(n-1)\pi x}{P}\right)\right]dx$$

$$\Rightarrow b_n=\frac{1}{P}\left[\frac{\frac{-\text{Cos}(n+1)\pi x}{P}}{\frac{(n+1)\pi}{P}}-\frac{\frac{\text{Cos}(n-1)\pi x}{P}}{\frac{(n-1)\pi}{P}}\right]_0^P$$

$$\Rightarrow b_n=\frac{1}{P}\left[\frac{-P}{(n+1)\pi}\text{Cos}\frac{(n+1)\pi x}{P}-\frac{P}{(n-1)\pi}\text{Cos}\frac{(n-1)\pi x}{P}\right]_0^P$$

$$\Rightarrow b_n=\frac{-1}{\pi}\left[\frac{1}{n+1}\text{Cos}\frac{(n+1)\pi x}{P}+\frac{1}{n-1}\text{Cos}\frac{(n-1)\pi x}{P}\right]_0^P$$

$$\Rightarrow b_n=\frac{-1}{\pi}\left[\frac{1}{n+1}\text{Cos}(n+1)\pi+\frac{1}{n-1}\text{Cos}(n-1)\pi-\frac{\text{Cos}0}{n+1}-\frac{\text{Cos}\,0}{n-1}\right]$$

$$\Rightarrow b_n=\frac{-1}{\pi}\left[\frac{-\text{Cos}\,n\pi}{n+1}-\frac{\text{Cos}\,n\pi}{n-1}-\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$\Rightarrow b_n=\frac{-1}{\pi}\left[\frac{-(-1)^n}{n+1}-\frac{(-1)^n}{n-1}-\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\left(\frac{1}{n+1}+\frac{1}{n-1}\right)(-1)^n+\left(\frac{1}{n+1}+\frac{1}{n-1}\right)\right]$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{2n}{n^2-1}(-1)^n+\frac{2n}{n^2-1}\right]$$

$$\boxed{b_n=\frac{2n}{\pi(n^2-1)}\left((-1)^n+1\right), \qquad n\neq1}$$

For $n=1$, we have

$$b_1=\frac{2}{P}\int_0^P f(x)\text{Sin}\left(\frac{\pi x}{P}\right)dx$$

$$\Rightarrow b_1=\frac{2}{P}\int_0^P \text{Cos}\left(\frac{\pi x}{P}\right)\text{Sin}\left(\frac{\pi x}{P}\right)dx$$

$$\Rightarrow b_1=\frac{1}{P}\int_0^P 2.\text{Sin}\left(\frac{\pi x}{P}\right).\text{Cos}\left(\frac{\pi x}{P}\right)dx$$

$$\Rightarrow b_1=\frac{1}{P}\int_0^P \text{Sin}\left(\frac{2\pi x}{P}\right)dx$$

$$\Rightarrow b_1=\frac{1}{P}\left[\frac{-\text{Cos}\left(\frac{2\pi x}{P}\right)}{\frac{2\pi}{P}}\right]_0^P$$

$$\Rightarrow b_1=\frac{1}{2\pi}[\text{Cos}\,2\pi-\text{Cos}\,0]=0$$

$$\Rightarrow \boxed{b_1=0}$$

Eq (1) can be written as

$$\text{Cos}\left(\frac{\pi x}{P}\right)=b_1\,\text{Sin}\left(\frac{\pi x}{P}\right)+\sum_{n=1}^{\infty} b_n\,\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow \text{Cos}\left(\frac{\pi x}{P}\right)=0.\text{Sin}\left(\frac{\pi x}{P}\right)+\sum_{n=2}^{\infty}\frac{2n}{\pi(n^2-1)}\left((-1)^n+1\right)\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow \text{Cos}\left(\frac{\pi x}{P}\right)=\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n\left((-1)^n+1\right)}{n^2-1}.\text{Sin}\left(\frac{n\pi x}{P}\right)$$

$$\Rightarrow \text{Cos}\left(\frac{\pi x}{P}\right)=\frac{2}{\pi}\left[\frac{2.2}{2^2-1}\text{Sin}\left(\frac{2\pi x}{P}\right)+\frac{4.2}{4^2-1}\text{Sin}\left(\frac{4\pi x}{P}\right)+\frac{6.2}{6^2-1}\text{Sin}\left(\frac{6\pi x}{P}\right)+\ldots\ldots\right]$$

$$\Rightarrow \text{Cos}\left(\frac{\pi x}{P}\right)=\frac{2}{\pi}\left[\frac{4}{3}\text{Sin}\left(\frac{2\pi x}{P}\right)+\frac{8}{15}\text{Sin}\left(\frac{4\pi x}{P}\right)+\frac{12}{35}\text{Sin}\left(\frac{6\pi x}{P}\right)+\cdots\right]$$

$$\Rightarrow \text{Cos}\left(\frac{\pi x}{P}\right)=\frac{8}{\pi}\left[\frac{1}{3}\text{Sin}\left(\frac{2\pi x}{P}\right)+\frac{2}{15}\text{Sin}\left(\frac{4\pi x}{P}\right)+\frac{3}{35}\text{Sin}\left(\frac{6\pi x}{P}\right)+\cdots\right]$$

NOTE: The book answer is wrong.

QUESTION 13:

Express $f(x)=\text{Sin}\left(\frac{\pi x}{L}\right)$, $0<x<L$ by a Fourier cosine series.

SOLUTION: Given that

$$f(x)=\text{Sin}\left(\frac{\pi x}{L}\right),\ 0<x<L$$

Since the Fourier cosine series does not involve sine terms, so we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.\text{Cos}\left(\frac{n\pi x}{P}\right) \quad (\because \text{Half-reriod}=P=L)$$

OR $\sin\left(\frac{\pi x}{L}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)\longrightarrow(1)$

Where $a_0=\frac{2}{P}\int_0^P f(x)dx$

$$\Rightarrow a_0=\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)dx$$

$$\Rightarrow a_0=\frac{2}{L}\left[\frac{-\cos\left(\frac{\pi x}{L}\right)}{\frac{\pi}{L}}\right]_0^L$$

$$\Rightarrow a_0=\frac{-2}{\pi}\left[\cos\left(\frac{\pi x}{L}\right)\right]_0^L$$

$$\Rightarrow a_0=\frac{-2}{\pi}[\cos\pi-\cos 0]$$

$$\Rightarrow a_0=\frac{-2}{\pi}[-1-1]=\frac{4}{\pi}$$

$$\Rightarrow \boxed{a_0=\frac{4}{\pi}}$$

$$a_n=\frac{2}{P}\int_0^P f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow a_n=\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right).\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow a_n=\frac{1}{L}\int_0^L 2\cos\left(\frac{n\pi x}{L}\right).\sin\left(\frac{\pi x}{L}\right)dx$$

$$\Rightarrow a_n=\frac{1}{L}\int_0^L\left[\sin\frac{(n+1)}{L}\pi x-\sin\frac{(n-1)}{L}\pi x\right]dx$$

$$\Rightarrow a_n=\frac{1}{L}\left[\frac{\frac{-\cos(n+1)\pi x}{L}}{\frac{(n+1)\pi}{L}}+\frac{\frac{\cos(n-1)\pi x}{L}}{\frac{(n-1)\pi}{L}}\right]_0^L$$

$$\Rightarrow a_n=\frac{1}{L}\left[\frac{-L}{(n+1)\pi}.\cos\frac{(n+1)}{L}\pi x+\frac{L}{(n-1)\pi}\cos\frac{(n-1)}{L}\pi x\right]_0^L$$

$$\Rightarrow a_n=\left[\frac{-1}{(n+1)\pi}\cos\frac{(n+1)}{L}\pi x+\frac{1}{(n-1)\pi}\cos\frac{(n-1)}{L}\pi x\right]_0^L$$

$$\Rightarrow a_n=\frac{-1}{(n+1)\pi}\cos(n+1)\pi+\frac{1}{(n-1)\pi}\cos(n-1)\pi+$$

$$+\frac{1}{(n+1)\pi}.\cos 0-\frac{1}{(n-1)\pi}.\cos 0$$

$$\Rightarrow a_n=\frac{(-1)(-\cos n\pi)}{(n+1)\pi}+\frac{1(-\cos n\pi)}{(n-1)\pi}+\frac{1}{(n+1)\pi}-\frac{1}{(n-1)\pi}$$

$$\Rightarrow a_n=\frac{\cos n\pi}{(n+1)\pi}-\frac{\cos n\pi}{(n-1)\pi}+\frac{1}{(n+1)\pi}-\frac{1}{(n-1)\pi}$$

$$\Rightarrow a_n=\frac{(-1)^n}{(n+1)\pi}-\frac{(-1)^n}{(n-1)\pi}+\frac{1}{(n+1)\pi}-\frac{1}{(n-1)\pi}$$

$$\Rightarrow a_n=\frac{(-1)^n}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]+\frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$\Rightarrow a_n=\left(\frac{(-1)^n}{\pi}+\frac{1}{\pi}\right)\left(\frac{1}{n+1}-\frac{1}{n-1}\right)$$

$$\Rightarrow a_n=\left(\frac{(-1)^n+1}{\pi}\right)\left(\frac{-2}{n^2-1}\right)$$

$$\Rightarrow \boxed{a_n=\frac{-2\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)},\qquad n\neq 1}$$

For n=1, we have

$$a_1=\frac{2}{P}\int_0^P \sin\left(\frac{\pi x}{L}\right).\cos\left(\frac{\pi x}{P}\right)dx$$

$$\Rightarrow a_1=\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)dx$$

$$\Rightarrow a_1=\frac{1}{L}\int_0^L 2\sin\left(\frac{\pi x}{L}\right).\cos\left(\frac{\pi x}{L}\right)dx$$

$$\Rightarrow a_1=\frac{1}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)dx$$

$$\Rightarrow a_1=\frac{1}{L}\left[\frac{-\cos\left(\frac{2\pi x}{L}\right)}{\frac{2\pi}{L}}\right]_0^l$$

$$\Rightarrow a_1=\frac{-1}{2\pi}\left[\cos\left(\frac{2\pi x}{L}\right)\right]_0^L$$

$$\Rightarrow a_1=\frac{-1}{2\pi}[\cos 2\pi-\cos 0]=0$$

$$\Rightarrow \boxed{a_1=0}$$

Eq (1)$\Rightarrow$

$$\sin\left(\frac{\pi x}{L}\right)=\frac{a_0}{2}+a_1\cos\left(\frac{\pi x}{L}\right)+\sum_{n=2}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \sin\left(\frac{\pi x}{L}\right)=\frac{2}{\pi}+0.\cos\left(\frac{\pi x}{L}\right)+\sum_{n=2}^{\infty}\frac{-2\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)}\cos\left(\frac{n\pi x}{L}\right)$$

Since Fou...

$$\Rightarrow \sin\left(\frac{\pi x}{L}\right)=\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}\cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \sin\left(\frac{\pi x}{L}\right)=\frac{2}{\pi}-\frac{2}{\pi}\left[\frac{2}{2^2-1}\cos\left(\frac{2\pi x}{L}\right)+\frac{2}{4^2-1}\cos\left(\frac{4\pi x}{L}\right)+\cdots\right]$$

$$\Rightarrow \sin\left(\frac{\pi x}{L}\right)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{3}\cos\left(\frac{2\pi x}{L}\right)+\frac{1}{15}\cos\left(\frac{4\pi x}{L}\right)+\cdots\right] \text{Ans.}$$

QUESTION 14:

Let $f(x)=\cos x$ for $0<x<\pi$. If $f(x)$ is odd and periodic with period 2π. Show that the Fourier series for $f(x)$ is given by $f(x)=\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m.\sin(2mx)}{4m^2-1}$

SOLUTION: Given that $f(x)=\cos x, \quad 0<x<\pi$

Since $f(x)$ is odd with period $=2L=2\pi$, so its Fourier series involve only the sine terms, so we have

$$f(x)=\sum_{n=1}^{\infty}b_n.\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow f(x)=\sum_{n=1}^{\infty}b_n\sin(nx)\rightarrow(1) \quad (\because L=\pi)$$

Where $b_n=\frac{2}{L}\int_0^L f(x).\sin\left(\frac{n\pi x}{L}\right)dx$

$$\Rightarrow b_n=\frac{2}{\pi}\int_0^{\pi}\cos x.\sin(nx)dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_0^{\pi}2\sin(n\pi).\cos x.\,dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x+\sin(n-1)x]dx$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1}-\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$\Rightarrow b_n=\frac{-1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}-\frac{\cos 0}{n+1}-\frac{\cos 0}{n-1}\right]$$

$$\Rightarrow b_n=\frac{-1}{\pi}\left[\frac{\cos n\pi.\cos\pi}{n+1}+\frac{\cos n\pi.\cos\pi}{n-1}-\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{(-1)^n(-1)}{n+1}+\frac{(-1)^n(-1)}{n-1}-\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{(-1)^n}{n+1}+\frac{(-1)^n}{n-1}+\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$\Rightarrow b_n=\frac{1}{\pi}\left[\frac{(-1)^n+1}{n+1}+\frac{(-1)^n+1}{n-1}\right]$$

$$\Rightarrow b_n=\frac{1}{\pi}\left((-1)^n+1\right)\left(\frac{1}{n+1}+\frac{1}{n-1}\right)$$

$$\Rightarrow b_n=\frac{1}{\pi}\left((-1)^n+1\right)\left(\frac{2n}{n^2-1}\right), \quad n\neq 1$$

$$\Rightarrow b_n=\frac{2n\left((-1)^n+1\right)}{\pi\left(n^2-1\right)}, \quad n\neq 1$$

$$\Rightarrow b_n=\begin{cases}0 & \text{for } n=1,3,5,\cdots\\ \frac{4n}{\pi\left(n^2-1\right)} & \text{for } n=2,4,6\cdots\end{cases}$$

For n=1, we have

$$b_1=\frac{2}{\pi}\int_0^{\pi}\cos x.\sin x\,dx$$

$$\Rightarrow b_1=\frac{1}{\pi}\int_0^{\pi}2\sin x.\cos x\,dx$$

$$\Rightarrow b_1=\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$\Rightarrow b_1=\frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}=0$$

$$\Rightarrow \boxed{b_1=0}$$

$$\text{Thus} \Rightarrow b_n=\begin{cases}0 & \text{for } n=1,3,5,7\cdots\\ \frac{4n}{\pi\left(n^2-1\right)} & \text{for } n=2,4,6\cdots\end{cases}$$

Putting the valves in (1), we have

$$\cos x=\sum_{n=2,4,6\ldots}^{\infty}\frac{4n}{\pi\left(n^2-1\right)}\sin nx$$

$$\Rightarrow \cos x=\frac{4}{\pi}\sum_{n=2,4,6\ldots}^{\infty}\left(\frac{n}{n^2-1}\right)\sin nx$$

Substituting n=2m

So that $m=1,2,3,\cdots$ for $n=2,4,6,\cdots$

Then

$$\cos x=\frac{4}{\pi}\sum_{n=2,4,6\ldots}^{\infty}\left(\frac{n}{n^2-1}\right)\sin nx=\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{2m}{(2m)^2-1}\sin 2m\pi$$

$$\Rightarrow \cos x=\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\sin 2m\pi$$

Which is the required result.