11·3

1423

(Exercise 11·3)

(1) Take Laplace Transform and Apply
$\mathcal{L}\{f(t)\} = sF(s) - f(0)$
and it calculated
(2) Apply initial condition
(3) above eq. will be converted into algebric Eq in $Y(s)$
(4) apply inverse Laplace Transform

Use the Laplace transform method to solve the following initial value problems:

Q1 $\frac{dy}{dt} - ky = ce^{kt}$; $y(0) = 0$

Sol: Given eq. is

$$\frac{dy}{dt} - ky = ce^{kt} \quad \text{——— (1)}$$

taking Laplace transform of both sides of (1) ~~1st step~~

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} - k\mathcal{L}\{y(t)\} = c\,\mathcal{L}\{e^{kt}\}$$

$$sY(s) - y(0) - kY(s) = c\cdot\frac{1}{s-k}$$

$$(s-k)Y(s) = \frac{c}{s-k} \quad ; \; y(0) = 0 \quad \text{~~2nd step~~}$$

$$Y(s) = \frac{c}{(s-k)^2} \quad \text{3rd}$$

then $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{c}{(s-k)^2}\right\}$ 4th step

$$y(t) = c\,\mathcal{L}^{-1}\left\{\frac{1}{(s-k)^2}\right\}$$

$$\boxed{y(t) = cte^{kt}} \quad \text{is the req. soln}$$

Q2 $\frac{dy}{dt} + 4y = 2e^{t} - 4e^{-t}$; $y(0) = 0$

Sol: Given eq. is

$$\frac{dy}{dt} + 4y = 2e^{t} - 4e^{-t} \quad \text{——— (1)}$$

taking Laplace transform of both sides of (1).

1424

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{y(t)\} = 2\mathcal{L}\{e^{t}\} - 4\mathcal{L}\{e^{-t}\}$$

$$sY(s) - y(0) + 4Y(s) = 2\cdot\frac{1}{s-1} - 4\cdot\frac{1}{s+1}$$

$$(s+4)Y(s) = \frac{2}{s-1} - \frac{4}{s+1} \qquad \because y(0) = 0$$

$$= \frac{2s+2-4s+4}{(s-1)(s+1)}$$

$$(s+4)Y(s) = \frac{-2s+6}{(s-1)(s+1)}$$

$$\Rightarrow \quad Y(s) = \frac{-2s+6}{(s-1)(s+1)(s+4)} \quad \text{———} \quad (A)$$

Consider

$$\frac{-2s+6}{(s-1)(s+1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+4}$$

or $-2s+6 = A(s+1)(s+4) + B(s-1)(s+4) + C(s-1)(s+1)$ —— I

For A, put $s = 1$ in I

$$-2+6 = A(1+1)(1+4)$$

$$4 = A(2)(5) \Rightarrow \boxed{A = \frac{2}{5}}$$

For B, put $s = -1$ in I

$$2+6 = B(-2)(3)$$

$$8 = B(-6) \Rightarrow \boxed{B = -\frac{4}{3}}$$

For C, put $s = -4$ in I

$$8+6 = C(-5)(-3)$$

$$14 = 15C \Rightarrow \boxed{C = \frac{14}{15}}$$

1425 83

S.

$$\frac{-2S+6}{(S-1)(S+1)(S+4)} = \frac{2}{5(S-1)} - \frac{4}{3(S+1)} + \frac{14}{15(S+4)}$$

Put in (A)

$$Y(S) = \frac{2}{5(S-1)} - \frac{4}{3(S+1)} + \frac{14}{15(S+4)}$$

Now

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{S+1}\right\} + \frac{14}{15}\mathcal{L}^{-1}\left\{\frac{1}{S+4}\right\}$$

$$y(t) = \frac{2}{5}e^{t} - \frac{4}{3}e^{-t} + \frac{14}{15}e^{-4t}$$ is req. soln.

Q3 $\frac{dy}{dt} + y = f(t)$ where $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 5 & \text{if } t \ge 1 \end{cases}$; $y(0)=0$

Soln Given eq. is

$$\frac{dy}{dt} + y = f(t) \quad \text{———} \quad (1)$$

taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$$

$$SY(s) - y(0) + Y(S) = \mathcal{L}\{5u_1(t)\}$$

$$(S+1)Y(S) = 5\frac{e^{-s}}{s}$$

$$Y(s) = 5\frac{e^{-s}}{s(s+1)}$$

$$= 5e^{-s}\left[\frac{1}{s(s+1)}\right]$$

$$= 5e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$\because y(0)=0$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^\infty e^{-st}(5)\,dt$$

$$= 0 + 5\int_1^\infty e^{-st}\,dt$$

$$= 5\left.\frac{e^{-st}}{-s}\right|_1^\infty$$

$$= -\frac{5}{s}\left\{0 - e^{-s}\right\}$$

$$= \frac{5}{s}e^{-s}$$

1426

34

$$Y(s) = 5\frac{e^{-s}}{s} - 5\frac{e^{-s}}{s+1}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = 5\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} - 5\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+1}\right\}$$

$$y(t) = 5u_1(t) - 5u_1(t)\,e^{-(t-1)}$$

$$= 5u_1(t)\left(1 - e^{-(t-1)}\right)$$ is req. soln.

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m/ Q4. $\frac{dy}{dt} + 2y = f(t)$ where $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$ $y(0) = 0$

Sol. Given eq. is

$$\frac{dy}{dt} + 2y = f(t) \quad \text{——— (1)}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Here $f(t) = t - t\,u_1(t)$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$$= t - (t-1+1)u_1(t)$$

$$= \int_0^1 e^{-st}\,t\,dt + 0$$

$$= t - u_1(t)(t-1) - u_1(t)$$

$$= t\frac{e^{-st}}{-s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt$$

so from (1)

$$\frac{dy}{dt} + 2y = t - u_1(t)(t-1) - u_1(t)$$

$$= \frac{e^{-s}}{-s} - 0 + \frac{1}{s}\int_0^1 e^{-st}\,dt$$

taking Laplace transform of both sides

$$\frac{e^{-s}}{-s} + \frac{1}{s}\frac{e^{-st}}{-s}\Big|_0^1$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{t - u_1(t)(t-1) - u_1(t)\}$$

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$\frac{e^{-s}}{-s} - \frac{1}{s^2}\left(e^{-s} - 1\right)$$

or $$(s+2)Y(s) = \frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$\frac{-e^{-s}}{s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}$$

$$(s+2)Y(s) = \frac{1}{s^2} - e^{-s}\left[\frac{1+s}{s^2}\right]$$

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{e^{-s}}{s}$$

1427

85

$$Y(s) = \frac{1}{s^2(s+2)} - e^{-s}\left(\frac{s+1}{s^2(s+2)}\right) \quad \text{——— (A)}$$

Consider

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

or $1 = As(s+2) + B(s+2) + Cs^2$

For B put $s = 0$

$1 = B(2) \Rightarrow \boxed{B = \frac{1}{2}}$

For C, put $s = -2$

$1 = C(4) \Rightarrow \boxed{C = \frac{1}{4}}$

For A, equating coff. of s^2

$A + C = 0$

$A + \frac{1}{4} = 0 \Rightarrow \boxed{A = -\frac{1}{4}}$

So $\frac{1}{s^2(s+2)} = \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}$

Now Consider

$$\frac{s+1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

or $s+1 = As(s+2) + B(s+2) + Cs^2$

For B, put $s = 0$

$1 = 2B \Rightarrow \boxed{B = \frac{1}{2}}$

For C put $s = -2$

$-1 = C(4) \Rightarrow \boxed{C = -\frac{1}{4}}$

For A, equating coff. of s^2

$A + C = 0 \Rightarrow A - \frac{1}{4} = 0$ or $\boxed{A = \frac{1}{4}}$

So

$$\frac{s+1}{s^2(s+2)} = \frac{1}{4s} + \frac{1}{2s^2} - \frac{1}{4(s+2)}$$

Putting in (A)

$$Y(s) = \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} - e^{-s}\left(\frac{1}{4s} + \frac{1}{2s^2} - \frac{1}{4(s+2)}\right)$$

$$Y(s) = \frac{-1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} - \frac{e^{-s}}{4s} - \frac{e^{-s}}{2s^2} + \frac{e^{-s}}{4(s+2)}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{-1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s^2}\right\}$$
$$+ \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1}{s+2}\right\}$$

$$y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}u_1(t) - \frac{1}{2}u_1(t)(t-1) + \frac{1}{4}u_1(t)\cdot e^{-2(t-1)}$$

$$= -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}u_1(t)\left(1 - 2t + e^{-2(t-1)}\right) \text{ is req. soln.}$$

Q5 $\quad \frac{dy}{dt} = \cos t + \int_0^t y(u)\cos(t-u)\,du \qquad y(0) = 1$

Sol. Given eq. is

$$\frac{dy}{dt} = \cos t + \int_0^t y(u)\cdot\cos(t-u)\,du \quad \text{——— (1)}$$

Let $f(t) = y(t)$ & $g(t) = \cos t$

then $f * g = \int_0^t y(u)\cdot\cos(t-u)\,du$

then

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t y(u)\cdot\cos(t-u)\,du\right\}$$

$$= Y(s)\cdot G(s)$$

where $Y(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{y(t)\}$ & $G(s) = \mathcal{L}\{g(t)\} = \frac{s}{s^2+1}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t)\, f(t-a)$

1429

37

So
$$\mathcal{L}\left\{\int_0^t y(u)\cdot \cos(t-u)\,du\right\} = Y(s)\cdot\frac{s}{s^2+1}$$

taking Laplace transform of both sides of ①

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{\cos t\} + \mathcal{L}\left\{\int_0^t y(u)\cdot\cos(t-u)\,du\right\}$$

$$sY(s) - y(0) = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$$

$$sY(s) - 1 = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$$

$$sY(s) - Y(s)\cdot\frac{s}{s^2+1} = \frac{s}{s^2+1} + 1$$

$$Y(s)\left(s - \frac{s}{s^2+1}\right) = \frac{s+s^2+1}{s^2+1}$$

$$Y(s)\left(\frac{s^3+\cancel{s}-\cancel{s}}{s^2+1}\right) = \frac{s^2+s+1}{s^2+1}$$

$$Y(s)(s^3) = s^2+s+1$$

$$Y(s) = \frac{s^2+s+1}{s^3}$$

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}$$

then $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$

$y(t) = 1 + t + \frac{1}{2}t^2$ is req. sol.

Q6) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = e^t$ $\quad y(0) = 1,\ y'(0) = 0$

Sol. Given $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = e^t$ ——— ①

1432

9.

(Q7)

$$\frac{d^2y}{dt^2} + y = \cos t \qquad y(0) = 0 \;,\; y'(0) = -1$$

Sol. Given eq. is

$$\frac{d^2y}{dt^2} + y = \cos t \quad \text{———} \quad (1)$$

taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\cos t\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{s}{s^2+1}$$

$$s^2Y(s) - s(0) + 1 + Y(s) = \frac{s}{s^2+1}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+1} - 1$$

$$Y(s) = \frac{s}{(s^2+1)^2} - \frac{1}{s^2+1}$$

$$Y(s) = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) - \frac{1}{s^2+1}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{(-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Rightarrow y(t) = \frac{1}{2}t\sin t - \sin t$$ is req. soln.

(Q8) $\frac{d^2y}{dt^2} + y = 4t\sin t \qquad y(0) = 0 = y'(0)$

Sol. Given

$$\frac{d^2y}{dt^2} + y = 4t\sin t \quad \text{———} \quad (1)$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{4t\sin t\}$$

1433

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4(-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$s^2Y(s) - 0 - 0 + Y(s) = -4 \cdot \frac{-1}{(s^2+1)^2} \cdot 2s$$

$$(s^2+1)Y(s) = \frac{8s}{(s^2+1)^2}$$

$$Y(s) = \frac{8s}{(s^2+1)^2} \cdot \frac{1}{s^2+1} \quad \text{or} \quad Y(s) = \frac{8s}{(s^2+1)^3}$$

Let $F(s) = \frac{8s}{(s^2+1)^2}$ & $G(s) = \frac{1}{s^2+1}$ $\qquad \mathcal{L}\{t f(t)\} = (-1)^1 \frac{d}{ds} \mathcal{L}\{f(t)\}$

then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{8s}{(s^2+1)^2}\right\} = 4\mathcal{L}^{-1}\left\{(-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} = 4t\sin t = f(t)$

& $\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = g(t)$

then

$$\mathcal{L}^{-1}\{Y(s)\} = f(t) * g(t)$$

$$\text{or } y(t) = \int_0^t 4u\sin u \sin(t-u)\,du$$

$$= -2\int_0^t -2u\sin u \cdot \sin(t-u)\cdot du$$

$$= -2\int_0^t u[\cos(u+t-u) - \cos(u-t+u)]\,du$$

$$= -2\int_0^t u[\cos t - \cos(2u-t)]\,du$$

$$= 2\int_0^t u[\cos(2u-t) - \cos t]\,du$$

$$= 2\int_0^t \underbrace{u\cdot\cos(2u-t)\,du}_{\text{integ. by parts}} - 2\int_0^t u\cos t\,du$$

1434

$$= 2\left[\left| u \cdot \frac{\sin(2u-t)}{2} \right|_0^t - \int_0^t \frac{\sin(2u-t)}{2} \cdot 1\, du \right] - 2\cos t \int_0^t u\, du$$ 12

$$= \left| u \sin(2u-t) \right|_0^t - \int_0^t \sin(2u-t)\, du - 2\cos t \left| \frac{u^2}{2} \right|_0^t$$

$$= t\sin(2t-t) - 0 - \left| \frac{\cos(2u-t)}{2} \right|_0^t - 2\cos t \left(\frac{t^2}{2} \right)$$

$$= t\sin t - \frac{1}{2}\left[\cos(2t-t) - \cos(-t) \right] - t^2\cos t$$

$$= t\sin t - \frac{1}{2}\left(\cancel{\cos t} - \cancel{\cos t} \right) - t^2\cos t$$

$y(t) = t\sin t - t^2\cos t$ is req. soln.

Q9 $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = 20e^{-t}\cos t \qquad y(0) = 0 = y'(0)$

Soln. Given eq. is

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = 20e^{-t}\cos t \quad \text{———} \quad (1)$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} - 2\mathcal{L}\left\{ \frac{dy}{dt} \right\} = 20\mathcal{L}\{ e^{-t}\cos t \}$$

$$s^2Y(s) - s\,y(0) - y'(0) - 2\left(sY(s) - y(0) \right) = 20\frac{s+1}{(s+1)^2 + (1)^2}$$

$$s^2Y(s) - 0 - 0 - 2\left(sY(s) - 0 \right) = 20\frac{s+1}{(s+1)^2 + (1)^2}$$

$$(s^2 - 2s)Y(s) = 20\frac{s+1}{s^2+2s+2}$$

$$Y(s) = 20\frac{s+1}{(s^2-2s)(s^2+2s+2)}$$

$$Y(s) = \frac{20s+20}{s(s-2)(s^2+2s+2)} \quad \text{———} \quad (A)$$

1435

Consider

$$\frac{20S+20}{S(S-2)(S^2+2S+2)} = \frac{A}{S} + \frac{B}{S-2} + \frac{CS+D}{S^2+2S+2}$$

$$\Rightarrow 20S+20 = A(S-2)(S^2+2S+2) + BS(S^2+2S+2) + (CS+D)(S^2-2S)$$

For A, put $S=0$

$$20 = A(-2)(2) \Rightarrow \boxed{A=-5}$$

For B, put $S=2$

$$40+20 = B(2)(4+4+2)$$

$$60 = B(20) \Rightarrow \boxed{B=3}$$

From above eq.

$$20S+20 = A(S^3+\cancel{2S^2}+2S-\cancel{2S^2}-4S-4) + B(S^3+2S^2+2S) + (CS^3-2CS^2+DS^2-2DS)$$

$$20S+20 = A(S^3-2S-4) + B(S^3+2S^2+2S) + (CS^3-2CS^2+DS^2-2DS)$$

Comparing coff. of S^3 & S^2

$$A+B+C = 0 \quad \text{——— I}$$

$$2B-2C+D = 0 \quad \text{——— II}$$

$$\text{I} \Rightarrow -5+3+C=0 \Rightarrow \boxed{C=2}$$

$$\text{II} \Rightarrow 2(3)-2(2)+D=0$$

$$6-4+D=0$$

$$2+D=0 \Rightarrow \boxed{D=-2}$$

So

$$\frac{20S+20}{S(S-2)(S^2+2S+2)} = -\frac{5}{S} + \frac{3}{S-2} + \frac{2S-2}{S^2+2S+2}$$

Put in (A)

$$Y(S) = -\frac{5}{S} + \frac{3}{S-2} + \frac{2(S-1)}{S^2+2S+2}$$

1436

$$Y(s) = -\frac{5}{s} + \frac{3}{s-2} + \frac{2(s+1)-4}{(s+1)^2+(1)^2}$$

Then

$$\mathcal{L}^{-1}\{Y(s)\} = -5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{2(s+1)-4}{(s+1)^2+(1)^2}\right\}$$

$$y(t) = -5(1) + 3e^{2t} + 2\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+(1)^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(1)^2}\right\}$$

or $y(t) = -5 + 3e^{2t} + 2e^{-t}\cos t - 4e^{-t}\sin t$ is req. soln.

Q.10 $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 12e^{-3t}\sin 2t$. $\quad y(0) = 1,\ y'(0) = 0$

Soln. Given eq. is

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 12e^{-3t}\sin 2t \quad \text{——— (1)}$$

taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dt}\right\} - 4\mathcal{L}\{y(t)\} = 12\mathcal{L}\{e^{-3t}\sin 2t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) - 4Y(s) = 12\frac{2}{(s+3)^2+(2)^2}$$

$$s^2Y(s) - s - 0 - 3(sY(s) - 1) - 4Y(s) = \frac{24}{s^2+6s+13}$$

$$s^2Y(s) - s - 3sY(s) + 3 - 4Y(s) = \frac{24}{s^2+6s+13}$$

$$(s^2-3s-4)Y(s) - s + 3 = \frac{24}{s^2+6s+13}$$

$$\Rightarrow (s^2-3s-4)Y(s) = s - 3 + \frac{24}{(s^2+6s+13)}$$

$$\Rightarrow Y(s) = \frac{s-3}{(s^2-3s-4)} + \frac{24}{(s^2-3s-4)(s^2+6s+13)}$$

From above eq.

$$24 = A(s^3+6s^2+13s+s^2+6s+13)+B(s^3+6s^2+13s-4s^2-24s-52)$$
$$+(Cs+D)(s^2-3s-4)$$

$$24 = A(s^3+7s^2+19s+13)+B(s^3+2s^2-11s-52)+C(s^3-3s^2-4s)$$
$$+D(s^2-3s-4)$$

Comparing coffs. of s^3 & s^2

$$A+B+C=0 \text{ ——— I}$$
$$7A+2B-3C+D=0 \text{ ——— II}$$

$$\text{I} \Rightarrow \frac{24}{265}-\frac{3}{5}+C=0$$

$$C = \frac{3}{5}-\frac{24}{265}$$
$$= \frac{159-24}{265}$$
$$C = \frac{135}{265} \quad \text{or} \quad \boxed{C = \frac{27}{53}}$$

$$\text{II} \Rightarrow 7\left(\frac{24}{265}\right)+2\left(-\frac{3}{5}\right)-3\left(\frac{27}{53}\right)+D=0$$

$$\frac{168}{265}-\frac{6}{5}-\frac{81}{53}+D=0$$

$$D = \frac{81}{53}+\frac{6}{5}-\frac{168}{265}$$
$$= \frac{405+318-168}{265}$$
$$D = \frac{555}{265} \quad \text{or} \quad \boxed{D = \frac{111}{53}}$$

So

$$\frac{24}{(s-4)(s+1)(s^2+6s+13)} = \frac{\frac{24}{265}}{s-4}+\frac{-\frac{3}{5}}{s+1}+\frac{\frac{27}{53}s+\frac{111}{53}}{s^2+6s+13}$$

1437

or $Y(s) = \dfrac{s-3}{(s-4)(s+1)} + \dfrac{24}{(s-4)(s+1)(s^2+6s+13)}$ —— (A) 95

Consider

$$\frac{s-3}{(s-4)(s+1)} = \frac{A}{s-4} + \frac{B}{s+1}$$

or $s-3 = A(s+1) + B(s-4)$

For A, put $s = 4$

$4-3 = A(4+1) \Rightarrow \boxed{A = \frac{1}{5}}$

For B, put $s = -1$

$-1-3 = B(-1-4)$

$-4 = -5B \Rightarrow \boxed{B = \frac{4}{5}}$

So

$$\frac{s-3}{(s-4)(s+1)} = \frac{1}{5(s-4)} + \frac{4}{5(s+1)}$$

Now Consider

$$\frac{24}{(s-4)(s+1)(s^2+6s+13)} = \frac{A}{s-4} + \frac{B}{s+1} + \frac{Cs+D}{s^2+6s+13}$$

$\Rightarrow 24 = A(s+1)(s^2+6s+13) + B(s-4)(s^2+6s+13) + (Cs+D)(s-4)(s+1)$

For A, put $s = 4$

$24 = A(5)(16+24+13)$

$24 = A(5)(53)$

$\boxed{A = \frac{24}{265}}$

For B, Put $s = -1$

$24 = B(-5)(1-6+13)$

$24 = B(-5)(8) \Rightarrow B = -\frac{24}{40}$

or $\boxed{B = -\frac{3}{5}}$

1439

$$\frac{24}{(s-4)(s+1)(s^2+6s+13)} = \frac{24}{265(s-4)} - \frac{3}{5(s+1)} + \frac{27s+111}{53(s^2+6s+13)} \quad 97$$

Put in (A)

$$Y(s) = \frac{1}{5(s-4)} + \frac{4}{5(s+1)} + \frac{24}{265(s-4)} - \frac{3}{5(s+1)} + \frac{27s+111}{53(s^2+6s+13)}$$

$$= \left(\frac{1}{5} + \frac{24}{265}\right)\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{1}{53}\,\frac{27s+111}{(s+3)^2+(2)^2}$$

$$= \left(\frac{53+24}{265}\right)\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{1}{53}\cdot\frac{27(s+3)+111-81}{(s+3)^2+(2)^2}$$

$$= \frac{77}{265}\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{1}{53}\,\frac{27(s+3)+30}{(s+3)^2+(2)^2}$$

$$Y(s) = \frac{77}{265}\cdot\frac{1}{s-4} + \frac{1}{5(s+1)} + \frac{27}{53}\cdot\frac{s+3}{(s+3)^2+(2)^2} + \frac{30}{53}\,\frac{1}{(s+3)^2+(2)^2}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{77}{265}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{27}{53}\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+(2)^2}\right\} + \frac{15}{53}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+(2)^2}\right\}$$

$$y(t) = \frac{77}{265}e^{4t} + \frac{1}{5}e^{-t} + \frac{27}{53}e^{-3t}\cos 2t + \frac{15}{53}e^{-3t}\sin 2t$$

Q11) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = u_3(t)$ $\qquad y(0)=0,\ y'(0)=1$

Sol. Given eq. is

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = u_3(t) \quad \text{———— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 4\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{u_3(t)\}$$

$$s^2Y(s) - s\,y(0) - y'(0) - 4\left(sY(s) - y(0)\right) + 4Y(s) = \frac{e^{-3s}}{s}$$

$$s^2Y(s) - 0 - 1 - 4sY(s) + 4Y(s) = \frac{e^{-3s}}{s}$$

1440

$$(s^2-4s+4)Y(s) = 1+\frac{e^{-3s}}{s}$$ 98

$$(s-2)^2 Y(s) = 1+\frac{e^{-3s}}{s}$$

then $$Y(s) = \frac{1}{(s-2)^2} + \frac{e^{-3s}}{s(s-2)^2}$$

$$\Rightarrow Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + e^{-3s}\left[\frac{1}{s(s-2)^2}\right] \quad \text{——— (A)}$$

Consider

$$\frac{1}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\Rightarrow 1 = A(s-2)^2 + Bs(s-2) + Cs$$

For A ; put $s=0$

$$1 = A(4) \Rightarrow \boxed{A=\frac{1}{4}}$$

For C, put $s=2$

$$1 = C(2) \Rightarrow \boxed{C=\frac{1}{2}}$$

For B, Comparing Coff. of s^2

$$A+B = 0$$

$$\frac{1}{4}+B = 0 \Rightarrow \boxed{B=-\frac{1}{4}}$$

So $$\frac{1}{s(s-2)^2} = \frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2}$$

Put in (A)

$$Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + e^{-3s}\left[\frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2}\right]$$

$$Y(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) + \frac{1}{4}\cdot\frac{e^{-3s}}{s} - \frac{1}{4}\cdot e^{-3s}\cdot\frac{1}{s-2} + \frac{1}{2}e^{-3s}\cdot\frac{1}{(s-2)^2}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{(-1)^1\cdot\frac{d}{ds}\left(\frac{1}{s-2}\right)\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{(s-2)^2}\right\}$$

1441

$$y(t) = te^{2t} + \frac{1}{4}U_3(t) - \frac{1}{4}U_3(t).e^{2(t-3)} + \frac{1}{2}U_3(t).(t-3).e^{2(t-3)}$$

$$y(t) = te^{2t} + \frac{1}{4}U_3(t)\left[1 - e^{2(t-3)} + 2(t-3).e^{2(t-3)}\right]$$ is req. soln.

Q12 $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t)$ where $f(t) = \begin{cases} 0 & \text{if } 0<t<2 \\ 3 & \text{if } 2<t<5 \\ 0 & \text{if } t>5 \end{cases}$

$y(0) = 0 = y'(0)$

Sol. Given eq. is

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = f(t) \quad \text{———} \quad (1)$$

Now f(t) satisfying the above conditions can be re-written as

$$f(t) = 3(U_2(t) - U_5(t))$$

or $f(t) = 3U_2(t) - 3U_5(t)$

Put value in (1)

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 3U_2(t) - 3U_5(t)$$

Taking Laplace transform of both sides

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{y(t)\} = 3\mathcal{L}\{U_2(t)\} - 3\mathcal{L}\{U_5(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = 3\frac{e^{-2s}}{s} - 3\frac{e^{-5s}}{s}$$

$$s^2Y(s) - 3sY(s) + 2Y(s) = 3\frac{e^{-2s}}{s} - 3\frac{e^{-5s}}{s}$$

$$(s^2 - 3s + 2)Y(s) = \frac{3e^{-2s}}{s} - \frac{3e^{-5s}}{s}$$

$$\Rightarrow Y(s) = \frac{3e^{-2s}}{s(s^2-3s+2)} - \frac{3e^{-5s}}{s(s^2-3s+2)}$$

1442

$$Y(s) = 3e^{-2s}\left[\frac{1}{s(s-1)(s-2)}\right] - 3e^{-5s}\left[\frac{1}{s(s-1)(s-2)}\right] \quad \text{——— (A)}$$

Consider

$$\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\Rightarrow \quad 1 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

For A, put $s = 0$

$$1 = A(-1)(-2) \quad \Rightarrow \boxed{A = \tfrac{1}{2}}$$

For B; put $s = 1$

$$1 = B(1)(-1) \quad \Rightarrow \boxed{B = -1}$$

For C, put $s = 2$

$$1 = C(2)(2-1)$$

$$1 = 2C \quad \Rightarrow \boxed{C = \tfrac{1}{2}}$$

So

$$\frac{1}{s(s-1)(s-2)} = \frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}$$

Put in (A)

$$Y(s) = 3e^{-2s}\left[\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right] - 3e^{-5s}\left[\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right]$$

$$Y(s) = \frac{3}{2}\cdot\frac{e^{-2s}}{s} - \frac{3e^{-2s}}{s-1} + \frac{3}{2}\cdot\frac{e^{-2s}}{s-2} - \frac{3}{2}\cdot\frac{e^{-5s}}{s} + \frac{3e^{-5s}}{s-1} - \frac{3}{2}\cdot\frac{e^{-5s}}{s-2}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-2}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s-1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s-2}\right\}$$

$$y(t) = \frac{3}{2}U_2(t) - 3U_2(t)e^{t-2} + \frac{3}{2}U_2(t)e^{2(t-2)} - \frac{3}{2}U_5(t) + 3U_5(t)e^{t-5} - \frac{3}{2}U_5(t)e^{2(t-5)}$$

$$y(t) = \frac{3}{2}U_2(t)\left[1 - 2e^{t-2} + e^{2(t-2)}\right] - \frac{3}{2}U_5(t)\left[1 - 2e^{t-5} + e^{2(t-5)}\right]$$

is req. soln.

11·3-19

1443

Q13 $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2(t-3)U_3(t)$ $\quad y(0) = 2,\ y'(0) = 1$

Sol. Given eq. is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2(t-3)U_3(t) \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 2\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{y(t)\} = 2\mathcal{L}\{U_3(t)\cdot(t-3)\}$$

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + Y(s) = 2\cdot\frac{e^{-3s}}{s^2}$$

$$s^2Y(s) - 2s - 1 + 2\left(sY(s) - 2\right) + Y(s) = 2\frac{e^{-3s}}{s^2}$$

$$s^2Y(s) - 2s - 1 + 2sY(s) - 4 + Y(s) = 2\frac{e^{-3s}}{s^2}$$

$$(s^2+2s+1)Y(s) - 2s - 5 = 2\frac{e^{-3s}}{s^2}$$

$$(s+1)^2Y(s) = 2s+5 + 2\frac{e^{-3s}}{s^2}$$

$$\Rightarrow Y(s) = \frac{2s+5}{(s+1)^2} + \frac{2e^{-3s}}{s^2(s+1)^2}$$

$$\text{or } Y(s) = \frac{2s+5}{(s+1)^2} + e^{-3s}\left[\frac{2}{s^2(s+1)^2}\right] \quad \text{——— (A)}$$

Consider

$$\frac{2s+5}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$\Rightarrow 2s+5 = A(s+1) + B$$

For B, put $s = -1$

$$-2+5 = B \quad \Rightarrow \boxed{B = 3}$$

Comparing coff. of s

$$\boxed{A = 2}$$

So $\dfrac{2s+5}{(s+1)^2} = \dfrac{2}{s+1} + \dfrac{3}{(s+1)^2}$ 102

Now Consider

$$\frac{2}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$\Rightarrow 2 = As(s+1)^2 + B(s+1)^2 + Cs^2(s+1) + Ds^2 \quad \text{———— } (\alpha)$$

For B put $s=0$

$$2 = B(1) \Rightarrow \boxed{B=2}$$

For D, put $s=-1$

$$2 = D(1) \Rightarrow \boxed{D=2}$$

Comparing Coff. of s^3 & s^2

$$A + C = 0 \quad \text{———— I}$$

$$2A + B + C + D = 0 \quad \text{———— II}$$

$\text{I} \Rightarrow A = -C$

Put in II

$$-2C + B + C + D = 0$$

$$B - C + D = 0$$

$$2 - C + 2 = 0 \Rightarrow \boxed{C=4}$$

Put in I

$$A + 4 = 0 \Rightarrow \boxed{A=-4}$$

So $\dfrac{2}{s^2(s+1)^2} = -\dfrac{4}{s} + \dfrac{2}{s^2} + \dfrac{4}{s+1} + \dfrac{2}{(s+1)^2}$

Put in (A)

$$Y(s) = \frac{2}{s+1} + \frac{3}{(s+1)^2} + e^{-3s}\left[-\frac{4}{s} + \frac{2}{s^2} + \frac{4}{s+1} + \frac{2}{(s+1)^2}\right]$$

1445

or $Y(s) = \frac{2}{s+1} + \frac{3}{(s+1)^2} - 4\frac{e^{-3s}}{s} + 2\frac{e^{-3s}}{s^2} + 4\frac{e^{-3s}}{(s+1)} + 2\frac{e^{-3s}}{(s+1)^2}$

then

$$\mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)}\right\}$$
$$+ 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\}$$

$$y(t) = 2e^{-t} + 3te^{-t} - 4U_3(t) + 2tU_3(t) + 4U_3(t)e^{-(t-3)} + 2U_3(t)(t-3).e^{-(t-3)}$$

$$y(t) = 2e^{-t} + 3te^{-t} + U_3(t)\left[-4 + 2t + 4e^{-(t-3)} + 2(t-3).e^{-(t-3)}\right]$$
is req. soln.

Q14 $\frac{d^2y}{dt^2} + y = \begin{cases} \cos t & \text{if } 0 \le t < \pi/2 \\ 0 & \text{if } \frac{\pi}{2} < t < \infty \end{cases}$ $y(0) = 3$, $y'(0) = -1$

Soln: Given eq. is

$$\frac{d^2y}{dt^2} + y = f(t) \quad \text{———} \quad (1)$$

where $f(t) = \begin{cases} \cos t & \text{if } 0 \le t < \pi/2 \\ 0 & \text{if } \frac{\pi}{2} < t < \infty \end{cases}$

Now $f(t)$ can be written as

$$f(t) = \cos t + U_{\pi/2}(t)\sin(t - \pi/2)$$

Put in (1)

$$\frac{d^2y}{dt^2} + y = \cos t + U_{\pi/2}(t).\sin(t - \pi/2)$$

taking Laplace transform of both sides

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{U_{\pi/2}(t).\sin(t - \pi/2)\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{s}{s^2+1} + \frac{e^{-\pi/2\, s}}{s^2+1}$$

1446

$$s^2Y(s) - 3s + 1 + Y(s) = \frac{s}{s^2+1} + \frac{e^{-\frac{\pi}{2}s}}{s^2+1}$$ 1.4

$$(s^2+1)Y(s) - 3s + 1 = \frac{s}{s^2+1} + \frac{e^{-\pi/2\, s}}{s^2+1}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+1} + \frac{e^{-\pi/2\, s}}{s^2+1} + 3s - 1$$

$$\text{or} \quad Y(s) = \frac{s}{(s^2+1)^2} + \frac{e^{-\pi/2\, s}}{(s^2+1)^2} + \frac{3s-1}{(s^2+1)}$$

$$= \frac{3s-1}{s^2+1} + \frac{s}{(s^2+1)^2} + \frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}$$

$$Y(s) = \frac{3s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{2}(-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right) + \frac{e^{-\frac{\pi}{2}s}}{(s^2+1)^2}$$

Then

$$\mathcal{L}^{-1}\{Y(s)\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{(-1)^1\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$
$$+ \mathcal{L}^{-1}\left\{\frac{e^{-\pi/2\, s}}{(s^2+1)^2}\right\}$$

$$y(t) = 3\cos t - \sin t + \frac{1}{2}t\sin t + U_{\pi/2}(t)\, f(t-\pi/2)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

$$f(t) = \frac{1}{2}(\sin t - t\cos t) \qquad \because \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$$

$$\Rightarrow f(t-\pi/2) = \frac{1}{2}\left[\sin(t-\pi/2) - (t-\pi/2)\cos(t-\pi/2)\right]$$

Put in above eq.

$$y(t) = 3\cos t - \sin t + \frac{1}{2}t\sin t + U_{\pi/2}(t)\cdot\frac{1}{2}\left[\sin(t-\pi/2) - (t-\tfrac{\pi}{2})\cos(t-\pi/2)\right]$$

$$= 3\cos t - \sin t + \frac{1}{2}t\sin t - \frac{1}{2}U_{\pi/2}(t)\left[\sin(\pi/2 - t) + (t-\pi/2)\cos(\pi/2 - t)\right]$$

1447

or
$$y(t) = 3\cos t - \sin t + \frac{1}{2} t \sin t - \frac{1}{2} U_{\pi/2}(t)\left(\cos t + (t - \pi/2)\sin t\right)$$
is req. soln.

Q15 $\frac{d^2y}{dt^2} + 4y = \sin t - U_{2\pi}(t)\cdot\sin t$ $\quad y(0) = 0 = y'(0)$

Sol. Given eq. is

$$\frac{d^2y}{dt^2} + 4y = \sin t - U_{2\pi}(t)\sin t \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{U_{2\pi}(t)\sin t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4\left(sY(s) - y(0)\right) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$s^2Y(s) + 4sY(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$(s^2+4s)Y(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

$$\text{or } Y(s) = \frac{1}{(s^2+1)(s^2+4)} - e^{-2\pi s}\frac{1}{(s^2+1)(s^2+4)} \quad \text{——— (A)}$$

Consider

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow 1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

$$1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

Comparing Coffs. of both sides

$$A + C = 0 \quad \text{——— I}$$

48

1·6

$B + D = 0$ ———— II

$4A + C = 0$ ———— III

$4B + D = 1$ ———— IV

I $\Rightarrow A = -C$

Put in III

$-4C + C = 0$

$-3C = 0 \Rightarrow \boxed{C = 0}$

Put in I

$A + 0 = 0 \Rightarrow \boxed{A = 0}$

II $\Rightarrow B = -D$

Put in IV

$4(-D) + D = 1$

$-3D = 1 \Rightarrow \boxed{D = -\frac{1}{3}}$

Put in II

$B - \frac{1}{3} = 0 \Rightarrow \boxed{B = \frac{1}{3}}$

So

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{0.s + \frac{1}{3}}{s^2+1} + \frac{0.s - \frac{1}{3}}{s^2+4}$$

$$= \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}$$

Put in (A)

$$Y(s) = \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)} - e^{-2\pi s}\left(\frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}\right)$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}$$

$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} - \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1^2}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{2}{s^2+2^2}\right\}$$

1449

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}U_{2\pi}(t)\sin(t-2\pi) + \frac{1}{6}U_{2\pi}(t)\sin 2(t-2\pi)$$

$$= \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}U_{2\pi}(t)\left[2\sin t - \sin 2t\right]$$

$$= \frac{1}{6}(2\sin t - \sin 2t)(1 - U_{2\pi}(t))$$ is req. soln.

Q16 $\frac{d^3y}{dt^3} - 4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 6y = te^t$ $\quad y(0) = 0 = y'(0)$ & $y''(0) = 1$

Soln. Given eq. is

$$\frac{d^3y}{dt^3} - 4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 6y = te^t \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} - 4\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} + 6\mathcal{L}\{y(t)\} = \mathcal{L}\{te^t\}$$

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] - 4\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) + y(0)\right] + 6Y(s) = -\frac{d}{ds}\left(\frac{1}{s-1}\right)$$

$$\left[s^3Y(s) - 0 - 0 - 1\right] - 4\left[s^2Y(s) - 0 - 0\right] + sY(s) + 0 + 6Y(s) = -\frac{-1}{(s-1)^2}$$

$$s^3Y(s) - 1 - 4s^2Y(s) + sY(s) + 6Y(s) = \frac{1}{(s-1)^2}$$

$$(s^3 - 4s^2 + s + 6)Y(s) = 1 + \frac{1}{(s-1)^2}$$

Solve by S.D.

-1	1	-4	1	6
	0	-1	5	-6
	1	-5	6	0

So $s = -1$

Hence $(s+1)$ is one factor

So above eq. becomes

$$(s+1)(s^2 - 5s + 6)Y(s) = \frac{(s-1)^2 + 1}{(s-1)^2}$$

1450

$$(s+1)(s-2)(s-3)\,Y(s) = \frac{s^2-2s+2}{(s-1)^2}$$ 1.8

or $$Y(s) = \frac{s^2-2s+2}{(s-1)^2(s+1)(s-2)(s-3)} \quad \text{———} \; (2)$$

We resolve it into partial fractions.

Consider

$$\frac{s^2-2s+2}{(s-1)^2(s+1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s-2)} + \frac{E}{(s-3)}$$

Multiplying both sides by $(s-1)^2(s+1)(s-2)(s-3)$

$$s^2-2s+2 = A(s-1)(s+1)(s-2)(s-3) + B(s+1)(s-2)(s-3) + C(s-1)^2(s-2)(s-3) + D(s-1)^2(s+1)(s-3) + E(s-1)^2(s+1)(s-2)$$

For B, put $s = 1$

$$1-2+2 = B(2)(-1)(-2)$$

$$1 = 4B \Rightarrow \boxed{B = \frac{1}{4}}$$

For C put $s = -1$

$$1+2+2 = C(4)(-3)(-4)$$

$$5 = 48C \Rightarrow \boxed{C = \frac{5}{48}}$$

For D, put $s = 2$

$$4-4+2 = D(1)(3)(-1)$$

$$2 = -3D \Rightarrow \boxed{D = -\frac{2}{3}}$$

For E, put $s = 3$

$$9-6+2 = E(4)(4)(1) \quad \text{or} \quad 5 = 16E$$

$$\Rightarrow \boxed{E = 5/16}$$

For A, Comparing coff. of s^4

$$A + C + D + E = 0$$

or $$A + \frac{5}{48} - \frac{2}{3} + \frac{5}{16} = 0$$

$$A = \frac{2}{3} - \frac{5}{48} - \frac{5}{16}$$

$$= \frac{32 - 5 - 15}{48}$$

$$A = \frac{12}{48}$$

$$\Rightarrow \boxed{A = \frac{1}{4}}$$

So from B

$$Y(s) = \frac{1}{4(s-1)} + \frac{1}{4(s-1)^2} + \frac{5}{48(s+1)} - \frac{2}{3(s-2)} + \frac{5}{16(s-3)}$$

then

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \frac{5}{48}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$y(t) = \frac{1}{4}e^{t} + \frac{1}{4}te^{t} + \frac{5}{48}e^{-t} - \frac{2}{3}e^{2t} + \frac{5}{16}e^{3t}$$ is req. soln.

Q17 $\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 7\frac{dy}{dt} - 3y = 20\sin t$ where

$y(0) = 0 = y'(0)$, $y''(0) = -2$

Sol. Given eq. is

$$\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 7\frac{dy}{dt} - 3y = 20\sin t \quad \text{—— (1)}$$

Taking Laplace transform of both sides of (1)

1452

110

$$\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} - 5\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 7\mathcal{L}\left\{\frac{dy}{dt}\right\} - 3\mathcal{L}\{y(t)\} = \mathcal{L}\{20\sin t\}$$

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] - 5\left[s^2Y(s) - sy(0) - y'(0)\right] + 7\left[sY(s) - y(0)\right] - 3Y(s) = 20\,\mathcal{L}\{\sin t\}$$

$$\left[s^3Y(s) - 0 - 0 + 2\right] - 5\left[s^2Y(s) - 0 - 0\right] + 7\left[sY(s) - 0\right] - 3Y(s) = 20\cdot\frac{1}{s^2+1}$$

$$s^3Y(s) + 2 - 5s^2Y(s) + 7sY(s) - 3Y(s) = \frac{20}{s^2+1}$$

$$(s^3 - 5s^2 + 7s - 3)\,Y(s) = -2 + \frac{20}{s^2+1}$$

Solve by S.D.

Hence we have from above

1	1	−5	7	−3
	0	1	−4	3
1	1	−4	3	0
	0	1	−3	
	1	−3	0	

$$(s-1)(s-1)(s-3)\,Y(s) = -2 + \frac{20}{s^2+1}$$

$$Y(s) = \frac{-2(s^2+1)+20}{(s-1)^2(s-3)(s^2+1)}$$

$$= \frac{-2(s^2+1-10)}{(s-1)^2(s-3)(s^2+1)}$$

$$= \frac{-2(s^2-9)}{(s-1)^2(s-3)(s^2+1)}$$

$$= \frac{-2\cancel{(s-3)}(s+3)}{(s-1)^2\cancel{(s-3)}(s^2+1)}$$

$$Y(s) = \frac{-2s-6}{(s-1)^2(s^2+1)} \qquad (2)$$

1453

Consider

$$\frac{-2s-6}{(s-1)^2(s^2+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1}$$

Multiplying both sides by $(s-1)^2(s^2+1)$

$$-2s-6 = A(s-1)(s^2+1) + B(s^2+1) + (Cs+D)(s-1)^2 \quad \text{———} \quad ③$$

For B, put $s=1$.

$$-8 = B(2) \implies \boxed{B=-4}$$

Now ③ $\implies$

$$-2s-6 = A(s^3+s-s^2-1) + B(s^2+1) + (Cs+D)(s^2-2s+1)$$

$$-2s-6 = A(s^3-s^2+s-1) + B(s^2+1) + Cs^3 - 2Cs^2 + Cs + Ds^2 - 2Ds + D$$

Comparing Coffs. of s^3, s^2, s, Const

$$A+C = 0 \quad \text{———} \quad \mathrm{I}$$

$$-A+B-2C+D = 0 \quad \text{———} \quad \mathrm{II}$$

$$A+C-2D = -2 \quad \text{———} \quad \mathrm{III}$$

$$-A+B+D = -6 \quad \text{———} \quad \mathrm{IV}$$

From I $A+C=0$, Put in III

$$0-2D = -2 \implies \boxed{D=1}$$

Put in IV

$$-A-4+1 = -6$$

$$-A-3 = -6$$

$$-A = -6+3$$

$$-A = -3$$

$$\boxed{A=3}$$

Put in I $3+C=0 \implies \boxed{C=-3}$

Hence from ②

$$Y(s) = \frac{3}{(s-1)} - \frac{4}{(s-1)^2} + \frac{-3s+1}{(s^2+1)}$$

1454

Now

$$L^{-1}\{Y(s)\} = 3L^{-1}\left\{\frac{1}{s-1}\right\} - 4L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{-3s+1}{s^2+1}\right\}$$

$$y(t) = 3L^{-1}\left\{\frac{1}{s-1}\right\} - 4L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{-3s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$y(t) = 3L^{-1}\left\{\frac{1}{s-1}\right\} - 4L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - 3L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$y(t) = 3e^{t} - 4te^{t} - 3\cos t + \sin t$ is req. soln

Q18 $\left(\frac{d^2}{dt^2} + 6\frac{d}{dt} + 7\right)^2 y = 0$, $y(0) = 0 = y'(0) = y''(0)$, $y'''(0) = 4\sqrt{2}$

Sol. Given eq. is

$$\left(\frac{d^2}{dt^2} + 6\frac{d}{dt} + 7\right)^2 y = 0$$

or $\left[\left(\frac{d^2}{dt^2}\right)^2 + \left(6\frac{d}{dt}\right)^2 + (7)^2 + 2\left(\frac{d^2}{dt^2}\right)\left(6\frac{d}{dt}\right) + 2\left(6\frac{d}{dt}\right)(7) + 2\left(\frac{d^2}{dt^2}\right)(7)\right] y = 0$

$$\left[\frac{d^4}{dt^4} + 36\frac{d^2}{dt^2} + 49 + 12\frac{d^3}{dt^3} + 84\frac{d}{dt} + 14\frac{d^2}{dt^2}\right] y = 0$$

$$\frac{d^4y}{dt^4} + 36\frac{d^2y}{dt^2} + 49y + 12\frac{d^3y}{dt^3} + 84\frac{dy}{dt} + 14\frac{d^2y}{dt^2} = 0$$

$$\frac{d^4y}{dt^4} + 12\frac{d^3y}{dt^3} + 50\frac{d^2y}{dt^2} + 84\frac{dy}{dt} + 49y = 0 \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

$$L\left\{\frac{d^4y}{dt^4}\right\} + 12L\left\{\frac{d^3y}{dt^3}\right\} + 50L\left\{\frac{d^2y}{dt^2}\right\} + 84L\left\{\frac{dy}{dt}\right\} + 49L\{y(t)\} = 0$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] + 12[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)]$$
$$+ 50(s^2Y(s) - sy(0) - y'(0)) + 84[sY(s) - y(0)] + 49Y(s) = 0$$

1455

$$\left[s^4 Y(s) - 0 - 0 - 0 - 4\sqrt{2}\right] + 12\left[s^3 Y(s) - 0 - 0 - 0\right] + 50\left[s^2 Y(s) - 0 - 0\right]$$ 113

$$+ 84\left[sY(s) - 0\right] + 49\,Y(s) = 0$$

$$s^4 Y(s) - 4\sqrt{2} + 12 s^3 Y(s) + 50 s^2 Y(s) + 84 s Y(s) + 49\, Y(s) = 0$$

$$(s^4 + 12 s^3 + 50 s^2 + 84 s + 49)\, Y(s) = 4\sqrt{2}$$

$$(s^2 + 6s + 7)^2\, Y(s) = 4\sqrt{2}$$

$$\Rightarrow \quad Y(s) = \frac{4\sqrt{2}}{(s^2+6s+7)^2} \quad \text{———} \; (\alpha)$$

To find $\mathcal{L}^{-1}\{Y(s)\}$, we use Convolution theorem.

Take $F(s) = \dfrac{1}{s^2+6s+7}$

& $G(s) = \dfrac{1}{s^2+6s+7}$

then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+6s+7}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+9-2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2-(\sqrt{2})^2}\right\}$$

$$= \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+3)^2-(\sqrt{2})^2}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-3t} \sinh\sqrt{2}t$$

So $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Also $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+6s+7}\right\}$

$$= \mathcal{L}^{-1}\{G(s)\} = g(t).$$

1456

114

$$\mathcal{L}^{-1}\left\{F(s)\cdot G(s)\right\} = f(t) * g(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+6s+7)^2}\right\} = \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t \frac{1}{\sqrt{2}}\cdot e^{-3u}\sinh\sqrt{2}u\cdot\frac{1}{\sqrt{2}}e^{-3(t-u)}\cdot\sinh\sqrt{2}(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-3t}\cdot\sinh\sqrt{2}u\cdot\sinh\sqrt{2}(t-u)\,du$$

$$= \frac{1}{4}\int_0^t e^{-3t}\left[2\sinh\sqrt{2}u\cdot\sinh\sqrt{2}(t-u)\right]du$$

$$= \frac{1}{4}\int_0^t e^{-3t}\left[\cosh\left(\sqrt{2}u+\sqrt{2}(t-u)\right)-\cosh\left(\sqrt{2}u-\sqrt{2}(t-u)\right)\right]du$$

$$= \frac{1}{4}\int_0^t e^{-3t}\left[\cosh\sqrt{2}t-\cosh(2\sqrt{2}u-\sqrt{2}t)\right]du$$

$$= \frac{e^{-3t}}{4}\int_0^t\left[\cosh\sqrt{2}t-\cosh(2\sqrt{2}u-\sqrt{2}t)\right]du$$

$$= \frac{e^{-3t}}{4}\int_0^t\cosh\sqrt{2}t\,du - \frac{e^{-3t}}{4}\int_0^t\cosh(2\sqrt{2}u-\sqrt{2}t)\,du$$

$$= \frac{e^{-3t}}{4}\cosh\sqrt{2}t\,\Big|u\Big|_0^t - \frac{e^{-3t}}{4}\left|\frac{\sinh(2\sqrt{2}u-\sqrt{2}t)}{2\sqrt{2}}\right|_0^t$$

$$= \frac{e^{-3t}}{4}\cosh\sqrt{2}t\,(t-0) - \frac{e^{-3t}}{8\sqrt{2}}\left[\sinh(2\sqrt{2}t-\sqrt{2}t)-\sinh(-\sqrt{2}t)\right]$$

$$= \frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}t-\frac{1}{2\sqrt{2}}\left(\sinh\sqrt{2}t+\sinh\sqrt{2}t\right)\right]$$

$$= \frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}t-\frac{1}{2\sqrt{2}}\sinh\sqrt{2}t-\frac{1}{2\sqrt{2}}\sinh\sqrt{2}t\right]$$

1457

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+6s+7)^2}\right\} = \frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}\,t - \frac{2}{2\sqrt{2}}\sinh\sqrt{2}\,t\right]$$

$$\mathcal{L}^{-1}\left\{\frac{4\sqrt{2}}{(s^2+6s+7)^2}\right\} = 4\sqrt{2}\cdot\frac{e^{-3t}}{4}\left[t\cosh\sqrt{2}\,t - \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t\right]$$

$$\mathcal{L}^{-1}\{Y(s)\} = \sqrt{2}\,e^{-3t}\left[t\cosh\sqrt{2}\,t - \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t\right]$$

$$y(t) = e^{-3t}\left[\sqrt{2}\,t\cosh\sqrt{2}\,t - \sinh\sqrt{2}\,t\right]$$ is req. soln.

Q19 $\frac{d^4y}{dt^4} + 5\frac{d^2y}{dt^2} + 4y = 1 - U_\pi(t)$; $y(0) = 0 = y'(0) = y''(0) = y'''(0)$

Sol. Given eq. is

$$\frac{d^4y}{dt^4} + 5\frac{d^2y}{dt^2} + 4y = 1 - U_\pi(t) \quad \text{———} (1)$$

Taking Laplace transform of both sides of (1)

$$\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} + 5\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{1 - U_\pi(t)\}$$

$$\left[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right] + 5\left[s^2Y(s) - sy(0) - y'(0)\right] + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$(s^4Y(s) - 0 - 0 - 0 - 0) + 5(s^2Y(s) - 0 - 0) + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$s^4Y(s) + 5s^2Y(s) + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$(s^4 + 5s^2 + 4)Y(s) = \frac{1 - e^{-\pi s}}{s}$$

$$(s^2+1)(s^2+4)Y(s) = \frac{1 - e^{-\pi s}}{s}$$

1458

$$Y(s) = (1-e^{-\pi s})\left(\frac{1}{s(s^2+1)(s^2+4)}\right) \quad \text{——— ①}$$

Consider

$$\frac{1}{s(s^2+1)(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{s^2+4}$$

Multiplying both sides by $s(s^2+1)(s^2+4)$

$$1 = A(s^2+1)(s^2+4) + (Bs+C)s(s^2+4) + (Ds+E)(s)(s^2+1) \quad \text{——— Ⓐ}$$

For A, put $s=0$ in Ⓐ

$$1 = A(1)(4) \Rightarrow \boxed{A = \frac{1}{4}}$$

From Ⓐ

$$1 = A(s^4+5s^2+4) + (Bs+C)(s^3+4s) + (Ds+E)(s^3+s)$$

Equating Coffs. of like powers of s

$$A+B+D=0 \quad \text{——— I}$$

$$C+E=0 \quad \text{——— II}$$

$$5A+4B+D=0 \quad \text{——— III}$$

$$4C+E=0 \quad \text{——— IV}$$

II $\Rightarrow$ $C=-E$

Put in IV

$$-4E+E=0$$

$$-3E=0 \Rightarrow \boxed{E=0}$$

Hence $\boxed{C=0}$

Subt. I from III

$$4A+3B=0$$

$$4(1/4)+3B=0$$

$$1+3B=0$$

$$\boxed{B=-1/3}$$

Available at
www.mathcity.org

Put in I

$$\frac{1}{4} - \frac{1}{3} + D = 0$$

$$-\frac{1}{12} + D = 0$$

$$\boxed{D = \frac{1}{12}}$$

S.

$$Y(s) = (1 - e^{-\pi s})\left[\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right]$$

$$Y(s) = \left(\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right) - e^{-\pi s}\left(\frac{1}{4s} - \frac{s}{3(s^2+1)} + \frac{s}{12(s^2+4)}\right)$$

Then

$$L^{-1}\{Y(s)\} = \frac{1}{4}L^{-1}\{\frac{1}{s}\} - \frac{1}{3}L^{-1}\{\frac{s}{s^2+(1)^2}\} + \frac{1}{12}L^{-1}\{\frac{s}{(s)^2+(2)^2}\} - \frac{1}{4}L^{-1}\{e^{-\pi s}\cdot\frac{1}{s}\}$$
$$+ \frac{1}{3}L^{-1}\{e^{-\pi s}\cdot\frac{s}{s^2+(1)^2}\} - \frac{1}{12}L^{-1}\{\frac{s}{(s)^2+(2)^2}\}$$

$$y(t) = \left(\frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t\right) - \frac{1}{4}U_\pi(t) + \frac{1}{3}U_\pi(t)\cos(t-\pi) - \frac{1}{12}U_\pi(t)\cos 2(t-\pi)$$

$$= \left(\frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t\right) - U_{\pi(t)}\left(\frac{1}{4} - \frac{1}{3}\cos(t-\pi) + \frac{1}{12}\cos 2(t-\pi)\right)$$

$y(t) = f(t) - U_\pi(t) f(t-\pi)$ where $f(t) = \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t$

is req. soln.

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220 $\quad t\frac{d^2y}{dt^2} + (t-1)\frac{dy}{dt} - y = 0$

$$y(0) = 5, \quad y(\infty) = 0$$

Sol. Given eq. is

$$t\frac{d^2y}{dt^2} + (t-1)\frac{dy}{dt} - y = 0 \quad \text{——— (1)}$$

Taking Laplace transform of both sides of (1)

1460

$$\mathcal{L}\left\{t\frac{d^2y}{dt^2}\right\} + \mathcal{L}\left\{(t-1)\frac{dy}{dt}\right\} - \mathcal{L}\{y(t)\} = 0$$ 118

$$\mathcal{L}\left\{t\frac{d^2y}{dt^2}\right\} + \mathcal{L}\left\{t\frac{dy}{dt}\right\} - \mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\{y(t)\} = 0$$

$$-\frac{d}{ds}\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - \frac{d}{ds}\mathcal{L}\left\{\frac{dy}{dt}\right\} - \left(sY(s)-y(0)\right) - Y(s) = 0$$

$$\because \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$$

$$-\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) - \frac{d}{ds}\left(sY(s) - y(0)\right) - sY(s) + y(0) - Y(s) = 0$$

$$-\frac{d}{ds}\left(s^2Y(s) - 5s - y'(0)\right) - \frac{d}{ds}\left(sY(s) - 5\right) - sY(s) + 5 - Y(s) = 0$$

$$-\left[s^2Y'(s) + 2sY(s) - 5\right] - \left[sY'(s) + Y(s)\right] - sY(s) + 5 - Y(s) = 0$$

$$-s^2Y'(s) - 2sY(s) + 5 - sY'(s) - Y(s) - sY(s) + 5 - Y(s) = 0$$

$$-(s^2+s)Y'(s) - 3sY(s) - 2Y(s) + 10 = 0$$

$$-(s^2+s)Y'(s) - (3s+2)Y(s) + 10 = 0$$

$$(s^2+s)Y'(s) + (3s+2)Y(s) = 10$$

$$Y'(s) + \frac{3s+2}{s^2+s}Y(s) = \frac{10}{s^2+s}$$

$$Y'(s) + \frac{3s+2}{s(s+1)}Y(s) = \frac{10}{s(s+1)} \quad \text{———} \quad (2)$$

It is a linear diff. eq. in $Y(s)$

$$\text{I.F.} = e^{\int \frac{3s+2}{s(s+1)}ds}$$

Consider $\dfrac{3s+2}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

1461

119

Multiplying both sides by $S(S+1)$

$$3S+2 = A(S+1) + BS$$

For A, put $S=0$

$$2 = A(1) \Rightarrow \boxed{A=2}$$

For B, put $S=-1$

$$-3+2 = B(-1)$$

$$-1 = -B \Rightarrow \boxed{B=1}$$

So
$$\frac{3S+2}{S(S+1)} = \frac{2}{S} + \frac{1}{S+1}$$

So
$$\text{I.F.} = e^{\int\left(\frac{2}{S}+\frac{1}{S+1}\right)dS} = e^{2\ln S+\ln(S+1)} = e^{\ln S^2+\ln(S+1)} = e^{\ln S^2(S+1)} = S^2(S+1)$$

Multiplying both sides of (2) by I.F. $S^2(S+1)$

$$d\left(Y(s)\cdot S^2(S+1)\right) = \frac{10}{S(S+1)}\cdot S^2(S+1)\,dS$$

$$\int d\left(Y(s)\cdot S^2(S+1)\right) = \int 10S\,dS$$

$$Y(S)\cdot S^2(S+1) = 10\frac{S^2}{2} + C$$

or
$$Y(S)\cdot S^2(S+1) = 5S^2 + C$$

$$Y(S) = \frac{5S^2+C}{S^2(S+1)}$$

$$Y(S) = \frac{5}{S+1} + \frac{C}{S^2(S+1)}$$

or
$$Y(S) = \frac{5}{S+1} + C\left(\frac{1}{S^2(S+1)}\right) \quad \text{—— (3)}$$

Consider

1462

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$\Rightarrow 1 = As(s+1) + B(s+1) + Cs^2$

For B, put $s = 0$.

$1 = B(1) \Rightarrow \boxed{B = 1}$

For C, put $s = -1$

$1 = C(-1)^2 \Rightarrow \boxed{C = 1}$

For A, Comparing Coff. of s^2

$A + C = 0$

$\Rightarrow A + 1 = 0 \Rightarrow \boxed{A = -1}$

So $$\frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

Hence from (3)

$$Y(s) = \frac{5}{s+1} + c\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right)$$

$$= \frac{5}{s+1} - \frac{c}{s} + \frac{c}{s^2} + \frac{c}{s+1}$$

$$Y(s) = \frac{c+5}{s+1} - c\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

Now $$\mathcal{L}^{-1}\{Y(s)\} = (c+5)\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - c\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2}\right\}$$

$$y(t) = (c+5)e^{-t} - c(1-t)$$

Now $y(\infty) = 0 \Rightarrow y \to 0$ as $t \to \infty$

So $0 = 0 - c(1-t) \Rightarrow c(1-t) = 0 \Rightarrow c = 0$

Hence $y(t) = 5e^{-t}$ is req. soln.

1463

Q21 $\frac{dx}{dt} - x - 3y = 0$

$\frac{dy}{dt} - 5x - 3y = 0$ $\qquad x(0) = 2,\ y(0) = 1$

Sol. Given eqs. are

$$\left.\begin{aligned}\frac{dx}{dt} - x - 3y &= 0\\ \frac{dy}{dt} - 5x - 3y &= 0\end{aligned}\right]$$

Taking Laplace Transform of above eqs.

$$\left.\begin{aligned}\mathcal{L}\{\tfrac{dx}{dt}\} - \mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\} &= 0\\ \mathcal{L}\{\tfrac{dy}{dt}\} - 5\mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\} &= 0\end{aligned}\right]$$

or $$\left.\begin{aligned}sX(s) - x(0) - X(s) - 3Y(s) &= 0\\ sY(s) - y(0) - 5X(s) - 3Y(s) &= 0\end{aligned}\right]$$

$$\left.\begin{aligned}sX(s) - 2 - X(s) - 3Y(s) &= 0\\ sY(s) - 1 - 5X(s) - 3Y(s) &= 0\end{aligned}\right]$$

or $$\left.\begin{aligned}(s-1)X(s) - 3Y(s) - 2 &= 0\\ -5X(s) + (s-3)Y(s) - 1 &= 0\end{aligned}\right]$$

Solve by Cramer's rule

$$\frac{X(s)}{\begin{vmatrix}-3 & -2\\ s-3 & -1\end{vmatrix}} = \frac{-Y(s)}{\begin{vmatrix}s-1 & -2\\ -5 & -1\end{vmatrix}} = \frac{1}{\begin{vmatrix}s-1 & -3\\ -5 & s-3\end{vmatrix}}$$

$$\frac{X(s)}{3+2s-6} = \frac{-Y(s)}{-s+1-10} = \frac{1}{s^2-4s+3-15}$$

1464

$$\frac{X(s)}{2s-3} = \frac{-Y(s)}{-s-9} = \frac{1}{s^2-4s-12}$$

$$\frac{X(s)}{2s-3} = \frac{Y(s)}{s+9} = \frac{1}{s^2-4s-12}$$

$$\Rightarrow X(s) = \frac{2s-3}{s^2-4s-12}$$

$$\& \quad Y(s) = \frac{s+9}{s^2-4s-12}$$

or $$X(s) = \frac{2s-3}{(s+2)(s-6)} \quad \text{------} \quad (A)$$

$$Y(s) = \frac{s+9}{(s+2)(s-6)} \quad \text{------} \quad (B)$$

Consider $$\frac{2s-3}{(s+2)(s-6)} = \frac{A}{s+2} + \frac{B}{s-6}$$

$$\Rightarrow 2s-3 = A(s-6) + B(s+2)$$

For A, put $s = -2$

$$-4-3 = A(-8) \Rightarrow \boxed{A = \frac{7}{8}}$$

For B, put $s = 6$

$$12-3 = B(6+2) \Rightarrow \boxed{B = \frac{9}{8}}$$

So $$\frac{2s-3}{(s+2)(s-6)} = \frac{7}{8(s+2)} + \frac{9}{8(s-6)}$$

Now Consider

$$\frac{s+9}{(s+2)(s-6)} = \frac{A}{s+2} + \frac{B}{s-6}$$

1465

123

$$\Rightarrow s+9 = A(s-6) + B(s+2)$$

For A, put $s = -2$

$$-2+9 = A(-8) \Rightarrow \boxed{A = -\frac{7}{8}}$$

For B, put $s = 6$

$$6+9 = B(6+2) \Rightarrow \boxed{B = \frac{15}{8}}$$

So

$$\frac{s+9}{(s+2)(s-6)} = -\frac{7}{8(s+2)} + \frac{15}{8(s-6)}$$

∴ Put values in eqs. (A) & (B)

$$\left.\begin{aligned} X(s) &= \frac{7}{8(s+2)} + \frac{9}{8(s-6)} \\ Y(s) &= -\frac{7}{8(s+2)} + \frac{15}{8(s-6)} \end{aligned}\right]$$

Taking inverse Laplace transform

$$\mathcal{L}^{-1}\{X(s)\} = \frac{7}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{9}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{7}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{15}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$$

or

$$\left.\begin{aligned} x(t) &= \frac{7}{8}e^{-2t} + \frac{9}{8}e^{6t} \\ y(t) &= -\frac{7}{8}e^{-2t} + \frac{15}{8}e^{6t} \end{aligned}\right] \text{ is req. soln.}$$

Q22 $\frac{dx}{dt} - 4x - 5y = e^{-4t}$ $x(0) = 0$

$\frac{dy}{dt} + 4x + 4y = e^{4t}$ $y(0) = 0$

1466

1241

Sol: Given eqs. are

$$\left.\begin{aligned}\frac{dx}{dt} - 4x - 5y &= e^{-4t}\\ \frac{dy}{dt} + 4x + 4y &= e^{4t}\end{aligned}\right]$$

Taking Laplace transform of both sides of above eqs.

$$\left.\begin{aligned}\mathcal{L}\left\{\frac{dx}{dt}\right\} - 4\mathcal{L}\{x(t)\} - 5\mathcal{L}\{y(t)\} &= \mathcal{L}\{e^{-4t}\}\\ \mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{x(t)\} + 4\mathcal{L}\{y(t)\} &= \mathcal{L}\{e^{4t}\}\end{aligned}\right]$$

or
$$\left.\begin{aligned}SX(S) - x(0) - 4X(S) - 5Y(S) &= \frac{1}{S+4}\\ SY(S) - y(0) + 4X(S) + 4Y(S) &= \frac{1}{S-4}\end{aligned}\right]$$

$$\left.\begin{aligned}(S-4)X(S) - 5Y(S) &= \frac{1}{S+4}\\ 4X(S) + (S+4)Y(S) &= \frac{1}{S-4}\end{aligned}\right]$$

$$\left.\begin{aligned}(S-4)X(S) - 5Y(S) - \frac{1}{S+4} &= 0\\ 4X(S) + (S+4)Y(S) - \frac{1}{S-4} &= 0\end{aligned}\right]$$

Solve by cramer's rule

$$\frac{X(S)}{\begin{vmatrix}-5 & -\frac{1}{S+4}\\ S+4 & \frac{-1}{S-4}\end{vmatrix}} = \frac{-Y(S)}{\begin{vmatrix}S-4 & \frac{-1}{S+4}\\ 4 & \frac{-1}{S-4}\end{vmatrix}} = \frac{1}{\begin{vmatrix}S-4 & -5\\ 4 & S+4\end{vmatrix}}$$

$$\frac{X(S)}{\frac{5}{S-4} + 1} = \frac{-Y(S)}{-1 + \frac{4}{S+4}} = \frac{1}{S^2 - 16 + 20}$$

1467

125

$$\frac{X(s)}{\frac{s+s-4}{s+4}} = \frac{-Y(s)}{\frac{-s-4+4}{s+4}} = \frac{1}{s^2+4}$$

$$\frac{X(s)}{\frac{s+1}{s-4}} = \frac{Y(s)}{\frac{s}{s+4}} = \frac{1}{s^2+4}$$

$$\Rightarrow X(s) = \frac{s+1}{(s-4)(s^2+4)} \quad \text{———} \quad (A)$$

$$\& \quad Y(s) = \frac{s}{(s+4)(s^2+4)} \quad \text{———} \quad (B)$$

Consider $\dfrac{s+1}{(s-4)(s^2+4)} = \dfrac{A}{s-4} + \dfrac{Bs+C}{s^2+4}$

$$\Rightarrow s+1 = A(s^2+4) + (Bs+C)(s-4)$$

For A, put $s = 4$

$$4+1 = A(16+4)$$

$$5 = 20A \quad \Rightarrow \boxed{A = \frac{1}{4}}$$

Equating coffs. of s^2 & s

$$A + B = 0 \quad \text{———} \quad \mathrm{I}$$

$$-4B + C = 1 \quad \text{———} \quad \mathrm{II}$$

$\mathrm{I} \Rightarrow \frac{1}{4} + B = 0 \Rightarrow \boxed{B = -\frac{1}{4}}$

$\mathrm{II} \Rightarrow -4\left(-\frac{1}{4}\right) + C = 1$

$$1 + C = 1 \quad \Rightarrow \boxed{C = 0}$$

S. $\dfrac{s+4}{(s-4)(s^2+4)} = \dfrac{1}{4(s-4)} + \dfrac{-\frac{1}{4}s}{s^2+4}$

$$= \frac{1}{4(s-4)} - \frac{s}{4(s^2+4)}$$

Consider

$$\frac{S}{(S+4)(S^2+4)} = \frac{A}{S+4} + \frac{BS+C}{S^2+4}$$

$$\Rightarrow \quad S = A(S^2+4) + (BS+C)(S+4)$$

For A, put $S = -4$

$$-4 = A(16+4)$$

$$-4 = 20A \quad \Rightarrow \boxed{A = \frac{-1}{5}}$$

Equating Coffs. of S^2 & S

$$A + B = 0 \quad \text{——— I}$$

$$4B + C = 1 \quad \text{——— II}$$

$$\text{I} \Rightarrow \quad -\frac{1}{5} + B = 0 \quad \Rightarrow \boxed{B = \frac{1}{5}}$$

$$\text{II} \Rightarrow \quad 4\left(\frac{1}{5}\right) + C = 1$$

$$C = 1 - \frac{4}{5}$$

$$\boxed{C = \frac{1}{5}}$$

So

$$\frac{S}{(S+4)(S^2+4)} = \frac{-1}{5(S+4)} + \frac{\frac{1}{5}S + \frac{1}{5}}{S^2+4}$$

$$= \frac{-1}{5(S+4)} + \frac{S+1}{5(S^2+4)}$$

Put values in (A) & (B)

$$X(S) = \frac{1}{4(S+4)} - \frac{S}{4(S^2+4)}$$

$$Y(S) = \frac{-1}{5(S+4)} + \frac{S+1}{5(S^2+4)}$$

Taking inverse Laplace transform of above eqs.

1469

$$\left.\begin{aligned} L^{-1}\{X(s)\} &= \frac{1}{4}L^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{4}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} \\ L^{-1}\{Y(s)\} &= \frac{-1}{5}L^{-1}\left\{\frac{1}{s+4}\right\} + \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{s^2+4}\right\} \end{aligned}\right]$$

$$\left.\begin{aligned} \text{or } L^{-1}\{X(s)\} &= \frac{1}{4}L^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{4}L^{-1}\left\{\frac{s}{(s)^2+(2)^2}\right\} \\ L^{-1}\{Y(s)\} &= -\frac{1}{5}L^{-1}\left\{\frac{1}{s+4}\right\} + \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{1}{10}L^{-1}\left\{\frac{2}{s^2+2^2}\right\} \end{aligned}\right]$$

$$\left.\begin{aligned} x(t) &= \frac{1}{4}e^{4t} - \frac{1}{4}\cos 2t \\ y(t) &= -\frac{1}{5}e^{-4t} + \frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t \end{aligned}\right\} \text{ is req. soln.}$$

Q23 $2\frac{dx}{dt} + \frac{dy}{dt} - x - y = 1$ $\qquad x(0)=0$

$\frac{dx}{dt} + \frac{dy}{dt} + 2x - y = t$ $\qquad y(0) = 0$

Sol: Given eqs. are

$$\left.\begin{aligned} 2\frac{dx}{dt} + \frac{dy}{dt} - x - y &= 1 \\ \frac{dx}{dt} + \frac{dy}{dt} + 2x - y &= t \end{aligned}\right]$$

Taking Laplace transform of both sides of above eqs.

$$\left.\begin{aligned} 2L\left\{\frac{dx}{dt}\right\} + L\left\{\frac{dy}{dt}\right\} - L\{x(t)\} - L\{y(t)\} &= L\{1\} \\ L\left\{\frac{dx}{dt}\right\} + L\left\{\frac{dy}{dt}\right\} + 2L\{x(t)\} - L\{y(t)\} &= L\{t\} \end{aligned}\right]$$

$$\left.\begin{aligned} 2(sX(s) - x(0)) + sY(s) - y(0) - X(s) - Y(s) &= \frac{1}{s} \\ sX(s) - x(0) + sY(s) - y(0) + 2X(s) - Y(s) &= \frac{1}{s^2} \end{aligned}\right]$$

1470

128

$$\left.\begin{aligned} 2sX(s) + sY(s) - X(s) - Y(s) &= \frac{1}{s} \\ sX(s) + sY(s) + 2X(s) - Y(s) &= \frac{1}{s^2} \end{aligned}\right]$$

$$(2s-1)X(s) + (s-1)Y(s) = \frac{1}{s} \quad \text{———} \quad (A)$$

$$(s+2)X(s) + (s-1)Y(s) = \frac{1}{s^2} \quad \text{———} \quad (B)$$

Subt. (B) from (A)

$$(2s-1-s-2)X(s) = \frac{1}{s} - \frac{1}{s^2}$$

$$(s-3)X(s) = \frac{s-1}{s^2}$$

$$\boxed{X(s) = \frac{s-1}{s^2(s-3)}}$$

Put in (B)

$$(s+2)\frac{(s-1)}{s^2(s-3)} + (s-1)Y(s) = \frac{1}{s^2}$$

$$\frac{s+2}{s^2(s-3)} + Y(s) = \frac{1}{s^2(s-1)}$$

$$Y(s) = \frac{1}{s^2(s-1)} - \frac{s+2}{s^2(s-3)}$$

$$= \frac{1}{s^2}\left[\frac{1}{s-1} - \frac{s+2}{s-3}\right]$$

$$= \frac{1}{s^2}\left[\frac{s-3-(s-1)(s+2)}{(s-1)(s-3)}\right]$$

$$= \frac{1}{s^2}\left[\frac{\cancel{s}-3-s^2-\cancel{s}+2}{(s-1)(s-3)}\right]$$

$$= \frac{-s^2-1}{s^2(s-1)(s-3)}$$

1471

$$Y(s) = -\frac{s^2+1}{s^2(s-1)(s-3)} \quad \text{——— ②}$$

1-9

Consider

$$\frac{s-1}{s^2(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3}$$

$$\Rightarrow s-1 = As(s-3) + B(s-3) + Cs^2$$

For B, put $s=0$

$$-1 = -3B \Rightarrow \boxed{B = \frac{1}{3}}$$

For C, put $s=3$

$$2 = 9C \Rightarrow \boxed{C = \frac{2}{9}}$$

Equating coff. of s^2

$$A + C = 0$$

$$A + \frac{2}{9} = 0$$

$$\boxed{A = -\frac{2}{9}}$$

$$\frac{s-1}{s^2(s-3)} = -\frac{2}{9s} + \frac{1}{3s^2} + \frac{2}{9(s-3)}$$

Now consider

$$\frac{-s^2-1}{s^2(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3}$$

$$\Rightarrow -s^2-1 = As(s-1)(s-3) + B(s-1)(s-3) + Cs^2(s-3) + Ds^2(s-1)$$

For B, put $s=0$

$$-1 = B(-1)(-3) \Rightarrow \boxed{B = -\frac{1}{3}}$$

For C, put $s=1$

$$-1-1 = C(-2) \Rightarrow \boxed{C = 1}$$

1472

For D, put $s = 3$

$$-9-1 = D(9)(2)$$

$$-10 = 18D \Rightarrow \boxed{D = -\frac{5}{9}}$$

Equating Coff. of s^3

$$A + C + D = 0$$

$$\Rightarrow A + 1 - \frac{5}{9} = 0$$

$$A + \frac{4}{9} = 0 \Rightarrow \boxed{A = -\frac{4}{9}}$$

So

$$\frac{-s^2-1}{s^2(s-1)(s-3)} = -\frac{4}{9s} - \frac{1}{3s^2} + \frac{1}{s-1} - \frac{5}{9(s-3)}$$

Hence

$$X(s) = -\frac{2}{9s} + \frac{1}{3s^2} + \frac{2}{9(s-3)}$$

$$\& \quad Y(s) = -\frac{4}{9s} - \frac{1}{3s^2} + \frac{1}{s-1} - \frac{5}{9(s-3)}$$

Taking inverse Laplace transform of above eqs.

$$L^{-1}\{X(s)\} = -\frac{2}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{2}{9}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$L^{-1}\{Y(s)\} = -\frac{4}{9}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{5}{9}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$x(t) = -\frac{2}{9} + \frac{1}{3}t + \frac{2}{9}e^{3t}$$

$$y(t) = -\frac{4}{9} - \frac{1}{3}t + e^{t} - \frac{5}{9}e^{3t}$$ is req. soln.

Q24 *

$$\frac{dx}{dt} + \frac{dy}{dt} = t$$

$$\frac{d^2x}{dt^2} - y = e^{-t}$$

$$x(0) = 3, \quad x'(0) = -2, \quad y(0) = 0$$

1473

Sol. Given eqs. are

$$\frac{dx}{dt} + \frac{dy}{dt} = t$$

$$\frac{d^2x}{dt^2} - y = e^{-t}$$

Taking Laplace transform of both sides of above eqs.

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{t\}$$

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-t}\}$$

$$sX(s) - x(0) + sY(s) - y(0) = \frac{1}{s^2}$$

$$s^2X(s) - sx(0) - x'(0) - Y(s) = \frac{1}{s+1}$$

$$sX(s) - 3 + sY(s) - 0 = \frac{1}{s^2}$$

$$s^2X(s) - 3s + 2 - Y(s) = \frac{1}{s+1}$$

or

$$sX(s) + sY(s) = \frac{1}{s^2} + 3$$

$$s^2X(s) - Y(s) = \frac{1}{s+1} + 3s - 2$$

$$sX(s) + sY(s) = \frac{1+3s^2}{s^2}$$

$$s^2X(s) - Y(s) = \frac{1+(3s-2)(s+1)}{s+1}$$

$$sX(s) + sY(s) = \frac{1+3s^2}{s^2} \quad \text{——— (A)}$$

$$s^2X(s) - Y(s) = \frac{3s^2+s-1}{s+1} \quad \text{——— (B)}$$

Multiplying eq. (A) by s^2 & (B) by s

1474

$$s\,X(s) + s^3 Y(s) = 1+3s^2 \quad \text{———} \ (A)$$

$$s^3 X(s) - s\,Y(s) = \frac{3s^3+s^2-s}{s+1} \quad \text{———} \ (B)$$

Subt. (B) from (A)

$$s^3 Y(s) + s\,Y(s) = 3s^2+1 - \frac{3s^3+s^2-s}{s+1}$$

$$(s^3+s)Y(s) = \frac{(3s^2+1)(s+1)-(3s^3+s^2-s)}{(s+1)}$$

$$s(s^2+1)Y(s) = \frac{\cancel{3s^3}+3s^2+s+1-\cancel{3s^3}-s^2+s}{s+1}$$

$$s(s^2+1)Y(s) = \frac{2s^2+2s+1}{s+1}$$

$$\Rightarrow Y(s) = \frac{2s^2+2s+1}{s(s+1)(s^2+1)}$$

Now

Consider $\dfrac{2s^2+2s+1}{s(s+1)(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{Cs+D}{s^2+1}$

$$\Rightarrow 2s^2+2s+1 = A(s+1)(s^2+1) + Bs(s^2+1) + (Cs+D)(s^2+s)$$

For A, put $s=0$

$$1 = A(1) \quad \Rightarrow \boxed{A=1}$$

For B, put $s=-1$

$$2-2+1 = B(-1)(1+1)$$

$$1 = -2B \quad \Rightarrow \boxed{B=-\frac{1}{2}}$$

Comparing coeffs. of s^3 & s^2

$$A+B+C=0 \quad \text{———} \ \mathrm{I}$$

$$A+C+D=2 \quad \text{———} \ \mathrm{II}$$

$$\mathrm{I} \Rightarrow 1-\frac{1}{2}+C=0$$

1475

$\frac{1}{2} + C = 0 \Rightarrow \boxed{C = -\frac{1}{2}}$

$\text{II} \Rightarrow 1 - \frac{1}{2} + D = 2$

$\frac{1}{2} + D = 2$

$D = 2 - \frac{1}{2} \Rightarrow \boxed{D = \frac{3}{2}}$

So $Y(s) = \frac{1}{s} - \frac{1}{2(s+1)} + \frac{-\frac{1}{2}s + \frac{3}{2}}{s^2+1}$

$Y(s) = \frac{1}{s} - \frac{1}{2(s+1)} - \frac{s-3}{2(s^2+1)}$

Applying $\mathcal{L}^{-1}$ on both sides

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{s}{s^2+1} - \frac{3}{s^2+1}\}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{s}{s^2+1^2}\} + \frac{3}{2}\mathcal{L}^{-1}\{\frac{1}{s^2+1^2}\}$

$\boxed{y(t) = 1 - \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{3}{2}\sin t}$

Diff. w.r.t. t

$\frac{dy}{dt} = \frac{1}{2}e^{-t} + \frac{1}{2}\sin t + \frac{3}{2}\cos t$

Put in given eq

$\frac{dx}{dt} + \frac{1}{2}e^{-t} + \frac{1}{2}\sin t + \frac{3}{2}\cos t = t$

$\frac{dx}{dt} = t - \frac{1}{2}e^{-t} - \frac{1}{2}\sin t - \frac{3}{2}\cos t$

Integ. w.r.t. t

$x(t) = \frac{t^2}{2} + \frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{3}{2}\sin t + C$

But $x(0) = 3$

So $3 = 0 + \frac{1}{2} + \frac{1}{2} - 0 + C$

or $3 = 1 + c \Rightarrow \boxed{c = 2}$

So reqd. soln. is

$$\left.\begin{aligned} x(t) &= \frac{t^2}{2} + \frac{1}{2}e^{-t} \ [illegible] \cos t - \frac{1}{2}\sin t + 2 \\ y(t) &= 1 - \frac{1}{2}e^{-t} - \ [illegible] + \frac{3}{2}\sin t \end{aligned}\right]$$

Q25 $\dfrac{dx}{dt} + 2\dfrac{d^2y}{dt^2} = e^{-t}$

$\dfrac{dx}{dt} + 2x - y = 1$ $\qquad x(0) = 0 = y(0) = y'(0)$

Sol. Given eqs. are

$$\left.\begin{aligned} \frac{dx}{dt} + 2\frac{d^2y}{dt^2} &= e^{-t} \\ \frac{dx}{dt} + 2x - y &= 1 \end{aligned}\right]$$

Taking Laplace transform of both sides of above eqs.

$$\left.\begin{aligned} \mathcal{L}\left\{\frac{dx}{dt}\right\} + 2\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= \mathcal{L}\{e^{-t}\} \\ \mathcal{L}\left\{\frac{dx}{dt}\right\} + 2\mathcal{L}\{x(t)\} - \mathcal{L}\{y(t)\} &= \mathcal{L}\{1\} \end{aligned}\right]$$

$$\left.\begin{aligned} sX(s) - x(0) + 2\left(s^2Y(s) - sy(0) - y'(0)\right) &= \frac{1}{s+1} \\ sX(s) - x(0) + 2X(s) - Y(s) &= \frac{1}{s} \end{aligned}\right]$$

or
$$\left.\begin{aligned} sX(s) - 0 + 2s^2Y(s) &= \frac{1}{s+1} \\ sX(s) - 0 + 2X(s) - Y(s) &= \frac{1}{s} \end{aligned}\right]$$

$$\left.\begin{aligned} sX(s) + 2s^2Y(s) &= \frac{1}{s+1} \quad \text{—— (A)} \\ (s+2)X(s) - Y(s) &= \frac{1}{s} \quad \text{—— (B)} \end{aligned}\right]$$

Multiplying eq. (B) by s^2

$$s X(s) + 2s^2 Y(s) = \frac{1}{s+1} \quad \text{———} \quad (A)$$

$$2s^2(s+2) X(s) - 2s^2 Y(s) = 2s \quad \text{———} \quad (B)$$

Adding (A) + (B)

$$\left[s + 2s^2(s+2)\right] X(s) = 2s + \frac{1}{s+1}$$

$$s\left(1 + 2s(s+2)\right) X(s) = \frac{2s^2+2s+1}{s+1}$$

$$s\left(1 + 2s^2 + 4s\right) X(s) = \frac{2s^2+2s+1}{s+1}$$

$$X(s) = \frac{2s^2+2s+1}{s(s+1)(2s^2+4s+1)}$$

Now Consider

$$\frac{2s^2+2s+1}{s(s+1)(2s^2+4s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{2s^2+4s+1}$$

$$\Rightarrow 2s^2+2s+1 = A(s+1)(2s^2+4s+1) + Bs(2s^2+4s+1) + (Cs+D)(s^2+s)$$

For A, put $s = 0$

$$1 = A(1) \Rightarrow \boxed{A = 1}$$

For B, put $s = -1$

$$2 - 2 + 1 = B(-1)(2-4+1)$$

$$1 = B(-1)(-1) \Rightarrow \boxed{B = 1}$$

Comparing Coff of s^3 & s^2

$$2A + 2B + C = 0 \quad \text{———} \quad \mathrm{I}$$

$$5A + B + D = 2 \quad \text{———} \quad \mathrm{II}$$

$$\mathrm{I} \Rightarrow 2(1) + 2(1) + C = 0 \Rightarrow \boxed{C = -4}$$

11·3-54

II $\Rightarrow$ $5+1+D=2$

$6+D=2 \Rightarrow \boxed{D=-4}$

So

$$\frac{2s^2+2s+1}{s(s+1)(2s^2+4s+1)} = \frac{1}{s}+\frac{1}{s+1}-\frac{-4s-4}{2s^2+4s+1}$$

$$= \frac{1}{s}+\frac{1}{s+1}-\frac{4(s+1)}{(2s^2+4s+1)}$$

So

$$X(s) = \frac{1}{s}+\frac{1}{s+1}-\frac{2(s+1)}{s^2+2s+\frac{1}{2}}$$

$$= \frac{1}{s}+\frac{1}{s+1}-\frac{2(s+1)}{s^2+2s+1+\frac{1}{2}-1}$$

$$= \frac{1}{s}+\frac{1}{s+1}-\frac{2(s+1)}{(s+1)^2-\frac{1}{2}}$$

$$X(s) = \frac{1}{s}+\frac{1}{s+1}-\frac{2(s+1)}{(s+1)^2-(\frac{1}{\sqrt{2}})^2}$$

Applying $\mathcal{L}^{-1}$ on both sides of eq.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{\tfrac{1}{s}\}+\mathcal{L}^{-1}\{\tfrac{1}{s+1}\}-2\,\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2-(\frac{1}{\sqrt{2}})^2}\right\}$$

$$\boxed{x(t) = 1+e^{-t}-2e^{-t}\cosh(\tfrac{1}{\sqrt{2}}t)}$$

Diff. w.r.t. t, we get

$$\frac{dx}{dt} = -e^{-t}+2e^{-t}\cosh\left(\frac{1}{\sqrt{2}}t\right)-2\,e^{-t}\sinh\left(\frac{1}{\sqrt{2}}t\right)\cdot\frac{1}{\sqrt{2}}$$

Put in given 2nd eq