CHAPTER # 5

THE DIFFERENTIAL OPERATORS

LIST OF FORMULAS OF Differential operators.

① $\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ (Nebla or del)

② Divergence of a vector. (Divergence Dot)

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3$$

where $\vec{F} = [F_1, F_2, F_3]$

③ $\text{div}(\vec{F} \pm \vec{g}) = \text{div}\vec{F} \pm \text{div}\,\vec{g}$

, $\text{div}(c\vec{F}) = c\,\text{div}\vec{F}$

④ Laplace operator:- ∇^2 is called laplace operator

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

⑤ Laplacian of a Scalar function f.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

⑥ Laplace's equation

$\nabla^2 f = 0$ is called laplace equation.

P.T.O

⑦ Harmonic function

If a function satisfies laplaces equation it is called Harmonic function.

⑧ Gradient of a scalar function $f = \vec{\nabla} f$ (scalar product)

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right].$$

⑨ Directional derivative

Derivative of a scalar function in the direction of a vector $\vec{a}$.

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot (\text{grad } f)$$

⑩ Level surface: The set of points satisfying

....

⑪ Unit normal vector to a level surface at the point.

$$\hat{n} = \frac{(\text{grad})_P}{|\text{grad}|_P}$$

⑫ $\text{grad}(fg) = f \text{ grad } g + g \text{ grad } f$

⑬ $\text{grad}\left(\frac{f}{g}\right) = \frac{g \text{ grad } f - f \text{ grad } g}{g^2}$

⑭ $\text{grad } r^n = n r^{n-2} \vec{r}.$

The Differential operators CH # 5

This chapter deals with the differential operators like gradient, divergence and Curl.

The vectors differential operator $\vec{\nabla}$
(Read as "nabla" or "del").

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

or $\vec{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$

Divergence of a vector

let $\vec{F} = [f_1, f_2, f_3]$

Then the scalar $\vec{\nabla}\cdot\vec{F}$ is called divergence of $\vec{F}$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot[f_1, f_2, f_3]$$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

PREPARED BY
AMIR TAIMOR MOHMAND
M.Sc (Mathematics)
University of Peshawar.
Dated: 27-12-2007.

Ex-1/80 Calculate $\operatorname{div}\vec{F}$ if

$$\vec{F}(x,y,z) = e^{xy}\cos z\,\vec{i} + e^{yz}\cos x\,\vec{j} + e^{xz}\cos y\,\vec{k}$$

Sol:- $\vec{F}(x,y,z) = e^{xy}\cos z\,\vec{i} + e^{yz}\cos x\,\vec{j} + e^{xz}\cos y\,\vec{k}.$

Here $f_1 = e^{xy}\cos z$, $f_2 = e^{yz}\cos x$, $f_3 = e^{xy}\cos y.$

$\boxed{\dfrac{\partial f_1}{\partial x} = e^{xy}\cos z\cdot y}$, $\boxed{\dfrac{\partial f_2}{\partial y} = e^{yz}\cos x\cdot z}$, $\boxed{\dfrac{\partial f_3}{\partial z} = e^{xy}\cos y\cdot x}$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div}\vec{F} = y\,e^{xy}\cos z + z\,e^{yz}\cos x + x\,e^{zx}\cos y$$ Ans

---

EXERCISE # 5.2

Q1/80 Calculate the divergence of the following vector fields.

$$\vec{F} = (x^2 - xy^2)\vec{i} + (y^2 - yz^2)\vec{j} + (z^2 - zx^2)\vec{k}.$$

Sol:- Here

$f_1 = x^2 - xy^2$, $\dfrac{\partial f_1}{\partial x} = 2x - y^2$

$f_2 = y^2 - yz^2$, $\dfrac{\partial f_2}{\partial y} = 2y - z^2$

$f_3 = z^2 - zx^2$, $\dfrac{\partial f_3}{\partial z} = 2z - x^2.$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div}\vec{F} = 2x - y^2 + 2y - z^2 + 2z - x^2 = 2(x+y+z) - x^2 - y^2 - z^2$$

$$\boxed{\operatorname{div}\vec{F} = 2(x+y+z) - (x^2+y^2+z^2)}$$ Ans

Q2/80 Calculate the divergence of the following vector field. $\vec{F} = x^2yz\,\vec{i} + xy^2z\,\vec{j} + xyz^2\,\vec{K}$.

Sol:- Here

$F_1 = x^2yz$ | $F_2 = xy^2z$ | $F_3 = xyz^2$

$\frac{\partial F_1}{\partial x} = 2xyz$ | $\frac{\partial F_2}{\partial y} = 2xyz$ | $\frac{\partial F_3}{\partial z} = 2xyz$.

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 2xyz + 2xyz + 2xyz$$

$$\boxed{\text{div}\vec{F} = 6xyz} \quad \text{Ans}$$

Q3/80 Calculate the divergence of the following vector fields. $\vec{F} = x\sin xy\,\vec{i} + y\sin yz\,\vec{j} + z\sin zx\,\vec{K}$.

Sol:- Here

$F_1 = x\sin xy$

$\frac{\partial F_1}{\partial x} = x(\cos xy)\cdot y + \sin xy\cdot 1$

$\boxed{\frac{\partial F_1}{\partial x} = xy\cos xy + \sin xy}$

$F_2 = y\sin yz$

$\frac{\partial F_2}{\partial y} = y(\cos yz)\cdot z + \sin yz\cdot 1$

$\boxed{\frac{\partial F_2}{\partial y} = yz\cos yz + \sin yz}$

$F_3 = z\sin zx$

$\frac{\partial F_3}{\partial z} = z(\cos zx)x + \sin zx\cdot 1$

$\boxed{\frac{\partial F_3}{\partial z} = zx\cos zx + \sin zx}$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= (xy\cos xy + \sin xy) + (yz\cos yz + \sin yz) + zx(\cos zx + \sin zx)$$

$$\text{div}\vec{F} = xy\cos xy + yz\cos yz + zx\cos zx + \sin xy + \sin yz + \sin zx.$$

Ans

Q4/80 Calculate the divergence of the following vector field

$$\vec{F} = \frac{x}{x+y+z}\vec{i} + \frac{y}{x+y+z}\vec{j} + \frac{z}{x+y+z}\vec{k}.$$

Sol:- Here $f_1 = \frac{x}{x+y+z}$

$$\frac{\partial f_1}{\partial x} = \frac{(x+y+z)1 - x(1)}{(x+y+z)^2} = \frac{\cancel{x}+y+z-\cancel{x}}{(x+y+z)^2} = \frac{y+z}{(x+y+z)^2}$$

$$\boxed{\frac{\partial f_1}{\partial x} = \frac{y+z}{(x+y+z)^2}}$$

$$f_2 = \frac{y}{x+y+z}$$

$$\frac{\partial f_2}{\partial y} = \frac{(x+y+z)\cdot 1 - y(1)}{(x+y+z)^2} = \frac{x+\cancel{y}+z-\cancel{y}}{(x+y+z)^2} = \frac{x+z}{(x+y+z)^2}.$$

$$\boxed{\frac{\partial f_2}{\partial y} = \frac{x+z}{(x+y+z)^2}}$$

$$f_3 = \frac{z}{x+y+z}$$

$$\frac{\partial f_3}{\partial z} = \frac{(x+y+z)\cdot 1 - z(1)}{(x+y+z)^2} = \frac{x+y+\cancel{z}-\cancel{z}}{(x+y+z)^2}$$

$$\boxed{\frac{\partial f_3}{\partial z} = \frac{x+y}{(x+y+z)^2}}$$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{y+z}{(x+y+z)^2} + \frac{x+z}{(x+y+z)^2} + \frac{x+y}{(x+y+z)^2} = \frac{y+z+x+z+x+y}{(x+y+z)^2}$$

P.T.O

$$= \frac{2x+2y+2z}{(x+y+z)^2} = \frac{2(x+y+z)}{(x+y+z)^2} = \frac{2}{(x+y+z)}$$

$$\boxed{\operatorname{div}\vec{F} = \frac{2}{x+y+z}}$$ Ans

Q5/80 Calculate the divergence of the vector field
$\vec{F} = (x-y)^2\vec{i} + (y-z)^2\vec{j} + (z-x)^2\vec{k}$.

Sol:- $f_1 = (x-y)^2$ $\quad f_2 = (y-z)^2$ $\quad f_3 = (z-x)^2$

$\frac{\partial f_1}{\partial x} = 2(x-y)\cdot 1$ $\quad \frac{\partial f_2}{\partial y} = 2(y-z)\cdot 1$ $\quad \frac{\partial f_3}{\partial z} = 2(z-x)\cdot 1$

$\boxed{\frac{\partial f_1}{\partial x} = 2(x-y)}$ $\quad \boxed{\frac{\partial f_2}{\partial y} = 2(y-z)}$ $\quad \boxed{\frac{\partial f_3}{\partial z} = 2(z-x)}$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div}\vec{F} = 2(x-y) + 2(y-z) + 2(z-x)$$

$$\operatorname{div}\vec{F} = 2x - 2y + 2y - 2z + 2z - 2x$$

$$\boxed{\operatorname{div}\vec{F} = 0}$$ Ans

imp Ch-05

Q6/80 Calculate the divergence of the vector field.

$$\vec{F} = \frac{z-x}{(x^2+y^2+z^2)}\vec{i} + \frac{x-y}{(x^2+y^2+z^2)}\vec{j} + \frac{y-z}{(x^2+y^2+z^2)}\vec{k}$$

Sol:- $f_1 = \dfrac{z-x}{x^2+y^2+z^2}$

$$\frac{\partial f_1}{\partial x} = \frac{x^2+y^2+z^2(-1)-(z-x)(2x)}{(x^2+y^2+z^2)^2} = \frac{-x^2-y^2-z^2-2zx+2x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial f_1}{\partial x} = \frac{x^2-y^2-z^2-2zx}{(x^2+y^2+z^2)^2}$$

$$f_2 = \frac{x-y}{x^2+y^2+z^2}$$

$$\frac{\partial f_2}{\partial y} = \frac{(x^2+y^2+z^2)(-1)-(x-y)2y}{(x^2+y^2+z^2)^2} = \frac{-x^2-y^2-z^2-2xy+2y^2}{(x^2+y^2+z^2)^2}$$

$$\boxed{\frac{\partial f_2}{\partial y} = \frac{-x^2+y^2-z^2-2xy}{(x^2+y^2+z^2)^2}}$$

$$f_3 = \frac{y-z}{x^2+y^2+z^2}$$

$$\frac{\partial f_3}{\partial z} = \frac{(x^2+y^2+z^2)(-1)-(y-z)(2z)}{(x^2+y^2+z^2)^2} = \frac{-x^2-y^2-z^2-2yz+2z^2}{(x^2+y^2+z^2)^2}$$

$$\boxed{\frac{\partial f_3}{\partial z} = \frac{-x^2-y^2+z^2-2yz}{(x^2+y^2+z^2)^2}}$$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{x^2-y^2-z^2-2zx}{(x^2+y^2+z^2)^2} + \frac{-x^2+y^2-2xy-z^2}{(x^2+y^2+z^2)^2} + \frac{-x^2-y^2+z^2-2xz}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2-y^2-z^2-2zx-x^2+y^2-z^2-2xy-x^2-y^2+z^2-2yz}{(x^2+y^2+z^2)^2}$$

$$= \frac{-x^2-y^2-z^2-2zx-2xy-2yz}{(x^2+y^2+z^2)^2}$$

$$= \frac{-(x^2+y^2+z^2+2zx+2xy+2yz)}{(x^2+y^2+z^2)^2}$$

$$= \frac{-(x+y+z)^2}{(x^2+y^2+z^2)^2} = -\left[\frac{x+y+z}{x^2+y^2+z^2}\right]^2$$

$$\operatorname{div}\vec{F} = -\left[\frac{(x+y+z)}{x^2+y^2+z^2}\right]^2 \quad \underline{Ans}$$

Q7/80) Find divergence of the vector field.

$\vec{F} = e^{xy}(\cos xy\,\vec{i} + \sin yz\,\vec{j} + z\sin zx\,\vec{k})$.

Sol:- $\vec{F_1} = e^{xy}\cos xy\,\vec{i} + e^{xy}\sin yz\,\vec{j} + e^{xy}z\sin zx\,\vec{k}$

$f_1 = e^{xy}\cos xy$

$$\frac{\partial f_1}{\partial x} = e^{xy}(-\sin xy\cdot y) + (\cos xy)\cdot e^{xy}\cdot y$$

$f_2 = e^{xy}\sin yz$

$$\frac{\partial f_2}{\partial y} = e^{xy}(\cos yz\cdot z) + \sin yz\cdot e^{xy}\,x$$

$f_3 = \underline{\underline{e}}^{xy}\, z\, \underline{\sin zx}$ · (Note this point)

$$\frac{\partial f_3}{\partial z} = e^{xy}\left[(zx\cos zx) + \sin zx\cdot 1\right]$$

Ch-05

$$\operatorname{div}\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div}\vec{f} = -ye^{xy}\sin xy + y\cos y\, e^{xy} + z\, e^{xy}\cos yz + x\sin yz\, e^{xy} + zx\, e^{xy}\cos zx + e^{xy}\sin zx.$$

$$\boxed{\operatorname{div}\vec{f} = e^{xy}\left(-y\sin xy + y\cos xy + z\cos yz + z\cos yz + x\sin yz + zx\cos zx + \sin zx\right).}$$ Ans

(i/79) Prove $\operatorname{div}(\vec{f}+\vec{g}) = \operatorname{div}\vec{f} + \operatorname{div}\vec{g}$.

Sol:- let $\vec{f} = [f_1, f_2, f_3]$, $\vec{g} = [g_1, g_2, g_3]$.

$$\vec{f}+\vec{g} = [f_1+g_1, f_2+g_2, f_3+g_3]$$

$$\operatorname{div}(\vec{f}+\vec{g}) = \operatorname{div}[f_1+g_1, f_2+g_2, f_3+g_3]$$

$$= \vec{\nabla}\cdot[f_1+g_1, f_2+g_2, f_3+g_3]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot[f_1+g_1, f_2+g_2, f_3+g_3]$$

$$= \frac{\partial}{\partial x}(f_1+g_1) + \frac{\partial}{\partial y}(f_2+g_2) + \frac{\partial}{\partial z}(f_3+g_3)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial g_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial g_3}{\partial z}$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + \left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}\right)$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot[f_1, f_2, f_3] + \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot[g_1, g_2, g_3]$$

$$= \vec{\nabla}\cdot\vec{f} + \vec{\nabla}\cdot\vec{g}$$

$$\boxed{\operatorname{div}(\vec{f}+\vec{g}) = \operatorname{div}\vec{f} + \operatorname{div}\vec{g}}$$ Ans

Q8/80 Find $\operatorname{div}(\vec{f}+\vec{g})$ if

$\vec{f} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{g} = x\sin x\vec{i} + y\cos y\vec{j} + z\sec z\vec{k}$.

Sol:- $\vec{f} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{f_1} = x \qquad f_2 = y \qquad f_3 = z$

$\frac{\partial f_1}{\partial x} = 1 \qquad \frac{\partial f_2}{\partial y} = 1 \qquad \frac{\partial f_3}{\partial z} = 1$

$\operatorname{div}\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_3}{\partial y} + \frac{\partial f_3}{\partial z}$

$\operatorname{div}\vec{f} = 1+1+1$

$\boxed{\operatorname{div}\vec{f} = 3}$

$\vec{g} = x\sin x\vec{i} + y\cos y\vec{j} + z\sec z\vec{k}$

$g_1 = x\sin x$

$\frac{\partial g_1}{\partial x} = x\cos x + \sin x\cdot 1$

$g_2 = y\cos y$

$\frac{\partial g_2}{\partial y} = y(-\sin y) + \cos y\cdot 1$

$g_3 = z\sec z$

$\frac{\partial g_3}{\partial z} = z\cdot\sec z\tan z + \sec z\cdot 1$

$\operatorname{div}\vec{g} = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}$

$\operatorname{div}\vec{g} = x\cos x + \sin x + (-y\sin y + \cos y) + (z\sec z\tan z + \sec z)$

$\operatorname{div}(\vec{f}+\vec{g}) = \operatorname{div}\vec{f} + \operatorname{div}\vec{g}$

$\operatorname{div}(\vec{f}+\vec{g}) = 3 + x\cos x + \sin x - y\sin y + \cos y + z\sec z\tan z + \sec z$

Ans.

LAPLACE OPERATOR

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

$$\nabla^2 = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

∇^2 is called the laplace operator.

LAPLACIAN OF A SCALAR FUNCTION f

$\nabla^2 f$ is called laplacian of f

Hence $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

LAPLACE'S EQUATION

The equation $\nabla^2 f = 0$ is known as laplace's operator.

HARMONIC FUNCTION

If $\nabla^2 f = 0$

Then the function f is said to be harmonic.

Q10/81 Find the Laplacian of f

(a) $f(x,y,z) = x^2yz + xy^2z + xyz^2$

Sol:- $f(x,y,z) = x^2yz + xy^2z + xyz^2$

$$\frac{\partial f}{\partial x} = 2xyz + y^2z + yz^2$$

$$\frac{\partial^2 f}{\partial x^2} = 2yz + 0 + 0 = 2yz$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 2yz}$$

$$\frac{\partial f}{\partial y} = x^2z + 2xyz + xz^2$$

$$\frac{\partial^2 f}{\partial y^2} = 0 + 2xz + 0 = 2xz$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = 2xz}$$

$$\frac{\partial f}{\partial z} = x^2y + xy^2 + 2xyz$$

$$\frac{\partial^2 f}{\partial z^2} = 0 + 0 + 2xy$$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = 2xy}$$

Laplacian of f

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = 2yz + 2xz + 2xy$$

$$\boxed{\nabla^2 f = 2(yz + xz + xy)}$$ Ans

(b) $f(x,y,z) = yz\cos x + zx\cos y + xy\cos z$.

Sol:- $\nabla^2 f = ?$

Since we know that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{--- (1)}$$

So,
$$\frac{\partial f}{\partial x} = yz(-\sin x) + z(1)\cos y + y\cos z$$

$$\frac{\partial^2 f}{\partial x^2} = -yz\cos x + 0 + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = -yz\cos x}$$

$$\frac{\partial f}{\partial y} = (1)\, z\cos x + zx(-\sin y) + x(1)\cos z$$

$$\frac{\partial^2 f}{\partial y^2} = 0 - zx\cos y + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial y^2} = -zx\cos y}$$

$$\frac{\partial f}{\partial z} = y(1)\cos x + (1)\, x\cos y + xy(-\sin z)$$

$$\frac{\partial^2 f}{\partial z^2} = 0 + 0 - xy\cos z \Rightarrow \boxed{\frac{\partial^2 f}{\partial z^2} = -xy\cos z}$$

Putting all these three values in equation (1)

$$\nabla^2 f = -yz\cos x - zx\cos y - xy\cos z$$

$$\boxed{\nabla^2 f = -(yz\cos x + zx\cos y + xy\cos z)}$$

Ans

Q11/81) If $f = a_1x^2 + a_2y^2 + a_3z^2 + b_1yz + b_2zx + b_3xy$.
where a_1, a_2, a_3 and b_1, b_2, b_3 are all constants.
Show that the Laplacian of f is zero (May be asked as constant)

Sol:- $f = a_1x^2 + a_2y^2 + a_3z^2 + b_1yz + b_2zx + b_3xy$

$$\frac{\partial f}{\partial x} = 2a_1x + 0 + 0 + 0 + b_2z + b_3y$$

$$\frac{\partial^2 f}{\partial x^2} = 2a_1 + 0 + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = 2a_1}$$

$$\frac{\partial f}{\partial y} = 0 + 2a_2y + 0 + 0 + 0 + b_3x$$

$$\frac{\partial f^2}{\partial y^2} = 2a_2 + 0 \Rightarrow \boxed{\frac{\partial f^2}{\partial y^2} = 2a_2}$$

$$\frac{\partial f}{\partial z} = 0 + 0 + 2a_3z + 0 + 0$$

$$\frac{\partial^2 f}{\partial z^2} = 2a_3 + 0 + 0 \Rightarrow \boxed{\frac{\partial f^2}{\partial z^2} = 2a_3}$$

Laplacian of f

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 2a_1 + 2a_2 + 2a_3$$

$$\boxed{\nabla^2 f = 2(a_1 + a_2 + a_3)} \quad \text{Ans}$$

GRADIENT OF A SCALAR FUNCTION

$\vec{\nabla} f$ is called gradient of the scalar function f.

$$\text{grad } f = \vec{\nabla} f \quad \text{where} \quad \vec{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

v.v. imp

Q12/81 Prove that $\text{div}(f\vec{\nabla} g) = f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g$

Sol:- L.H.S

$$\text{div}(f\vec{\nabla} g) = \text{div}\left[f\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] g\right]$$

$$= \text{div}\left[f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\right]$$

$$= \text{div}\left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right]$$

$$= \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$\boxed{\text{div}(f\vec{\nabla} g) = f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial f}} \quad (1)$$

R.H.S =

$$f(\nabla^2 g) + \vec{\nabla} f \cdot \vec{\nabla} g = f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \cdot \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]$$

$$f(\nabla^2 g) + \vec{\nabla} f \cdot \vec{\nabla} g = f\frac{\partial^2 g}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \quad (2)$$

(1) = (2)

So $\boxed{\text{div}(f\vec{\nabla} g) = f\vec{\nabla} g + \vec{\nabla} f \cdot \vec{\nabla} g}$ Proved

(Q13/81) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{a}$ is a constant prove

(i) $\operatorname{div}(\vec{a} \times \vec{r}) = 0$

Sol:- $\operatorname{div}(\vec{a} \times \vec{r}) = \vec{\nabla} \cdot (\vec{a} \times \vec{r})$

$= \vec{\nabla} \cdot \vec{a} \times \vec{r}$ (scalar triple product)

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$, where $\vec{a} = [a_1, a_2, a_3]$

Expanding by R_1

$$= \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 - 0 + 0 - 0 + 0 - 0$$

$$\boxed{\operatorname{div}(\vec{a} \times \vec{r}) = 0}$$ Proved

V. imp (ii) $\operatorname{div}[\vec{a} \times (\vec{r} \times \vec{a})] = 2a^2$

Sol:- $\operatorname{div}[\vec{a} \times (\vec{r} \times \vec{a})] = \operatorname{div}[(\vec{a} \cdot \vec{a})\vec{r} - (\vec{a} \cdot \vec{r})\vec{a}]$ (vector triple product)

$$= \operatorname{div}[a^2 \vec{r} - (\vec{a} \cdot \vec{r})\vec{a}]$$

$$= \operatorname{div} a^2 \vec{r} - \operatorname{div}(\vec{a} \cdot \vec{r})\vec{a} \quad \text{①}$$

$$= a^2 \operatorname{div} \vec{r} - \operatorname{div}(\vec{a} \cdot \vec{r})\vec{a}$$

$$\operatorname{div} \vec{r} = \vec{\nabla} \cdot \vec{r}$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [x, y, z]$$

P.T.O

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1$$

$$\operatorname{div} \vec{r} = \vec{\nabla} \cdot \vec{r} = 3.$$

$$(\vec{a}\cdot\vec{r})\vec{a} = \left([a_1, a_2, a_3]\cdot[x, y, z]\right)[a_1, a_2, a_3]$$

$$= (a_1 x + a_2 y + a_3 z)[a_1, a_2, a_3]$$

$$= \left[(a_1 x + a_2 y + a_3 z)a_1, (a_1 x + a_2 y + a_3 z)a_2, (a_1 x + a_2 y + a_3 z)a_3\right]$$

$$\operatorname{div}(\vec{a}\cdot\vec{r})\vec{a} = \vec{\nabla}\cdot(\vec{a}\cdot\vec{r})\vec{a}$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot\left[(a_1 x + a_2 y + a_3 z)a_1, (a_1 x + a_2 y + a_3 z)a_2, (a_1 x + a_2 y + a_3 z)a_3\right]$$

$$= \frac{\partial}{\partial x}(a_1^2 x + a_1 a_2 y + a_1 a_3 z) + \frac{\partial}{\partial y}(a_1 x + a_2^2 y + a_1 a_3 z)$$
$$+ \frac{\partial}{\partial z}(a_3 a_1 x + a_2 a_3 y + a_3^2 z)$$

$$= a_1^2 + 0 + 0 + 0 + a_2^2 + 0 + 0 + 0 + a_3^2$$

$$= a_1^2 + a_2^2 + a_3^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2$$

$$= |\vec{a}|^2$$

$$\operatorname{div}(\vec{a}\cdot\vec{r})\vec{a} = a^2$$

(1) $\Rightarrow$

$$\operatorname{div}[\vec{a}\times(\vec{r}\times\vec{a})] = a^2(3) - a^2 = 3a^2 - a^2$$

$$\operatorname{div}[\vec{a}\times(\vec{r}\times\vec{a})] = 2a^2 \quad \underline{\text{Ans}}$$

Q15/81 Show $\vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\nabla}g)=0$

Sol:- $\vec{\nabla}f=\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]f$

$$=\left[\frac{\partial}{\partial x}f,\frac{\partial}{\partial y}f,\frac{\partial}{\partial z}f\right]$$

$$\vec{\nabla}g=\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]g$$

$$=\left[\frac{\partial}{\partial x}g,\frac{\partial}{\partial y}g,\frac{\partial}{\partial z}g\right]$$

$\vec{\nabla}\cdot(\vec{\nabla}f\times\vec{\nabla}g)=\vec{\nabla}\cdot\vec{\nabla}f\times\vec{\nabla}g$ (scalar triple product)

$$=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}\end{vmatrix}$$

Expanding by R_1

$$=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)$$
$$+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right)$$
$$+\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$$

$$=\left(\frac{\partial f}{\partial y}\right)\frac{\partial^2 g}{\partial x\partial z}+\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z}-\left(\frac{\partial f}{\partial z}\right)\frac{\partial^2 g}{\partial x\partial y}$$
$$-\frac{\partial^2 f}{\partial x\partial z}\cdot\frac{\partial g}{\partial y}+\left(\frac{\partial f}{\partial z}\right)\frac{\partial^2 g}{\partial y\partial x}+\frac{\partial^2 f}{\partial y\partial z}\cdot\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}\cdot\frac{\partial^2 g}{\partial y\partial z}$$
$$-\frac{\partial^2 f}{\partial y\partial x}\frac{\partial g}{\partial z}+\frac{\partial f}{\partial x}\cdot\frac{\partial^2 g}{\partial z\partial y}+\frac{\partial^2 f}{\partial z\partial x}\cdot\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial z\partial x}$$
$$-\frac{\partial^2 f}{\partial z\partial y}\frac{\partial g}{\partial x}$$

$\vec{\nabla}\cdot(\vec{\nabla}f \times \vec{\nabla}g) = 0$ proved

DIRECTIONAL DERIVATIVES

Derivative of a scalar function in the direction of a vector $\vec{a}$

$$\boxed{\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|}\cdot(\text{grad } f)}$$

EXERCISE # 5.4

Find the derivative of f at p in direction of $\vec{a}$, where

Q1 77 $f = e^{x+y+z}$, $P(0,0,0)$, $\vec{a} = \vec{i} + 2\vec{j} - 2\vec{k}$

Sol:- Directional derivative of f

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|}\cdot \text{grad } f \quad \text{——— (1)}$$

$$f = e^{x+y+z}$$

$$\frac{\partial f}{\partial x} = e^{x+y+z}\cdot 1 = e^{0+0+0} = e^{0} = 1$$

$$\frac{\partial f}{\partial y} = e^{x+y+z};\ 1 = e^{0+0+0} = e^{0} = 1$$

$$\frac{\partial f}{\partial z} = e^{x+y+z},\ 1 = e^{0+0+0} = 1$$

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = [1, 1, 1]$$

$$① \Rightarrow \quad \frac{df}{ds} = \frac{\vec{i} + 2\vec{j} - 2\vec{k}}{|\vec{i} + 2\vec{j} - 2\vec{k}|} \cdot [1, 1, 1]$$

$$= \frac{[1, 2, -2]}{\sqrt{1+4+4}} \cdot [1, 1, 1]$$

$$= \frac{1(1) + 2(1) + (-2)(1)}{3}$$

$$\boxed{\frac{df}{ds} = \frac{1}{3}} \quad \underline{\text{Ans}}$$

Q2/77 $\quad f = e^{yz}\cos x + e^{zx}\cos y + e^{xy}\cos z$

$P(\pi/6, \pi/3, 0)$, $\quad \vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$

Sol:- Directional derivative of f

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f \quad \text{———} ①$$

$$f = e^{yz}\cos x + e^{zx}\cos y + e^{xy}\cos z$$

$$\frac{\partial f}{\partial x} = e^{yz}(-\sin x) + \cos y \, e^{zx}(\cdot z) + \cos z \, e^{xy} \cdot y$$

$$= e^{\pi/3 \cdot 0}(-\sin \pi/6) + \cos \pi/3 \left(e^{0 \cdot \pi/6} \cdot 0\right) + \cos 0 \, e^{\pi/6 \cdot \pi/3} \cdot \pi/3$$

$$= 1(-1/2) + \left(\tfrac{1}{2}\right)(0) + 1 \cdot e^{\pi^2/18} \cdot \pi/3$$

$$\boxed{\frac{\partial f}{\partial x} = -\frac{1}{2} + \frac{\pi}{3} e^{\frac{\pi^2}{18}}}$$

$$\frac{\partial f}{\partial y} = \left(e^{yz} \cdot z\right) \cos x + e^{zx}\left(-\sin y\right) + \left(e^{xy} \cdot x\right) \cos z$$

$$= \left(e^{\pi/3 \cdot 0} \cdot 0\right) \cos \pi/6 + e^{0 \cdot \pi/6}\left(-\sin \pi/3\right) + \left(e^{\frac{\pi}{6} \cdot \frac{\pi}{3}} \cdot \pi/6\right) \cos 0$$

$$= 1\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} e^{\frac{\pi^2}{18}}$$

$$\boxed{\frac{\partial f}{\partial y} = \frac{-\sqrt{3}}{2} + \frac{\pi}{6} e^{\pi^2/18}}$$

$$\frac{\partial f}{\partial z} = \left(e^{yz} \cdot y\right) \cos x + \left(e^{zx} \cdot x\right) \cos y + e^{xy}\left(-\sin z\right)$$

$$= \left(e^{\pi/3 \cdot 0} \cdot \pi/3\right) \cos \pi/6 + \left(e^{0 \cdot \frac{\pi}{6}} \cdot \frac{\pi}{6}\right) \cos \frac{\pi}{3} + e^{\frac{\pi}{6} \cdot \pi/3}\left(-\sin 0\right)$$

$$= \frac{\pi}{3}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\left(\frac{1}{2}\right)$$

$$\boxed{\frac{\partial f}{\partial z} = \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12}}$$

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$= \left[-\frac{1}{2} + \frac{\pi}{3} e^{\pi^2/18}, -\frac{\sqrt{3}}{2} + \frac{\pi}{6} e^{\pi^2/18}, \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12}\right]$$

$$= \frac{[3, 2, -1]}{\sqrt{3^2 + 2^2 + 1}} \cdot \left[-\frac{1}{2} + \frac{\pi}{3} e^{\pi^2/18}, \frac{-\sqrt{3}}{2}, \frac{\pi}{2\sqrt{3}} + \frac{\pi}{12}\right]$$

$$= \frac{1}{\sqrt{14}}\left[\frac{-3}{2} + \pi e^{\pi^2/18} - \sqrt{3} + \frac{\pi}{3} e^{\pi^2/18} - \frac{\pi}{2\sqrt{3}} - \frac{\pi}{12}\right]$$

$$\frac{df}{ds} = \frac{1}{\sqrt{14}}\left[-\frac{3}{2} - \sqrt{3} + \pi e^{\pi^2/18} + \frac{\pi}{3} e^{\pi^2/18} - \pi\left(\frac{1}{2\sqrt{3}} + \frac{1}{12}\right)\right]$$

Ans

Q3 / 78 $f = \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}$, $P(1,-1,1)$

$\vec{a} = \vec{i} - \vec{j} + \vec{k}$

Sol:- Directional derivative is

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot (\text{grad } f) \quad \text{--- (1)}$$

$$f = \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}$$

$$\frac{\partial f}{\partial x} = \frac{-yz}{x^2} + \frac{z}{y} + \frac{y}{z} = -\frac{(-1)(1)}{1^2} + \frac{1}{-1} + \frac{-1}{1}$$

$$\boxed{\frac{\partial f}{\partial x} = -1}$$

$$\frac{\partial f}{\partial y} = \frac{z}{x} - \frac{zx}{y^2} + \frac{x}{z} = \frac{1}{1} - \frac{1 \cdot 1}{(-1)^2} + \frac{1}{1}$$

$$\boxed{\frac{\partial f}{\partial y} = 1}$$

$$\frac{\partial f}{\partial z} = \frac{y}{x} + \frac{x}{y} - \frac{xy}{z^2} = \frac{-1}{1} + \frac{1}{-1} - \frac{1(-1)}{1^2}$$

$$\boxed{\frac{\partial f}{\partial z} = -1}$$

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\text{grad } f = [-1, 1, -1]$$

P.T.O

① ⟹

$$\frac{df}{ds} = \frac{[1,-1,1]}{\sqrt{1^2+1^2+1^2}} \cdot [-1,1,-1]$$

$$= \frac{1(-1)+(-1)(1)+1(-1)}{\sqrt{3}} = \frac{-3}{\sqrt{3}} = \frac{-\sqrt{3}\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}}$$

$$\boxed{\frac{df}{ds} = -\sqrt{3}} \quad \text{Ans}$$

(24/78) $f = x^2+y^2+z^2$, $p(2,2,2)$, $\vec{a} = \vec{i}+2\vec{j}-3\vec{K}$

Sol:- $\dfrac{df}{ds} = \dfrac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f$ ——— ①

$$f = x^2+y^2+z^2$$

$$\frac{\partial f}{\partial x} = 2x+0+0 = 2(2) = 4$$

$$\frac{\partial f}{\partial y} = 0+2y+0 = 2(2) = 4$$

$$\frac{\partial f}{\partial z} = 0+0+2z = 2(2) = 4$$

$$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\text{grad } f = [4,4,4]$$

So $$\frac{df}{ds} = \frac{\vec{i}+2\vec{j}-3\vec{K}}{\sqrt{1^2+2^2+(-3)^2}} \cdot \text{grad } f$$

$$= \frac{(\vec{i}+2\vec{j}-3\vec{K}) \cdot [4,4,4]}{\sqrt{1+4+9}} = \frac{4\vec{i}+8\vec{j}-12\vec{K}}{\sqrt{14}}$$

Ch-05

Q6/78 $f = x^2yz + 4yz^2$, $P(1,-2,1)$, $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

Sol:-

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f \longrightarrow (1)$$

Do?

Q11/78 Find the rate of change of $f = x^2 + yz$ at $(3, 1, -5)$ in the direction of $\vec{i} + 2\vec{j} - \vec{k}$.

Sol:- Here $f = x^2 + yz$, $P(3,1,-5)$, $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$

Rate of change of f in the direction of vector $\vec{a}$ is directional derivative.

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f \longrightarrow (1)$$

$$f = x^2 + yz$$

$$\frac{\partial f}{\partial x} = 2x = 2(3) = 6$$

$$\frac{\partial f}{\partial y} = z = -5$$

P.T.O

$\frac{\partial f}{\partial z} = y = 1$

$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f$

$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [6, -5, 1]$

(1) $\Rightarrow$

$\frac{df}{ds} = \frac{[1, 2, -1]}{\sqrt{1^2 + 2^2 + (-1)^2}} \cdot [6, -5, 1]$

$= \frac{(1)(6) + (2)(-5) + (-1)(1)}{\sqrt{1+4+1}} = \frac{6-10-1}{\sqrt{6}}$

$\boxed{\frac{df}{ds} = \frac{-5}{\sqrt{6}}}$ Ans

(Ex-1 / 76) Find the directional derivative of the $f = x^2 + yz$ in the direction of $\vec{i} + 2\vec{j} - \vec{k}$ at the point $P(3, 1, -5)$.

Sol:- Since $\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f$ ~ (1)

$\text{grad } f = \vec{\nabla} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

$\frac{\partial f}{\partial x} = 2x + 0 = 2(3) = 6$

$\frac{\partial f}{\partial y} = z = -5$

$\frac{\partial f}{\partial z} = 0 + y = 1$

P.T.O

$$\frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot \text{grad } f$$

$$= \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{(1)^2+(2)^2+(-1)^2}} \cdot (6, -5, 1)$$

$$= \frac{\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{1+4+1}} \cdot (6, -5, 1)$$

$$= \frac{6\vec{i} - 10\vec{j} - 1\vec{k}}{\sqrt{6}} \quad \underline{\text{Ans}}$$

Level Surface Of a Scalar Function.

let us suppose that $f(x, y, z)$ denote a scalar function; Then for a constant c, the set of points satisfying the equation

$f(x, y, z) = c$

is called a level surface of f.

Unit normal vector to a level surface at the point P.

$$\hat{n} = \frac{(\text{grad } f)_P}{|\text{grad} f|_P}$$

where $(\text{grad } f)_P$ is called the given point.

Find the unit normal vector to the given level surface at the given point P.

Q.7 / 78 $x^3 - y^3 + z^3 - 3xyz = 14$, $P(2, 1, -1)$

Sol:- Here $f = x^3 - y^3 + z^3 - 3xyz$

$$\frac{\partial f}{\partial x} = 3x^2 - 0 + 0 - 3yz = 3(2)^2 - 3(1)(-1) = 15$$

$$\frac{\partial f}{\partial y} = 0 - 3y^2 + 0 - 3xz = -3(1)^2 - 3(2)(-1) = 3$$

$$\frac{\partial f}{\partial z} = 0 - 0 + 3z^2 - 3xy = 3(-1)^2 - 3\cdot 2\cdot 1 = -3$$

$$(\text{grad } f)_P = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]_P = [15, 3, -3] = 3[5, 1, -1]$$

Unit ~~vector~~ normal vector to level surface at pt P.

$$\hat{n} = \frac{(\text{grad} f)_P}{(|\text{grad} f|)_P} = \frac{3[5, 1, -1]}{3\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\cancel{3}(5\vec{i} + \vec{j} - \vec{k})}{\cancel{3}\sqrt{27}}$$

$$\boxed{\hat{n} = \frac{5\vec{i} + \vec{j} - \vec{k}}{3\sqrt{3}}}$$ Ans

PREPARED BY

AMIR TAIMUR MOHMAND

M.Sc (Mathematics)

University of Peshawar.

Dated: 04-01-2008.

Q8/78 $\quad x+y+z=1$, $P(4,2,-5)$

Sol:- $f = x+y+z$

$$\frac{\partial f}{\partial x} = 1 , \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial z} = 1$$

$$(\text{grad} f)_P = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]_P = [1,1,1]$$

$$|\text{grad} f| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

Unit normal vector to the level surface at point P.

$$\hat{n} = \frac{(\text{grad})_P}{(|\text{grad} f|)_P} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}} \quad \underline{\text{Ans}}$$

Ex-3/77 Find unit normal vector at the point $Q(2,1,-1)$ to the level surface.

$$x^3+y^3+z^3-3xyz = 14.$$

Sol:- Let $f(x,y,z) = x^3+y^3+z^3-3xyz$

$$\hat{n} = \frac{(\text{grad} f)_P}{|\text{grad}|_P} \quad \text{------ } (1)$$

So, $\text{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

$$\frac{\partial f}{\partial x} = 3x^2-3yz = 3(2)^2-3(1)(-1) = 15$$

$$\frac{\partial f}{\partial y} = 3y^2-3xz = 3(1)^2-3(2)(-1) = 9$$

$$\frac{\partial f}{\partial z} = 3z^2 - 3xy = 3(-1)^2 - 3(2)(1) = -3$$

$$\hat{n} = \frac{15\vec{i} + 9\vec{j} - 3\vec{k}}{\sqrt{(15)^2 + (9)^2 + (-3)^2}} = \frac{3(5\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{9 \times 45}}$$

$$\hat{n} = \frac{5\vec{i} + 3\vec{j} - \vec{k}}{3\sqrt{5}}$$ Ans

Q9/78 $yz + zx + xy = 5$, $P(-1,1,1)$, $\hat{n} = ?$

Sol: $f(x,y,z) = yz + zx + xy$

$$\frac{\partial f}{\partial x} = z + y = 1 + 1 = 2$$

$$\frac{\partial f}{\partial y} = z + x = 1 - 1 = 0$$

$$\frac{\partial f}{\partial z} = y + x = 1 - 1 = 0$$

$$\hat{n} = \frac{2\vec{i} + 0\vec{j} + 0\vec{k}}{\sqrt{2^2 + 0^2 + 0^2}} = \frac{2\vec{i}}{2}$$

$$\hat{n} = \vec{i}$$ Ans

Q10/78 $x^2 + y^2 + z^2 = 6$, $P(1,1,1)$

Sol: $f = x^2 + y^2 + z^2$

$$\frac{\partial f}{\partial x} = 2x = 2(1) = 2$$

$$\frac{\partial f}{\partial y} = 2y = 2(1) = 2$$

$$\frac{\partial f}{\partial z} = 2z = 2(1) = 2$$

$$\frac{\cancel{(\text{grad } f)_p}}{\cancel{(|\text{grad } f|)_p}} \quad (\text{grad } f)_p = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [2,2,2]$$

$$(|\text{grad } f|)_p = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$$

Unit normal vector to the level surface at the given point.

$$\hat{n} = \frac{(\text{grad } f)_p}{(|\text{grad } f|)_p} = \frac{2\vec{i}+2\vec{j}+2\vec{k}}{2\sqrt{3}}$$

$$\hat{n} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}} \quad \text{Ans}$$

Theorem 75 Prove that grad $(f+g)$ = grad f + grad g.

proof:- $\text{grad}(f+g) = \vec{\nabla}(f+g)$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right](f+g)$$

$$= \left[\frac{\partial}{\partial x}(f+g), \frac{\partial}{\partial y}(f+g), \frac{\partial}{\partial z}(f+g)\right]$$

$$= \left[\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}+\frac{\partial g}{\partial y}, \frac{\partial f}{\partial z}+\frac{\partial g}{\partial z}\right]$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] + \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]f + \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]g$$

$$= \vec{\nabla} f + \vec{\nabla} g$$

$\text{grad}(f+g) = \text{grad } f + \text{grad } g$. proved.

Theorem 76 Prove that grad $(fg) = f\,\text{grad}\,g + g\,\text{grad}\,f$

Proof:- $\text{grad}(fg) = \vec{\nabla}(fg)$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right](fg) = \left[\frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg), \frac{\partial}{\partial z}(fg)\right]$$

$$= \left[f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g, f\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}g, f\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}g\right]$$

$$= \left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right] + \left[\frac{\partial f}{\partial x}g, \frac{\partial f}{\partial y}g, \frac{\partial f}{\partial z}g\right]$$

$$= f\left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right] + g\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$= f\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]g + g\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]f$$

$$= f\,\vec{\nabla}g + g\,\vec{\nabla}f$$

$\text{grad}(fg) = f\,\text{grad}\,g + g\,\text{grad}\,f$. <u>proved</u>

Q12 78 Prove that

(i) $\text{grad}(fgh) = gh\,\text{grad}\,f + hf\,\text{grad}\,g + fg\,\text{grad}\,h$.

Sol:- $\text{grad}(fgh) = \text{grad}[\underline{f}(\underline{gh})]$

$$= f\,\text{grad}(gh) + gh\,\text{grad}\,f.$$

$$= f[g\,\text{grad}\,h + h\,\text{grad}\,g] + gh\,\text{grad}\,f$$

$$= fg\,\text{grad}\,h + fh\,\text{grad}\,g + gh\,\text{grad}\,f$$

rewritting in the reverse order we get

grad $(fgh) =$ gh grad f + hf grad g + fg grad h.

ii) $\text{grad}\left(\frac{f}{g}\right) = \frac{g \,\text{grad}\, f - f \,\text{grad}\, g}{g^2}$.

Sol:- $\text{grad}\left(\frac{f}{g}\right) = \text{grad}\; f\,(g)^{-1}$

$= f \,\text{grad}\, g^{-1} + g^{-1}\,\text{grad}\, f$

$= f\, \vec{\nabla} g^{-1} + \frac{1}{g}\,\text{grad}\, f$

$= f\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] g^{-1} + \frac{1}{g}\,\text{grad}\, f$

$= f\left[\frac{\partial}{\partial x} g^{-1}, \frac{\partial}{\partial y} g^{-1}, \frac{\partial}{\partial z} g^{-1}\right] + \frac{1}{g}\,\text{grad}\, f$

$= f\left[-g^{-2}\frac{\partial g}{\partial x}, -g^{-2}\frac{\partial g}{\partial y}, -g^{-2}\frac{\partial g}{\partial z}\right] + \frac{1}{g}\,\text{grad}\, f$

$= f\left(-\frac{1}{g^2}\right)\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] g + \frac{1}{g}\,\text{grad}\, f$

$= -\frac{f}{g^2}\vec{\nabla} g + \frac{1}{g}\,\text{grad}\, f$

$= -\frac{f}{g^2}\,\text{grad}\, g + \frac{1}{g}\,\text{grad}\, f$

$= \frac{-f\,\text{grad}\, g + g\,\text{grad}\, f}{g^2}$

$\text{grad}\left(\frac{f}{g}\right) = \frac{g\,\text{grad}\, f - f\,\text{grad}\, g}{g^2}$ proved

v. imp

Ex-4 / 77 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then evaluate grad r^n.

Sol: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = |\vec{r}| = (x^2+y^2+z^2)^{1/2}$$

$$r^n = \left[(x^2+y^2+z^2)^{1/2}\right]^n = (x^2+y^2+z^2)^{n/2}$$

$$\text{grad } r^n = \text{grad } (x^2+y^2+z^2)^{n/2} = \vec{\nabla}(x^2+y^2+z^2)^{n/2}$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right](x^2+y^2+z^2)^{n/2}$$

$$= \left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}, \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}, \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}\right]$$

$$= \left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x), \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2y), \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2z)\right]$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}[x, y, z]$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= n\left[(x^2+y^2+z^2)^{1/2}\right]^{n-2}\vec{r}$$

$$= n\, r^{n-2}\,\vec{r} \qquad \because (x^2+y^2+z^2)^{1/2} = r$$

$$\boxed{\text{grad } r^n = n\, r^{n-2}\,\vec{r}}$$ proved

Ch-05

ii/79 prove $div(c\vec{f}) = c \, div \, f$, c is constant.

Sol:- let $\vec{f} = [f_1, f_2, f_3]$

$$div(c\vec{f}) = \vec{\nabla} \cdot c[f_1, f_2, f_3]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [cf_1, cf_2, cf_3]$$

$$= \frac{\partial}{\partial x} cf_1 + \frac{\partial}{\partial y} cf_2 + \frac{\partial}{\partial z} cf_3$$

$$= c\left[\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}\right]$$

$$= c\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [f_1, f_2, f_3]$$

$div(c\vec{f}) = c \, div \vec{f}$. proved.

iii/79 prove $div(\phi\vec{f}) = \phi \, div f + grad \, \phi \cdot \vec{f}$.

let $\vec{f} = [f_1, f_2, f_3]$

$$div(\phi\vec{f}) = \vec{\nabla} \cdot \phi\vec{f} = \vec{\nabla} \cdot \phi[f_1, f_2, f_3]$$

$$= \vec{\nabla} \cdot [\phi f_1, \phi f_2, \phi f_3] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [\phi f_1, \phi f_2, \phi f_3]$$

$$= \frac{\partial}{\partial x}(\phi f_1) + \frac{\partial}{\partial y}(\phi f_2) + \frac{\partial}{\partial z}(\phi f_3)$$

$$= \phi \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial x} f_1 + \phi \frac{\partial f_2}{\partial y} + \frac{\partial}{\partial y} \phi f_2 + \phi \frac{\partial f_3}{\partial z} + \frac{\partial \phi}{\partial z} f_3$$

$$= \phi \frac{\partial f_1}{\partial x} + \phi \frac{\partial f_2}{\partial y} + \phi \frac{\partial f_3}{\partial z} + \frac{\partial \phi}{\partial x} f_1 + \frac{\partial \phi}{\partial x} f_1 + \frac{\partial \phi}{\partial y} f_2 + \frac{\partial \phi}{\partial z} f_3$$

$$\operatorname{div}(\phi\vec{F}) = \phi\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)+\left[\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right]\cdot[F_1,F_2,F_3]$$

$$= \phi\,\vec{\nabla}\cdot\vec{F} + \vec{\nabla}\phi\cdot\vec{F}$$

$\operatorname{div}(\phi\vec{F}) = \phi\operatorname{div}\vec{F} + \operatorname{grad}\phi\cdot\vec{F}$. proved.

CURL OF A VECTOR

let $\vec{F} = [F_1, F_2, F_3]$ be a vector, Then $\vec{\nabla}\times\vec{F}$ is called curl of $\vec{F}$ and denoted by Curl $\vec{F}$.

$$\operatorname{Curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Ex 83) Calculate Curl $\vec{F}$ at $(1,1,1)$ if

$$F(x,y,z) = x^2y\,\vec{i} + (4xz+y^2)\vec{j} + (5z^2+xy^2)\vec{k}.$$

Sol:- $\operatorname{Curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 4xz+y^2 & 5z^2+xy^2 \end{vmatrix}$

By R_1

$$\operatorname{Curl}\vec{F} = \vec{i}\left[\frac{\partial}{\partial y}(5z^2+xy^2) - \frac{\partial}{\partial z}(4xz+y^2)\right]$$

$$+\vec{j}\left[\frac{\partial}{\partial z}(x^2-y)-\frac{\partial}{\partial x}(5z^2+xy^2)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(4xz+y^2)-\frac{\partial}{\partial y}(x^2y)\right]$$

$$=\vec{i}\left[(0+2xy)-(4x+0)\right]+\vec{j}\left[0-(0+y^2)\right]$$

$$+\vec{k}\left[(4z+0)-x^2\right]$$

at $P(1,1,1)$.

$$\left(\text{Curl}\,\vec{F}\right)_P=(2\cdot1-4\cdot1)\vec{i}+\vec{j}\left(-(1)^2\right)+\vec{k}(4\cdot1-1^2)$$

$$\left(\text{Curl}\,\vec{F}\right)_P=-2\vec{i}-\vec{j}+3\vec{k}\quad \underline{\text{Ans}}$$

EXERCISE #5.4

Calculate Curl $\vec{F}$ if

Q1/84 $\vec{F}=(y^2+z^2)\vec{i}+(z^2+x^2)\vec{j}+(x^2+y^2)\vec{k}$.

Sol:- $\text{Curl}\,\vec{F}=\vec{\nabla}\times\vec{F}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ y^2+z^2&z^2+x^2&x^2+y^2\end{vmatrix}$

$$=\vec{i}\left[\frac{\partial}{\partial y}(x^2+y^2)-\frac{\partial}{\partial z}(z^2+x^2)\right]+\vec{j}\left[\frac{\partial}{\partial z}(y^2+z^2)-\frac{\partial}{\partial x}(x^2+y^2)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(z^2+x^2)-\frac{\partial}{\partial y}(y^2+z^2)\right]$$

$$=\vec{i}\left[2y-2z\right]+\vec{j}\left[2z-2x\right]+\vec{k}\left[2x-2y\right]$$

$$\text{Curl}\,\vec{F}=2(y-z)\vec{i}+2(z-x)\vec{j}+2(x-y)\vec{k}.\quad\underline{\text{Ans}}$$

Q2/84 $\vec{F} = (x^2+y^2+z^2)^{-\frac{1}{2}}(yz\vec{i} + zx\vec{j} + xy\vec{K})$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2+z^2)^{-\frac{1}{2}}yz & (x^2+y^2+z^2)^{-\frac{1}{2}}zx & (x^2+y^2+z^2)^{-\frac{1}{2}}xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}xy - \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}yz\right]$$

$$+\vec{j}\left[\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}yz + \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}xy\right]$$

$$+\vec{K}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}zx - \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}yz\right]$$

$$= \vec{i}\Big[x\left\{(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 1 + y\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}(2y)\right\}$$

$$- y\left\{(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 1 + z\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}2z\right\}$$

$$+\vec{j}\Big[x\left\{(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 1 + z\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2z\right.$$

$$- y\left\{(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 1 + x\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}2x\right\}\Big]$$

$$+\vec{K}\Big[z\left\{(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 1 + x\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)\right. \cancel{z}$$

$$- z\left\{(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 1 + y\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2y\right.\Big]$$

$$= \vec{i}\,(x^2+y^2+z^2)^{-\frac{3}{2}}\left[x\{x^2+y^2+z^2-y^2\} - y\{x^2+y^2+\cancel{z^2}-\cancel{z^2}\}\right]$$

$$+\vec{j}\,(x^2+y^2+z^2)^{-\frac{3}{2}}\left[y\{x^2+y^2+z^2-z^2\} - y\{\cancel{x^2}+y^2+z^2-\cancel{x^2}\}\right]$$

$$+\vec{K}\,(x^2+y^2+z^2)^{-\frac{3}{2}}\left[z\{x^2+y^2+z^2\} - x^2\{-z\{x^2+\cancel{y^2}+z^2-\cancel{y^2}\}\right]$$

$$= (x^2+y^2+z^2)^{-3/2}\left[x(\cancel{x^2}+z^2-\cancel{x^2}-y^2)\right]\vec{i} + y(x^2+\cancel{y^2}-\cancel{y^2}-z^2)\vec{j}$$
$$+ z(y^2+z^2-x^2-z^2)\vec{k}\,]$$

$$= (x^2+y^2+z^2)^{-3/2}\left[x(z^2-y^2)\vec{i} + y(x^2-z^2)\vec{j} + z(y^2-x^2)\vec{k}\right.$$

Ans

Q 3 / 84 $\vec{F} = yz\cos x\,\vec{i} + zx\cos y\,\vec{j} + xy\cos z\,\vec{k}$

$$\text{Curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz\cos x & zx\cos y & xy\cos z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy\cos z) - \frac{\partial}{\partial z}(zx\cos y)\right] + \vec{j}\left[\frac{\partial}{\partial z}(yz\cos x) - \frac{\partial}{\partial x}(xy\cos z)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(zx\cos y) - \frac{\partial}{\partial y}(yz\cos x)\right]$$

$$= \vec{i}\left[x\cos z - x\cos y\right] + \vec{j}\left[y\cos x - y\cos z\right]$$
$$+ \vec{k}\left[z\cos y - z\cos x\right].$$

$$\text{Curl}\,\vec{F} = x(\cos z - \cos y)\vec{i} + y(\cos x - \cos z)\vec{j} + z(\cos y - \cos x)\vec{k}$$

Ans

Q4/84 $\vec{F} = x(y^2+z^2)\vec{i} + y(z^2+x^2)\vec{j} + z(x^2+y^2)\vec{k}$.

$\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F}$.

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(y^2+z^2) & y(z^2+x^2) & z(x^2+y^2) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y} z(x^2+y^2) - \frac{\partial}{\partial z} y(z^2+x^2)\right] + \vec{j}\left[\frac{\partial}{\partial x} x(y^2+z^2) - \frac{\partial}{\partial x} z(x^2+y^2)\right] + \vec{k}\left[\frac{\partial}{\partial x} y(z^2+x^2) - \frac{\partial}{\partial y} x(y^2+z^2)\right]$$

$$\text{Curl}\,\vec{F} = [z(\cancel{x^2}+2y) - y(2z+\cancel{x^2})]\,\vec{i} + [\cancel{x}(y^2+z^2) - z(2x+y^2)]\,\vec{j} + [y(z^2+2x) - x(2y+z^2)]\,\vec{k}$$

$$= [x^2z + 2yz - 2yz - x^2y]\,\vec{i} + [\cancel{xy^2+xz^2-2xz-zy^2}]\,\vec{j} + \vec{k}\,[yz^2 + 2xy - 2xy - xz^2]$$

$$\text{Curl}\,\vec{F} = x^2(z-y)\,\vec{i} + y^2(x-z)\,\vec{j} + z^2(y-x)\,\vec{k}$$

Ans

Q5/84 $\vec{F} = yz\log x\,\vec{i} + zx\log y\,\vec{j} + xy\log z\,\vec{k}$

$$\text{Curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz\log x & zx\log y & xy\log z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy\log z) - \frac{\partial}{\partial z}(zx\log y)\right]$$

$$+\vec{j}\left[\frac{\partial}{\partial z}(yz\log x) - \frac{\partial}{\partial x}(xy\log z)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(zx\log y) - \frac{\partial}{\partial y}(yz\log x)\right]$$

$$= [x\log z - x\log y]\vec{i} + [y\log x - y\log z]\vec{j} + [z\log y - z\log x]\vec{k}$$

$$= x(\log z - \log y)\vec{i} + y(\log x - \log z)\vec{j} + z(\log y - \log x)\vec{k}$$

$$\text{Curl}\,\vec{F} = x\log\frac{z}{y}\,\vec{i} + y\log\frac{x}{z}\,\vec{j} + z\log\frac{y}{x}\,\vec{k}$$

Ans

v.v.imp

(VII/84) Prove $\vec{\nabla}\times(b\vec{f}+c\vec{g}) = b\vec{\nabla}\times\vec{f}+c\,\vec{\nabla}\times\vec{g}$

Proof: $\vec{f}=[f_1, f_2, f_3]$, $\vec{g}=[g_1, g_2, g_3]$

$$\vec{\nabla}\times(b\vec{f}+c\vec{g}) = \vec{\nabla}\times\left[b[f_1, f_2, f_3]+c[g_1, g_2, g_3]\right]$$

$$= \vec{\nabla}\times\left[[bf_1, bf_2, bf_3]+[cg_1, cg_2, cg_3]\right]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\times\left[bf_1+cg_1, bf_2+cg_2, bf_3+cg_3\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bf_1+cg_1 & bf_2+cg_2 & bf_3+cg_3 \end{vmatrix}$$

By R_1

$$= \vec{i}\left[\frac{\partial}{\partial y}(bf_3+cg_3)-\frac{\partial}{\partial z}(bf_2+cg_2)\right]+\vec{j}\left[\frac{\partial}{\partial z}(bf_1+cg_1)-\frac{\partial}{\partial x}(bf_3+cg_3)\right]+\vec{k}\left[\frac{\partial}{\partial x}(bf_2+cg_2)-\frac{\partial}{\partial y}(bf_1+cg_1)\right]$$

$$= \vec{i}\left[b\frac{\partial f_3}{\partial y}+c\frac{\partial g_3}{\partial y}-b\frac{\partial f_2}{\partial z}-c\frac{\partial g_2}{\partial z}\right]$$
$$+\vec{j}\left[b\frac{\partial f_1}{\partial z}+c\frac{\partial g_1}{\partial z}-b\frac{\partial f_3}{\partial x}-c\frac{\partial g_3}{\partial x}\right]$$
$$+\vec{k}\left[b\frac{\partial f_2}{\partial x}+c\frac{\partial g_2}{\partial x}-b\frac{\partial f_1}{\partial y}-c\frac{\partial g_1}{\partial y}\right]$$

$$= \vec{i}\left[b\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\right] + \vec{i}\left[c\left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right)\right]$$

$$+\vec{j}\left[b\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right] + \vec{j}\left[c\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right)\right]$$

$$+\vec{k}\left[b\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) + \vec{k}\left[c\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_1}{\partial x}\right)\right]\right.$$

$$= b\left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\vec{k}\right]$$

$$+c\left[\left(\frac{\partial g_1}{\partial y} - \frac{\partial g_2}{\partial z}\right)\vec{i} + \left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right)\vec{j} + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial x}\right)\vec{k}\right]$$

$$\vec{\nabla}\times(b\vec{f} + c\vec{g}) = b\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} + c\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$\vec{\nabla}\times(b\vec{f} + c\vec{g}) = b\,\vec{\nabla}\times\vec{f} + c\,\vec{\nabla}\times\vec{g}$$ <u>Proved</u>

---

v.v.imp (viii) prove $\vec{\nabla}\times(\phi\vec{f}) = \phi\,\vec{\nabla}\times\vec{f} + (\vec{\nabla}\phi)\times\vec{f}$

proof:- $\vec{f} = [f_1, f_2, f_3]$

$$\vec{\nabla}\times(\phi\vec{f}) = \vec{\nabla}\times[\phi[f_1, f_2, f_3]]$$

$$= \vec{\nabla}\times[\phi f_1, \phi f_2, \phi f_3]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi f_1 & \phi f_2 & \phi f_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(\phi f_3) - \frac{\partial}{\partial z}(\phi f_2)\right] + \vec{j}\left[\frac{\partial}{\partial z}(\phi f_1) - \frac{\partial}{\partial x}(\phi f_3)\right] + \vec{K}\left[\frac{\partial}{\partial x}(\phi f_2) - \frac{\partial}{\partial y}(\phi f_1)\right]$$

$$= \vec{i}\left[\phi\frac{\partial f_3}{\partial y} + \frac{\partial \phi}{\partial y} f_3 - \phi\frac{\partial f_2}{\partial z} - \frac{\partial \phi}{\partial z} f_2\right] + \vec{j}\left[\phi\frac{\partial f_1}{\partial z} + \frac{\partial \phi}{\partial z} f_1 - \phi\frac{\partial f_3}{\partial x} - \frac{\partial \phi}{\partial x} f_3\right] + \vec{K}\left[\phi\frac{\partial f_2}{\partial x} + \frac{\partial \phi}{\partial x} f_2 - \phi\frac{\partial f_1}{\partial y} - \frac{\partial \phi}{\partial y} f_1\right] \quad (1)$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{F} = \vec{i}\left[\frac{\partial}{\partial y} f_3 - \frac{\partial f_2}{\partial z}\right] + \vec{j}\left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right] + \vec{K}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$\phi\vec{\nabla}\times\vec{F} = \vec{i}\left[\phi\frac{\partial f_3}{\partial y} - \phi\frac{\partial f_2}{\partial z}\right] + \vec{j}\left[\phi\frac{\partial f_1}{\partial z} - \phi\frac{\partial f_3}{\partial x}\right] + \vec{K}\left[\phi\frac{\partial f_2}{\partial x} - \phi\frac{\partial f_1}{\partial y}\right] \quad (2)$$

$$(\vec{\nabla}\phi)\times\vec{F} = \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right]\times\left[f_1, f_2, f_3\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$(\vec{\nabla}\phi)\times\vec{F} = \vec{i}\left[f_3\frac{\partial\phi}{\partial y} - f_2\frac{\partial\phi}{\partial z}\right] + \vec{j}\left[f_1\frac{\partial\phi}{\partial z} - f_3\frac{\partial\phi}{\partial x}\right] + \vec{k}\left[f_2\frac{\partial\phi}{\partial x} - f_1\frac{\partial\phi}{\partial y}\right] \quad (3)$$

(2) + (3)

$$\phi\vec{\nabla}\times\vec{F} + (\vec{\nabla}\phi)\times\vec{F} = \vec{i}\left[\phi\frac{\partial f_3}{\partial y} + \frac{\partial\phi}{\partial y}f_3 - \phi\frac{\partial f_2}{\partial z} - \frac{\partial\phi}{\partial z}f_3\right] + \vec{j}\left[\phi\frac{\partial f_1}{\partial z} + \frac{\partial\phi}{\partial z}f_1 - \phi\frac{\partial f_3}{\partial x} - \frac{\partial\phi}{\partial x}f_3\right] + \vec{k}\left[\phi\frac{\partial f_2}{\partial x} + \frac{\partial\phi}{\partial x}f_2 - \phi\frac{\partial f_1}{\partial y} - \frac{\partial\phi}{\partial y}f_1\right] \quad (4)$$

(1) = (4)

So

$$\vec{\nabla}\times(\phi\vec{F}) = \phi\,\vec{\nabla}\times\vec{F} + (\vec{\nabla}\phi)\times\vec{F}$$

proved