CHAPTER#3 VECTOR CALCULUS

EXERCISE # 3.1

Q1/38 Calculate $|f(t)|$ if $\vec{f}(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + 8\vec{k}$

Sol:- $\vec{f}(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + 8\vec{k}$

$$|\vec{f}(t)| = \sqrt{\sin^2 t + \cos^2 t + 8^2} = \sqrt{1+64}$$

$$\boxed{|\vec{f}(t)| = \sqrt{65}}$$ Ans

Q3/38 If $\vec{f}(t)$ and $\vec{g}(t)$ are vector functions, then which of (i) $\vec{f}(t)+\vec{g}(t)$ (ii) $\vec{f}(t)\cdot\vec{g}(t)$ and (iii) $\vec{f}(t)\times\vec{g}(t)$ is a vector functions or a scalar functions?

Sol:- (i) let $\vec{f}(t) = [f_1, f_2, f_3]$

$$\vec{g}(t) = [g_1, g_2, g_3]$$

$$\vec{f}(t)+\vec{g}(t) = [f_1, f_2, f_3] + [g_1, g_2, g_3]$$

$$\boxed{\vec{f}(t)+\vec{g}(t) = [f_1+g_1, f_2+g_2, f_3+g_3]}$$

$\therefore$ $\vec{f}(t)+\vec{g}(t)$ is a vector function.

P.T.O

(ii) $\vec{f}(t)\cdot\vec{g}(t) = [f_1, f_2, f_3]\cdot[g_1, g_2, g_3]$

$$\boxed{\vec{f}(t)\cdot\vec{g}(t) = f_1 g_1 + f_2 g_2 + f_3 g_3}$$

$\therefore$ $\vec{f}(t)\cdot\vec{g}(t)$ is a scalar function.

(iii) $\vec{f}(t)\times\vec{g}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$

$$\vec{f}(t)\times\vec{g}(t) = \vec{i}\,[f_2 g_3 - f_3 g_2] + \vec{j}\,[f_3 g_1 - f_1 g_3] + \vec{k}\,[f_1 g_2 - f_2 g_1]$$

$\therefore$ $\vec{f}(t)\times\vec{g}(t)$ is a vector function.

(Q4/38) If $\vec{f}(t) = \vec{i} + (2\tan t)\vec{j} + (2\tan^2 t)\vec{k}$. Find $|\vec{f}(t)|$.

Sol:- $\vec{f}(t) = \vec{i} + (2\tan t)\vec{j} + (2\tan^2 t)\vec{k}$

$$|\vec{f}(t)| = \sqrt{1^2 + (2\tan t)^2 + (2\tan^2 t)^2}$$

$$|\vec{f}(t)| = \sqrt{1 + 4\tan^2 t + 4\tan^4 t}$$

$$= \sqrt{(1 + 2\tan^2 t)^2}$$

$$\boxed{|\vec{f}(t)| = 1 + 2\tan^2 t}$$ Ans

Q5/38 Given $\vec{f}(t) = t^2\vec{i} + (t-1)\vec{j} + (t+t^2+1)\vec{k}$
and $\vec{g}(t) = (t^2+1)\vec{i} + t\vec{j} - \vec{k}$, Find
$\vec{f}(t)\cdot\vec{g}(t)$ and $\vec{f}(t)\times\vec{g}(t)$

Sol:- (i) $\vec{f}(t) = t^2\vec{i} + (t-1)\vec{j} + (t+t^2+1)\vec{k}$

$$\vec{g}(t) = (t^2+1)\vec{i} + t\vec{j} - \vec{k}$$

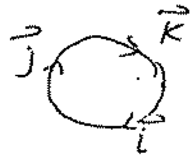
$$\vec{f}(t)\cdot\vec{g}(t) = [t^2, t-1, t^2+t+1]\cdot[t^2+1, t, -1]$$

$$= t^2(t^2+1) + (t-1)t + (t^2+t+1)(-1)$$

$$= t^4 + t^2 + \cancel{t^2} - t - \cancel{t^2} - t - 1$$

$$\boxed{\vec{f}(t)\cdot\vec{g}(t) = t^4 + t^2 - 2t - 1}$$ Ans

(ii) $\vec{f}(t)\times\vec{g}(t)$



$$\vec{f}(t)\times\vec{g}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & t-1 & t+t^2+1 \\ t^2+1 & t & -1 \end{vmatrix}$$

$$= \vec{i}\{-1(t-1) - t(t+t^2+1)\} + \vec{j}\{(t^2+1)(t+t^2+1) + t^2\} + \vec{k}\{t^3 - (t-1)(t^2+1)\}$$

$$= \vec{i}(-t+1-t^2-t^3-t) + \vec{j}(t^3+t^4+t^2+t+t^2+1+t^2) + \vec{k}(t^3-(t^3+t-t^2-1))$$

$$= \vec{i}(-t^3-t^2-2t+1) + \vec{j}(t^4+t^3+3t^2+t+1) + \vec{k}(\cancel{t^3}-\cancel{t^3}-(t-t^2-1))$$

$$\vec{f}(t)\times\vec{g}(t) = \vec{i}(-t^3-t^2-2t+1) + \vec{j}(t^4+t^3+3t^2+t+1) + \vec{k}(t^2-t+1)$$ Ans

EXERCISE # 3.4 v.v. important.

Q 1 / 47 If $\vec{F}(t) = (2t+1)\vec{i} + (3-2t^2)\vec{j} + (t^2+1)\vec{k}$

and $\vec{g}(t) = (3+2t^2)\vec{i} + (3t+1)\vec{j} + (2t-t^3)\vec{k}$.

Calculate $\frac{d}{dt}(\vec{F}+\vec{g})$ ...

Sol:- $\vec{F}(t) = (2t+1)\vec{i} + (3-2t)\vec{j} + (t^2+1)\vec{k}$

$$\boxed{\vec{F}'(t) = 2\vec{i} + (-4t)\vec{j} + 2t\,\vec{k}}$$

$\vec{g}(t) = (3+2t^2)\vec{i} + (3t+1)\vec{j} + (2t-t^3)\vec{k}$

$$\boxed{\vec{g}'(t) = 4t\,\vec{i} + 3\vec{j} + (2-3t^2)\vec{k}}$$

$$\frac{d}{dt}(\vec{F}+\vec{g}) = \frac{d}{dt}\vec{F} + \frac{d}{dt}\vec{g}$$

$$= \vec{F}'(t) + \vec{g}'(t)$$

NOTE

$\vec{F}'(t) = \frac{d}{dt}\vec{F}(t)$

Dash notation is used for Derivative w.r.t "t" in this book.

$$= 2\vec{i} - 4t\vec{j} + 2t\vec{k} + 4t\,\vec{i} + 3\vec{j} + (2-3t^2)\vec{k}$$

$$= (2+4t)\vec{i} + (-4t+3)\vec{j} + (2t+2-3t^2)\vec{k}$$

$$\boxed{\frac{d}{dt}(\vec{F}+\vec{g}) = (2+4t)\vec{i} + (3-4t)\vec{j} + (2+2t-3t^2)\vec{k}}$$

Ans

Ex - 1 / 46 If $\vec{F}(t) = (2t+3)\vec{i} + \sin 2t\,\vec{j} + \log t\,\vec{k}$ and

$\vec{g}(t) = (t^3-6)\vec{i} - 2t\,\vec{j} + (3t^2+5)\vec{k}$

Calculate $\frac{d}{dt}(\vec{F}+\vec{g})$.

Sol:- $\vec{F}(t) = (2t+3)\vec{i} + \sin 2t\,\vec{j} + \log t\,\vec{k}$.

$$\vec{F}'(t) = 2\vec{i} + \cos 2t\,\vec{j} + \left(\frac{1}{t}\right)\vec{k}$$

$$\vec{g}(t) = (t^3 - t)\vec{i} - 2t\,\vec{j} + (3t^2+5)\vec{k}$$

$$\boxed{\vec{g}'(t) = 3t^2\vec{i} - 2\vec{j} + 6t\,\vec{k}}$$

$$\frac{d}{dt}(\vec{F}+\vec{g}) = f'(t) + g'(t)$$

$$= 2\vec{i} + 3t^2\vec{i} + 2\cos 2t\,\vec{j} - 2\vec{j} + \frac{1}{t}\vec{k} + 6t\vec{k}$$

$$\boxed{\frac{d}{dt}(\vec{F}+\vec{g}) = (3t^2+2)\vec{i} + 2(\cos 2t - 1)\vec{j} + \left(6t + \frac{1}{t}\right)\vec{k}}$$

FORMULAS

S = Scalar.

① $\frac{d}{dt}[S\vec{F}] = S\,\vec{F}' + S'\,\vec{F}$

② $\frac{d}{dt}[\vec{F}\cdot\vec{g}] = \vec{F}\cdot\vec{g}' + \vec{F}'\cdot\vec{g}$

③ $\frac{d}{dt}[\vec{F}\times\vec{g}] = \vec{F}\times\vec{g}' + \vec{F}'\times\vec{g}$

④ $\frac{d}{dt}\left(\frac{\vec{F}}{S}\right) = \frac{S\vec{F}' - \vec{F}S'}{S^2}$

v.v.imp

Q2/47 If $\vec{F}(t) = e^{2t}\vec{i} + e^{-2t}\vec{j} + \log t\,\vec{k}$. Find the derivative of $\dfrac{\vec{F}}{te^{t}}$

Sol:- $\vec{F}(t) = e^{2t}\vec{i} + e^{-2t}\vec{j} + \log t\,\vec{k}$

$\vec{F}'(t) = 2e^{2t}\vec{i} - 2e^{-2t}\vec{j} + \frac{1}{t}\vec{k}$

$$\frac{d}{dt}\left(\frac{\vec{F}}{te^{t}}\right) = \frac{(te^{t})\vec{F}' - \vec{F}\frac{d}{dt}(te^{t})}{(te^{t})^{2}}$$

$$= \frac{1}{t^{2}e^{2t}}\left[(te^{t})\left(2e^{2t}\vec{i} - 2e^{-2t}\vec{j} + \frac{1}{t}\vec{k}\right) - (te^{t} + e^{t}\cdot 1)\left(e^{2t}\vec{i} + e^{-2t}\vec{j} + \log t\,\vec{k}\right)\right]$$

$$= \frac{e^{t}}{t^{2}e^{2t}}\left[2te^{2t}\vec{i} - 2te^{-2t}\vec{j} + e^{t}\vec{k} - (t+1)\left(e^{2t}\vec{i} + e^{-2t}\vec{j} + \log t\,\vec{k}\right)\right]$$

$$= \frac{1}{t^{2}e^{t}}\left[\begin{array}{l} 2te^{2t}\vec{i} - 2te^{-2t}\vec{j} + \vec{k} \\ -(t+1)e^{2t}\vec{i} - (t+1)e^{-2t}\vec{j} - (t+1)\log t\,\vec{k} \end{array}\right]$$

$$= \frac{1}{t^{2}e^{t}}\left[(2te^{2t} - te^{2t} - e^{2t})\vec{i} - (2t+t+1)e^{-2t}\vec{j} + \{1-(t+1)\log t\}\vec{k}\right]$$

$$= \frac{1}{t^{2}e^{t}}\left[(te^{2t} - e^{2t})\vec{i} - (3t+1)e^{-2t}\vec{j} + \{1-(t+1)\log t\}\vec{k}\right]$$

$$= \frac{(t-1)e^{2t}\vec{i}}{t^{2}e^{t}} - \frac{(3t+1)e^{-2t}}{t^{2}e^{t}}\vec{j} + \left\{\frac{1-(t+1)\log t}{t^{2}e^{t}}\right\}\vec{k}$$

$$\frac{d}{dt}\left(\frac{\vec{F}}{te^{t}}\right) = \frac{(t-1)e^{t}\vec{i}}{t^{2}} - \frac{(3t+1)e^{-2t}}{t^{2}}\vec{j} + \frac{e^{-t}\{1-(t+1)\log t\}\vec{k}}{t^{2}}$$

Ans

(Ex-3/46) Show that if $\vec{h}(t) = \dfrac{\vec{f}(t)}{g(t)}$ where $\vec{f}(t)$ is a vector function and $g(t)$ is a scalar non-zero function, then $\vec{h}'(t) = \dfrac{g(t)\vec{f}'(t) - g'(t)\vec{f}(t)}{g^2(t)}$

Sol:- $\vec{h}(t) = \vec{f}(t)[g(t)]^{-1}$

$$\vec{h}'(t) = \vec{f}(t)(-1)[g(t)]^{-2} g'(t) + [g(t)]^{-1}\vec{f}'(t)$$

$$= \frac{-\vec{f}(t)\, g'(t)}{[g(t)]^2} + \frac{\vec{f}'(t)}{g(t)}$$

$$= \frac{-\vec{f}(t)\, g'(t) + \vec{f}'(t)\, g(t)}{[g(t)]^2}$$

$$\vec{h}'(t) = \frac{\vec{f}'(t)\, g(t) - \vec{f}(t)\, g'(t)}{[g(t)]^2} \quad \underline{\text{Ans}}$$

(Q3/47) If $\vec{f}(t) = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, Show that $\vec{f} \times \vec{f}' = \omega\, \vec{a} \times \vec{b}$

Sol:- $\vec{f}(t) = \vec{a}\cos\omega t + \vec{b}\sin\omega t$

$$\boxed{\vec{f}'(t) = \vec{a}(-\omega \sin\omega t) + \vec{b}\,\omega\cos\omega t}$$

Now

$$\vec{f}(t) \times \vec{f}'(t) = (\vec{a}\cos\omega t + \vec{b}\sin\omega t) \times \omega(-\vec{a}\sin\omega t + \vec{b}\cos\omega t)$$

$$= \omega(\vec{a}\cos\omega t + \vec{b}\sin\omega t) \times (-\vec{a}\sin\omega t + \vec{b}\cos\omega t)$$

P.T.O

$= \omega[-\vec{a}\times\vec{a}\cos\omega t\sin\omega t + \vec{a}\times\vec{b}\cos^2\omega t - \vec{b}\times\vec{a}\sin^2\omega t$
$+ \vec{b}\times\vec{b}\sin\omega t\cos\omega t]$

$= \omega[0 + \vec{a}\times\vec{b}\cos^2\omega t + \vec{a}\times\vec{b}\sin^2\omega t + 0]$

As $\vec{a}\times\vec{a} = \vec{b}\times\vec{b} = 0$

$= \omega\,\vec{a}\times\vec{b}(\cos^2\omega t + \sin^2\omega t)$

$= \omega\,\vec{a}\times\vec{b}(1)$

$\Rightarrow \vec{f}(t)\times\vec{f}'(t) = \omega\,\vec{a}\times\vec{b}$ proved.

Q6/48. Prove that if $\vec{f}(t)$ is a vector function of t, $\frac{d}{dt}(\vec{f}\times\vec{f}') = \vec{f}\times\vec{f}''$

Sol:- $\frac{d}{dt}(\vec{f}\times\vec{f}') = \vec{f}\times\vec{f}'' + \vec{f}'\times\vec{f}'$

$= \vec{f}\times\vec{f}'' + 0 \quad \because \vec{f}'\times\vec{f}' = 0$

$$\boxed{\frac{d}{dt}(\vec{f}\times\vec{f}') = \vec{f}\times\vec{f}''}$$

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Q 7/48 If $\vec{F}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\,\vec{k}$

Prove that

(i) $|\vec{F}'|^2 = a^2 + b^2$ (ii) $|\vec{F}' \times \vec{F}''|^2 = a^2(a^2+b^2)$

(iii) $\vec{F}' \cdot \vec{F}'' \times \vec{F}''' = a^2 b$.

Sol:- $\vec{F} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\,\vec{k}$

$$\vec{F}' = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + b\vec{k}$$

$$\vec{F}'' = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0$$

$$\vec{F}''' = a\sin t\,\vec{i} - a\cos t\,\vec{j}$$

(i)

$$|\vec{F}'| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}$$ Squaring

$$|\vec{F}'|^2 = a^2\sin^2 t + a^2\cos^2 t + b^2$$

$$|\vec{F}'|^2 = a^2(\sin^2 t + \cos^2 t) + b^2$$

$$|\vec{F}'|^2 = a^2 + b^2. \text{ Ans.}$$

(ii) To find $|\vec{F}' \times \vec{F}''|^2$

$$\vec{F}' \times \vec{F}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$ Expanding by R_1

$$|\vec{F}' \times \vec{F}''| = \vec{i}\,[0 + ab\sin t] + \vec{j}\,[-ab\cos t - 0] + \vec{k}\,[a^2\sin^2 t + a^2\cos^2 t]$$

$$= ab\sin t\,\vec{i} - ab\cos t\,\vec{j} + a^2(\sin^2 t + \cos^2 t)\,\vec{k}$$

P.T.O

$$= \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + (a^2)^2}$$

Squaring

$$|\vec{r}' \times \vec{r}''|^2 = a^2b^2(\sin^2 t + \cos^2 t) + a^4$$

$$|\vec{r}' \times \vec{r}''|^2 = a^2(a^2 + b^2)$$

(iii) To find $\vec{r}' \cdot \vec{r}'' \times \vec{r}'''$

$$\vec{r}' \cdot \vec{r}'' \times \vec{r}''' = \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= \cancel{(-a\sin t)(0-0)} + \cancel{a\cos t(0-0)} + b(a^2\cos^2 t + a^2\sin^2 t)$$

$$= a^2 b \quad \text{Ans}$$

FORMULAS

① Scalar triple :- $(\vec{a} \cdot \vec{b} \times \vec{c}) = \begin{vmatrix} \text{Component of 1st} \\ \text{'' \quad '' \quad 2nd} \\ \text{'' \quad '' \quad 3rd} \end{vmatrix}$

② Vector triple :- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

③ Dot & cross i.e '$\cdot$' & '$\times$' are interchangable in scalar triple.

(Q8/48) Show that, For a vector function $\vec{F}(t)$

$$\frac{d}{dt}\left(\vec{F}\cdot\vec{F}'\times\vec{F}''\right) = \vec{F}\cdot\vec{F}''\times\vec{F}'''$$

Sol:- $\frac{d}{dt}\left[\vec{F}\cdot\vec{F}'\times\vec{F}''\right] = \frac{d}{dt}\left[\vec{F}\cdot\left(\vec{F}'\times\vec{F}''\right)\right]$

$$= \vec{F}\cdot\frac{d}{dt}\left(\vec{F}'\times\vec{F}''\right) + \vec{F}'\cdot\vec{F}'\times\vec{F}''$$

$$= \vec{F}\cdot\left[\vec{F}'\times\vec{F}''' + \vec{F}''\times\vec{F}''\right] + \vec{F}'\times\vec{F}'\cdot\vec{F}''$$

$$= \vec{F}\cdot\left[\vec{F}'\times\vec{F}''' + 0\right] + 0$$

as $\vec{F}''\times\vec{F}'' = 0$ & $\vec{F}'\times\vec{F}' = 0$.

$$\frac{d}{dt}\left(\vec{F}\cdot\vec{F}'\times\vec{F}''\right) = \vec{F}\cdot\vec{F}'\times\vec{F}''' \quad \text{Ans}$$

NOTE:- There is no concept about $\frac{\text{vector}}{\text{vector}}$

(Q9/48) Differentiate $\frac{(\vec{F}+\vec{a})}{(\vec{F}^2+\vec{a}^2)}$ (scalar) and $\frac{(\vec{F}\times\vec{a})}{(\vec{F}\cdot\vec{a})^2}$ (scalar) where $\vec{F}$ is a vector function of t and $\vec{a}$ is a constant vector.

Sol:- (i) $$\frac{d}{dt}\left(\frac{\vec{F}+\vec{a}}{\vec{F}^2+\vec{a}^2}\right) = \frac{(\vec{F}^2+\vec{a}^2)\frac{d}{dt}(\vec{F}+\vec{a}) - (\vec{F}+\vec{a})\frac{d}{dt}(\vec{F}^2+\vec{a}^2)}{(\vec{F}^2+\vec{a}^2)^2}$$

$$= \frac{1}{(\vec{F}^2+\vec{a}^2)^2}\left[(\vec{F}^2+\vec{a}^2)(\vec{F}'+0) - (\vec{F}+\vec{a})\frac{d}{dt}(\vec{F}\cdot\vec{F}+\vec{a}^2)\right.$$

P.T.O

$$= \frac{1}{(\vec{r}^2+\vec{a}^2)^2}\left[(\vec{r}^2+\vec{a}^2)\vec{r}' - (\vec{r}+\vec{a})[\vec{r}\cdot\vec{r}'+\vec{r}\cdot\vec{r}'+0]\right]$$

As $\frac{d}{dt}(\vec{a}^2)=0$.

$$= \frac{(\vec{r}^2+\vec{a}^2)\vec{r}' - 2\vec{r}\cdot\vec{r}'(\vec{r}+\vec{a})}{(\vec{r}^2+\vec{a}^2)^2}$$

(ii) $$\frac{d}{dt}\left(\frac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}\right) = \frac{\vec{r}\cdot\vec{a}\,\frac{d}{dt}[\vec{r}\times\vec{a}] - \vec{r}\times\vec{a}\,\frac{d}{dt}(\vec{r}\cdot\vec{a})}{(\vec{r}\cdot\vec{a})^2}$$

$$= \frac{\vec{r}\cdot\vec{a}\cdot\vec{r}'\times\vec{a} - \vec{r}\times\vec{a}\cdot\vec{r}'\cdot\vec{a}}{(\vec{r}\cdot\vec{a})^2}$$

$$= \frac{(\vec{r}\cdot\vec{a})(\vec{r}'\times\vec{a}) - (\vec{r}\times\vec{a})(\vec{r}'\cdot\vec{a})}{(\vec{r}\cdot\vec{a})^2}$$

v.v.imp

Q 10 / 48 Differentiate $\dfrac{(\vec{i}+t\vec{j}+t^2\vec{k})}{|\vec{i}+t\vec{j}+t^2\vec{k}|}$

Sol:-
$$\frac{d}{dt}\frac{(\vec{i}+t\vec{j}+t^2\vec{k})}{|\vec{i}+t\vec{j}+t^2\vec{k}|} = \frac{d}{dt}\frac{(\vec{i}+t\vec{j}+t^2\vec{k})}{\sqrt{1^2+t^2+t^4}}$$

$$= \frac{\sqrt{1+t^2+t^4}\,\frac{d}{dt}(1.\vec{i}+t\vec{j}+t^2\vec{k}) - (\vec{i}+t\vec{j}+t^2\vec{k})\frac{d}{dt}(1+t^2+t^4)^{1/2}}{(\sqrt{1+t^2+t^4})^2}$$

$$= \frac{\sqrt{1+t^2+t^4}\,(\vec{j}+2t\vec{k}) - \frac{2t+4t^3}{2\sqrt{1+t^2+t^4}}(\vec{i}+t\vec{j}+t^2\vec{k})}{(1+t^2+t^4)}$$

$$= \frac{1}{1+t^2+t^4}\left[\sqrt{1+t^2+t^4}\,(\vec{j}+2t\vec{k}) - \frac{t+2t^3}{\sqrt{1+t^2+t^4}}(\vec{i}+t\vec{j}+t^2\vec{k})\right]$$

$$= \frac{(1+t^2+t^4)(\vec{j}+2t\vec{k}) - (t+2t^3)(\vec{i}+t\vec{j}+t^2\vec{k})}{(1+t^2+t^4)\sqrt{1+t^2+t^4}}$$

$$= \frac{(1+t^2+t^4)\vec{j} - t(t+2t^3)\vec{j} + (1+t^2+t^4)2t\vec{k} - t^2(t+2t^3)\vec{k} - (t+2t^3)\vec{i}}{(1+t^2+t^4)^{3/2}}$$

$$= \frac{1}{(1+t^2+t^4)^{3/2}}\left[(1+t^2+t^4-t^2-2t^4)\vec{j} + \left(\begin{matrix}2t+2t^3+2t^5\\ -t^3-2t^5\end{matrix}\right)\vec{k} - (t+2t^3)\vec{i}\right]$$

$$= \frac{(1-t^4)\vec{j} + (2t+t^3)\vec{k} - (t+2t^3)\vec{i}}{(1+t^2+t^4)^{3/2}}$$

$$= \frac{-(t+2t^3)\vec{i} + (1-t^4)\vec{j} + (2t+t^3)\vec{k}}{(1+t^2+t^4)^{3/2}}$$

v.v.imp

(Q 11 / 48) If $\vec{F}$ is a vector function of t, prove that $\frac{d}{dt}\left(\frac{\vec{F}}{|\vec{F}|}\right) = \frac{[\vec{F}'(\vec{F}\cdot\vec{F}) - \vec{F}(\vec{F}\cdot\vec{F}')]}{(\vec{F}\cdot\vec{F})^{3/2}}$

Sol:- $|\vec{F}| = \sqrt{|\vec{F}|^2} = \sqrt{\vec{F}\cdot\vec{F}}$

$$\frac{d}{dt}\left(\frac{\vec{F}}{|\vec{F}|}\right) = \frac{d}{dt}\left(\frac{\vec{F}}{\sqrt{\vec{F}\cdot\vec{F}}}\right)$$

$$= \frac{1}{(\sqrt{\vec{F}\cdot\vec{F}})^2}\left[\sqrt{\vec{F}\cdot\vec{F}}\,\vec{F}' - \vec{F}\frac{d}{dt}(\vec{F}\cdot\vec{F})^{1/2}\right]$$

$$= \frac{1}{\vec{F}\cdot\vec{F}}\left[\sqrt{\vec{F}\cdot\vec{F}}\,\vec{F}' - \vec{F}\frac{1}{2\sqrt{\vec{F}\cdot\vec{F}}}\frac{d}{dt}(\vec{F}\cdot\vec{F})\right]$$

$$= \frac{1}{\vec{F}\cdot\vec{F}}\left[\sqrt{\vec{F}\cdot\vec{F}}\,\vec{F}' - \frac{\vec{F}}{2\sqrt{\vec{F}\cdot\vec{F}}}(\vec{F}\cdot\vec{F}' + \vec{F}'\cdot\vec{F})\right]$$

$$= \frac{1}{\vec{F}\cdot\vec{F}}\left[\sqrt{\vec{F}\cdot\vec{F}}\,\vec{F}' - \frac{\vec{F}}{2\sqrt{\vec{F}\cdot\vec{F}}}\cdot(2\vec{F}\cdot\vec{F}')\right]$$

$$= \frac{1}{\vec{F}\cdot\vec{F}}\left[\frac{(\vec{F}\cdot\vec{F})\vec{F}' - \vec{F}(\vec{F}\cdot\vec{F}')}{\sqrt{\vec{F}\cdot\vec{F}}}\right]$$

$$= \frac{\vec{F}'(\vec{F}\cdot\vec{F}) - \vec{F}(\vec{F}\cdot\vec{F}')}{(\vec{F}\cdot\vec{F})^{3/2}}$$

proved

Q 12/48 If $\vec{f}, \vec{g}$ and $\vec{h}$ are vectors functions of t, prove that

(i) $\frac{d}{dt}(\vec{f}\cdot\vec{g}\times\vec{h}) = \vec{f}'\cdot\vec{g}\times\vec{h} + \vec{f}\cdot\vec{g}'\times\vec{h} + \vec{f}\cdot\vec{g}\times\vec{h}'$.

(ii) $\frac{d}{dt}[\vec{f}\times(\vec{g}\times\vec{h})] = \vec{f}'\times(\vec{g}\times\vec{h}) + \vec{f}\times(\vec{g}'\times\vec{h}) + \vec{f}\times(\vec{g}\times\vec{h}')$

Sol:- (i) $\frac{d}{dt}[\vec{f}\cdot\vec{g}\times\vec{h}] = \vec{f}'\cdot\vec{g}\times\vec{h} + \vec{f}\cdot\vec{g}'\times\vec{h} + \vec{f}\cdot\vec{g}\times\vec{h}'$.

$$\frac{d}{dt}[\vec{f}\cdot(\vec{g}\times\vec{h})] = \vec{f}\cdot\frac{d}{dt}(\vec{g}\times\vec{h}) + \vec{f}'\cdot(\vec{g}\times\vec{h})$$

$$= \vec{f}\cdot(\vec{g}\times\vec{h}' + \vec{g}'\times\vec{h}) + \vec{f}'\cdot\vec{g}\times\vec{h}$$

$$= \vec{f}\cdot\vec{g}\times\vec{h}' + \vec{f}\cdot\vec{g}'\times\vec{h} + \vec{f}'\cdot\vec{g}\times\vec{h}$$

write in reverse order

$$= \vec{f}'\cdot\vec{g}\times\vec{h} + \vec{f}\cdot\vec{g}'\times\vec{h} + \vec{f}\cdot\vec{g}\times\vec{h}'.$$

(ii) 2nd part

$$\frac{d}{dt}[\vec{f}\times(\vec{g}\times\vec{h})] = \vec{f}\times\frac{d}{dt}(\vec{g}\times\vec{h}) + (\vec{g}\times\vec{h})\times\vec{f}'$$

$$= \vec{f}\times[\vec{g}\times\vec{h}' + h\times\vec{g}'] + (\vec{g}\times\vec{h})\times\vec{f}'$$

$$= \vec{f}\times\vec{g}\times\vec{h}' + \vec{f}\times\vec{h}\times\vec{g}' + \vec{g}\times\vec{h}\times\vec{f}'$$

$$= \vec{f}'\times(\vec{g}\times\vec{h}) + \vec{f}\times(\vec{g}'\times\vec{h}) + \vec{f}\times(\vec{g}\times\vec{h}')$$

Ans

Q 13/48 If $\vec{f}$, $\vec{g}$ and $\vec{h}$ are vector functions of a scalar variable "t" and if $\vec{f}' = \vec{h} \times \vec{f}$ and $\vec{g}' = \vec{h} \times \vec{g}$, prove that

$$\frac{d}{dt}(\vec{f} \times \vec{g}) = \vec{h} \times (\vec{f} \times \vec{g}).$$

Sol:- $\frac{d}{dt}(\vec{f} \times \vec{g}) = \vec{f} \times \vec{g}' + \vec{f}' \times \vec{g}$

$$= \vec{f} \times \vec{g}' - \vec{g} \times \vec{f}'$$

$$= \vec{f} \times (\vec{h} \times \vec{g}) - \vec{g} \times (\vec{h} \times \vec{f})$$

$$= (\vec{f} \cdot \vec{g})\vec{h} - (\vec{f} \cdot \vec{h})\vec{g} - [(\vec{g} \cdot \vec{f})\vec{h} - (\vec{g} \cdot \vec{h})\vec{f}]$$

$$= \cancel{(\vec{f} \cdot \vec{g})\vec{h}} - (\vec{f} \cdot \vec{h})\vec{g} - \cancel{(\vec{f} \cdot \vec{g})\vec{h}} + (\vec{g} \cdot \vec{h})\vec{f}$$

$$= (\vec{h} \cdot \vec{g})\vec{f} - (\vec{h} \cdot \vec{f})\vec{g}$$

$$\frac{d}{dt}(\vec{f} \times \vec{g}) = \vec{h} \times (\vec{f} \times \vec{g})$$ proved

v.v.imp

Q14 / 48 If $\vec{u}(t)$ is a unit vector, Show that

$$\vec{u}\cdot\left(\vec{u}+\frac{d^2\vec{u}}{dt^2}\right)+\left(\frac{d\vec{u}}{dt}\right)^2=1.$$

Sol:- $|\vec{u}|=1$

Now $\vec{u}.\vec{u} = |\vec{u}|^2 = 1^2 = 1$

$\vec{u}.\vec{u} = 1$

Differentiate w.r.t.t

$$\vec{u}\frac{d\vec{u}}{dt}+\frac{d\vec{u}}{dt}.\vec{u}=0$$

$$\vec{u}.\frac{d\vec{u}}{dt}+\vec{u}.\frac{d\vec{u}}{dt}=0$$

$\vec{a}.\vec{a}=a^2$

$$2\,\vec{u}\,\frac{d\vec{u}}{dt}=0$$

Again Differentiate w.r.t.t.

$$\vec{u}.\frac{d^2\vec{u}}{dt^2}+\frac{d\vec{u}}{dt}.\frac{d\vec{u}}{dt}=0$$

$$\vec{u}.\frac{d^2\vec{u}}{dt^2}+\left(\frac{d\vec{u}}{dt}\right)^2=0$$

It is because of required proof.

Adding $\vec{u}.\vec{u}$ on both sides

$$\vec{u}.\vec{u}+\vec{u}.\frac{d^2\vec{u}}{dt^2}+\left(\frac{d\vec{u}}{dt}\right)^2=0+\vec{u}.\vec{u}$$

$$\vec{u}.\left(\vec{u}+\frac{d^2\vec{u}}{dt^2}\right)+\left(\frac{d\vec{u}}{dt}\right)^2=1$$

proved.

Ex-2/46 If ω, a, b are constant and if $\vec{F}(t) = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, Show that $\vec{F}''(t) + \omega^2\vec{F}(t) = 0$.

Sol:- $\vec{F}(t) = \vec{a}\cos\omega t + \vec{b}\sin\omega t$

$$\vec{F}'(t) = -\vec{a}\,\omega\sin\omega t + \vec{b}\,\omega\cos\omega t$$

$$\vec{F}''(t) = -\vec{a}\,\omega^2\cos\omega t - \vec{b}\,\omega^2\sin^2\omega t$$

$$= -\omega^2[\vec{a}\cos\omega t + \vec{b}\sin\omega t]$$

$$\vec{F}''(t) + \omega^2\vec{F}(t) = -\vec{a}\,\omega^2\cos\omega t - \vec{b}\,\omega^2\sin^2\omega t + \omega^2(\vec{a}\cos\omega t + \vec{b}\sin\omega t)$$

$\vec{F}''(t) + \omega^2\vec{F}(t) = 0$ proved.

v. imp

Ex-4/47 Prove that the derivative of a vector $\vec{a}$ of <u>constant magnitude</u> is orthogonal to $\vec{a}$.

Sol:- $\dfrac{d}{dt}(\vec{a}\cdot\vec{a}) = \vec{a}\cdot\dfrac{d\vec{a}}{dt} + \vec{a}\,\dfrac{d\vec{a}}{dt}$

Target: $\boxed{\dfrac{d\vec{a}}{dt}\cdot\vec{a} = 0}$

$$\frac{d}{dt}|\vec{a}|^2 = 2\,\vec{a}\cdot\left(\frac{d\vec{a}}{dt}\right) \longrightarrow (1)$$

Since magnitude of $\vec{a}$ is constant

$$\therefore \frac{d}{dt}|\vec{a}|^2 = 0$$

$$0 = 2\vec{a}\cdot\frac{d\vec{a}}{dt} \Rightarrow \vec{a}\cdot\frac{d\vec{a}}{dt} = 0$$

$$\Rightarrow \frac{d\vec{a}}{dt}\cdot\vec{a} = 0$$

So derivative of $\vec{a}$ is orthogonal to $\vec{a}$.

v. imp

(Q5/48) If $\vec{F}(t) = (1-\cos t)\vec{i} + (t-\sin t)\vec{j} + (\sin t - t\cos t)\vec{k}$

and $\vec{g}(t) = \sin t\,\vec{i} + \cos t\,\vec{j}$; calculate

(i) $\frac{d}{dt}(\vec{F}\cdot\vec{g})$ (ii) $\frac{d}{dt}(\vec{F}\times\vec{g})$.

Sol:- $\vec{F}(t) = (1-\cos t)\vec{i} + (t-\sin t)\vec{j} + (\sin t - t\cos t)\vec{k}$

$\vec{F}'(t) = \sin t\,\vec{i} + (1-\cos t)\vec{j} + [\cos t - \{t(-\sin t) + \cos t \cdot 1\}]$

$\vec{F}'(t) = \sin t\,\vec{i} + (1-\cos t)\vec{j} + [\cancel{\cos t} + t\sin t - \cancel{\cos t}]$

$$\boxed{\vec{F}'(t) = \sin t\,\vec{i} + (1-\cos t)\vec{j} + t\sin t\,\vec{k}}$$

$\vec{g}(t) = \sin t\,\vec{i} + \cos t\,\vec{j}$

$$\boxed{\vec{g}'(t) = \cos t\,\vec{i} - \sin t\,\vec{j}}$$

(i) To find $\frac{d}{dt}(\vec{F}\cdot\vec{g})$.

$\frac{d}{dt}(\vec{F}\cdot\vec{g}) = \vec{F}\cdot\vec{g}' + \vec{F}'\cdot\vec{g}$

$= [1-\cos t,\ t-\sin t,\ \sin t - t\cos t]\cdot[\cos t, -\sin t, 0]$

$+ [\sin t, 1-\cos t, t\sin t]\cdot[\sin t, \cos t, 0]$

$= (1-\cos t)\cos t - (t-\sin t)\sin t + 0 + \sin^2 t + (1-\cos t)\cos t + 0$

$= \cos t - \cos^2 t - t\sin t + \sin^2 t + \sin^2 t + \cos t - \cos^2 t$

$$= 2\cos t - 2\cos^2 t + 2\sin^2 t - t\sin t$$

$$= 2\cos t - 2(\cos^2 t + \sin^2 t) - t\sin t$$

$$\boxed{\frac{d}{dt}(\vec{F}\cdot\vec{g}) = 2\cos t - 2\cos 2t - t\sin t}$$ Ans

(ii) To find $\frac{d}{dt}(\vec{F}\times\vec{g})$.

$$\frac{d}{dt}(\vec{F}\times\vec{g}) = \vec{F}\times\vec{g}' + \vec{F}'\times\vec{g} \quad \text{——} (1)$$

$$\vec{F}\times\vec{g}' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1-\cos t & t-\sin t & \sin t - t\cos t \\ \cos t & -\sin t & 0 \end{vmatrix}$$ Expanding by R_1

$$= \vec{i}\,[0 + \sin t(\sin t - t\cos t)] + \vec{j}\,[\cos t(\sin t - t\cos t)] + \vec{k}\,[-\sin t(1-\cos t) - \cos t(t-\sin t)]$$

$$\vec{F}\times\vec{g}' = (\sin^2 t - t\sin t\cos t)\vec{i} + (\sin t\cos t - t\cos^2 t)\vec{j} + (-\sin t + \sin t\cos t - t\cos t + \sin t\cos t)\vec{k}$$

$$\boxed{\vec{F}\times\vec{g}' = (\sin^2 t - t\sin t\cos t)\vec{i} + (\sin t\cos t - t\cos^2 t)\vec{j} + (\sin 2t - \sin t - t\cos t)\vec{k}}$$

$$\vec{F}'\times\vec{g} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin t & 1-\cos t & t\sin t \\ \sin t & \cos t & 0 \end{vmatrix}$$ Expanding by R_1

$$= \vec{i}\,[0 - t\sin t\cos t] + \vec{j}\,[t\sin^2 t] + \vec{k}\,[\sin t\cos t - \sin t + \sin t\cos t]$$

$$= -t\sin t\cos t\,\vec{i} + t\sin^2 t\,\vec{j} + (\sin 2t - \sin t)\,\vec{k}.$$

① $\Rightarrow$

$$\frac{d}{dt}(\vec{F}\times\vec{g}) = (\sin^2 t - t\sin t\cos t)\vec{i} + (\sin t\cos t - t\cos t)\vec{j} + (\sin 2t - \sin t - t\cos t)\vec{k}$$
$$- \sin t\cos t\,\vec{i} + t\sin 2t\,\vec{j} + (\sin 2t - \sin t)\vec{k}$$

$$= (\sin^2 t - 2t\sin t\cos t)\vec{i} + [\sin t\cos t - t(\cos^2 t - \sin^2 t)]\vec{j} + [2\sin t - 2\sin t - t\cos t]\vec{k}$$

$$= (\sin^2 t - t\sin 2t)\vec{i} + (\tfrac{1}{2}\sin 2t - t\cos 2t)\vec{j} + (2\sin 2t - 2\sin t - t\cos t)\vec{k} \quad \text{Ans}$$

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V.imp

Q4/47 If $\vec{f}(t) = (3t^2+1)\vec{i} + (2t^3-1)\vec{j} + (2t^2+3t^3)\vec{k}$

and $\vec{g}(t) = t\vec{i} + (t^2-2t)\vec{j} + (3t-t^3)\vec{k}$

Calculate $\frac{d}{dt}(\vec{f}\cdot\vec{g})$ and $\frac{d}{dt}(\vec{f}\times\vec{g})$.

Do?

Ex-5 / 47 If $\vec{F}(t) = t\,\vec{i} - t^2\,\vec{j} + t^3\,\vec{k}$ and

$\vec{g}(t) = \sin t\,\vec{i} + \cos t\,\vec{j}$, Calculate

(i) $\frac{d}{dt}(\vec{F}\cdot\vec{g})$ (ii) $\frac{d}{dt}(\vec{F}\times\vec{g})$.

Do?

Do?

Definition <u>Derivative of a vector function</u>

$$\vec{F}'(t) = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$

$$\frac{d}{dt}\vec{h}(t) = \lim_{\Delta t \to 0} \frac{\vec{h}(t+\Delta t) - \vec{h}(t)}{\Delta t}$$

v.imp (ii/45) Prove $\frac{d}{dt}(\vec{F}+\vec{g}) = \frac{d}{dt}\vec{F} + \frac{d}{dt}\vec{g}$.

$\vec{h}(t) = \vec{h}$ short

Sol:- Let $\vec{h} = \vec{F} + \vec{g}$

$$\vec{h}(t) = \vec{F}(t) + \vec{g}(t)$$

$$\vec{h}(t+\Delta t) = \vec{F}(t+\Delta t) + \vec{g}(t+\Delta t)$$

$$\frac{d}{dt}\vec{h}(t) = \lim_{\Delta t \to 0} \frac{\vec{h}(t+\Delta t) - \vec{h}(t)}{\Delta t}$$

$$\frac{d}{dt}\vec{h}(t) = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) + \vec{g}(t+\Delta t) - [\vec{F}(t) + \vec{g}(t)]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t) + \vec{g}(t+\Delta t) - \vec{g}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{\vec{g}(t+\Delta t) - \vec{g}(t)}{\Delta t}$$

$$\frac{d}{dt}\vec{h}(t) = \frac{d}{dt}\vec{F}(t) + \frac{d}{dt}\vec{g}(t)$$

$$\frac{d}{dt}\left(\vec{F}(t) + \vec{g}(t)\right) = \frac{d}{dt}\vec{F}(t) + \frac{d}{dt}\vec{g}(t)$$

$$\frac{d}{dt}(\vec{F}+\vec{g}) = \frac{d}{dt}\vec{F} + \frac{d}{dt}\vec{g}$$ proved

(i/45) $\frac{d}{dt}\vec{a} = 0$ where $\vec{a}$ is constant.

Sol:- Let $\vec{h} = \vec{a}$

$$\vec{h}(t) = \vec{a}$$

$$\vec{h}(t+\Delta t) = \vec{a}$$

$$\frac{d}{dt}\vec{h}(t) = \lim_{\Delta t\to 0}\frac{\vec{h}(t+\Delta t) - \vec{h}(t)}{\Delta t}$$

$$\frac{d}{dt}\vec{h}(t) = \lim_{\Delta t\to 0}\frac{\vec{a} - \vec{a}}{\Delta t} = \lim_{\Delta t\to 0} 0.$$

$$\boxed{\frac{d}{dt}\vec{a} = 0}$$ proved.

v.imp (k/45) $\frac{d}{dt}(\vec{F}\times\vec{g}) = \vec{F}\times\frac{d}{dt}\vec{g} + \frac{d\vec{F}}{dt}\times\vec{g}$

Sol:- let $\vec{h} = \vec{F}\times\vec{g}$

$$\vec{h}(t) = \vec{F}(t)\times\vec{g}(t)$$

$$\vec{h}(t+\Delta t) = \vec{F}(t+\Delta t)\times\vec{g}(t+\Delta t)$$

$$\frac{d}{dt}\vec{h}(t) = \lim_{\Delta t\to 0}\frac{\vec{h}(t+\Delta t) - \vec{h}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{\vec{F}(t+\Delta t)\times\vec{g}(t+\Delta t) - \vec{F}(t)\times\vec{g}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{\vec{F}(t+\Delta t)\times\vec{g}(t+\Delta t) - \vec{F}(t)\times\vec{g}(t)}{\Delta t}$$

P.T.O

(27)

(introduction) $- \vec{F}(t) \times \vec{g}(t)$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) \times \vec{g}(t+\Delta t) - \vec{F}(t+\Delta t) \times \vec{g}(t) + \vec{F}(t+\Delta t) \times \vec{g}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) \times [\vec{g}(t+\Delta t) - \vec{g}(t)] - [\vec{F}(t+\Delta t) - \vec{F}(t)] \times \vec{g}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \vec{F}(t+\Delta t) \times \lim_{\Delta t \to 0} \frac{\vec{g}(t+\Delta t) - \vec{g}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t)\vec{F}(t)}{\Delta t} \times \lim_{\Delta t \to 0} \vec{g}(t)$$

$$= \vec{F}(t+0) \times \frac{d}{dt}(\vec{g}(t)) + \frac{d}{dt}\vec{F}(t) \times \vec{g}(t)$$

$$= \vec{F}(t) \times \frac{d}{dt}\vec{g}(t) + \frac{d}{dt}\vec{F}(t) \times \vec{g}(t)$$

$$\frac{d}{dt}(\vec{F} \times \vec{g}) = \vec{F} \times \left(\frac{d}{dt}\vec{g}\right) + \left(\frac{d}{dt}\vec{F}\right) \times \vec{g}$$

imp. (iv) 45 Prove that $\frac{d}{dt}(\vec{F} \cdot \vec{g}) = \vec{F} \cdot \left(\frac{d}{dt}\vec{g}\right) + \left(\frac{d}{dt}\vec{F}\right) \cdot \vec{g}$.

let $h = \vec{F} \cdot \vec{g}$ & $h(t+\Delta t) = F(t+\Delta t) \cdot g(t+\Delta t)$

$$\frac{d}{dt}\vec{h}(t) = \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} = \frac{\vec{F}(t+\Delta t) \cdot \vec{g}(t+\Delta t) - \vec{F}(t)\vec{g}(t)}{\Delta t}$$

introduction.

$$= \frac{\vec{F}(t+\Delta t) \cdot \vec{g}(t+\Delta t) - \vec{F}(t) \cdot \vec{g}(t) - \vec{F}(t+\Delta t) \cdot \vec{g}(t) + \vec{F}(t+\Delta t) \cdot \vec{g}(t)}{\Delta t}$$

$$= \frac{F(t+\Delta t) \cdot [g(t+\Delta t) - g(t)] - [F(t+\Delta t) - F(t)] \cdot g(t)}{\Delta t}$$

$= \cancel{\lim_{\Delta t \to 0} F(t+\Delta t) \frac{d}{dt} \frac{g(t+\Delta t) - g(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}}$

P. T. O.

$$= \lim_{\Delta t\to 0} \vec{f}(t+\Delta t)\cdot \frac{d}{dt}\frac{\vec{g}(t+\Delta t)-\vec{g}(t)}{\Delta t} + \lim_{\Delta t\to 0} \frac{\vec{f}(t+\Delta t)-\vec{f}(t)}{\Delta t}\cdot \lim_{\Delta t\to 0}\vec{g}(t)$$

$$= f(t)\cdot \frac{d}{dt}\vec{g}(t) + \frac{d}{dt}\vec{f}(t)\cdot\vec{g}(t)$$

$$\frac{d}{dt}(\vec{f}\cdot\vec{g}) = \vec{f}\cdot\left(\frac{d}{dt}\vec{g}\right) + \left(\frac{d}{dt}\vec{f}\right)\cdot\vec{g}.$$ proved.

$\frac{iii}{45}$ Prove that $\frac{d}{dt}(u\vec{f}) = u\frac{d\vec{f}}{dt} + \left(\frac{du}{dt}\right)\vec{f}$.

let $\vec{h}(t) = u(t).\vec{f}(t)$

& $\vec{h}(t+\Delta t) = u(t+\Delta t).f(t+\Delta t)$

$$\frac{d}{dt}\vec{h}(t) = \frac{h(t+\Delta t)\vec{f}(t+\Delta t) - u(t)\vec{f}(t)}{\Delta t}$$

introduction.

$$= \frac{u(t+\Delta t)f(t+\Delta t) - u(t)\vec{f}(t) - u(t+\Delta t)f(t) + u(t+\Delta t)f(t)}{\Delta t}$$

$$= \frac{u(t+\Delta t)[f(t+\Delta t) - f(t)]}{\Delta t} + \frac{[u(t+\Delta t) - u(t)]f(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0} u(t+\Delta t)\cdot \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t} + \lim_{\Delta t\to 0}\frac{u(t+\Delta t)-u(t)}{\Delta t}\lim_{\Delta t\to 0} f(t)$$

$$= u(t)\cdot\frac{d}{dt}\vec{f}(t) + \frac{du}{dt}\cdot\vec{f}(t)$$

$$= u\frac{d}{dt}\vec{f} + \left(\frac{du}{dt}\right)\vec{f}.$$

Proved

EXERCISE 3.3

Q1/44 If $\vec{F}(t) = (t^2+2t-1)\vec{i} + (3t^2-2)\vec{j} + (5-6t)\vec{k}$

Find $\vec{F}'(t)$ and $\vec{F}''(t)$.

Sol:- $\vec{F}(t) = (t^2+2t-1)\vec{i} + (3t^2-2)\vec{j} + (5-6t)\vec{k}$.

$$\vec{F}'(t) = (2t+2)\vec{i} + (6t)\vec{j} - 6\vec{k}$$

$$\vec{F}''(t) = 2\vec{i} + 6\vec{j}$$

Q2/44 If $\vec{F}(t) = \cos t\,\vec{i} + t\sin t\,\vec{j} + t^2\sec t\,\vec{k}$

calculate $|\vec{F}'(t)|$.

Sol:- $\vec{F}(t) = \cos t\,\vec{i} + t\sin t\,\vec{j} + t^2\sec t\,\vec{k}$

$$\vec{F}'(t) = -\sin t\,\vec{i} + (t\cos t + 1\cdot\sin t)\vec{j} + (t^2\sec t\tan t + 2t\sec t)\vec{k}$$

$$\vec{F}'(t) = -\sin t\,\vec{i} + (t\cos t + \sin t)\vec{j} + (t^2\sec t\cdot\tan t + 2t\sec t)\vec{k}$$

$$|\vec{F}'(t)| = \sqrt{(-\sin t)^2 + (t\cos t + \sin t)^2 + (t^2\sec t\cdot\tan t + 2t\sec t)^2}$$

$$= \sqrt{\sin^2 t + t^2\cos^2 t + \sin^2 t + 2t\cos t\sin t +}$$

Do?

Q3/44 If $\vec{F}(t) = t\hat{i} + t^2\vec{j} + \left(\frac{1}{t}\right)\vec{k}$, find $f''(t)$.

Sol. $\vec{F}(t) = t\vec{i} + t^2\vec{j} + \left(\frac{1}{t}\right)\vec{k}$,

$$\vec{F}'(t) = \vec{i} + 2t\vec{j} - \frac{1}{t^2}\vec{k} \quad \text{Ans.}$$

Q4/44 Find $\vec{F}'(t)$, if $\vec{F}(t) = (6t^3+1)\vec{i} + (t^2-1)\vec{j} + (2t^3+5t^2+3)\vec{k}$.

Sol. $\vec{F}(t) = (6t^3+1)\vec{i} + (t^2-1)\vec{j} + (2t^3+5t^2+3)\vec{k}$

$$\vec{F}'(t) = 18t^2\vec{i} + 2t\vec{j} + (6t^2+10t)\vec{k} \quad \text{Ans}$$

Q5/44 If $\vec{F}(t) = \cos t\,\vec{i} - \sin t\,\vec{j} + 8\vec{k}$, show that $\vec{F}'(t)\cdot\vec{F}''(t) = 0$.

Sol. $\vec{F}(t) = \cos t\,\vec{i} - \sin t\,\vec{j} + 8\vec{k}$

$$\vec{F}'(t) = -\sin t\,\vec{i} - \cos t\,\vec{j}$$

$$\vec{F}''(t) = -\cos t\,\vec{i} + \sin t\,\vec{j}$$

$$\vec{F}'(t)\cdot\vec{F}''(t) = [-\sin t, -\cos t]\cdot[-\cos t, \sin t]$$

$$= (-\sin t)(-\cos t) + (-\cos t)(\sin t)$$

$$= \cancel{\sin t\cos t} - \cancel{\cos t\sin t}$$

$$\vec{F}'(t)\cdot\vec{F}''(t) = 0 \quad \text{proved}$$

(Q6/44) If $\vec{F}(t) = \left(\sin\frac{1}{t}\right)\vec{i} + \left(\cos\frac{1}{t}\right)\vec{j} + \left(\frac{2}{t}\right)\vec{K}$.

Calculate $\frac{d}{dt}[\vec{F}(t)]$.

Sol:- $\vec{F}(t) = \sin\frac{1}{t}\,\vec{i} + \cos\frac{1}{t}\,\vec{j} + \frac{2}{t}\,\vec{K}$

$$\frac{d}{dt}\vec{F}(t) = \cos\frac{1}{t}\cdot\frac{d}{dt}\frac{1}{t}\,\vec{i} - \sin\frac{1}{t}\cdot\frac{d}{dt}\frac{1}{t}\,\vec{j} + 2\frac{d}{dt}t^{-1}\vec{K}$$

$$= -\cos\frac{1}{t}\cdot\frac{1}{t^2}\,\vec{i} + \sin\frac{1}{t}\cdot\frac{1}{t^2} - 2\frac{1}{t^2}.$$

$$= -\frac{1}{t^2}\cos\frac{1}{t}\,\vec{i} + \frac{1}{t^2}\sin\frac{1}{t}\,\vec{j} - 2\frac{1}{t^2}\vec{K}.$$

Ans

(Q7/44) Given that $\vec{F}(t) = \vec{i}\left(\frac{t^2+1}{t}\right) + \left(\frac{\vec{j}}{1+t}\right) + \vec{K}t$.

Calculate $\frac{d}{dt}\vec{F}(t)$ and $\vec{F}(t)\cdot\vec{F}'(t)$.

Sol:- $\vec{F}(t) = \vec{i}\left(\frac{t^2+1}{t}\right) + \left(\frac{\vec{j}}{1+t}\right) + \vec{K}t$.

$$\vec{F}(t) = \left(\frac{t^2}{t} + \frac{1}{t}\right)\vec{i} + \frac{1}{1+t}\vec{j} + t\vec{K}$$

$$\vec{F}(t) = \left(t + \frac{1}{t}\right)\vec{i} + \frac{1}{1+t}\vec{j} + t\vec{K}$$

$$\vec{F}'(t) = \left(1 - \frac{1}{t^2}\right)\vec{i} + \frac{-1}{(1+t)^2}\vec{j} + \vec{K}$$

$$\vec{F}'(t) = \left(\frac{t^2-1}{t^2}\right)\vec{i} - \frac{1}{(1+t)^2}\vec{j} + \vec{K}$$

$$\vec{F}(t)\cdot\vec{F}'(t) = \left[\frac{t^2+1}{t}, \frac{1}{1+t}, t\right]\cdot\left[\frac{t^2-1}{t^2}, \frac{-1}{(1+t)^2}, 1\right]$$

$$= \left(\frac{t^2+1}{t}\right)\left(\frac{t^2-1}{t^2}\right) + \left(\frac{1}{1+t}\right)\left(\frac{-1}{(1+t)^2}\right) + (t \cdot 1).$$

$$\vec{F}(t)\cdot\vec{F}'(t) = \frac{t^4-1}{t^3} - \frac{1}{(1+t)^3} + t \quad \underline{\text{Ans}}.$$

(Q 8 / 44) If $\vec{F}(t) = (t-\sin t)\vec{i} + (1-\cos t)\vec{j} + (t\sin t + \cos t)\vec{k}$

Find the components of $\vec{F}'(t)$ & $\vec{F}''(t)$ at $t=0$ and $t=\pi/2$.

Sol:- $\vec{F}(t) = (t-\sin t)\vec{i} + (1-\cos t)\vec{j} + (t\sin t + \cos t)\vec{k}$

$$\vec{F}'(t) = (1-\cos t)\vec{i} + \sin t\,\vec{j} + \{t\cos t + \sin t\cdot 1 - \sin t\}\vec{k}$$

$$\vec{F}''(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + (-t\sin t + \cos t\cdot 1)\vec{k}$$

at $t=0$

$$\vec{F}'(0) = (1-\cos 0)\vec{i} + \sin 0\,\vec{j} + (0\cdot\cos 0)\vec{k}.$$

$$\vec{F}'(0) = (1-1)\vec{i} + 0\vec{j} + 0\vec{k}$$
$$= 0\vec{i} + 0\vec{j} + 0\cdot\vec{k}$$

$$\vec{F}'(0) = [0, 0, 0].$$

$$\vec{F}''(0) = \sin 0\,\vec{i} + \cos 0\,\vec{j} + (0+\cos 0)\vec{k}$$
$$= 0\vec{i} + 1\vec{j} + 1\vec{k}$$

$$\vec{F}''(0) = [0, 1, 1].$$

at $t=\pi/2$

$$\vec{F}'(\pi/2) = (1-\cos\pi/2)\vec{i} + \sin\pi/2\,\vec{j} + (\tfrac{\pi}{2}\cos\pi/2)\vec{k}$$
$$= (1-0)\vec{i} + 1\vec{j} + (\tfrac{\pi}{2}\cdot 0)\vec{k}$$

P.T.O

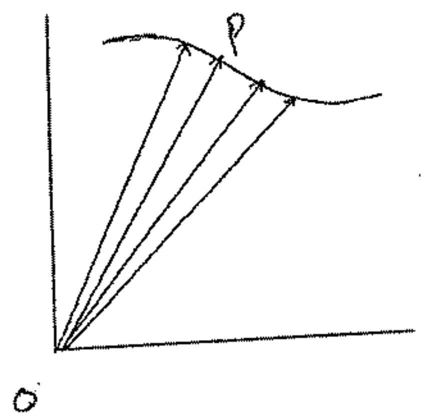
$\vec{F}'(\pi/2) = [1, 1, 0]$

$\vec{F}''(\pi/2) = \sin\frac{\pi}{2}\,\vec{i} + \cos\frac{\pi}{2}\,\vec{j} + \left(-\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)\vec{k}$

$= 1\cdot\vec{i} + 0\cdot\vec{j} + \left(-\frac{\pi}{2} + 0\right)\vec{k}$

$\vec{F}''(\pi/2) = [1, 0, -\pi/2]$. Ans

POSITION VECTOR



Tip of position vector of a point describes the curve C.

Position vector of a point P at any time t.

$\vec{OP} = \vec{F}(t) = x\vec{i} + y\vec{j} + z\vec{k}$.

Q9/44 For the curve eliminate t. (all parts)

Given that $\vec{F}(t)$ is the position vector of a point p whose position varies with t, describe the curve traced out by p in each of the following cases.

(i) $\vec{F}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$.

$$\left.\begin{aligned}\vec{F}(t) &= \cos t\,\vec{i} + \sin t\,\vec{j}\\ \vec{F}(t) &= x\vec{i} + y\vec{j}\end{aligned}\right\}\text{ Comparing}$$

$$\left.\begin{aligned}x &= \cos t\\ y &= \sin t\end{aligned}\right\}\text{ Squaring and adding.}$$

$$x^2 + y^2 = \cos^2 t + \sin^2 t$$

$$\boxed{x^2 + y^2 = 1}$$

(ii) $\vec{F}(t) = t\vec{i} + t^2\vec{j}$

$$\left.\begin{aligned}\vec{F}(t) &= t\vec{i} + t^2\vec{j}\\ &= x\vec{i} + y\vec{j}\end{aligned}\right\}\text{ Comparing}$$

$x = t$, $\boxed{y = t^2}$

put in (1)

$$\boxed{y = x^2}$$

(35)

v.imp (iii) 44 $\vec{F}(t) = \left(\frac{3t}{1+t^3}\right)\vec{i} + \left(\frac{3t^2}{1+t^3}\right)\vec{j}$

$$\left.\begin{aligned}\vec{F}(t) &= \left(\frac{3t}{1+t^3}\right)\vec{i} + \left(\frac{3t^2}{1+t^3}\right)\vec{j} \\ &= x\vec{i} + y\vec{j}\end{aligned}\right\} \text{Comparing}$$

$$x = \frac{3t}{1+t^3} \text{ ——— (I)} \quad \& \quad y = \frac{3t^2}{1+t^3} \text{ ——— (II)}$$

(I) ÷ by (II)

$$\frac{x}{y} = \frac{\frac{3t}{1+t^3}}{\frac{3t^2}{1+t^3}} = \frac{3t}{\cancel{1+t^3}} \times \frac{\cancel{1+t^3}}{3t^2}$$

$$\frac{x}{y} = \frac{1}{t} \Rightarrow \boxed{t = \frac{y}{x}} \text{ put in (I)}$$

$$x = \frac{3 \cdot y}{x} \times \frac{1}{1+\left(\frac{y}{x}\right)^3} = \frac{3y}{x} \times \frac{1}{1+\frac{y^3}{x^3}}$$

$$x = \frac{3y}{\cancel{x}} \times \frac{x^{\cancel{3}}}{x^3+y^3}$$

$$\frac{x}{1} = \frac{3y\,x^2}{x^3+y^3}$$

$$\cancel{x}(x^3+y^3) = 3y\,x^{\cancel{2}}$$

$$\boxed{x^3+y^3 = 3xy}$$

Ans

(iv/44) $\vec{F}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$

$$\left.\begin{aligned}\vec{F}(t) &= t\vec{i} + t^2\vec{j} + t^3\vec{k}\\ &= x\vec{i} + y\vec{j} + z\vec{k}\end{aligned}\right\}\text{ Comparing}$$

$x = t$, $y = t^2$, $z = t^3$

$\boxed{y = x^2}$, $z = t^2 \times t$

$\boxed{z = yx}$.

(v/44) $\vec{F}(t) = (a\cos t)\vec{i} + (\cos t)\vec{j} + c\vec{k}$

$$\left.\begin{aligned}\vec{F}(t) &= (a\cos t)\vec{i} + (\cos t)\vec{j} + c\vec{k}\\ &= x\vec{i} + y\vec{j} + z\vec{k}\end{aligned}\right\}\text{ Comparing}$$

$$\left.\begin{aligned}x = a\cos t &\Rightarrow \boxed{\frac{x}{a} = \cos t}\\ y = b\sin t &\Rightarrow \boxed{y/b = \sin t}\end{aligned}\right\}\text{ Squaring and adding}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Ans

EXERCISE # 3.6

Q1/54 Integrate $\vec{F}(t)$ if $\vec{F}(t)$ is $t^2\vec{i}+(\frac{1}{t^2})\vec{j}+4\vec{K}$.

Sol:- $\vec{F}(t) = t^2\vec{i} + \frac{1}{t^2}\vec{j} + 4\vec{K}$

$$\int \vec{F}(t)\,dt = \int (t^2\vec{i} + t^{-2}\vec{j} + 4\vec{K})\,dt$$

$$= \frac{t^3}{3}\vec{i} + \frac{t^{-1}}{-1}\vec{j} + 4t\vec{K} + c$$

$$\boxed{\int \vec{F}(t)\,dt = \frac{t^3}{3}\vec{i} - \frac{1}{t}\vec{j} + 4t\vec{K} + c}$$

Q2/54 Integrate

$\vec{F}(t) = t\vec{i} + \frac{1}{t}\vec{j} + \vec{K}$

$$\int \vec{F}(t)\,dt = \int (t\vec{i} + \frac{1}{t}\vec{j} + \vec{K})\,dt$$

$$\boxed{\int \vec{F}(t)\,dt = \frac{t^2}{2}\vec{i} + \log t + t\vec{K} + c}$$

INTEGRATION BY PARTS

$$\int (\text{1st} \cdot \text{2nd})\,dt = \text{1st} \cdot \text{integral of 2nd} - \int \left(\frac{d}{dt}\,\text{1st}\right)(\text{integral of 2nd})\,dt.$$

Q3/54 Integrate

$\cos t\,\vec{i} + (t\sec^2 t + \tan t)\vec{j} - \sin t\,\vec{k}$.

Sol:- $\int \vec{F}(t)\,dt = \int [\cos t\,\vec{i} + (t\sec^2 t + \tan t)\vec{j} - \sin t\,\vec{k}]\,dt$

~~integrate~~ integrate by parts

$\cancel{\int \cos t\,\vec{i}\,dt + \int (t\sec^2 t + \tan t)\,dt}$

$= \sin t\,\vec{i} + \left(t\tan t - \cancel{\int 1\cdot\tan t\,dt} + \cancel{\int \tan t\,dt}\right)\vec{j} + \cos t\,\vec{k} + \vec{C}$

$\int \vec{F}(t)\,dt = \sin t\,\vec{i} + t\tan t\,\vec{j} + \cos t\,\vec{k} + C$.

Q4/54 $\vec{F}(t) = \left(\frac{1}{t}\right)\vec{i} + \left(\frac{1}{t^2}\right)\vec{j} + \left(\frac{1}{t^3}\right)\vec{k}$

Sol:- $\vec{F}(t) = \left(\frac{1}{t}\right)\vec{i} + \frac{1}{t^2}\vec{j} + \frac{1}{t^3}\vec{k}$

$\int \vec{F}(t)\,dt = \ln t\,\vec{i} + \int t^{-2}\vec{j}\,dt + \int t^{-3}\,dt\,\vec{k}$

$= \ln t\,\vec{i} + \frac{t^{-1}}{-1}\vec{j} + \frac{t^{-2}}{-2}\vec{k}$

$\int \vec{F}(t)\,dt = \ln t\,\vec{i} - \frac{1}{t}\vec{j} - \frac{1}{2t^2}\vec{k} + C$

Ans

Q5/54 Integrate

$$\vec{F}(t) = (t^2+1)\,\vec{i} + (t^3+t^2+3)\,\vec{j} + (2-t)\,\vec{k}.$$

$$\int \vec{F}(t)\,dt = \int (t^2+1)\,dt\,\vec{i} + \int (t^3+t^2+3)\,dt\,\vec{j} + \int (2-t)\,dt\,\vec{k}$$

$$= \left(\frac{t^3}{3} + t\right)\vec{i} + \left(\frac{t^4}{4} + \frac{t^3}{3} + 3t\right)\vec{j} + \left(2t - \frac{t^2}{2}\right)\vec{k} + c$$

Ans

Q13/55 Prove that (i) $\int [\vec{a}\cdot\vec{F}(t)]\,dt = \vec{a}\cdot\int \vec{F}(t)\,dt$

(ii) $\int [\vec{a}\times\vec{F}(t)]\,dt = \vec{a}\times\int \vec{F}(t)\,dt$.

where a is a constant vector:

Sol:- (i) $\int [\underline{\underline{\vec{a}}}\cdot\underline{\underline{\vec{F}}}(t)]\,dt$ integrating by parts.

$$= \vec{a}\int \vec{F}(t)\,dt - \int \left(\frac{d\vec{a}}{dt}\right)\cdot\left(\int \vec{F}(t)\,dt\right)dt$$

Since $\vec{a}$ is a constant ~~and~~ vectors

so $\frac{d\vec{a}}{dt} = 0$

$$= \vec{a}\cdot\int F(t)\,dt - 0$$

$$\int [\vec{a}\cdot\vec{F}(t)]\,dt = \vec{a}\cdot\int \vec{F}(t)\,dt$$

$$\int [\vec{a}\cdot\vec{F}(t)]dt = \vec{a}\int \vec{F}(t)\,dt.$$

(ii) $\int [\vec{a}\times\vec{F}(t)]\,dt$ by parts

$$= \vec{a}\times\left(\int \vec{F}(t)\,dt\right) - \int\left(\frac{d\vec{a}}{dt}\right)\times\left(\int\vec{F}(t)\,dt\right)dt$$

c

$$= \vec{a}\times\int\vec{F}(t)\,dt - 0$$

$$\int [\vec{a}\times\vec{F}(t)]\,dt = \vec{a}\times\int\vec{F}(t)\,dt.$$

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v.v. imp topic

DIFFERENTIAL EQUATION (D.E)

Any equation containing derivative is called Differential equation.

Ex $\quad \dfrac{d^2\vec{V}}{dt^2} - 5\dfrac{d\vec{V}}{dt} + 6\vec{V} = 0.$

SOLUTION OF D.E

To find a formula for dependent variable (say $\vec{V}$) in term of independent variable (t).

Ex: $\quad \dfrac{d^2\vec{V}}{dt^2} - 5\dfrac{d\vec{V}}{dt} + 6\vec{V} = 0$

A.E = Auxilary equation

A.E $\quad m^2 - 5m + 6 = 0$

$\quad (m-2)(m-3) = 0, \quad m = 2, 3$

$\vec{V} \to 1$
$\frac{d\vec{V}}{dt} \to m$
$\frac{d^2\vec{V}}{dt^2} \to m^2$

Solution $\quad \vec{V} = \vec{C_1}e^{2t} + \vec{C_2}e^{3t}$ Ans

Ex-2 Solve $\frac{d^2\vec{V}}{dt^2} + 4\frac{d\vec{V}}{dt} + 4\vec{V} = 0$

A.E $m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0$

$m = -2, -2$

$\vec{V} = (\vec{C}_1 + C_2 t)e^{-2t}$.

Remember this table by heart.

Roots of A.E	Solution.
① Different $m_1, m_2, m_3, \ldots$	$\vec{V} = \vec{C}_1 e^{m_1 t} + \vec{C}_2 e^{m_2 t} + \vec{C}_3 e^{m_3 t} + \ldots$
② Same	$\vec{V} = (\vec{C}_1 + \vec{C}_2 t + C_3 t^2 + \ldots)e^{(\text{same})t}$
③ Complex $m = \alpha + i\beta$	$\vec{V} = e^{\alpha t}\{\vec{C}_1 \cos\beta t + \vec{C}_2 \sin\beta t\}$

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Q10/54 Find the solution of the equations $\frac{d^2\vec{V}}{dt^2} = \pm \omega^2 V$ where $\vec{V}$ is a vector function of t, and ω is a constant scalar.

Sol:- $\frac{d^2\vec{V}}{dt^2} = \omega^2\vec{V}$

$$\frac{d^2\vec{V}}{dt^2} - \omega^2\vec{V} = 0$$

A.E $m^2 - \omega^2 = 0$

$(m-\omega)(m+\omega) = 0$

$m = \omega, -\omega$

$\vec{V} = \vec{c}_1 e^{\omega t} + \vec{c}_2 e^{-\omega t}$

$$\frac{d^2\vec{V}}{dt^2} = -\omega^2\vec{V}$$

$$\frac{d^2\vec{V}}{dt^2} + \omega^2\vec{V} = 0$$

A.E $m^2 + \omega^2 = 0$

$m^2 = \sqrt{-\omega^2}$

$m = \pm\sqrt{-\omega^2} = \pm i\omega$

$m = 0, \pm i\omega.$

$$\vec{V} = e^{0t}\{\vec{c}_1 \cos\omega t + \vec{c}_2 \sin\omega t\}$$

$$\boxed{\vec{V} = \vec{c}_1 \cos\omega t + \vec{c}_2 \sin\omega t}$$ Ans

v.imp Q9/54 Solve the equation $\frac{d^2\vec{V}}{dt^2} + 2\frac{d\vec{V}}{dt} + 4\vec{V} = 0$ where $\vec{V}$ is a constant function of t.

Sol:- $\frac{d^2\vec{V}}{dt^2} + 2\frac{d\vec{V}}{dt} + 4\vec{V} = 0$

A.E $m^2 + 2m + 4 = 0$

$m = \frac{-2 \pm \sqrt{4-16}}{4} = \frac{-2 \pm \sqrt{-4\times 3}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2}$

$\boxed{m = -1 \pm i\sqrt{3}}$

Solution

$$\vec{V} = e^{-t}\{\vec{c}_1 \cos\sqrt{3}.t + \vec{c}_2 \sin\sqrt{3}t\}$$ Ans

Ex-4 / S3 If $\vec{v}$ is a vector function of a scalar variable, solve the equation

$$\frac{d^2\vec{v}}{dt^2} + 2a\frac{d\vec{v}}{dt} + b\vec{v} = 0.$$

Sol:- $\frac{d^2\vec{v}}{dt^2} + 2a\frac{d\vec{v}}{dt} + b\vec{v} = 0$

A.E $m^2 + 2am + b = 0$

$$m = \frac{-2a \pm \sqrt{4a^2-4b}}{2} = \frac{-2a \pm 2\sqrt{a^2-b}}{2}$$

$$m = -a \pm \sqrt{a^2-b}$$

$$m = -a + \sqrt{a^2-b}\ ,\ -a-\sqrt{a^2-b}$$

Solution

$$\vec{v} = \vec{c}_1\, e^{(-a+\sqrt{a^2-b})t} + \vec{c}_2\, e^{(-a-\sqrt{a^2-b})t}$$ Ans

Initial Value Problems

In order to find integration constant some conditions are given, such problems are called initial value problems.

v.v.imp

Q 12 / 55 Find a vector $\vec{V}$ which satisfies

$$\frac{d^3\vec{v}}{dt^3} - \frac{d^2\vec{v}}{dt^2} - 2\frac{d\vec{v}}{dt} = 0$$

~~Strue~~ such that $\vec{v} = \vec{i}$, $\frac{d\vec{v}}{dt} = \vec{j}$ and $\frac{d^2\vec{v}}{dt^2} = \vec{k}$

For $t = 0$.

Sol:- $$\frac{d^3\vec{V}}{dt^3} - \frac{d^2\vec{V}}{dt^2} - 2\cdot\frac{d\vec{V}}{dt} = 0$$

$$m^3 - m^2 - 2m = 0$$

$$m(m^2 - m - 2) = 0$$

$$m(m-2)(m+1) = 0$$

$$m = 0, 2, -1$$

Solution:

$$\vec{V} = \vec{C_1} e^{0.t} + \vec{C_2} e^{2t} + \vec{C_3} e^{-1t}$$

$$\boxed{\vec{V} = \vec{C_1} + \vec{C_2} e^{2t} + \vec{C_3} e^{-t}} \quad (1)$$

$$\boxed{\frac{d\vec{V}}{dt} = 2\vec{C_2} e^{2t} - \vec{C_3} e^{-t}} \quad (2)$$

$$\boxed{\frac{d^2\vec{V}}{dt^2} = 4\vec{C_2} e^{2t} + \vec{C_3} e^{-t}} \quad (3)$$

By Applying given conditions ie $\vec{V} = \vec{i}$ for $t=0$.

(1) $\Rightarrow$ $$\boxed{\vec{i} = \vec{C_1} + \vec{C_2} + \vec{C_3}} \quad (I)$$

$\frac{d\vec{v}}{dt} = \vec{j}$ for $t = 0$

(2) $\Rightarrow$ $$\boxed{\vec{j} = 2\vec{C_2} - \vec{C_3}} \quad (II)$$

$$\frac{d^2\vec{v}}{dt^2} = \vec{k} \quad \text{for } t=0$$

$$③ \Rightarrow \boxed{\vec{k} = 4\vec{c}_2 + \vec{c}_3} \text{ — III}$$

$\text{II} + \text{III}$

$$2\vec{c}_2 - \vec{c}_3 = \vec{j}$$
$$4\vec{c}_2 + \vec{c}_3 = \vec{k} \quad \text{adding}$$
$$\overline{\quad 6\vec{c}_2 = \vec{j} + \vec{k} \quad}$$

$$\boxed{\vec{c}_2 = \frac{1}{6}(\vec{j} + \vec{k})}$$

$$\text{II} \Rightarrow \vec{c}_3 = 2\vec{c}_2 - \vec{j} = \frac{2}{6}(\vec{j} + \vec{k}) - \vec{j}$$

$$\vec{c}_3 = \frac{1}{3}\vec{j} + \frac{1}{3}\vec{k} - \vec{j}$$

$$\boxed{\vec{c}_3 = -\frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}}$$

$$\text{I} \Rightarrow \vec{c}_1 = \vec{i} - \vec{c}_2 - \vec{c}_3 = \vec{i} - \frac{1}{6}\vec{j} - \frac{1}{6}\vec{k} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$$

$$\boxed{\vec{c}_1 = \vec{i} + \frac{1}{2}\vec{j} - \frac{1}{2}\vec{k}}$$

① becomes

$$\vec{v} = \vec{i} + \frac{1}{2}\vec{j} - \frac{1}{2}\vec{k} + \frac{1}{6}\vec{j}e^{-t} + \frac{1}{6}\vec{k}e^{2t}$$
$$\qquad - \frac{2}{3}\vec{j}e^{-t} + \frac{1}{3}\vec{k}e^{-t}$$

$$\vec{v} = \vec{i} - \left(\frac{1}{2} + \frac{2}{3}e^{-t} - \frac{1}{6}e^{2t}\right)\vec{j} + \left(\frac{1}{2} + \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t}\right)\vec{k}$$

Ans

TYPE — SOLUTION OF D.E

$$\boxed{\frac{d^2\vec{V}}{dt^2} = \vec{F}(t)} \quad \text{OR Constant.}$$

Procedure: Integrate twice.

Ex-3/53 If $\vec{V}$ is a vector function of a scalar variable t, and $\vec{V}$ is a constant vector, solve the equation $\frac{d^2\vec{V}}{dt^2} = \vec{a}$.

Sol:- $\frac{d^2\vec{V}}{dt^2} = \vec{a}$

$\vec{V}'' = \vec{a}$

integrate

$\vec{V}' = \vec{a}t + \vec{c}_1$

integrate again

$$\boxed{\vec{V} = \vec{a}\frac{t^2}{2} + \vec{c}_1 t + \vec{c}_2}$$

vv. imp

Q 11/55 If $\vec{v}$ is a vector function, solve the equation $\frac{d^2\vec{v}}{dt^2} = at + b$, where $\vec{a}$ and $\vec{b}$ are constant, and both $\vec{v}(t)$, and $\vec{v}'(t)$ vanish at $t = 0$.

Solve:- $\frac{d^2\vec{v}}{dt^2} = \vec{a}t + \vec{b}$

$$\vec{v}'' = \vec{a}t + \vec{b}$$

integrate

$$\boxed{\vec{v}' = \frac{\vec{a}t^2}{2} + \vec{b}t + \vec{c}_1} \quad (1)$$

integrate

$$\boxed{\vec{v} = \frac{\vec{a}t^3}{6} + \frac{\vec{b}t^2}{2} + \vec{c}_1 t + \vec{c}_2} \quad (2)$$

$\vec{v}$ and $\vec{v}'$ vanish at $t = 0$

In mathematics vanish means $= 0$.

$(1) \Rightarrow \quad 0 = 0 + 0 + \vec{c}_1$

$\Rightarrow \boxed{\vec{c}_1 = 0}$

$(2) \Rightarrow \quad 0 = 0 + 0 + 0 + \vec{c}_2$

$\Rightarrow \boxed{\vec{c}_2 = 0}$

(2) becomes

$$\boxed{\vec{v} = \frac{\vec{a}t^3}{6} + \frac{\vec{b}t^2}{2}}$$

Ans

v.v. imp

(Q6/54) Determine the vector function which has $2\cos 2t\,\vec{i} + 2\sin 2t\,\vec{j} + 4\vec{k}$ as its deriative and $\vec{i} + \vec{j} + \vec{k}$ as value at $t = 0$.

Sol:- $\dfrac{d\vec{v}}{dt} = 2\cos 2t\,\vec{i} + 2\sin 2t\,\vec{j} + 4\vec{k}$

$\vec{v}' = 2\cos 2t\,\vec{i} + 2\sin 2t\,\vec{j} + 4\vec{k}$

integrate

$$\vec{v} = \frac{2\sin 2t}{2}\vec{i} - \frac{2\cos 2t}{2}\vec{j} + 4t\vec{k} + \vec{c}$$

$$\boxed{\vec{v} = \sin 2t\,\vec{i} - \cos 2t\,\vec{j} + 4t\vec{k} + \vec{c}} \quad (1)$$

$\vec{v} = \vec{i} + \vec{j} + \vec{k}$ at $t = 0$

$\vec{i} + \vec{j} + \vec{k} = 0 - \vec{j} + 0 + \vec{k}$

$$\boxed{\vec{c} = \vec{i} + 2\vec{j} + \vec{k}}$$

(1) $\Rightarrow$

$$\vec{v} = \sin 2t\,\vec{i} - \cos 2t\,\vec{j} + 4t\vec{k} + \vec{i} + 2\vec{j} + \vec{k}$$

$$\boxed{\vec{v} = (\sin 2t + 1)\vec{i} + (2 - \cos 2t)\vec{j} + (4t - 1)\vec{k}}$$

Ans

v.imp

Q 7 / 54 If $\vec{r}''(t) = 4\vec{i}$, and if $\vec{r}(t) = 0$, when $t = 0$ and $\vec{r}'(t) = 4\vec{j}$ when $t = 0$, Show that the tip of the position vector $\vec{r}(t)$ describes a parabola.

Sol:- (i) $\vec{r}''(t) = 4\vec{i}$

integrate

$$\boxed{\vec{r}'(t) = 4t\vec{i} + \vec{c}_1} \quad (1)$$

integrate

$$\boxed{\vec{r}(t) = 4t^2\vec{i} + \vec{c}_1 t + \vec{c}_2} \quad (2)$$

$\vec{r}(t) = 0$ when $t = 0$

(2) $\Rightarrow$ $0 = 0 + 0 + \vec{c}_2$

$\Rightarrow \boxed{\vec{c}_2 = 0}$

$\vec{r}'(t) = 4\vec{j}$ when $t = 0$

(1) $\Rightarrow$ $4\vec{j} = 0 + \vec{c}_1$

$\Rightarrow \boxed{\vec{c}_1 = 4\vec{j}}$

$\therefore \vec{r}(t) = 2t^2\vec{i} + 4t\vec{j} + 0$

$$\boxed{\vec{r}(t) = 2t^2\vec{i} + 4t\vec{j}}$$ Ans

p.T.o

(ii) To show that the tip of position vector describes parabola.

$$\vec{r}(t) = 2t^2\vec{i} + 4t\vec{j}$$
$$= x\vec{i} + y\vec{j}$$
Comparing

$x = 2t^2$, $y = 4t \Rightarrow \boxed{t = \frac{y}{4}}$

$$x = 2\left(\frac{y}{4}\right)^2 = 2\left(\frac{y^2}{16}\right)$$

$$\boxed{x = \frac{y^2}{8}}$$

$$y^2 = 8x$$

$$\Rightarrow \quad y^2 = 4(2)x$$

$$y^2 = 4(2)x$$

which is the equation of parabola.

Hence the tip of position vector describes parabola.

v.imp

Q8/54 Solve the equation $\vec{a} \times \frac{d^2\vec{v}}{dt^2} = \vec{b}$ where $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}$ and $\vec{b}$ are constant and $\vec{v}$ is a vector function of t.

Sol:- $\vec{a} \times \frac{d^2\vec{v}}{dt^2} = \vec{b}$

$$\vec{a} \times \vec{v}'' = \vec{b}$$

integrate

$$\vec{a} \times \vec{v}' = \vec{b}t + \vec{c}_1$$

integrate

$$\vec{a} \times \vec{v} = \frac{\vec{b}t^2}{2} + \vec{c}_1 t + \vec{c}_2$$

Taking cross product with $\vec{a}$

$$\vec{a} \times (\vec{a} \times \vec{v}) = \vec{a} \times \left(\frac{\vec{b}t^2}{2} + \vec{c}_1 t + \vec{c}_2\right)$$

$$(\vec{a}\cdot\vec{v})\vec{a} - (\vec{a}\cdot\vec{a})\vec{v} = \vec{a} \times \frac{\vec{b}t^2}{2} + \vec{a} \times \vec{c}_1 t + \vec{a} \times \vec{c}_2$$

$$(\vec{a}\cdot\vec{v})\vec{a} - \vec{a} \times \frac{\vec{b}t^2}{2} - \vec{a} \times \vec{c}_1 t - \vec{a} \times \vec{c}_2 = (\vec{a}\cdot\vec{a})\vec{v}$$

$$a^2\vec{v} = (\vec{a}\cdot\vec{v})\vec{a} + \vec{b} \times \vec{a}\frac{t^2}{2} + \vec{c}_1 a t$$

$$\boxed{\vec{v} = \left(\frac{\vec{a}\cdot\vec{v}}{a^2}\right)\vec{a} + \vec{b} \times \vec{a}\left(\frac{t^2}{2a^2}\right) + \vec{c}_1 \times a\frac{t}{a^2}}$$

Ans

EXERSISE # 3.1

Q1/38 Calculate $|\vec{F}(t)|$ if $\vec{F}(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + 8\vec{k}$.

Sol:- $\vec{F}(t) = \sin t\,\vec{i} + \cos t\,\vec{j} + 8\vec{k}$

$$|\vec{F}(t)| = \sqrt{\sin^2 t + \cos^2 t + (8)^2}$$

$$= \sqrt{1+64} = \sqrt{65} \quad \text{Ans.}$$

Q3/38 If $\vec{F}(t)$ and $\vec{g}(t)$ use vector functions, then which of (i) $\vec{F}(t) + \vec{g}(t)$ (ii) $\vec{F}(t) \cdot \vec{g}(t)$ (iii) $\vec{F}(t) \times \vec{g}(t)$ is a vector or a scalar functions?

Sol:- (i) $\vec{F}(t) = [f_1, f_2, f_3]$, $\vec{g}(t) = [g_1, g_2, g_3]$

$$\vec{F}(t) + \vec{g}(t) = [f_1 + g_1, f_2 + g_2, f_3 + g_3]$$

which is a vector function.

(ii) $\vec{F}(t) \cdot \vec{g}(t) = [f_1, f_2, f_3] \cdot [g_1, g_2, g_3]$

$$= f_1 \cdot g_1 + f_2 \cdot g_2 + f_3 \cdot g_3$$

which is a scalar function.

(iii) $\vec{F}(t) \times \vec{g}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$

$$= \vec{i}\,(f_2 g_3 - f_3 g_2) + \vec{j}\,(f_3 g_1 - f_1 g_3) + \vec{k}\,(f_1 g_2 - f_2 g_1)$$

which is a vector function.

EX 50 If $\vec{F}(s,t) = (ts^2 - 2t^2 s)\vec{i} + t^3 s^2 \vec{j} + t^2 s^3 \vec{k}$. Calculate the partial derivatives of $\vec{F}(s,t)$.

$$\vec{F}(s) = (2ts - 2t^2)\vec{i} + 2t^3 s\vec{j} + 3t^2 s^2 \vec{k}.$$

$$\vec{F}(t) = (s^2 - 4ts)\vec{i} + 3t^2 s^2 \vec{j} + 2ts^3 \vec{k}$$

Now we will have to find 2nd order partial derivatives.

$$\vec{F}(ss) = (2t - 0)\vec{i} + 2t^3 \vec{j} + 6t^2 s \vec{k}.$$

$$\vec{F}(tt) = (0 - 4s)\vec{i} + 6ts^2 \vec{j} + 2s^3 \vec{k}.$$

$$\vec{F}(st) = (2s - 4t)\vec{i} + 6t^2 s\vec{j} + 6ts^2 \vec{k}.$$

$$\vec{F}(ts) = (2s - 4t)\vec{i} + 6ts\vec{j} + 6ts^2 \vec{k}.$$

Ans