C.H # 2 VECTORS

APPLICATIONS OF VECTOR ALGEBRA.

Third year (B.Sc)

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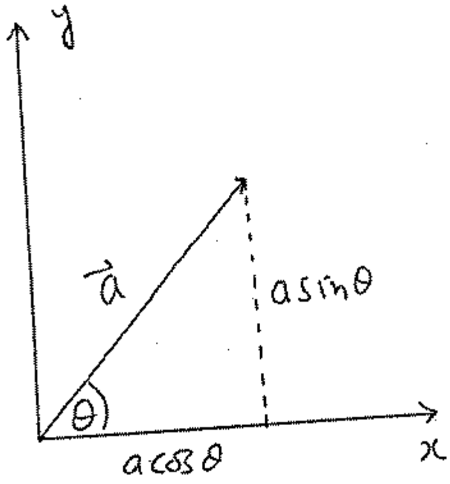
CHAPTER # 2

APPLICATIONS OF VECTOR ALGEBRA.

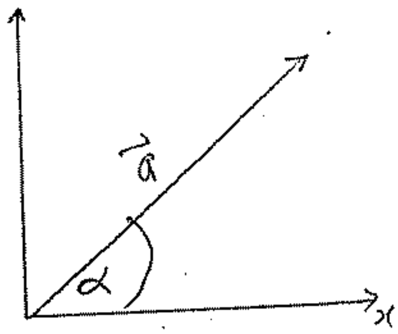
$\vec{a} = [a_1, a_2] = a_1\vec{i} + a_2\vec{j}$

$\vec{a} = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j}$

$\boxed{\vec{a} = a(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})}$



FORMULA FOR UNIT VECTOR



Let us consider $\vec{a}$ is a unit vector in the xy-plane making angle α with the positive direction of x-axis

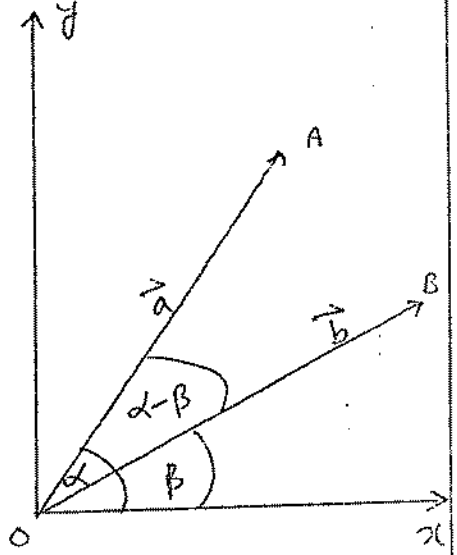
$\vec{a} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$

$\vec{a} = \cos(\text{angle})\,\vec{i} + \sin(\text{angle})\,\vec{j}$.

(Ex-1 / 28) Prove victorially that

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Sol:- Let $\vec{OA}$ and $\vec{OB}$ be two unit vectors in the xy-plane making angles α and β with the positive direction of x-axis.



$\vec{OA} = \vec{a} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$

$\vec{OB} = \vec{b} = \cos\beta\,\vec{i} + \sin\beta\,\vec{j}$

By taking dot product

$\vec{a}\cdot\vec{b} = [\cos\alpha, \sin\alpha]\cdot[\cos\beta, \sin\beta]$

$ab\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Since $\vec{a}$ & $\vec{b}$ are unit vectors

i.e $a = b = 1.$

$ab\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

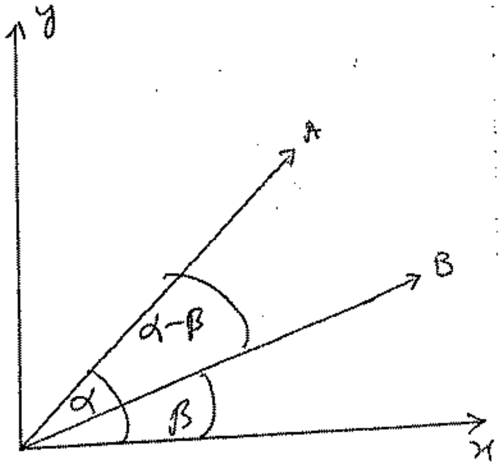
$1.1\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

proved

<u>Question</u> Using vectors prove

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$



proof:- let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two unit vectors in the xy-plane making angles α and β with the direction of x-axis

$$\overrightarrow{OA} = \vec{a} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$$

$$\overrightarrow{OB} = \vec{b} = \cos\beta\,\vec{i} + \sin\beta\,\vec{j}$$

By taking cross product

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \vec{i}[0] + \vec{j}[0-0] + \vec{k}[\cos\alpha\sin\beta - \sin\alpha\cos\beta]$$

$$|\vec{a}\times\vec{b}| = \sqrt{0^2 + 0^2 + [-(\sin\alpha\cos\beta - \cos\alpha\sin\beta)]^2}$$

$$ab\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Since $a = b = 1$ because $\vec{a}$ & $\vec{b}$ are unit vectors

So $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

<u>proved</u>.

Ch-02

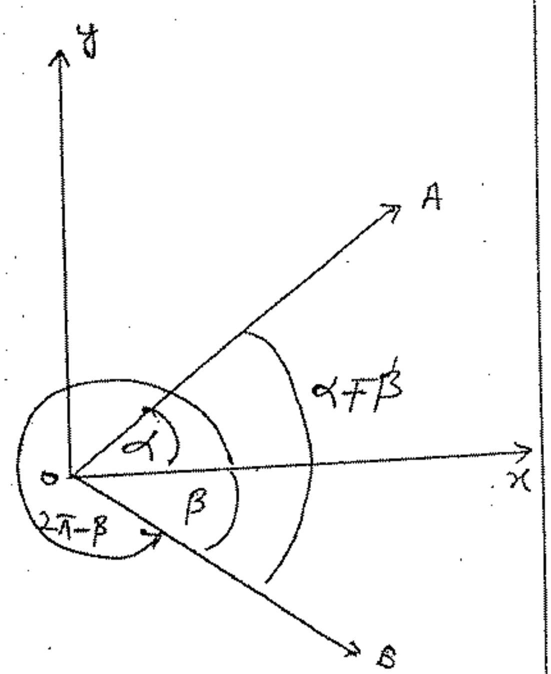
NOTE

Angle between two vectors is equal to angle between their directions.

Q1/35 By using vectors prove that

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

Proof:- Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two unit vectors in the xy-plane making angles α, $2\pi-\beta$ with the positive direction of x-axis.



$\overrightarrow{OA} = \vec{a} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$

$\overrightarrow{OB} = \vec{b} = \cos(2\pi-\beta)\,\vec{i} + \sin(2\pi-\beta)\,\vec{j}$

$\vec{b} = \cos\beta\,\vec{i} - \sin\beta\,\vec{j}$

Taking dot product

$\vec{a}\cdot\vec{b} = [\cos\alpha, \sin\alpha]\cdot[\cos\beta, -\sin\beta]$

$ab\cos(\alpha+\beta) = \cos\alpha\cos\beta + \sin\alpha(-\sin\beta)$

Since $\vec{a}$, $\vec{b}$ are unit vectors $\therefore a = b = 0$

$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$

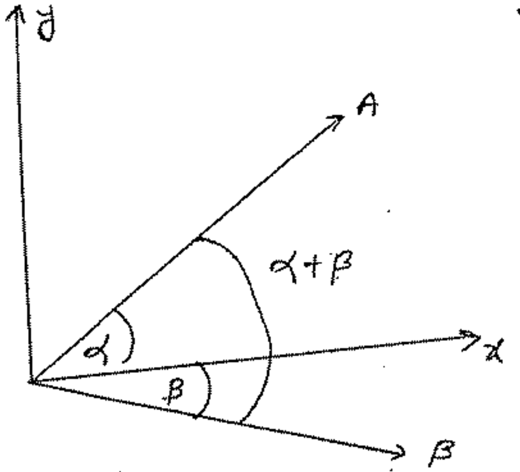
Q2/35 Prove that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Proof:- let $\vec{OA}$ and $\vec{OB}$ be two unit vectors in the xy-plane making angles α, $2\pi-\beta$ with the positive direction of x-axis

$\vec{OA} = \vec{a} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$

$\vec{OB} = \vec{b} = \cos(2\pi-\beta)\vec{i} + \sin(2\pi-\beta)\vec{j}$

$\vec{b} = \cos\beta\,\vec{i} - \sin\beta\,\vec{j}$



Taking cross product

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & -\sin\beta & 0 \end{vmatrix}$$ by R_1

$\vec{a}\times\vec{b} = \vec{i}[0-0] + \vec{j}[0-0] + \vec{k}[-\sin\beta\cos\alpha - \cos\beta\sin\alpha]$

$|\vec{a}\times\vec{b}| = \sqrt{0^2+0^2+(-(\sin\alpha\cos\beta+\cos\alpha\sin\beta))^2}$

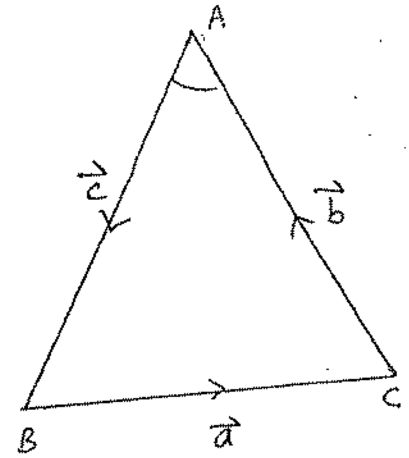
$ab\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$1\cdot 1\cdot \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$ proved

Ch-02

NOTE



$\overrightarrow{BC} = \vec{a}$

$\overrightarrow{CA} = \vec{b}$

$\overrightarrow{AB} = \vec{c}$

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$\overrightarrow{BC} = -\overrightarrow{AB} - \overrightarrow{CA}$

$\overrightarrow{BC} + \overrightarrow{AB} + \overrightarrow{CA} = 0$

$\vec{a} + \vec{c} + \vec{b}$

$\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \boxed{\vec{a} + \vec{b} + \vec{c} = 0}$

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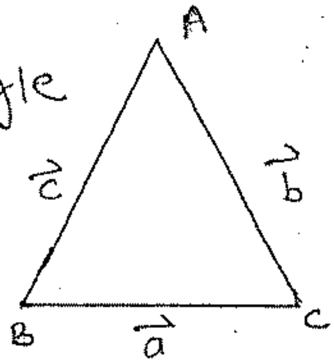
v.imp EX-2 / 28 Prove Sine ~~of~~ law of trigonometry.

with the usual notations, in a ΔABC

prove $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$



Proof:- let ABC be a triangle

$\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$

From the figure

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$\vec{a} = -\overrightarrow{AB} - \overrightarrow{CA}$

$\vec{a} = -\vec{c} - \vec{b}$

$\vec{a} + \vec{b} + \vec{c} = 0$

Taking cross product with $\vec{a}$

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = 0$

$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$

$0 + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$

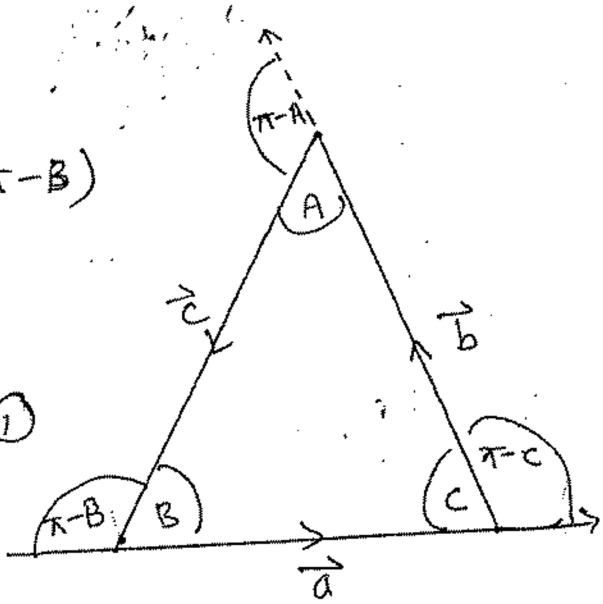
$\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

$|\vec{a} \times \vec{b}| = |\vec{c} \times \vec{a}|$

$ab\sin(\pi - C) = ca\sin(\pi - B)$

$b\sin C = c\sin B$

$$\boxed{\frac{\sin C}{c} = \frac{\sin B}{b}} \quad (1)$$



Again

$\vec{a} + \vec{b} + \vec{c} = 0$

Taking cross product with $\vec{b}$

$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = 0$

$\vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = 0$

$\vec{b} \times \vec{a} + 0 + \vec{b} \times \vec{c} = 0$

$\vec{b} \times \vec{a} = -\vec{b} \times \vec{a}$

$$\vec{b} \times \vec{c} = \vec{a} \times \vec{b}$$

$$|\vec{b} \times \vec{c}| = |\vec{a} \times \vec{b}|$$

$$bc \sin(\pi - A) = ab \sin(\pi - C)$$

$$c \sin A = a \sin C$$

$$\boxed{\frac{\sin A}{a} = \frac{\sin C}{c}} \quad (2)$$

Combining (2) and (1)

$$\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin B}{b}$$

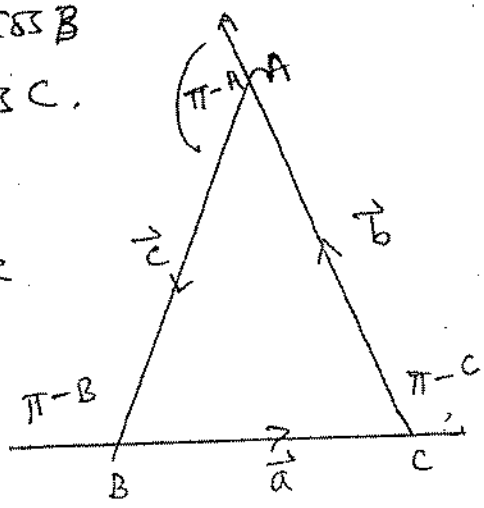
$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}} \quad \text{proved.}$$

Prove law of Cosine

(i) $a^2 = b^2 + c^2 - 2bc \cos A$

(ii) $b^2 = a^2 + c^2 - 2ac \cos B$

(iii) $c^2 = a^2 + b^2 - 2ab \cos C$.



Proof:- In a ΔABC

Let ABC be a triangle

$\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$

From the figure

(see next page)

$\vec{BC} = \vec{BA} + \vec{AC} = -\vec{AB} - \vec{CA}$

$\vec{a} = -\vec{c} - \vec{b}$

$\vec{a} = -(\vec{b} + \vec{c})$

Squaring both sides

$(\vec{a})^2 = [-(\vec{b} + \vec{c})]^2$

$= (\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{c})$

$= \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c}$

$= b^2 + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{c} + c^2$

$= b^2 + c^2 + 2bc\cos(\pi - A)$

$a^2 = b^2 + c^2 + 2bc(-\cos A)$

$\boxed{a^2 = b^2 + c^2 - 2bc\cos A.}$

(ii) $b^2 = a^2 + c^2 - 2ac\cos B$

In ΔABC

$\vec{CA} = \vec{CB} + \vec{BA}$

$\vec{b} = -\vec{BC} - \vec{AB}$

$\vec{b} = -(\vec{a} + \vec{c})$ squaring

$(\vec{b})^2 = [-(\vec{a} + \vec{c})]^2$

$(\vec{b})^2 = [(\vec{a} + \vec{c}) \cdot (\vec{a} + \vec{c})]$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{c}$$

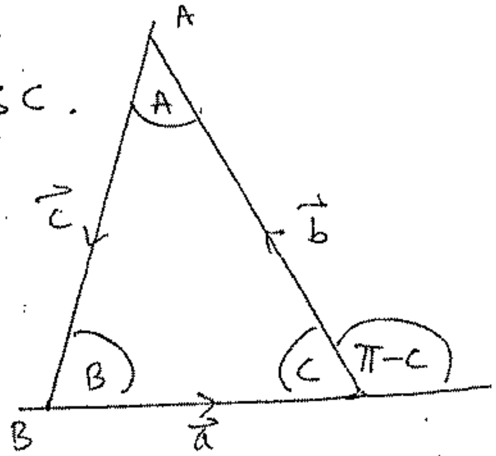
$$(\vec{b})^2 = a^2 + 2\vec{a}\cdot\vec{c} + c^2$$

$$b^2 = a^2 + 2ac\cos(\pi - B) + c^2$$

$\cos(\pi - B) = -\cos B$

$$b^2 = a^2 + c^2 - 2ac\cos(\pi - B)$$

(iii) $c^2 = a^2 + b^2 - 2ab\cos C.$



$\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$

let $\vec{AB} = \vec{AC} + \vec{CB}$

$$\vec{c} = -\vec{CA} - \vec{BC}$$

$$\vec{c} = -\vec{b} - \vec{a}$$

$$\vec{c} = -(\vec{b} + \vec{a})$$

Squarring.

$$(\vec{c})^2 = [-(\vec{b} + \vec{a})]^2 = [(\vec{b} + \vec{a})\cdot(\vec{b} + \vec{a})]$$

$$(\vec{c})^2 = \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{a}$$

$$(\vec{c})^2 = b^2 + 2\vec{a}\cdot\vec{b} + a^2$$

$$c^2 = a^2 + b^2 + 2ab\cos(\pi - c)$$

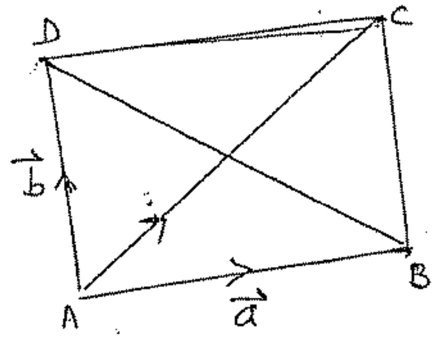
$$c^2 = a^2 + b^2 + 2ab(-\cos c)$$

$$c^2 = a^2 + b^2 - 2ab\cos c.$$

proved

Q9/27 Prove that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals.

Sol:- let ABCD be a llm



$\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$

So, that,

$\overrightarrow{DC} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$.

$AB^2 + BC^2 + DC^2 + AD^2 = \overrightarrow{AB}^2 + \overrightarrow{BC}^2 + \overrightarrow{DC}^2 + \overrightarrow{AD}^2$

$= \vec{a}^2 + \vec{b}^2 + \vec{a}^2 + \vec{b}^2$

$A^2 = \vec{A}^2$

$AB^2 + BC^2 + DC^2 + AD^2 = 2a^2 + 2b^2$ — ①

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$\overrightarrow{BD} = \overrightarrow{BC} - \overrightarrow{DC}$

$\boxed{\overrightarrow{BD} = \vec{b} - \vec{a}}$

$AC^2 + BD^2 = \overrightarrow{AC}^2 + \overrightarrow{BD}^2$

$AC^2 + BD^2 = (\vec{a} + \vec{b})^2 + (\vec{b} - \vec{a})^2$

$AC^2 + BD^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a}\vec{b} + \vec{b}^2 + \vec{a}^2 - 2\vec{b}\vec{a}$

$AC^2 + BD^2 = 2a^2 + 2b^2$ — ②

① = ②

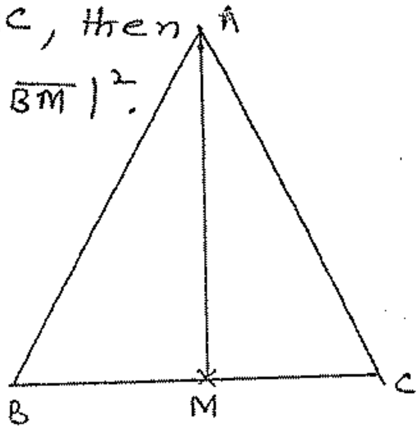
$AB^2 + BC^2 + DC^2 + AD^2 = AC^2 + BD^2$ proved

Diagonal

Non adjecent corners of a //m is called diagonals.

v.imp Ex-5 25 Prove that in a triangle ABC if M is the mid point of BC, then

$$|\overline{AB}|^2 + |\overline{AC}|^2 = 2|\overline{AM}|^2 + 2|\overline{BM}|^2.$$



Sol:- let ABC be a triangle M, is the mid point of BC.

$\therefore\ \overrightarrow{BM} = \overrightarrow{MC}$

Then

$\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MB}$

$\boxed{\overrightarrow{AB} = \overrightarrow{AM} - \overrightarrow{BM}}$

$\overrightarrow{AC} = \overrightarrow{AM} + \overrightarrow{MC}$

$\boxed{\overrightarrow{AC} = \overrightarrow{AM} + \overrightarrow{BM}}$

$$L.H.S = |\overline{AB}|^2 + |\overline{AC}|^2$$

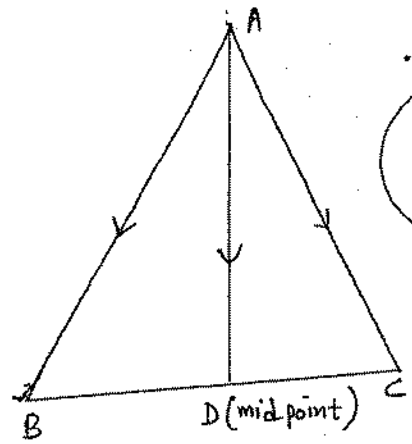
$$= \overrightarrow{AB}^2 + \overrightarrow{AC}^2 = (\overrightarrow{AM} - \overrightarrow{BM})^2 + (\overrightarrow{AM} + \overrightarrow{BM})^2$$

$$= \overrightarrow{AM}^2 + \overrightarrow{BM}^2 - \cancel{2\overrightarrow{AM}\cdot\overrightarrow{BM}} + \overrightarrow{AM}^2 + \overrightarrow{BM}^2 + \cancel{2\overrightarrow{AM}\cdot\overrightarrow{BM}}$$

$$= 2\,\overrightarrow{AM}^2 + 2\,\overrightarrow{BM}^2$$

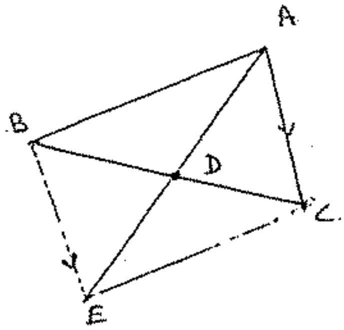
$$L.H.S = 2|\overline{AM}|^2 + 2|\overline{BM}|^2$$

proved



Special case of (λ, μ) theorem

$\overrightarrow{AD} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$



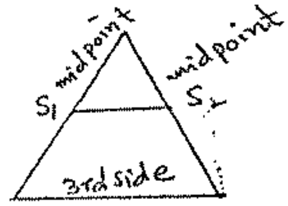
proof:- $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$

$2\overrightarrow{AD} = \overrightarrow{AD} + \overrightarrow{AC}$

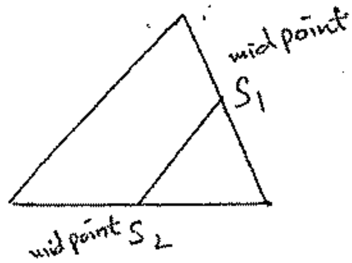
$\overrightarrow{AD} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$

Some points for next Question.

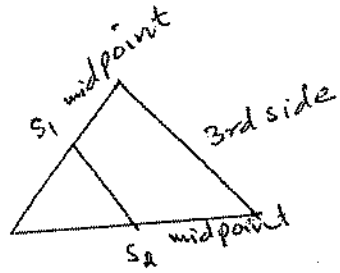
① (i)



(ii)



(iii)

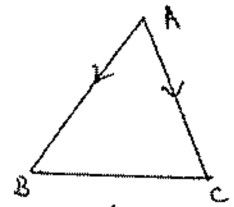


② Formula for parallel vectors

If $\vec{a} = K\vec{b}$

$\vec{a} \parallel \vec{b}$, K is any nonzero number.

③



$\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$

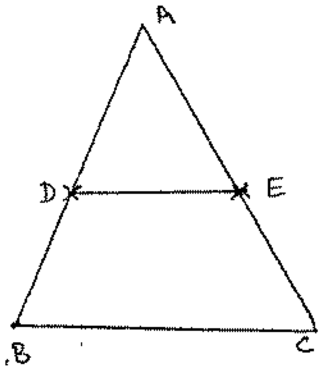
$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$

$\boxed{\overrightarrow{BC} = \vec{c} - \vec{b}}$

Ch-02

v.imp

Ex-1 / 21 Show that the straight line joining the mid points of the two sides of a triangle is ||el to the third sides and equal to one half of it.



Sol:- let us suppose we have a triangle ABC & D, E are its mid points of its sides AB and AC.

Let $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC}$

$\overrightarrow{BC} = -\vec{b} + \vec{c}$

$\boxed{\overrightarrow{BC} = \vec{c} - \vec{b}}$ —— (1)

$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE}$

$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$

$= \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$

$= \frac{1}{2}(-\overrightarrow{AB} + \overrightarrow{AC})$

$= \frac{1}{2}(-\vec{b} + \vec{c})$

$= \frac{1}{2}(\vec{c} - \vec{b}$

$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$ By using (1)

$\overline{AB}$ = line

$|\overline{AB}|$ = Length of line.

(i) ⇒ $|\overline{DE}| = \frac{1}{2}|\overline{BC}|$

(ii) $\overline{DE} \parallel \overline{BC}$.

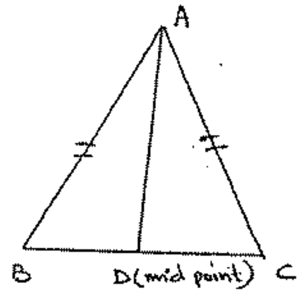
Q8/27 Prove that the median bisecting the base of an isoceles triangle is perpendicular to the base.

Isoceles triangle
That triangle having any two sides equal is called an isoceles triangle.

Sol:- To prove $\overline{AD} \perp \overline{BC}$

let ABC be an isoceles triangle.

Therefore
AB = AC



Suppose the median AD bisects the base at point D, so that D is the mid point of BC.

let $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$

$\overrightarrow{BC} = \vec{c} - \vec{b}$

If $\vec{a} \cdot \vec{b} = 0$
then $\vec{a} \perp ar\ \vec{b}$

By (λ, μ) theorem

$\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

$\boxed{\overrightarrow{AD} = \frac{1}{2}(\vec{b} + \vec{c})}$

P.T.O

Now

$\overrightarrow{AD} \cdot \overrightarrow{BC} = \frac{1}{2}(\vec{b}+\vec{c}) \cdot (\vec{c}-\vec{b})$

$\overrightarrow{AD} \cdot \overrightarrow{BC} = \frac{1}{2}(\vec{c}+\vec{b}) \cdot (\vec{c}-\vec{b})$

$\overrightarrow{AD} \cdot \overrightarrow{BC} = \frac{1}{2}(\vec{c}^2 - \vec{b}^2)$

$\overrightarrow{AD} \cdot \overrightarrow{BC} = \frac{1}{2}(\vec{b}^2 - \vec{b}^2)$

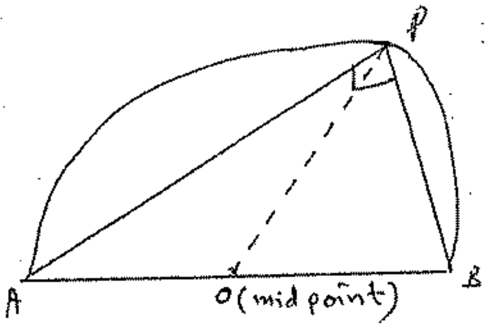
$\overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

$\Rightarrow \overline{AD} \perp \overline{BC}$ proved

$AB = AC$
$\overrightarrow{AB} = \overrightarrow{AC}$
$\vec{b} = \vec{c}.$

To prove $\angle APB = \pi/2$

let $\overset{\frown}{APB}$ be a semicircle of centre O and radius OP, P is any point on the semicircle.



$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$\overrightarrow{AP} = \overrightarrow{OP} + \overrightarrow{AO}$

$\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP}$

$\overrightarrow{BP} = -\overrightarrow{OB} + \overrightarrow{OP}$

$\overrightarrow{BP} = -\overrightarrow{AO} + \overrightarrow{OP}$

$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{AO}$

OP = AO radius.

$\overrightarrow{AP} \cdot \overrightarrow{BP} = (\overrightarrow{OP} + \overrightarrow{AO}) \cdot (\overrightarrow{OP} - \overrightarrow{AO})$

$= \overrightarrow{OP}^2 - \overrightarrow{AO}^2$

$= OP^2 - AO^2$

$\overrightarrow{AP}\cdot\overrightarrow{BP} = AO^2 - AO^2$

$\overrightarrow{AP}\cdot\overrightarrow{BP} = 0$

$\Rightarrow \overline{AP} \perp \overline{BP}$

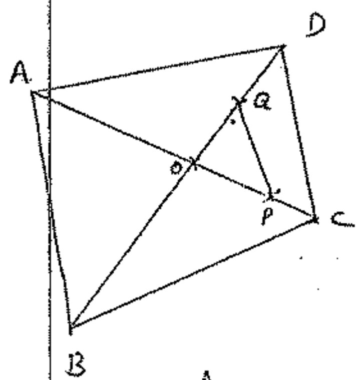
$\Rightarrow \angle APB = \frac{\pi}{2}.$

Q10/27) If P and Q are the mid points of the diagonals of a quadrilateral ABCD, Prove that $|AB|^2+|BC|^2+|CD|^2+|DA|^2$

$= |AC|^2+|BD|^2+4|PQ|^2.$

Sol:- Let ABCD be a quadrilateral.

Suppose P, Q are mid points of diagonals AC and BD



Suppose diagonals intersect at point O.

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, $\overrightarrow{OD} = \vec{d}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \vec{b} - \vec{a}$

Similarly

$\overrightarrow{BC} = \vec{c} - \vec{b}$

$\overrightarrow{CD} = \vec{d} - \vec{c}$

$\overrightarrow{DA} = \vec{a} - \vec{d}$

$\overrightarrow{AC} = \vec{c} - \vec{a}$

$\overrightarrow{BD} = \vec{d} - \vec{b}$

$$\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CD} + \overrightarrow{DQ}$$

$$= \frac{1}{2}\overrightarrow{AC} + \overrightarrow{CD} + \frac{1}{2}\overrightarrow{DB}$$

$$= \frac{1}{2}(\vec{c} - \vec{a}) + (\vec{d} - \vec{c}) + \frac{1}{2}(\vec{b} - \vec{d})$$

$$= \frac{1}{2}\left[\vec{c} - \vec{a} + 2\vec{d} - 2\vec{c} + \vec{b} - \vec{d}\right]$$

$$\boxed{\overrightarrow{PQ} = \frac{1}{2}\left[(\vec{b} + \vec{d}) - (\vec{a} + \vec{c})\right]}$$

$$L.H.S = |AB|^2 + |BC|^2 + |CD|^2 + |DA|^2$$

$$L.H.S = \overrightarrow{AB}^2 + \overrightarrow{BC}^2 + \overrightarrow{CD}^2 + \overrightarrow{DA}^2$$

$$L.H.S = (\vec{b} - \vec{a})^2 + (\vec{c} - \vec{b})^2 + (\vec{d} - \vec{c})^2 + (\vec{a} - \vec{d})^2$$

$$L.H.S = b^2 + a^2 - 2\vec{b}.\vec{a} + c^2 + b^2 - 2\vec{c}\vec{b} + d^2 + c^2 - 2\vec{d}.\vec{c} + a^2 + d^2 - 2\vec{a}.\vec{d}$$

$$L.H.S = 2a^2 + 2b^2 + 2c^2 + 2d^2 - 2\vec{a}\vec{b} - 2\vec{b}\vec{c} - 2\vec{c}\vec{d} - 2\vec{a}\vec{d} \quad (1)$$

$$R.H.S = |AC|^2 + |BD|^2 + 4|PQ|^2$$

$$= \overrightarrow{AC}^2 + \overrightarrow{BD}^2 + 4\overrightarrow{PQ}^2$$

$$= (\vec{c} - \vec{a})^2 + (\vec{d} - \vec{b})^2 + 4 \cdot \frac{1}{4}\left[(\vec{b} + \vec{d}) - (\vec{a} + \vec{c})\right]^2$$

P.T.O

$\therefore$

$$= (\vec{c}-\vec{a})^2 + (\vec{d}-\vec{b})^2 + \left[(\vec{b}+\vec{d})-(\vec{a}+\vec{c})\right]^2$$

$$= c^2+a^2-2\cancel{\vec{c}\cdot\vec{a}}+d^2+b^2-2\cancel{\vec{d}\cdot\vec{b}}-2(\vec{b}+\vec{d})\cdot(\vec{a}+\vec{c})$$

$$+c^2+a^2+2\cancel{\vec{a}\cdot\vec{c}}+d^2+b^2+2\cancel{\vec{b}\cdot\vec{d}}$$

$$R.H.S = 2a^2+2b^2+2c^2+2d^2-2\vec{b}\vec{a}-2\vec{b}\vec{c}-2\vec{d}\vec{a}-2\vec{d}\vec{c} \quad (2)$$

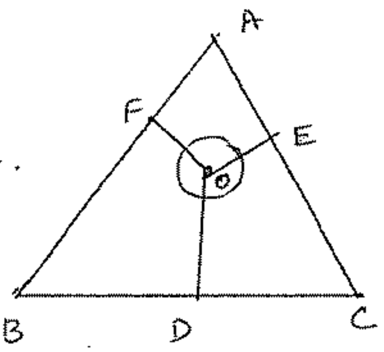
$$(1) = (2)$$

So,

$$|AB|^2+|BC|^2+|CD|^2+|DA|^2 = |AC|^2+|BD|^2+4|PQ|^2.$$

proved.

NOTE



Right bisector = $\perp$ar bisector.

Concurrent

Passing through a single point.

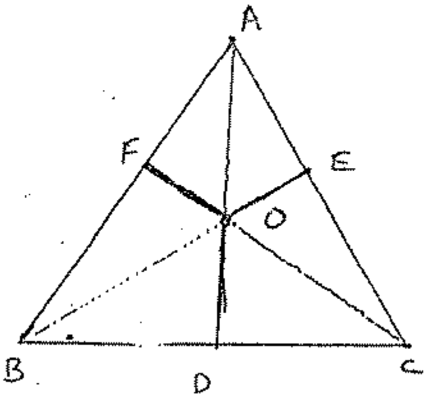
In order to prove three right bisector are concurent proceed as follows.

D, E, F are mid points.

Suppose right bisector OD and OE meet at point O.

Join O with F. If we prove that OF is also a right bisector. Then the result will be proved v.v.imp

Q11/27 Prove that the right bisector of the sides of a triangle are concurrent.



Proof:- Let ABC be a triangle and D, E, F be mid points of its sides. Suppose the right bisectors OD & OE intersect at point O.

Join O & F. In order to prove that the right bisectors are concurrent, we just prove that OF is also a right bisector. But OF is already bisector ($\because$ F is mid pt of AB)

$\therefore$ we only prove $OF \perp AB$

Target
$\overrightarrow{OF} \cdot \overrightarrow{AB} = 0$
$\frac{1}{2}(\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a})=0$

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$.

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\overrightarrow{OB} + \overrightarrow{OC}$$
$$= \overrightarrow{OC} - \overrightarrow{OB}$$
$$= \vec{c} - \vec{b}$$

Similarly on the same line

$$\overrightarrow{CA} = \vec{a} - \vec{c}$$
$$\overrightarrow{AB} = \vec{b} - \vec{a}$$

Since D, E, F are mid points i.e

By (λ, μ) theorem P.T.O

$$\overrightarrow{OD} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) = \frac{1}{2}(\vec{b} + \vec{c})$$

$$\overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OA}) = \frac{1}{2}(\vec{c} + \vec{a})$$

$$\overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(\vec{a} + \vec{b})$$

$\cancel{\overrightarrow{OB}}$

$\overline{OD} \perp \overline{BC}$

$\overrightarrow{OD} \cdot \overrightarrow{BC} = 0$

$\frac{1}{2}(\vec{b} + \vec{c})(\vec{c} - \vec{b}) = 0$

$(\vec{c} + \vec{b})(\vec{c} - \vec{b}) = 0$

$\vec{c}^{\,2} - \vec{b}^{\,2} = 0$ ——— (1)

$\overline{OE} \perp \overline{AC}$

$\overrightarrow{OE} \cdot \overrightarrow{AC} = 0$

$\frac{1}{2}(\vec{c} + \vec{a})(\vec{c} - \vec{a}) = 0$

$\vec{c}^{\,2} - \vec{a}^{\,2} = 0$ ——— (2)

Target

$\overrightarrow{OF} \cdot \overrightarrow{AB} = 0$

$\frac{1}{2}(\vec{a} + \vec{b})(\vec{b} - \vec{a}) = 0$

Subtract (2) from (1)

$\cancel{\vec{c}^{\,2}} - \vec{b}^{\,2} - \cancel{\vec{c}^{\,2}} + \vec{a}^{\,2} = 0$

$-(\vec{b}^{\,2} - \vec{a}^{\,2}) = 0$

$(\vec{b} + \vec{a})(\vec{b} - \vec{a}) = 0$ Multiply by $\frac{1}{2}$

$\boxed{\frac{1}{2}(\vec{b} + \vec{a})}\boxed{(\vec{b} - \vec{a})} = 0$

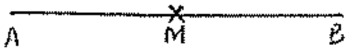
$\overrightarrow{OF} \cdot \overrightarrow{AB} = 0$

$\overline{OF} \perp \overline{AB}$

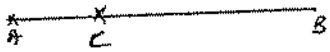
Hence OF is also a right bisector

i.e all the right bisectors are concurrent

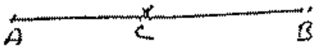
NOTE



$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$

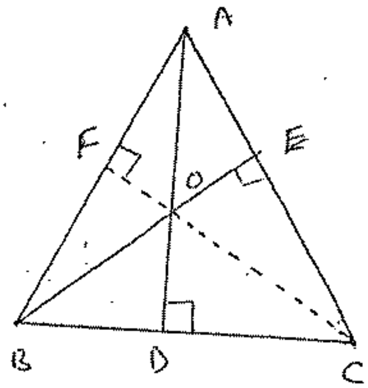


$\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$



$\overrightarrow{AC} = x\,\overrightarrow{AB}$.

Ex-6 25 Prove that the altitudes of a triangle are concurrent.



Proof:- let ABC be a triangle.

Suppose the altitudes AD and BE intersect at point O.

Join C and F passing through O.

In order to prove that the altitudes of triangle are concurrent, we will prove that CF is also an altitude i.e. $CF \perp AB$

Target
$CF \perp AB$
$\overrightarrow{CF}\cdot\overrightarrow{AB} = 0$

let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$\overrightarrow{BC} = -\overrightarrow{OB} + \overrightarrow{OC}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\overrightarrow{BC} = \vec{c} - \vec{b}$

Similarly $\overrightarrow{CA} = \vec{a} - \vec{c}$

$\overrightarrow{AB} = \vec{b} - \vec{a}$

$\overrightarrow{AO} = x\,\overrightarrow{AD}$

$\overrightarrow{AD} = \frac{1}{x}\overrightarrow{AO}$

$\overrightarrow{BO} = y\,\overrightarrow{BE}$

$\overrightarrow{BE} = \frac{1}{y}\overrightarrow{BO}$

$\overline{AD} \perp \overline{BC}$

$\overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

$\frac{1}{x} \cdot \overrightarrow{AO} \cdot \overrightarrow{BC} = 0$

$-\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$

$\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$

$\vec{a} \cdot (\vec{c} - \vec{b}) = 0$

$\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} = 0 \sim (1)$

$\overline{BE} \perp \overline{AC}$

$\overrightarrow{BC} \cdot \overrightarrow{AC} = 0$

$\frac{1}{y} \overrightarrow{BO} \cdot \overrightarrow{AC} = 0$

$-\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$\vec{b} \cdot (\vec{c} - \vec{a}) = 0$

$\vec{b} \cdot \vec{c} - \vec{b} \cdot \vec{a} = 0 \sim (2)$

(1) − (2)

$\vec{a} \cdot \vec{c} - \cancel{\vec{a} \cdot \vec{b}} - \vec{b} \cdot \vec{c} + \cancel{\vec{b} \cdot \vec{a}} = 0$

$-\vec{b} \cdot \vec{c} + \vec{a} \cdot \vec{c} = 0$

$-\vec{c}(\vec{b} - \vec{a}) = 0$

$-\overrightarrow{OC}(\vec{b} - \vec{a}) = 0$

$\overrightarrow{CO} \cdot \overrightarrow{AB} = 0$

$\frac{1}{2} \overrightarrow{CO} \cdot \overrightarrow{AB} = 0$

$\overrightarrow{CF} \cdot \overrightarrow{AB} = 0$

$\overline{CF} \perp \overline{AB}$

i.e. CF is also an altitude.

Hence all the altitudes are Concurrent

Ch-02

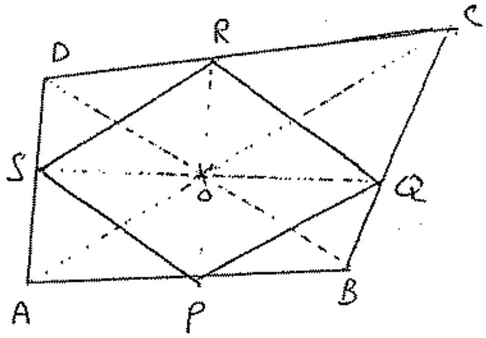
Ex-3 / 23 Prove that in a quadrilateral

(a) The straight lines joining the mid points of the sides taken in order form a parallelogram.

(b) The straight lines joining the mid points of the opposite sides bisect each other.

Sol:- Let ABCD be a quadrilateral.

Suppose P, Q, R, S are mid points of its sides. Taking O as the origin.



Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

$\overrightarrow{OC} = \vec{c}$, $\overrightarrow{OD} = \vec{d}$

① <u>To prove PQRS is a ||m</u>

Since P, Q, R, S are mid points, so by (λ, μ) theorem

$$\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(\vec{a} + \vec{b})$$

$$\overrightarrow{OQ} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) = \frac{1}{2}(\vec{b} + \vec{c})$$

$$\overrightarrow{OR} = \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD}) = \frac{1}{2}(\vec{c} + \vec{d})$$

$$\overrightarrow{OS} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OA}) = \frac{1}{2}(\vec{d} + \vec{a})$$

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\overrightarrow{PQ} = \frac{1}{2}(\vec{b}+\vec{c}) - \frac{1}{2}(\vec{a}+\vec{b})$

$\overrightarrow{PQ} = \frac{1}{2}(\vec{b}+\vec{c}-\vec{a}-\vec{b})$

$\boxed{\overrightarrow{PQ} = \frac{1}{2}(\vec{c}-\vec{a})}$

$\overrightarrow{SR} = \overrightarrow{SO} + \overrightarrow{OR}$

$\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS}$

$\overrightarrow{SR} = \frac{1}{2}(\vec{c}+\vec{d}) - \frac{1}{2}(\vec{d}+\vec{a})$

$\overrightarrow{SR} = \frac{1}{2}(\vec{c}+\cancel{\vec{d}}-\cancel{\vec{d}}-\vec{a})$

$\boxed{\overrightarrow{SR} = \frac{1}{2}(\vec{c}-\vec{a})}$

$\overrightarrow{PQ} = \overrightarrow{SR}$

$\vec{a} = k\vec{b}$

$\Rightarrow \vec{a} \parallel \vec{b}$

$\Rightarrow$ (i) $|\overrightarrow{PQ}| = |\overrightarrow{SR}|$ ①

(ii) $\overline{PQ} \parallel \overline{SR}$

$\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$

$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP}$

$\overrightarrow{PS} = \frac{1}{2}(\vec{d}+\vec{a}) - \frac{1}{2}(\vec{a}+\vec{b})$

$\overrightarrow{PS} = \frac{1}{2}(\vec{d}+\cancel{\vec{a}}-\cancel{\vec{a}}-\vec{b})$

$\boxed{\overrightarrow{PS} = \frac{1}{2}(\vec{d}-\vec{b})}$

$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

$= \frac{1}{2}(\vec{c}+\vec{d}) - \frac{1}{2}(\vec{b}+\vec{c})$

$= \frac{1}{2}(\cancel{\vec{c}}+\vec{d}-\vec{b}-\cancel{\vec{c}})$

$\boxed{\overrightarrow{QR} = \frac{1}{2}(\vec{d}-\vec{b})}$

$\overrightarrow{PS} = \overrightarrow{QR}$

$\Rightarrow$ (i) $|\overrightarrow{PS}| = |\overrightarrow{QR}|$ ②

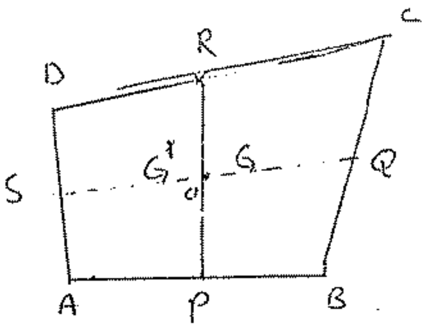
(ii) $\overline{PS} \parallel \overline{QR}$

So from ① and ② PQRS is a parallelogram.

Ex-3 / 23 (b) To find the lines bisect each other.

Let G = mid point of PR

G' = mid point of SQ



By (λ, μ) theorem

$$\vec{OG} = \frac{1}{2}(\vec{OP} + \vec{OR})$$

$$\vec{OG} = \frac{1}{2}\left[\frac{1}{2}(\vec{a}+\vec{b}) + \frac{1}{2}(\vec{c}+\vec{d})\right]$$

$$\boxed{\vec{OG} = \frac{1}{4}\left[\vec{a}+\vec{b}+\vec{c}+\vec{d}\right]} \quad (1)$$

Again by (λ, μ) theorem

$$\vec{OG'} = \frac{1}{2}(\vec{OS} + \vec{OQ})$$

$$\vec{OG'} = \frac{1}{2}\left[\frac{1}{2}(\vec{d}+\vec{a}) + \frac{1}{2}(\vec{b}+\vec{c})\right]$$

$$\boxed{\vec{OG'} = \frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})} \quad (11)$$

(1) = (11), So $\vec{OG} = \vec{OG'}$

$\Rightarrow G = G'$ So lines bisect each other.

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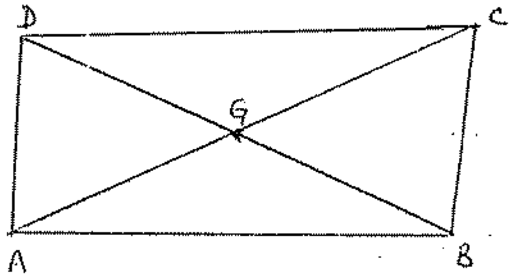
Dated : 07-01-2008.

Q4/26 Prove that the diagonals of a ||m bisect each other.

NOTE

The non adjacent Corners of a ||m are called diagonals.

Sol:-



Let ABCD be a ||m.

Suppose the diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point G.

$\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$

So, that $\overrightarrow{DC} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$

$\overrightarrow{AG} = x\overrightarrow{AC}$

$\overrightarrow{BG} = y\overrightarrow{BD}$.

From the figure

$\overrightarrow{AB} = \overrightarrow{AG} + \overrightarrow{GB} = \overrightarrow{AG} - \overrightarrow{BG}$

$\overrightarrow{AB} = x\overrightarrow{AC} - y\overrightarrow{BD}$

$\overrightarrow{AB} = x(\overrightarrow{AB} + \overrightarrow{BC}) - y(\overrightarrow{BC} + \overrightarrow{CD})$

$\overrightarrow{AB} = x(\overrightarrow{AB} + \overrightarrow{BC}) - y(\overrightarrow{BC} - \overrightarrow{DC})$

$\vec{a} = x(\vec{a} + \vec{b}) - y(\vec{b} - \vec{a})$

$\vec{a} = x\vec{a} + x\vec{b} - y\vec{b} + y\vec{a}$

$\vec{a} = (x+y)\vec{a} + (x-y)\vec{b}$ ——— (2)

$\vec{a} =$

Comparing Scalars on both sides.

$$\left.\begin{array}{ll}\vec{a}: & x+y=1 \\ \vec{b}: & x-y=0\end{array}\right\} \text{adding}$$

$$2x = 1 \Rightarrow \boxed{x = \tfrac{1}{2}}$$

$$y = x$$

$$\boxed{y = 1/2}$$

① $\Rightarrow$ $\overrightarrow{AG_1} = \frac{1}{2}\overrightarrow{AC}$

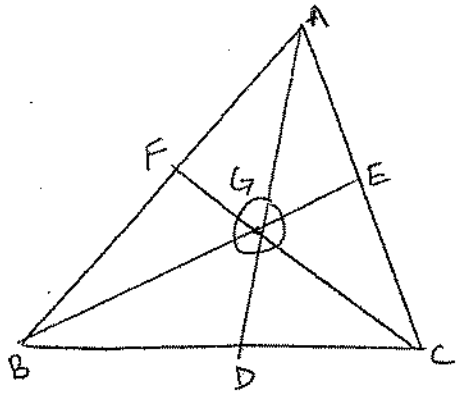
$\Rightarrow$ G is mid point of AC.

$\overrightarrow{BG} = \frac{1}{2}\overrightarrow{BC}$

$\Rightarrow$ G is mid point of BD.

Hence the diagonals of a //m intersect each other.

Centriod



$$\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AD}$$

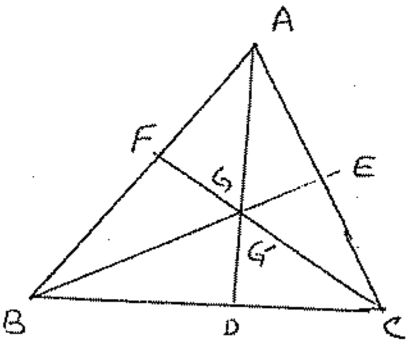
$$\overrightarrow{BG} = \frac{2}{3}\overrightarrow{BE}$$

$$\overrightarrow{CG} = \frac{2}{3}\overrightarrow{CF}.$$

(Ex-2.22) Show that the medians of a triangle trisect each other and are concurrent

Sol:-

(i) To prove medians of triangle trisect each other.



let ABC be a triangle D, E, F are mid points of its sides.

Suppose the medians AD and BE intersect at point G.

let $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$

From figure

$$\overrightarrow{AB} = \overrightarrow{AG} + \overrightarrow{GB}$$

$$\boxed{\begin{aligned}\overrightarrow{AG} &= x\,\overrightarrow{AD}\\ \overrightarrow{BG} &= y\,\overrightarrow{BE}\end{aligned}} \quad (1)$$

$$\overrightarrow{AB} = \overrightarrow{AG} - \overrightarrow{BG}$$

$$\overrightarrow{AB} = x\,\overrightarrow{AD} - y\,\overrightarrow{BE}$$

$$\overrightarrow{AB} = x\left[\left(\tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}\right)\right] - y\left(\overrightarrow{BA} + \overrightarrow{AE}\right)\Big]$$

$$\vec{b} = \frac{x}{2}\left[\overrightarrow{AB} + \overrightarrow{AC}\right] - y\left(-\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}\right)$$

$$\vec{b} = \frac{x}{2}\left(\vec{b} + \vec{c}\right) - y\left(-\vec{b} + \frac{\vec{c}}{2}\right)$$

$$\vec{b} = \frac{x}{2}\vec{b} + \frac{x}{2}\vec{c} + y\vec{b} - \frac{y}{2}\vec{c}$$

P.T.O

$$\vec{b} = \left(\frac{x}{2}+y\right)\vec{b} + \left(\frac{x}{2}-\frac{y}{2}\right)\vec{c}$$

Comparing scalars on both sides

$\vec{b}$: $\frac{x}{2} + y = 1 \quad \text{--- (2)}$

$\vec{c}$: $\frac{x}{2} - \frac{y}{2} = 0$

$$\boxed{x = y}$$

(2) $\Rightarrow \frac{x}{2} + x = 1$

$$\frac{3}{2}x = 1$$

$$\boxed{x = \frac{2}{3}}$$

$$y = x = \frac{2}{3}$$

(1) $\Rightarrow \overrightarrow{AG} = \frac{2}{3}\overrightarrow{AD}$

$\Rightarrow$ G tri sect $\overline{AD}$

$\overrightarrow{BG} = \frac{2}{3}\overrightarrow{BE}$

$\Rightarrow$ G trisect $\overline{BE}$

Hence the medians of a triangle trisect each other.

(ii) To prove medians are concurrent.

Suppose G' is a point on CF such that

$$\overrightarrow{CG'} = \frac{2}{3}\overrightarrow{CF}$$

$$\overrightarrow{AG'} = \overrightarrow{AC} + \overrightarrow{CG'} = \overrightarrow{AC} + \frac{2}{3}\overrightarrow{CF}$$

$$\overrightarrow{AG'} = \vec{c} + \frac{2}{3}(\overrightarrow{CA} + \overrightarrow{AF})$$

$$= \vec{c} + \frac{2}{3}\left[-\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB}\right]$$

$$\overrightarrow{AG'} = \vec{c} + \frac{2}{3}\left[-\vec{c} + \frac{1}{2}\vec{b}\right]$$

$$= \vec{c} - \frac{2}{3}\vec{c} + \frac{1}{3}\vec{b}$$

$$= \frac{1}{3}\vec{c} + \frac{1}{3}\vec{b}$$

$$= \frac{1}{3}(\vec{b} + \vec{c})$$

$\div$ & $\times$ by (2)

$$= \frac{2}{3}\left(\frac{\vec{b} + \vec{c}}{2}\right)$$

$$\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

$$\overrightarrow{AD} = \frac{1}{2}(\vec{b} + \vec{c})$$

$$\overrightarrow{AG'} = \frac{2}{3}\overrightarrow{AD}$$

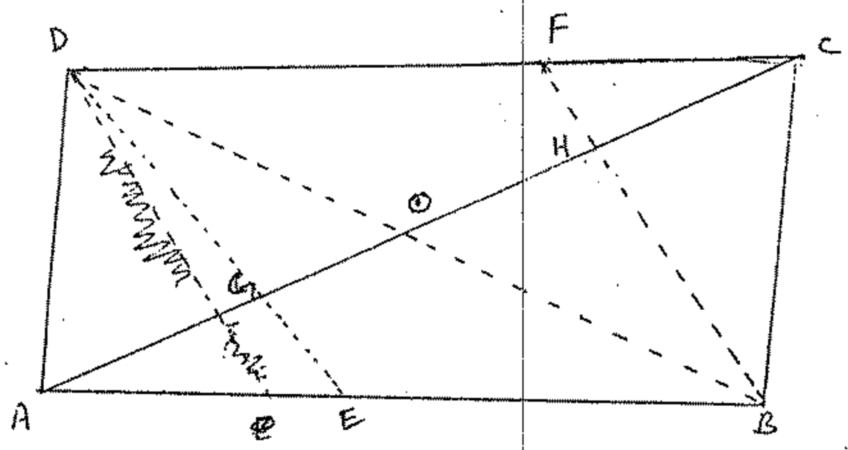
$$\overrightarrow{AG'} = \overrightarrow{AG}$$

$$\Rightarrow G' = G$$

Hence the medians are concurrent.

(Q5/26) Prove that the diagonals $\overline{AC}$ of a ||m ABCD is trisected by the straight lines joining B and D with the mid points of the opposite sides.

Sol:-



To prove that $\overline{AG} = \overline{GH} = \overline{HC} = \frac{1}{3} AC$

Let ABCD be a ||m. Suppose the diagonals AC and BD intersect at point O.

Consider $\triangle ABD$

Since E and O are mid points of the sides AB and BD

$\therefore$ AD and DE are medians

$\therefore$ $AG = \frac{2}{3} AO$ (property of centroid)

$AG = \frac{2}{3} \cdot \frac{1}{2} \cdot AC$

$\boxed{AG = \frac{1}{3}AC}$

Consider $\Delta ABCD$

Since F and O are mid points of the ~~triangle~~ sides CD and BD.

$\therefore$ BF and OC are medians.

$\therefore CH = \frac{2}{3}.OC$ (property of centroid)

$CH = \frac{2}{3}\left(\frac{1}{2}AC\right)$

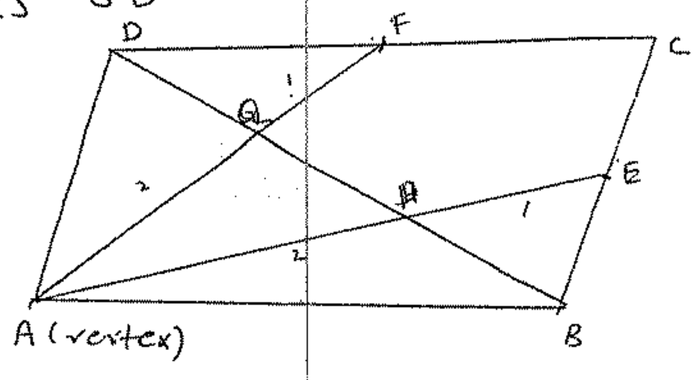
$\boxed{CH = \frac{2}{3}\left(\frac{1}{2}AC\right)}$

Similarly $\boxed{GH = \frac{1}{3}AC}$

Hence the diagonals AC is trisected by the lines BF and DE.

Q6 / 26 In a ||m ABCD, the vertex A is joined with the mid points of $\overline{BC}$ and $\overline{CD}$. Show that these lines trisect the diagonals $\overline{BD}$

Sol:-



Do?

Consider the ||m ABCD is shown. Take "A" is the origin.

let $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AD} = \vec{d}$, then $\overrightarrow{DC} = \vec{b}$, $\overrightarrow{BC} = \vec{d}$

Now $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{b} + \vec{d}$

As E, F are the mid points of $\overrightarrow{BC}$ & $\overrightarrow{DC}$

$\overrightarrow{AE}$ means P.V of "E" w.r.t A

$\therefore \overrightarrow{AE} = \dfrac{\text{P.V of } B + \text{P.V of } C}{2} = \dfrac{\overrightarrow{AB} + \overrightarrow{AC}}{2} = \dfrac{\vec{b} + \vec{b} + \vec{d}}{2} = \dfrac{2\vec{b} + \vec{d}}{2}$

& $\overrightarrow{AF} = \dfrac{\text{P.V of } D + \text{P.V of } C}{2} = \dfrac{\overrightarrow{AD} + \overrightarrow{AC}}{2} = \dfrac{\vec{d} + b + \vec{d}}{2} = \dfrac{\vec{b} + 2\vec{d}}{2}$

let "p" is the point of trisection of $\overrightarrow{AE}$ & $\overrightarrow{BD}$

then P.V of "p" w.r.t A & E must be the same as P.V of "p" w.r.t B & D.

Now P.V of "p" w.r.t A & E is

$= \dfrac{2(\text{P.V of } E) + (\text{P.V of } A)}{2+1}$

$= \dfrac{2\left(\frac{2\vec{b} + \vec{d}}{2}\right) + 1(0)}{2+1} = \dfrac{2\vec{b} + \vec{d}}{3}$

using $\dfrac{k_1 x_2 + k_2 x_1}{k_1 + k_2}$ Ratio formula.

P.V of "p" w.r.t $\vec{B}$ & $\vec{D} = \dfrac{2(\vec{b}) + 1(\vec{d})}{2+1} = \dfrac{2\vec{b} + \vec{d}}{3}$

Hence "p" is the point of trisection of $\overrightarrow{AE}$ & $\overrightarrow{BD}$

let "Q" be the point of trisection of AF & BD.

Then P.V of "Q" w.r.t A, F as well as w.r.t B, D must be the same.

When we say P.V of B, it means $\overrightarrow{AB}$, B w.r.t "A" As "A" is origin.

Now P.V of "Q" w.r.t A & F $= \dfrac{1(0) + 2\left(\frac{\vec{b} + 2\vec{d}}{2}\right)}{1+2} = \dfrac{\vec{b} + 2\vec{d}}{3}$

P.V. of "Q" w.r.t B & D $= \dfrac{1(\text{P.V of } B) + 2(\text{P.V of } D)}{1+2}$

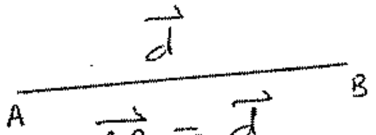
$= \dfrac{1(\vec{b}) + 2(\vec{d})}{3} = \dfrac{\vec{b} + 2\vec{d}}{3}$

Hence "Q" is the point of trisection of $\overrightarrow{AF}$ & $\overrightarrow{BD}$

Hence the result.

WORK DONE BY A CONSTANT FORCE IN MOVING A PARTICLE FROM POINT A TO POINT B.

$$W = \vec{F} \cdot \vec{d}$$



$$\vec{AB} = \vec{d}$$

Ex-1 (31) Find the work done by the force $2\vec{i} + 3\vec{j} + 4\vec{k}$ in moving a particle along the vector $3\vec{i} - 4\vec{j} + 5\vec{k}$.

Sol:- $\vec{F} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$\vec{d} = 3\vec{i} - 4\vec{j} + 5\vec{k}$

work done

$W = \vec{F} \cdot \vec{d} = [2, 3, 4] \cdot [3, -4, 5]$

$W = 2 \times (3) + 3(-4) + 4(5) = 6 - 12 + 20$

$\boxed{W = 14}$ Ans

---

v.imp

Ex-2 (31) Find the total work done by the forces $2\vec{i} - 3\vec{j} + 5\vec{k}$, $-\vec{i} + \vec{j} - \vec{k}$, $\vec{i} + \vec{k}$ in the displacement of a particle from the point $A(4, 3, 1)$ to the point $B(5, 1, 2)$

Sol:

P.T.O

$$\left.\begin{aligned}\vec{F_1} &= 2\vec{i} - 3\vec{j} + 5\vec{k} \\ \vec{F_2} &= -\vec{i} + \vec{j} - \vec{k} \\ \vec{F_3} &= \vec{i} \qquad\quad + k\end{aligned}\right\} \text{adding}$$

$$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3}$$

$$\boxed{\vec{F} = 2\vec{i} - 2\vec{j} + 5\vec{k}}$$

$$\vec{d} = \vec{AB} = [5,1,2] + [4,3,1] = [5-4, 1-3, 2-1]$$

$$\boxed{\vec{d} = [1,-2,1]} \Rightarrow \vec{d} = \vec{i} - 2\vec{j} + \vec{k}$$

$$W = \vec{F}\cdot\vec{d} = (2\vec{i} - 2\vec{j} + 5\vec{k})\cdot(\vec{i} - 2\vec{j} + \vec{k})$$

$$W = 2\times 1 + (-2)(-2) + 5(1)$$

$$\boxed{W = 11}$$ Ans

Q5 / 35 Find the total work done by the forces $\vec{i} + 2\vec{j} + 3\vec{k}$, $2\vec{i} - \vec{j}$ and $\vec{j} - 2\vec{k}$ in displacing a particle from $(-4,1,2)$ to $(5,0,3)$

Sol:- Since
$$\begin{aligned}\vec{F_1} &= \vec{i} + 2\vec{j} + 3\vec{k} \\ \vec{F_2} &= 2\vec{i} - \vec{j} \\ \vec{F_3} &= \vec{j} - 2\vec{k}\end{aligned}$$

$$\vec{F} = 3\vec{i} + 2\vec{j} + \vec{k}$$

$$\vec{d} = \overrightarrow{AB} = [5, 0, 3] - [-4, 1, 2]$$

$$\vec{d} = [5+4, -1, 1]$$

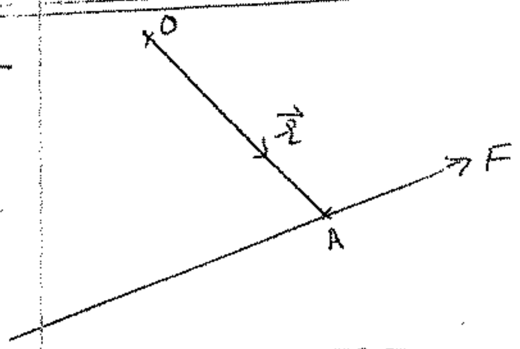
$$\vec{d} = 9\vec{i} - \vec{j} + \vec{k}.$$

$$W = \vec{F}\cdot\vec{d} = (3\vec{i} + 2\vec{j} + \vec{k})\cdot(9\vec{i} - \vec{j} + \vec{k})$$

$$W = 27 - 2 + 1$$

$$\boxed{W = 26} \quad \text{Ans}$$

VECTOR MOMENT



Moment of a force $\vec{F}$ about point O.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$\vec{r}$ = position vector of point O.

$$\boxed{\vec{r} = \overrightarrow{OA}}$$

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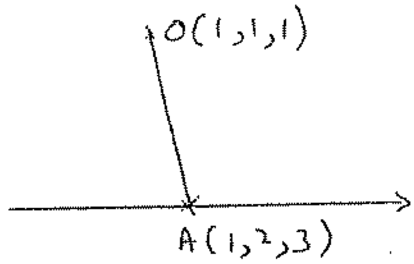
Q6/36 A force $\vec{F} = \vec{i} + 2\vec{j} - 3\vec{k}$ is aplied at $(1,2,3)$. Find its moment about $(1,1,1)$.



Sol:- $\vec{F} = \vec{i} + 2\vec{j} - 3\vec{k}$

$A = (1,2,3)$, $O = (1,1,1)$

$$\vec{r} = \vec{OA} = [1,2,3] - [1,1,1]$$

$$\vec{r} = [1-1, 2-1, 3-1] = [0,1,2]$$

$$\boxed{\vec{r} = 0\vec{i} + \vec{j} + 2\vec{k}}$$

Vector Moment of force about O

$$\vec{L} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{vmatrix}$$

$$\vec{L} = \vec{i}(-3-4) + \vec{j}(2-0) + \vec{k}(0-1)$$

$$\boxed{\vec{L} = -7\vec{i} + 2\vec{j} - \vec{k}}$$

Moment of $\vec{F}$ about O

$\cancel{\vec{L} = \vec{i}(-3-4)}$

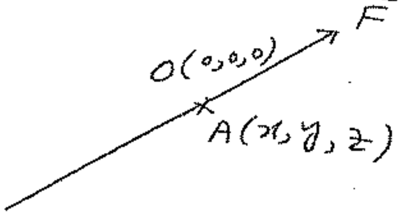
$$L = |\vec{L}| = \sqrt{(-7)^2 + (2)^2 + (-1)^2} = \sqrt{49+4+1}$$

$$\boxed{L = \sqrt{54}}$$ Ans

(Q7/36) Find the moment of the force $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ about the origin if (x, y, z) is any point on its line of action and show that the components of the vector moment about the axes are

$y F_z - z F_y$, $z F_x - x F_z$, $x F_y - y F_x$.

Sol:- $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$



$A(x, y, z)$, $O(0,0,0)$

$$\vec{r} = \vec{OA} = [x, y, z] - [0,0,0] = [x-0, y-0, z-0]$$

$$\vec{r} = [x, y, z]$$

Vector moment of the force $\vec{F}$ about O.

$$\vec{L} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{L} = \vec{i}(y F_z - z F_y) + \vec{j}(z F_x - x F_z) + \vec{k}(x F_y - y F_x)$$

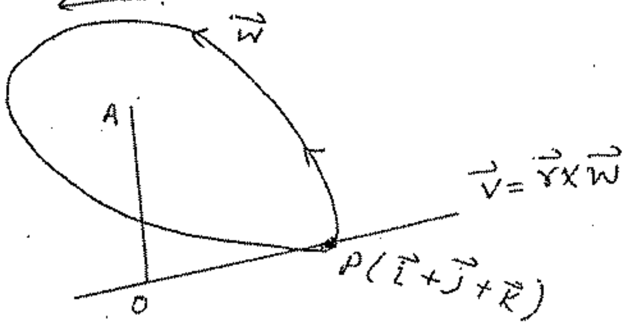
$\Rightarrow$ Components of vector moment about the axes are $y F_z - z F_y$, $z F_x - x F_z$, $x F_y - y F_x$

Moment of the force $\vec{F}$ about O.

$$L = |\vec{L}| = \sqrt{(y F_z - z F_y)^2 + (z F_x - x F_z)^2 + (x F_y - y F_x)^2}.$$

End of Ch-2

Ex-3 / 33 A rigid body is rotating at 3 radians/sec about an axis given by the vector $\vec{OA} = 3\vec{i} + 6\vec{j} + 2\vec{k}$. Find the velocity of the points P of the body whose position vector w.r.t O is $\vec{i} + \vec{j} + \vec{k}$.

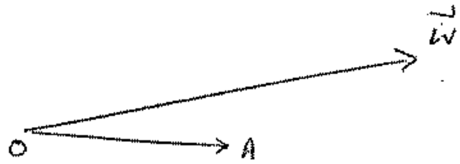


Sol:- $\vec{OA} = 3\vec{i} + 6\vec{j} + 2\vec{k}$

$\vec{OP} = \vec{i} + \vec{j} + \vec{k}$

As the direction of the angular velocity (OR)

As a unit vector in the direction of angular



$$\text{velocity} = \frac{3\vec{i} + 6\vec{j} + 2\vec{k}}{\sqrt{3^2 + 6^2 + 2^2}}$$

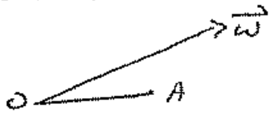
$$= \frac{3\vec{i} + 6\vec{j} + 2\vec{k}}{\sqrt{49}}$$

$$= \frac{1}{7}(3\vec{i} + 6\vec{j} + 2\vec{k})$$

$$\Rightarrow w = \underbrace{(3)}_{\text{given}} \left(\frac{3\vec{i} + 6\vec{j} + 2\vec{k}}{7}\right)$$

$\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

A vector divided by its magnitude is called a unit vector.



$\hat{OA} = \dfrac{\vec{OA}}{|\vec{OA}|}$

$\therefore$ vector = (magnitude)(direction)

$$\vec{a} = a\,\hat{a}$$

$$W = \frac{3}{7}(3\vec{i} + 6\vec{j} + 2\vec{k})$$

P. T. O

& $\vec{r} = \vec{OP} = \vec{i} + \vec{j} + \vec{k}$

$O(0,0,0)$ & $P(1,1,1)$

$\vec{r} = \vec{OP} = (1,1,1) - (0,0,0) = (1,1,1)$

$\vec{r} = \vec{i} + \vec{j} + \vec{k}$

Now $\vec{V} = \vec{r} \times \vec{W} = \frac{3}{7} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 6 & 2 \end{vmatrix}$

$\vec{V} = \frac{3}{7} \left\{ \vec{i} \begin{vmatrix} 1 & 1 \\ 6 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} \right\}$

$\vec{V} = \frac{3}{7} \left\{ (2-6)\vec{i} - \vec{j}(2-3) + (6-3)\vec{k} \right\}$

$\vec{V} = \frac{3}{7} \left\{ -4\vec{i} + \vec{j} + 3\vec{k} \right\}$

Magnitude of the velocity

$= |\vec{V}| = v = \frac{3}{7} \sqrt{(-4)^2 + (1)^2 + (3)^2}$

$v = \frac{3}{7} \sqrt{16+1+9}$

$\boxed{v = \frac{3\sqrt{26}}{7}}$

Ans

As $m\vec{a} = \vec{b}$

$|\vec{b}| = m|\vec{a}|$

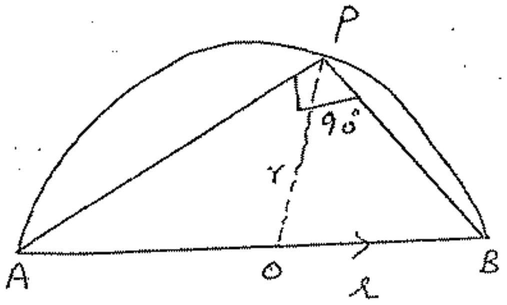
m is out of modulus it is scalar.

Ex-4 / 24 Prove that an angle inscribed in a semi-circle is a right angle.

Sol:-

Consider the figure,

Let O is the centre of $\overrightarrow{AB}$ i.e. diameter of the semi-circle as shown.



r = radius of semicircle

Semi-circle

$\angle P = 90°$ should be from condition.

Take any point "P" on the semi-circle.

From ΔAPO

$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$$
$$= \overrightarrow{OP} + \overrightarrow{AO}$$

& from ΔBPO

$$\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP}$$
$$= -\overrightarrow{OB} + \overrightarrow{OP}$$
$$= -\overrightarrow{AO} + \overrightarrow{OP}$$

$\Rightarrow -\overrightarrow{AO} + \overrightarrow{OP}$

$\Rightarrow \overrightarrow{OP} - \overrightarrow{AO}$

$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{AO}$$

$\because \overrightarrow{OB} = \overrightarrow{AO} = r$

$\overrightarrow{OB} = \overrightarrow{AO} = r$ because $\overrightarrow{OB}$ & $\overrightarrow{OA}$ represents radius of semi-circle & radius is equal at every point also direction of $\overrightarrow{AO}$ & $\overrightarrow{OB}$ are the same.

P.T.O

Now

$$\overrightarrow{AP}\cdot\overrightarrow{BP} = (\overrightarrow{OP}+\overrightarrow{AO})\cdot(\overrightarrow{OP}-\overrightarrow{AO})$$

$$= \overrightarrow{OP}^2 - \overrightarrow{AO}^2$$

$$= \overrightarrow{OP}^2 - \overrightarrow{OP}^2 = 0$$

$\therefore$ $\overrightarrow{OP} = \overrightarrow{AO} = r$ both are radius of semi-circle.

$\Rightarrow$ $\overrightarrow{AP}\cdot\overrightarrow{BP} = 0$

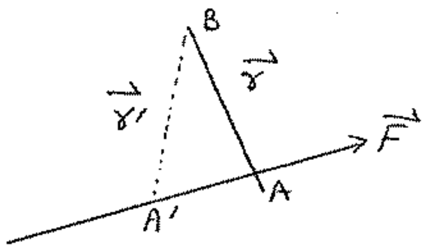
$\Rightarrow$ $\overrightarrow{AP} \perp \overrightarrow{BP}$

$\Rightarrow$ $\angle APB = 90^\circ$

Prove that vector morment is independent of the choice of A on the line of action of $\vec{F}$.

Proof: [viz you can take point A every where, the moment will be the same].

Consider the figure



let "A" be a point on the line of action of $\vec{F}$ and "A'" is another point on the line of action of $\vec{F}$, such that

$$\overrightarrow{BA} = \vec{r}$$
$$\overrightarrow{BA'} = \vec{r}'$$

P.T.O

from $\Delta ABA'$,

$$\vec{BA} = \vec{BA'} + \vec{A'A}$$

$$\Rightarrow \vec{r} = \vec{r}' + \vec{A'A} \quad \text{——— (1)}$$

let "M" be the moment of $\vec{F}$ due to A.

let "M'" be the moment of $\vec{F}$ due to A'.

Now $\vec{M} = \vec{r} \times \vec{F}$

OR $\vec{BA} \times \vec{F} = \vec{r} \times \vec{F}$

$$\vec{M} = \vec{r} \times \vec{F}$$

$$= (\vec{r}' + \vec{A'A}) \times \vec{F} \qquad \because \vec{r} = \vec{r}' + \vec{A'A} \text{ From (1)}$$

$$\vec{M} = \vec{r}' \times \vec{F} + \vec{A'A} \times \vec{F}$$

$$\vec{A'A} \times \vec{F} = 0 \qquad \because \vec{A'A} \times \vec{F} = 0$$

Because $\vec{A'A}$ & $\vec{F}$ are on the same line of action, therefore the angle is 0 and therefore cross product is zero because

$$\vec{A'A} \times \vec{F} = A'A\, F \sin 0 = 0$$

$$\Rightarrow \vec{A'A} \times \vec{F} = 0$$

$$\Rightarrow \vec{r}' \times \vec{F} = 0$$

$$\Rightarrow \vec{M}'$$

$\therefore \vec{BA'} \times \vec{F} = \vec{M}'$

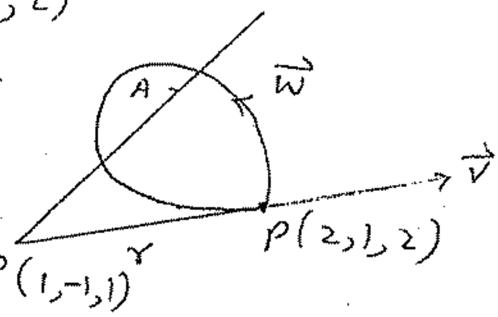
$\vec{BA} \times \vec{F} = \vec{M}$

$$\Rightarrow \vec{M} = \vec{M'}$$

Hence the vector moment of $\vec{F}$ is independent of the choice of A.

Q 8 / 36 A rigid body is rotating with an angular velocity of 4 radians per second about an axis passing through the point $(1,-1,1)$ and in the direction of $3\vec{i}+4\vec{j}$. Find the velocity of the particle at $(2,1,2)$.

Sol:- Here $O(1,-1,1)$, $P(2,1,2)$



& $\overrightarrow{OA} = 3\vec{i}+4\vec{j}$

As a unit vector in the direction of angular velocity is

$$= \frac{3\vec{i}+4\vec{j}}{\sqrt{3^2+4^2}} = \frac{1}{5}(3\vec{i}+4\vec{j})$$

$$\Rightarrow \text{Angular velocity} = (4)\left(\frac{3\vec{i}+4\vec{j}}{5}\right)$$

$$= \frac{4}{5}(3\vec{i}+4\vec{j})$$

$$\& \ \vec{r} = \overrightarrow{OP} = (2,1,2)-(1,-1,1)$$

$$\vec{r} = (1,2,1) = \vec{i}+2\vec{j}+\vec{k}$$

Now $$\vec{v} = \vec{r}\times\vec{w} = \frac{4}{5}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= \frac{4}{5}\left\{\vec{i}(0-4) - \vec{j}(0-3) + \vec{k}(4-6)\right\}$$

$$= \frac{4}{5}\left\{-4\vec{i}+3\vec{j}-2\vec{k}\right\}$$

Magnitude of the velocity

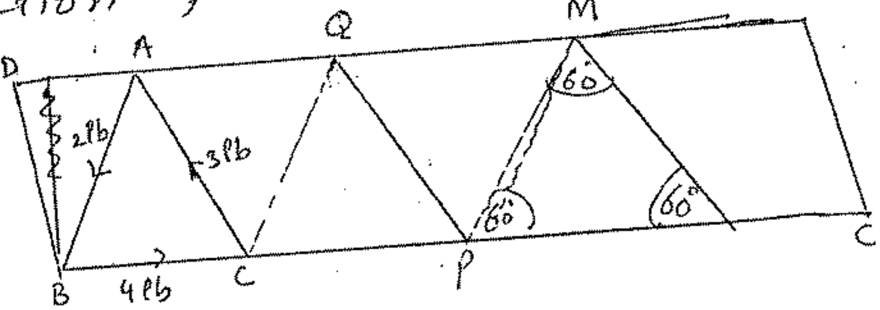
$$|\vec{v}| = v = \frac{4}{5}\sqrt{(-4)^2+(3)^2+(-2)^2}$$

$$= \frac{4}{5}\sqrt{16+9+4}$$

$$\boxed{v = \frac{4}{5}\sqrt{29}}$$ Ans

Q1 (2.6) Forces of 4lb (Pound), 3lb, 2lb act along the BC, CA, AB respectively of an equilateral triangle ABC (sides are equal). Find the magnitude, direction & line of the action of resultant.

Sol:



Resolving all the force along x-axis & y-axis

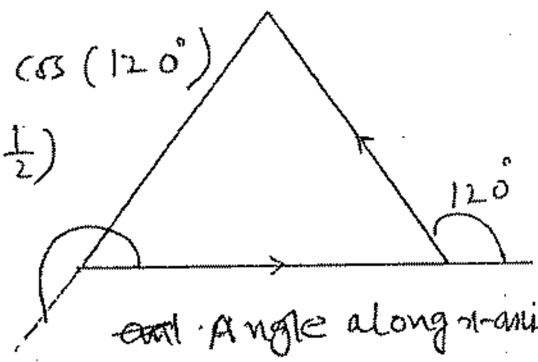
$$\vec{F_x} = 4\cos(0°)+3\cos(120°)+2\cos(-120°)$$

$$= 4(1)+3\left(-\tfrac{1}{2}\right)+2\cos(120°)$$

$$= 4+3\left(\tfrac{-1}{2}\right)+2\left(-\tfrac{1}{2}\right)$$

$$= 4-\frac{3}{2}-1 = \frac{3}{2}$$

$$\boxed{F_x = \frac{3}{2}}$$



~~and~~ Angle along x-axis.

$\therefore \cos(-\theta) = \cos\theta.$

&

$\vec{F_y} = 4\sin(0°)+3\sin(120°)+2\sin(-120°)$

$= 4(0) + 3\left(\frac{\sqrt{3}}{2}\right) - 2\sin(120°)$

$= 0 + \frac{3\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}$

$\vec{F_y} = \frac{\sqrt{3}}{2}$

Now Resultant $= R = \sqrt{F_x^2+y^2}$

$R = \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{\frac{12}{4}}$

$\boxed{R = \sqrt{3}\ lb}$

For direction & lines of action.

$4\vec{BC} + 3\vec{CA} + 2\vec{AB}$

∵ Because, forces 4 lb 3 lb, 2 lb acting along BC, CA & AB respectively.

$= 4\vec{BC}+3\vec{CA}+2\vec{AB}$

$= 4\vec{BC}+3\vec{CA}+2\vec{AB}$

$= 4\vec{BC}+3\vec{CA}+2(\vec{AD}+\vec{DB})$

∵ Because $\vec{DB} = \vec{AC}$ & ∥ to it

$= 4\vec{BC}+3\vec{CA}+2\vec{AD}+2\vec{DB}$

$= 4\vec{BC}+3\vec{CA}+2\vec{AD}+2\vec{AC}$

$= 4\vec{BC}+3\vec{CA}+2\vec{AD}-2\vec{CA}$

$= 4\vec{BC}+\vec{PQ}+\vec{QA}$ ∵ $2\vec{AD} = \vec{QA}$ see fig.

$= 4\vec{BC}+\vec{PA}$ $(\vec{PL}=4\vec{BC})$ ∵ $\vec{CA} = \vec{PQ}$ & ∥

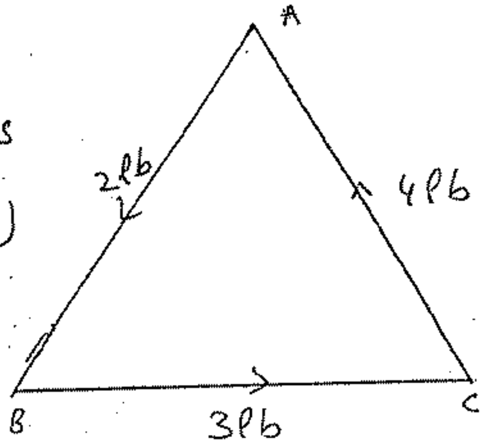
∵ $\vec{PA} = \vec{PQ}+\vec{QA}$ rem triangles are equilateral.

$= \vec{PL}+\vec{PA}$

$= \vec{PL}+\vec{LM} = \vec{PM}$ ∵ $\vec{PA} = \vec{LM}$ & ∥

Hence the resultant $F(\sqrt{3}\ lb)$ is along $\overleftrightarrow{PM}$ P.T.O

Q 22 / 26 Forces of 3lb, 4lb, 2lb act along the sides BC, CA, AB respectively of an equilateral triangle. Find the resultant completely.



Sol:- By resolving all the Forces along x-axis & y-axis

$F_x = 3\cos(0^\circ) + 4\cos(120^\circ) + 2\cos(-120^\circ)$

$F_x = 3(1) + 4(-\frac{1}{2}) + 2(-\frac{1}{2})$

$\boxed{F_x = 0}$

$F_y = 3\sin(0^\circ) + 4\sin(120^\circ) + 2\sin(-120^\circ)$

$F_y = 3\sin(0^\circ) + 4\sin(120^\circ) - 2\sin(120^\circ)$

$F_y = 3(0) + 4(\sqrt{3}/2) - 2(\sqrt{3}/2)$ $\quad \because \sin(-\theta) = -\sin\theta$

$F_y = 0 + 4\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} = 2\sqrt{3} - \sqrt{3}$

$\boxed{F_y = \sqrt{3}}$

Now Resultant(R) $= \sqrt{F_x^2 + F_y^2} = \sqrt{0^2 + (\sqrt{3})^2} = \sqrt{3}$

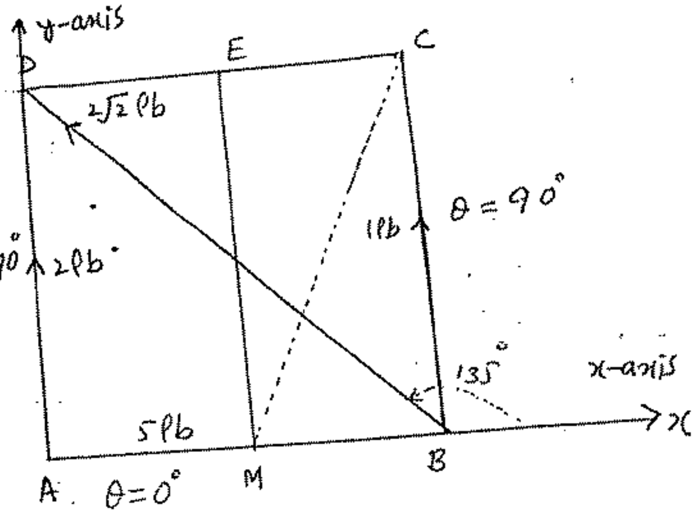
$\boxed{R = \sqrt{3}\ lb}$ i.e Resultant is completely along y-axis because x-components of force is zero.

As $\tan\theta = \frac{F_y}{F_x} = \frac{\sqrt{3}}{0} = \infty$ $\quad \because \tan(90^\circ) = \infty$

$\Rightarrow \theta = \tan^{-1}(\infty) = \theta = \pi/2$

Hence the resultant is $\sqrt{3}$ lb along y-axis or $\sqrt{3}$ lb making angle of "$\pi/2$" with x-axis.

Q3 / 2.6 ABCD is a square, Forces of 5 lb, 1 lb, 3 lb, $2\sqrt{2}$ lb act along AB, BC, AD, BD respectively. Find the resultant completely.



Sol:- Resolving into x-axis & y-axis

then

$Fx = 5\cos(0) + 1\cos 90° + 3\cos 90° + 2\sqrt{2}\cos 135°$

$Fx = 5(1) + 1(0) + 3(0) + 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)$

$Fx = 5 - 2 = 3$

$\boxed{Fx = 3}$

$Fy = 5\sin 0° + 1\sin 90° + 3\sin 90° + 2\sqrt{2}\sin 135°$

$Fy = 5(0) + 1(1) + 3(1) + 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1 + 3 + 2$

$\boxed{Fy = 6}$

Now, Resultant

$R^2 = x^2 + y^2 = 9 + 36 = 45$

$R = \sqrt{45}$

$\boxed{R = 3\sqrt{5}}$

For direction, as $2\sqrt{2}$ is in xy-plane.

The resolve part of the force $2\sqrt{2}$ lb along x-axis is $2\sqrt{2}\cos(135) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -2$

P.T.O

Ch-02

The resolve part of force $2\sqrt{2}$ lb along y-axis

$$2\sqrt{2}\sin(135) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$$

The resolve part of force $2\sqrt{2}$ lb along $\overrightarrow{AD}$ is

$2\overrightarrow{AD}$ or $2\overrightarrow{BC}$, so forces are $5\overrightarrow{AB}$, $\overrightarrow{BC}$, $3\overrightarrow{AD}$

$2\overrightarrow{BA}$, $2\overrightarrow{AD}$

Now

$$5\overrightarrow{AB} + \overrightarrow{BC} + 3\overrightarrow{AD} + 2\overrightarrow{BA} + 2\overrightarrow{AD} = 5\overrightarrow{AB} + \overrightarrow{BC} + 3\overrightarrow{AD} + 2\overrightarrow{BA} + 2\overrightarrow{BC}$$

$$= 5\overrightarrow{AB} + 3\overrightarrow{BC} + 3\overrightarrow{AD} - 2\overrightarrow{AB}$$

$$= 3\overrightarrow{AB} + 3\overrightarrow{BC} + 3\overrightarrow{AD}$$ $\because$ $2\overrightarrow{AD} = 2\overrightarrow{BC}$ are $\parallel$

$$= 3\overrightarrow{AB} + 6\overrightarrow{ME}$$

$\because 3\overrightarrow{BC} + 3\overrightarrow{AD} = 6\overrightarrow{ME}$.

where "M" & "E" are the mid points of $\overrightarrow{AB}$ & $\overrightarrow{CD}$.

$$= 3\overrightarrow{AB} + 6\overrightarrow{ME}$$

$$= 3(2\overrightarrow{MB}) + 6\overrightarrow{ME}$$ $\therefore 2\overrightarrow{MB} = \overrightarrow{AB}$ (see figure)

$$= 6\overrightarrow{MB} + 6\overrightarrow{ME}$$

$$= 6(\overrightarrow{MB} + \overrightarrow{ME})$$

$\because \overrightarrow{MB} + \overrightarrow{ME} = \overrightarrow{MC}$.

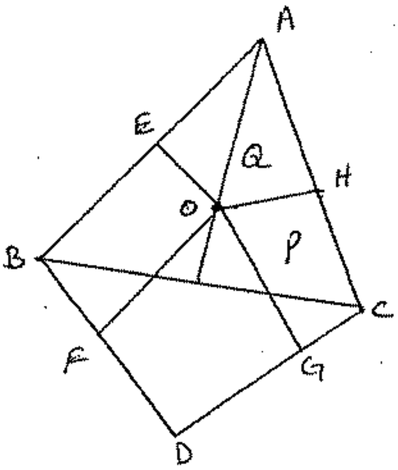
$$= 6\overrightarrow{MC}$$

Hence resultant is $3\sqrt{5}$ lb along $\overrightarrow{MC}$.

Q7/26) Prove that the straight lines joining the mid-points of the opposite edges of a tetrahedron bisect each other.

Sol:-

TETRAHEDREN
Four forces, four angles edges (sides) bisect (divide in 2 parts)



Consider the figure a tetrahedron is shown.

Take O is the origin & the position vectors of A, B, C & D with respect to O are $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ & $\overrightarrow{OD} = \vec{d}$

let E, F, G, H are the mid points of the sides $\overline{AB}$, $\overline{BD}$, $\overline{BC}$ & $\overline{CA}$ respectively.

As E is the mid point of $\overrightarrow{AB}$

P.T.O

$\therefore$ p.v of $E = \vec{OE} = \frac{\vec{a}+\vec{b}}{2}$

F is the mid point of $\overline{BD}$

p.v of F is $\vec{OF} = \frac{\vec{b}+\vec{d}}{2}$

G is the mid-point of $\overline{DC}$

p.v of G is $\vec{OG} = \frac{\vec{c}+\vec{d}}{2}$

& H is the mid-point of $\vec{CA}$

p.v of $H = \vec{OH} = \frac{\vec{c}+\vec{a}}{2}$

Now let P be the mid-point of $\vec{EG}$

then p.v of P is $\vec{OP} = \frac{\vec{OE}+\vec{OG}}{2} = \frac{\frac{\vec{a}+\vec{b}}{2}+\frac{\vec{c}+\vec{d}}{2}}{2}$

$$= \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4} \quad (1)$$

let Q be the mid-point of $\vec{EH}$

then p.v of Q is $\vec{OQ} = \frac{\vec{OF}+\vec{OH}}{2}$

$$= \frac{\frac{\vec{b}+\vec{d}}{2}+\frac{\vec{c}+\vec{a}}{2}}{2} = \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4} \quad (2)$$

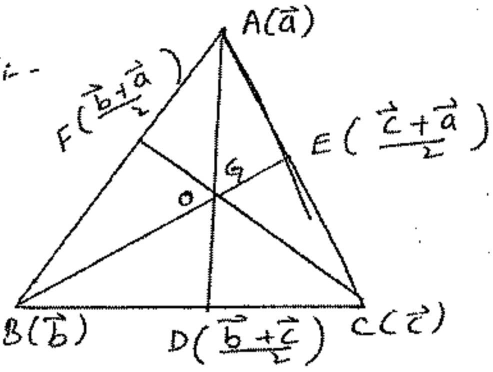
From (1) & (2) P & Q concides

Hence the result

Q12/27 Prove that the straight lines joining the vertices of a tetrahedron to the centriods of areas of the opposite faces are concurrent.

The point which divides medians in 2:1

Sol:-



Centriod

point of concurrency of medians.

Part—a

First we will find the p.v of the centriod of each face.

Consider the face ABC in fig A

The p.v of A, B, C are $\vec{a}, \vec{b}, \vec{c}$ w.r.t same origin O.

i.e $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$

Let D, E, F are the mid points of sides $\vec{BC}$, $\vec{CA}$ & $\vec{AB}$ respectively.

then $\vec{OD} = \dfrac{\vec{OB}+\vec{OC}}{2} = \dfrac{\vec{b}+\vec{c}}{2}$

$\vec{OE} = \dfrac{\vec{OC}+\vec{OA}}{2} = \dfrac{\vec{c}+\vec{a}}{2}$

$\vec{OF} = \dfrac{\vec{OA}+\vec{OB}}{2} = \dfrac{\vec{a}+\vec{b}}{2}$

Also medians are $\vec{AD}$, $\vec{BE}$ & $\vec{CF}$.

Let G be the centriod, then p.v of G w.r.t

A & D is $= \vec{OG} = \dfrac{2(\vec{OD}) + 1(\vec{OA})}{2+1}$

$= \dfrac{2\left(\frac{\vec{b}+\vec{c}}{2}\right) + 1(\vec{a})}{3} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

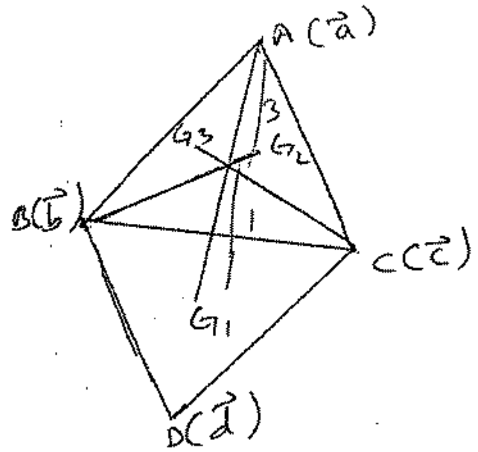
Similarly if we consider the other two sides, then p.v of G will be $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

Hence Centroid of ΔABC is $\vec{G} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

Similarly Centroid of ΔABD is $\vec{G} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

& " " ΔBCD " $\vec{G} = \dfrac{\vec{b}+\vec{c}+\vec{d}}{3}$

Part-b Now we will prove that straight line from vertices of a tetrahedron to the Centriod of opposite faces are concurrent.



Proof:- let G_1, G_2, G_3 be the Centriode of the faces ΔBCD, ΔACD & ΔABD respectively. then we will show that AG_1, BG_2 & CG_3 are concurrent.

For AG_1:- A point which divides AG_1 in the ratio $3:1$ is

P.T.O

is

$$= \frac{3(G_1)+1(A)}{3+1} = \cancel{\frac{3\left(\vec{d}+\vec{b}+\vec{c}\right)+1(\vec{a})}{4}}$$

$$= \frac{3\left(\frac{\vec{d}+\vec{b}+\vec{c}}{3}\right)+1(\vec{a})}{4}$$

$$= \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

Similarly the point which divides BG_2 & CG_3 in the ratio 3:1

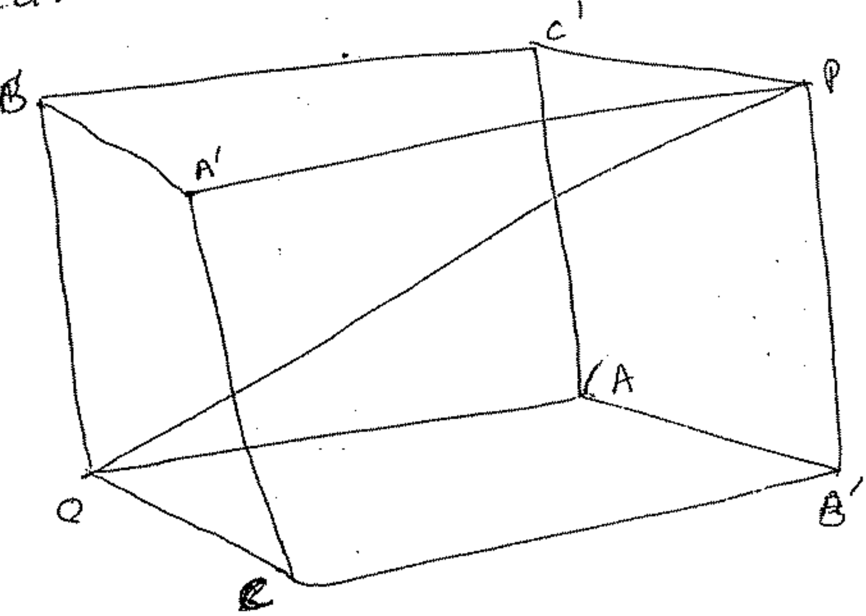
is $= \dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$

Hence the lines AG_1, BG_2, CG_3 are passing through the same point & are concurrent.

(Q13/27) Prove that the four diagonals of a parallelopiped and the straight lines joining the mid points of opposite edges are concurrent at their point of bisection.

proof:-

parallelopiped
Six faces



Proof:- Consider the figure a //piped or (cube) is shown. let p.v's of A, B, C w.r.t the origin O are

$\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ & $\vec{OC} = \vec{c}$

from ΔOAP

then $\vec{OP} = \vec{OA} + \vec{AP}$

but $\vec{AP} = \vec{AC'} + \vec{C'P} = \vec{OB} + \vec{OC}$.

$\therefore$ $\vec{AC}$ & $\vec{C'P}$ are //

& equal to $\vec{OB}$ & $\vec{OC}$.

$\therefore$ $\vec{OP} = \vec{OA} + \vec{OB} + \vec{OC}$

$\boxed{\vec{OP} = \vec{a} + \vec{b} + \vec{c}}$

from ΔOCA

$\vec{OA'} = \vec{OC} + \vec{CA'}$ $\quad\therefore$ $\vec{CA'}$ // $\vec{OB}$ & equal.

$= \vec{OC} + \vec{OB}$

$= \vec{c} + \vec{b}$

from $\Delta OAB'$

$\vec{OB'} = \vec{OA} + \vec{AB'} = \vec{OA} + \vec{OC}$ $\quad\therefore$ $\vec{AB'}$ // $\vec{OC}$ & equal.

$= \vec{a} + \vec{c}$

& P.T.O

& From $\triangle OAC'$

$$\overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{AC'} \qquad \because \overrightarrow{AC'} \parallel \overrightarrow{OB} \text{ \& equal.}$$
$$= \overrightarrow{OA} + \overrightarrow{OB}$$

$$\overrightarrow{OC'} = \vec{a} + \vec{b}$$

let E be the mid point of $A'C$, then

$$\overrightarrow{OE} = \frac{\text{P.V of } A' + \text{P.V of } C}{2}$$

$$\overrightarrow{OE} = \frac{\overrightarrow{OA'} + \overrightarrow{OC}}{2} = \frac{\vec{c} + \vec{b} + \vec{c}}{2} = \frac{2\vec{c} + \vec{b}}{2}$$

& let F be the mid-point of AC', then

$$\overrightarrow{OF} = \frac{\text{P.V of } A + \text{P.V of } C'}{2} = \frac{\overrightarrow{OA} + \overrightarrow{OC'}}{2}$$

$$\overrightarrow{OF} = \frac{\vec{a} + \vec{b} + \vec{a}}{2} = \frac{2\vec{a} + \vec{b}}{2}$$

Now let M' be the mid-point of $\overrightarrow{EF}$

then $$\overrightarrow{OM'} = \frac{\overrightarrow{OE} + \overrightarrow{OF}}{2} = \frac{\text{P.V of } E + \text{P.V of } F}{2}$$ because M' is mid point of E & F

$$\overrightarrow{OM'} = \frac{\overrightarrow{OE} + \overrightarrow{OF}}{2}$$

$$= \frac{\frac{2\vec{c} + \vec{b}}{2} + \frac{2\vec{a} + \vec{b}}{2}}{2} = \frac{2\vec{c} + \vec{b} + 2\vec{a} + \vec{b}}{4}$$

$$= \frac{\vec{a} + \vec{b} + \vec{c}}{2} \quad (1)$$

let M be the mid-point of $\overrightarrow{OP}$, then

P.T.O

$$\overrightarrow{OM} = \frac{p.v \text{ of } O + p.v \text{ of } P}{2}$$

$$= \frac{\overrightarrow{OO} + \overrightarrow{OP}}{2} = \frac{0 + (\vec{a}+\vec{b}+\vec{c})}{2} \quad \because \overrightarrow{OP} = \vec{a}+\vec{b}+\vec{c}$$

$$= \frac{\vec{a}+\vec{b}+\vec{c}}{2} \quad \text{--- (II)}$$

From (I) & (II) M & M′ concide.

$\Rightarrow$ The line $\overrightarrow{EF}$ & $\overrightarrow{OP}$ bisect each other

Similarly if we consider the other diagonals & sides, then diagonals & sides will bisect each other.

Hence the result.