VECTORS

MATHS – B

CH # 1 VECTORS

Third year (B·Sc)

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Ch-01

CHAPTER # 1 VECTORS

Define vector (Axiomatic definition of vectors)

A triple of numbers $[a_1, a_2, a_3]$ is called vector if the following axioms are satisfied.

1) Equality $[a_1, a_2, a_3] = [b_1, b_2, b_3]$

$$\Rightarrow a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3.$$

2) Scalar multiplication

$$k[a_1, a_2, a_3] = [ka_1, ka_2, ka_3]$$

3) Addition

$$[a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1+b_1, a_2+b_2, a_3+b_3].$$

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EXERCISE # 1.1

Q1/6 Show that operation of vector addition is commutative.

Sol:- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$\vec{a}+\vec{b} = [a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1+b_1, a_2+b_2, a_3+b_3]$

$\vec{a}+\vec{b} = [b_1+a_1, b_2+a_2, b_3+a_3]$; $a_1, a_2, a_3, b_1, b_2, b_3 \in R$

$\vec{a}+\vec{b} = [b_1, b_2, b_3] + [a_1, a_2, a_3]$

$\vec{a}+\vec{b} = \vec{b} + \vec{a}$

proved.

Q2/6 Show that vector addition is associative

i.e if $\vec{V_1} = [x_1, y_1, z_1]$, $\vec{V_2} = [x_2, y_2, z_2]$

$\vec{V_3} = [x_3, y_3, z_3]$ then $(\vec{V_1}+\vec{V_2})+\vec{V_3} = \vec{V_1} + (\vec{V_2}+\vec{V_3})$

Sol:-

$(\vec{V_1}+\vec{V_2})+\vec{V_3} = ([x_1, y_1, z_1] + [x_2, y_2, z_2]) + [x_3, y_3, z_3]$

$= [x_1+x_2, y_1+y_2, z_1+z_2] + [x_3, y_3, z_3]$

$= [x_1, y_1, z_1] + ([x_2, y_2, z_2] + [x_3, y_3, z_3])$

$= [x_1, y_1, z_1] + [x_2+x_3, y_2+y_3, z_2+z_3]$

$(\vec{V_1}+\vec{V_2})+\vec{V_3} = \vec{V_1} + (\vec{V_2}+\vec{V_3})$ proved.

Q3/6) Show that the distributive law $K[\vec{a}+\vec{b}] = K\vec{a} + K\vec{b}$ where K is a scalar holds.

Sol:- $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$$K(\vec{a}+\vec{b}) = K[a_1+b_1, a_2+b_2, a_3+b_3]$$

$$= [K(a_1+b_1), K(a_2+b_2), K(a_3+b_3)] \quad \text{(by def: of vectors)}$$

$$= [Ka_1+Kb_1, Ka_2+Kb_2, Ka_3+Kb_3]$$

$$= [Ka_1, Ka_2, Ka_3] + [Kb_1, Kb_2, Kb_3] \quad \text{(by def: of vectors)}$$

$$= K[a_1, a_2, a_3] + K[b_1, b_2, b_3]$$

$$K(\vec{a}+\vec{b}) = K\vec{a} + K\vec{b}.$$

* * *

NULL VECTOR OR ZERO VECTOR

$\vec{0} = [0,0,0]$ is called zero vector or null vector.

* * *

IDENTITY ELEMENT W.R.T ADDITION

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

i.e $\vec{0}$ is identity element w.r.t addition of vector.

Q4/6 Show that the null vector is the identity element with respect to operation of addition of vectors.

Sol: $\vec{a} = [a_1, a_2, a_3]$, $\vec{0} = [0, 0, 0]$

$\vec{a} + \vec{0} = [a_1, a_2, a_3] + [0, 0, 0] = [a_1+0, a_2+0, a_3+0]$

$\boxed{\vec{a} + \vec{0} = [a_1, a_2, a_3]}$ —— (1)

$\vec{0} + \vec{a} = [0, 0, 0] + [a_1, a_2, a_3] = [0+a_1, 0+a_2, 0+a_3]$

$\boxed{\vec{0} + \vec{a} = [a_1, a_2, a_3]}$ —— (2)

(1) = (2)

$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$

$\therefore$ $\vec{0}$ is the identity element w.r.t addition of vectors.

ADDITIVE INVERSE

$\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

ie $-\vec{v}$ is additive inverse of $\vec{v}$.

MAGNITUDE OF A VECTOR
(Length)

$\vec{a} = [a_1, a_2, a_3]$

$|\vec{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Q5/6 Given a vector $\vec{v} = [v_1, v_2, v_3]$, Show that the vector $(-1)\vec{v}$ is the additive inverse of $\vec{v}$. Hence $(-1)\vec{v}$ may be written as $-\vec{v}$.
(The vectors $\vec{v}$ and $-\vec{v}$ have the same magnitude and direction but are opposite to each other in sense)

Sol:- $\vec{v} = (v_1, v_2, v_3)$

$-\vec{v} = (-v_1, -v_2, -v_3)$

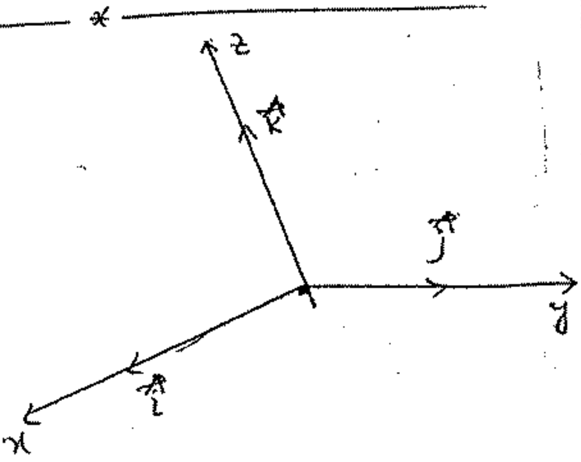
So, $\vec{v} + (-\vec{v}) = (v_1, v_2, v_3) + (-v_1, -v_2, -v_3)$

$= (v_1 - v_1, v_2 - v_2, v_3 - v_3) = (0, 0, 0)$

So $\vec{v} + (-\vec{v}) = \vec{0}$ proved.

* *

UNIT VECTOR



Vectors of magnitude 1 is called unit vectors.

$\hat{i}, \hat{j}, \hat{k}$ are in the figure unit vectors.

$\hat{i} = [1, 0, 0]$

$\hat{j} = [0, 1, 0]$

$\hat{k} = [0, 0, 1]$

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Prove

$$[a_1, a_2, a_3] = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

Sol:- As, $\vec{i} = [1,0,0], \vec{j} = [0,1,0], \vec{k} = [0,0,1]$

R.H.S $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = a_1[1,0,0] + a_2[0,1,0] + a_3[0,0,1]$$

$$= [a_1, 0, 0] + [0, a_2, 0] + [0, 0, a_3]$$

$$= [a_1+0+0,\ 0+a_2+0,\ 0+0+a_3]$$

$$\Rightarrow \boxed{a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = [a_1, a_2, a_3]} \quad \text{Ans.}$$

$$[a_1, a_2, a_3] = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

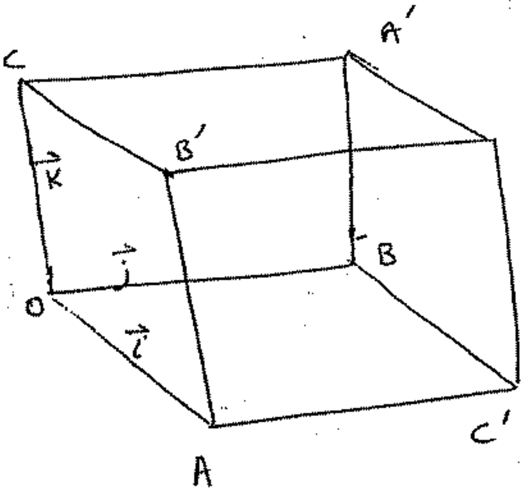
Component form ; $\vec{i}, \vec{j}, \vec{k}$ form.

(Ex-1/11) The unit vectors $\vec{i}, \vec{j}, \vec{k}$ are represented respectively by three edges $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ of the unit cube shown. Write down the expressions for vectors represented by the diagonals $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ of the cube. Find the lengths of the direction cosines of these diagonals.

Sol:- P.T.O

$\overrightarrow{OA} = \vec{i}$, $\overrightarrow{OB} = \vec{j}$, $\overrightarrow{OC} = \vec{k}$

(i) To find $\overline{AA'}, \overline{BB'}, \overline{CC'}$



$\overrightarrow{AA'} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BA'}$

$\overrightarrow{AA'} = -\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$

$\boxed{\overrightarrow{AA'} = -\vec{i} + \vec{j} + \vec{k}}$

$\overrightarrow{BB'} = \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AB'}$

$\overrightarrow{BB'} = -\overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC}$

$\overrightarrow{BB'} = -\vec{j} + \vec{i} + \vec{k}$

$\boxed{\overrightarrow{BB'} = \vec{i} - \vec{j} + \vec{k}}$

$\overrightarrow{CC'} = \overrightarrow{CO} + \overrightarrow{OB} + \overrightarrow{BC'}$

$\overrightarrow{CC'} = -\overrightarrow{OC} + \overrightarrow{OB} + \overrightarrow{OA}$

$\overrightarrow{CC'} = -\vec{k} + \vec{j} + \vec{i}$

$\boxed{\overrightarrow{CC'} = \vec{i} + \vec{j} + \vec{k}}$

DIRECTION COSINES OF VECTORS

Def:-

If $\vec{a} = [a_1, a_2, a_3]$ then the number

$$\frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|}$$

are called direction cosines of vector $\vec{a}$.

(ii) To Find lengths

$|AA'| = \sqrt{(-1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

$|BB'| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$

$|CC'| = \sqrt{(1)^2+(1)^2+(-1)^2} = \sqrt{3}$

(iii) To find direction cosines

Direction cosine of $\overrightarrow{AA'} = \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Direction cosine of $\overrightarrow{BB'} = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

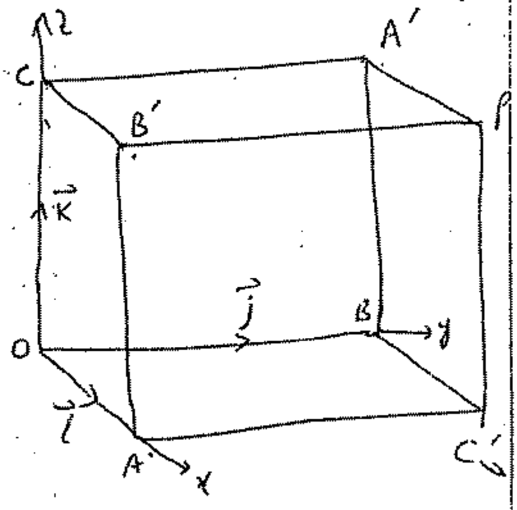
Direction cosine of $\overrightarrow{CC'} = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$.

EXERCISE # 1.2

Q 1 / 11 Three edges of a unit cube through the origin O represents the vectors $\vec{i}, \vec{j}, \vec{k}$ respectively. Write down the expressions for the vectors represented by

i) The diagonal of the cube through O,

ii) The diagonals of the three faces passing through O.



Sol:- $\overrightarrow{OA} = \vec{i}$

$\overrightarrow{OB} = \vec{j}$

$\overrightarrow{OC} = \vec{k}$

(i) To find diagonals of the cube through point O i.e $\overrightarrow{OP}$.

P.T.O

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BA'} + \overrightarrow{A'P}$$

$$= \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OA}$$

$$= \vec{j} + \vec{k} + \vec{i}$$

$$\boxed{\overrightarrow{OP} = \vec{i} + \vec{j} + \vec{k}}$$

(ii) To find diagonals of the three faces passing through O ie To find $\overrightarrow{OA'}, \overrightarrow{OB'}, \overrightarrow{OC'}$

$$\overrightarrow{OA'} = \overrightarrow{OB} + \overrightarrow{BA'} = \overrightarrow{OB} + \overrightarrow{OC}$$

$$\boxed{\overrightarrow{OA'} = \vec{j} + \vec{k}}$$

$$\overrightarrow{OB'} = \overrightarrow{OA} + \overrightarrow{AB'}$$

$$= \overrightarrow{OA} + \overrightarrow{OC}$$

$$\boxed{\overrightarrow{OB'} = \vec{i} + \vec{k}}$$

$$\overrightarrow{OC'} = \overrightarrow{OB} + \overrightarrow{BC'}$$

$$\overrightarrow{OC'} = \overrightarrow{OB} + \overrightarrow{OA}$$

$$\boxed{\overrightarrow{OC'} = \vec{i} + \vec{j}}$$

Given the vectors $\vec{a} = 3\vec{i} - 2\vec{j} + 4\vec{k}$
$\vec{b} = 2\vec{i} + \vec{j} + 3\vec{k}$ Find the magnitudes
and direction cosines of the vectors
(i) $\vec{a}$ (ii) $\vec{b}$ (iii) $\vec{a}+\vec{b}$ (iv) $\vec{a}-\vec{b}$ (v) $3\vec{a}-2\vec{b}$.

Sol:- $\vec{a} = 3\vec{i} - 2\vec{j} + 4\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + 3\vec{k}$

(v) $3\vec{a} - 2\vec{b} = 3(3\vec{i} - 2\vec{j} + 4\vec{k}) - 2(2\vec{i} + \vec{j} + 3\vec{k})$

$$= 9\vec{i} - 6\vec{j} + 12\vec{k}$$
$$-4\vec{i} - 2\vec{j} - 6\vec{k}$$

$$\boxed{3\vec{a} - 2\vec{b} = 5\vec{i} - 8\vec{j} + 6\vec{k}}$$

$|3\vec{a} - 2\vec{b}| = \sqrt{5^2 + (-8)^2 + 6^2} = \sqrt{25+64+36}$
$= \sqrt{125} = 5\sqrt{5}$

$$\boxed{|3\vec{a} - 2\vec{b}| = 5\sqrt{5}}$$

And the direction cosine of $3\vec{a} - 2\vec{b}$

is $\dfrac{5}{5\sqrt{5}}, \dfrac{-8}{5\sqrt{5}}, \dfrac{6}{5\sqrt{5}}$

(i) + (ii) + (iv) + (iii) Do ?

Define Scalar (dot) product of two vectors

let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

Then the scalar product of $\vec{a}$ & $\vec{b}$ is defined as follows

$$\vec{a}\cdot\vec{b} = [a_1, a_2, a_3]\cdot[b_1, b_2, b_3]$$

$$\boxed{\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

$\left(\frac{i}{12}\right)$ Prove that scalar product is commutative $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Proof:- $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a}\cdot\vec{b} = [a_1, a_2, a_3]\cdot[b_1, b_2, b_3]$$

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{a}\cdot\vec{b} = b_1 a_1 + b_2 a_2 + b_3 a_3$$

$$\vec{a}\cdot\vec{b} = [b_1, b_2, b_3]\cdot[a_1, a_2, a_3]$$

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$\Rightarrow$ Scalar product is commutative.

$\left(\frac{ii}{12}\right)$ Prove that the scalar product is distributive

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$

Proof :- $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, $\vec{c} = [c_1, c_2, c_3]$

$$\vec{a}\cdot(\vec{b}+\vec{c}) = [a_1, a_2, a_3]\cdot[b_1+c_1, b_2+c_2, b_3+c_3]$$

$$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$

$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$

$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$$

$$= [a_1, a_2, a_3]\cdot[b_1, b_2, b_3] + [a_1, a_2, a_3]\cdot[c_1, c_2, c_3]$$

$$\boxed{\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}}$$

$\Rightarrow$ Scalar product is distributive.

*

(iv) Prove that $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$

proof: $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$, $\vec{k} = [0, 0, 1]$

$$\vec{i}\cdot\vec{i} = [1,0,0]\cdot[1,0,0] = 1\cdot1 + 0\cdot0 + 0\cdot0 = 1$$

$$\vec{j}\cdot\vec{j} = [0,1,0]\cdot[0,1,0] = 0\cdot0 + 1\cdot1 + 0\cdot0 = 1$$

$$\vec{k}\cdot\vec{k} = [0,0,1]\cdot[0,0,1] = 0\cdot0 + 0\cdot0 + 1\cdot1 = 1.$$

(iii/12) $\vec{a}\cdot\vec{a} = a^2$, and we may also write $\vec{a}\cdot\vec{a}$ as $|a|^2$ or a^2.

Sol:- $\vec{a} = [a_1, a_2, a_3]$

$$\vec{a}\cdot\vec{a} = [a_1, a_2, a_3]\cdot[a_1, a_2, a_3]$$

$$= a_1\cdot a_1 + a_2\cdot a_2 + a_3\cdot a_3$$

$$= a_1^2 + a_2^2 + a_3^2$$

$$\vec{a}\cdot\vec{a} = \left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^2 = |\vec{a}|^2$$

$$\boxed{\vec{a}\cdot\vec{a} = a^2} \quad \text{Ans}$$

(v) $\vec{a} = (\vec{a}\cdot\vec{i})\vec{i} + (\vec{a}\cdot\vec{j})\vec{j} + (\vec{a}\cdot\vec{k})\vec{k}$

Sol:- $\vec{a} = [a_1, a_2, a_3] = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \longrightarrow (1)$

$$\vec{a}\cdot\vec{i} = [a_1, a_2, a_3]\cdot[1,0,0] = [a_1\cdot 1, a_2\cdot 0, a_3\cdot 0] = a_1$$

$$\vec{a}\cdot\vec{j} = [a_1, a_2, a_3]\cdot[0,1,0] = a_1\cdot 0 + a_2\cdot 1 + a_3\cdot 0 = a_2$$

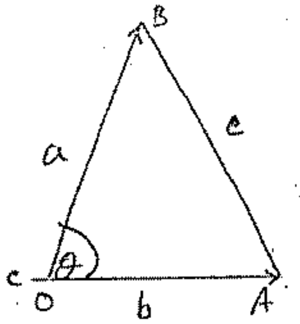
$$\vec{a}\cdot\vec{k} = [a_1, a_2, a_3]\cdot[0,0,1] = a_1\cdot 0 + a_2\cdot 0 + a_3\cdot 1 = a_3.$$

$(1) \Rightarrow$

$$\vec{a} = (\vec{a}\cdot\vec{i})\vec{i} + (\vec{a}\cdot\vec{j})\vec{j} + (\vec{a}\cdot\vec{k})\vec{k}$$

Ans

Law of Cosine



$$(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB)\cos\theta.$$

$$c^2 = b^2 + a^2 - 2bc\cos\angle A$$

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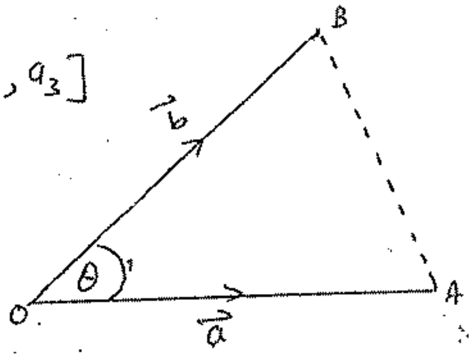
v.v.imp Prove that $\vec{a}\cdot\vec{b} = ab\cos\theta$, where θ is angle between the vector $\vec{a}$ & $\vec{b}$.



Proof:- let $\overrightarrow{OA} = \vec{a} = [a_1, a_2, a_3]$

$\overrightarrow{OB} = \vec{b} = [b_1, b_2, b_3]$.

Points joining A and B.

By applying law of cosine

$$(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB)\cos\theta$$

$$2(OA)(OB)\cos\theta = (OA)^2 + (OB)^2 - (AB)^2$$

$$2ab\cos\theta = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - |\overrightarrow{AB}|^2 \quad \left(\because a^2 = |\vec{a}|^2\right) \rightarrow (1)$$

$$\overrightarrow{OA} = [a_1, a_2, a_3]$$

$$|\overrightarrow{OA}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\boxed{|\overrightarrow{OA}|^2 = a_1^2 + a_2^2 + a_3^2} \quad (i)$$

Available at www.mathcity.org

$$\overrightarrow{OB} = [b_1, b_2, b_3]$$

$$|\overrightarrow{OB}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

$$\boxed{|\overrightarrow{OB}|^2 = b_1^2 + b_2^2 + b_3^2} \quad (ii)$$

P.T.O

$$\vec{AB} = \vec{AO} + \vec{OB} = -\vec{OA} + \vec{OB} = \vec{OB} - \vec{OA}$$

$$\vec{AB} = [b_1, b_2, b_3] - [a_1, a_2, a_3] = [b_1-a_1, b_2-a_2, b_3-a_3]$$

$$|\vec{AB}| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2}$$

$$|\vec{AB}|^2 = \left(\sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2}\right)^2$$

$$|\vec{AB}|^2 = (b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2$$

$$\boxed{|\vec{AB}|^2 = b_1^2 + a_1^2 - 2a_1b_1 + b_2^2 + a_2^2 - 2a_2b_2 + b_3^2 + a_3^2 - 2a_3b_3} \quad \text{(iii)}$$

putting values from (i), (ii) and (iii) in (1).

$$2ab\cos\theta = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - (a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2a_1b_1 - 2a_2b_2 - 2a_3b_3)$$

$$2ab\cos\theta = \cancel{a_1^2} + \cancel{a_2^2} + \cancel{a_3^2} + \cancel{b_1^2} + \cancel{b_2^2} + \cancel{b_3^2} - \cancel{a_1^2} - \cancel{a_2^2} - \cancel{a_3^2} - \cancel{b_1^2} - \cancel{b_2^2} - \cancel{b_3^2} + 2a_1b_1 + 2a_2b_2 + 2a_3b_3$$

$$ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3 = [a_1, a_2, a_3]\cdot[b_1, b_2, b_3]$$

$$ab\cos\theta = \vec{a}\cdot\vec{b}$$

$$\Rightarrow \vec{a}\cdot\vec{b} = ab\cos\theta. \text{ proved.}$$

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viii/14 If the angle θ b/w the vectors a and b is $\pi/2$, then $\vec{a}\cdot\vec{b}=0$. On the other hand, if $\vec{a}\cdot\vec{b}=0$, then one of the following will hold (a) atleast one of the vectors a, b is a zero vector OR

(b) The vectors a, b are non-zero vectors and the angle b/w them is a right angle.

when $a\cdot b=0$, the vectors a and b are said to be orthogonal to each other.

If $\vec{a}$ & $\vec{b}$ are orthogonal iff $\vec{a}\cdot\vec{b}=0$.

suppose $\vec{a}$ & $\vec{b}$ are orthogonal if $\theta=\pi/2$

$$\vec{a}\cdot\vec{b} = ab\cos\theta = ab\cos\pi/2 = ab(0)=0$$

$$\vec{a}\cdot\vec{b}=0$$

Conversly :- if $\vec{a}\cdot\vec{b}=0$

$\Rightarrow ab\cos\theta=0$

$\Rightarrow \cos\theta=0$

$\Rightarrow \theta=\cos^{-1}(0)=\pi/2$

$\Rightarrow \vec{a}$ & $\vec{b}$ are orthogonal

Define vector (cross) product of two vectors.

let $\vec{a}=[a_1,a_2,a_3]$, $\vec{b}=[b_1,b_2,b_3]$

Then the cross product of $\vec{a}$ and $\vec{b}$ is defined as follow $\vec{a}\times\vec{b}=[a_1,a_2,a_3]\times[b_1,b_2,b_3]$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ (Remember)

Properties of determinants

(1) If any row (column) of a determinant is zero then its value will be zero.

(2) If any two rows (columns) of a determinant are same, its value will be zero.

(3) Interchange of any two rows (columns) changes the sign of determinant.

(4) Transposition does not change the value of determinant.

(5) $$\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & k \end{vmatrix} = k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

i.e. We can take common from a single row (single column). (Don't mix it with scalar multiplication in matrices).

(6) $$\begin{vmatrix} a+\alpha & b & c \\ b+\beta & e & f \\ c+\gamma & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & e & f \\ c & h & k \end{vmatrix} + \begin{vmatrix} \alpha & b & c \\ \beta & e & f \\ \gamma & h & k \end{vmatrix}$$

(7) $$\begin{vmatrix} a & b & c \\ d & e & f \\ g+\alpha & h+\beta & k+\gamma \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ \alpha & \beta & \gamma \end{vmatrix}$$

($\frac{i}{14}$) Prove that vector product is anti Commutative.

Proof:- Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= - \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad R_{23}$$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\Rightarrow$ vector product is anti commutative.

(ii/14) The vector product of a vector with itself yields a zero vector, i.e. $\vec{a} \times \vec{a} = 0$.

More generally, if $\vec{b} = k\vec{a}$, then $\vec{a} \times \vec{b} = 0$

Sol:- $\vec{a} \times \vec{a} = 0$

$\vec{a} = [a_1, a_2, a_3]$

$$\vec{a} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= 0 \qquad (\because R_2 = R_3)$$

<u>2nd part</u> let $\vec{b} = k\vec{a}$ then to show $\vec{a} \times \vec{b} = 0$

$\vec{b} = k\vec{a} = k[a_1, a_2, a_3] = [ka_1, ka_2, ka_3]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ ka_1 & ka_2 & ka_3 \end{vmatrix}$$

$$= k \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= k(0) \qquad (\because R_2 = R_3)$$

$$\boxed{\vec{a} \times \vec{b} = 0}$$

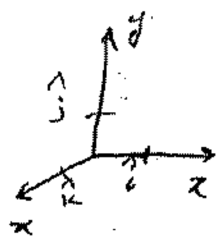
iii/14

$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

$\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$

$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$

$\hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$



Sol:- $\hat{i} = [1, 0, 0]$, $\hat{j} = [0, 1, 0]$, $\hat{k} = [0, 0, 1]$

$$\hat{i}\times\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 \quad \text{as } (R_2 = R_3)$$

$\Rightarrow \hat{i}\times\hat{i} = 0.$ Do the other parts.

Prove that vector product is distributive.

Proof:- $\boxed{\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}}$ (definition of distributive law)

$(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$

proof:- $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$\vec{c} = [c_1, c_2, c_3]$

$$\vec{a}\times(\vec{b}+\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$$

Using properties of determinant we have,

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}).$$

P.T.O

2nd part

$$(\vec{a}+\vec{b})\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

using properties of determinant we have

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$$ proved

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$\frac{v}{15}$ The vector $a\times b$ is orthogonal to both the vectors a and b. For $a\cdot(a\times b) = b\cdot(a\times b) = 0$.

Sol:- (i) $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ (ii) $\vec{b}\cdot(\vec{a}\times\vec{b}) = 0$

$\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, $\vec{c} = [c_1, c_2, c_3]$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ Expanding by row 1st

$$\vec{a}\times\vec{b} = \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(b_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)$$

$$\vec{a}\cdot(\vec{a}\times\vec{b}) = [a_1, a_2, a_3]\cdot[a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$$

$$= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1\cancel{a_2b_3} - a_1\cancel{a_3b_2} + a_2\cancel{a_3b_1} - a_1\cancel{a_2b_3} + a_1\cancel{a_3b_2} - a_2\cancel{a_3b_1}$$

$$\boxed{\vec{a}\cdot(\vec{a}\times\vec{b}) = 0}$$

P.T.O

$\Rightarrow \vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$.

Similarly $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

$\Rightarrow \vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$

Hence $\vec{a} \times \vec{b}$ is orthogonal to bothe the vectors $\vec{a}$ and $\vec{b}$.

V1/15 Prove that $|\vec{a} \times \vec{b}| = ab \sin\theta$, where $0 < \theta < \pi$ is the angle b/w the vectors $\vec{a}$ & $\vec{b}$.

Proof:- $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ Expand by R_1

$$\vec{a} \times \vec{b} = \hat{i}(a_2 b_3 - a_3 b_2) + \hat{j}(a_3 b_1 - a_1 b_3) + \hat{k}(a_1 b_2 - a_2 b_1)$$

$$|\vec{a} \times \vec{b}| = \sqrt{(a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2} \quad (A)$$

We know that $\vec{a} \cdot \vec{b} = ab\cos\theta$.

$$(\vec{a} \cdot \vec{b})^2 = a^2 b^2 \cos^2\theta = a^2 b^2 (1 - \sin^2\theta)$$

$$= a^2 b^2 - a^2 b^2 \sin^2\theta$$

$$a^2 b^2 \sin^2\theta = a^2 b^2 - (\vec{a} \cdot \vec{b})^2 \qquad \left(\begin{array}{l} a^2 = |\vec{a}|^2 \\ b^2 = |\vec{b}|^2 \end{array} \right.$$

$$a^2 b^2 \sin^2\theta = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

ch-01

$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2$$

$$= \cancel{a_1^2b_1^2} + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + \cancel{a_2^2b_2^2} + a_2^2b_3^2 + a_3^2b_1^2$$
$$+ a_3^2b_2^2 + \cancel{a_3^2b_3^2} - \cancel{a_1^2b_1^2} - \cancel{a_2^2b_2^2} - \cancel{a_3^2b_3^2} - 2a_1b_1a_2b_2$$
$$- 2a_2b_2a_3b_3 - 2a_3b_3a_1b_1$$

$$a^2b^2\sin^2\theta = (a_1b_2)^2 + (a_1b_3)^2 + (a_2b_1)^2 + (a_2b_3)^2$$
$$+ (a_3b_1)^2 + (a_3b_2)^2 - 2(a_1b_1)(a_2b_2)$$
$$- 2a_2b_2a_3b_3 - 2a_3b_3a_1b_1$$

$$a^2b^2\sin^2\theta = (a_1b_2)^2 + (a_2b_1)^2 - 2a_1b_1a_2b_2$$
$$+ (a_2b_3)^2 + (a_3b_2)^2 - 2a_2b_2a_3b_3$$
$$+ (a_1b_3)^2 + (a_3b_1)^2 - 2a_3b_3a_1b_1$$

$$a^2b^2\sin^2\theta = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2$$

Taking square root from both sides.

$$ab\sin\theta = \sqrt{(a_1b_2-a_2b_1)^2 + (a_2b_3-a_3b_2)^2 + (a_1b_3-a_3b_1)^2} \quad B$$

A = B

$$\Rightarrow \boxed{|\vec{a}\times\vec{b}| = ab\sin\theta}$$

proved

SCALAR TRIPLE PRODUCT
(Box Product)

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors then the dot product of $\vec{a}$ with $\vec{b}\times\vec{c}$ is called scalar triple product and is written as $\vec{a}\cdot(\vec{b}\times\vec{c})$ or simple $\vec{a}\cdot(\vec{b}\times\vec{c})$..

VECTOR TRIPLE PRODUCT

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors then the cross product of $\vec{a}$ with $\vec{b}\times\vec{c}$ is called vector triple product and is written as

$\vec{a}\times(\vec{b}\times\vec{c})$.

Q1/19 Taking $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$\vec{c} = [c_1, c_2, c_3]$ Show that

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof: $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$\vec{c} = [c_1, c_2, c_3]$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Ch-01

$$\vec{b}\times\vec{c} = \vec{i}\,(b_2c_3 - b_3c_2) + \vec{j}\,(b_3c_1 - b_1c_3) + \vec{k}\,(b_1c_2 - b_2c_1)$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = [a_1, a_2, a_3]\cdot[b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3,\ b_1c_2 - b_2c_1]$$

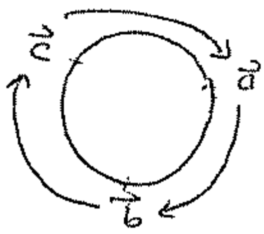
$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ proved.

FORMULA FOR SCALAR TRIPLE PRODUCT.

$$\text{Scalar triple product} = \begin{vmatrix} \text{Components of 1st vector} \\ \text{Components of 2nd vector} \\ \text{Components of 3rd vector.} \end{vmatrix}$$

$$\vec{a}\cdot\vec{b}\times\vec{c} = \vec{b}\cdot\vec{c}\times\vec{a} = \vec{c}\cdot\vec{a}\times\vec{b}$$



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(ii/16) Prove that cyclic permutation of the ~~vec~~ vectors in a scalar triple product leaves its value unchanged.

Proof:- $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$\vec{c} = [c_1, c_2, c_3]$

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad R_{12}$$

$$= +\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad R_{23}$$

$$\boxed{\vec{a}\cdot\vec{b}\times\vec{c} = \vec{b}\cdot\vec{c}\times\vec{a}} \quad (1)$$

Again

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad R_{13}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad R_{23} \quad \cancel{\vec{c}\cdot\vec{a}\times\vec{b}}$$

$$\boxed{\vec{a}\cdot\vec{b}\times\vec{c} = \vec{c}\cdot\vec{a}\times\vec{b}} \quad (2)$$

From ① and ②

$$\boxed{\vec{a}\cdot\vec{b}\times\vec{c} = \vec{b}\cdot\vec{c}\times\vec{a} = \vec{c}\cdot\vec{a}\times\vec{b}}$$

⇒ Cyclic permutation of the vectors in a scalar triple product leaves its value unchanged.

*

Prove that '×' & '·' can be interchanged in a scalar triple product.

Sol:- Since dot product is commutative

$$\vec{c}\cdot(\vec{a}\times\vec{b}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

$$\vec{a}\cdot\vec{b}\times\vec{c} = (\vec{a}\times\vec{b})\cdot\vec{c}$$ Ans

*

Q2/19 Show that $\hat{i}\cdot\hat{j}\times\hat{k} = 1$

Sol:- Since $\hat{i} = [1,0,0]$, $\hat{j} = [0,1,0]$

$\hat{k} = [0,0,1]$

$$\hat{i}\cdot(\hat{j}\times\hat{k}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ Expanding by R_1

$$= 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1(1-0) - 0 + 0 = 1$$

$$\Rightarrow \hat{i}\cdot(\hat{j}\times\hat{k}) = 1$$ Proved.

Q3/19 Twelve scalar triple products can be written involving the vectors $\vec{a}, \vec{b}, \vec{c}$. Write down each of these giving its value in terms of $\Delta = \vec{a} \cdot \vec{b} \times \vec{c}$.

$$\Delta = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad \text{--- (1)}$$

$$\Delta = \vec{b} \cdot (\vec{c} \times \vec{a}) \quad \text{--- (2)}$$

$$\Delta = \vec{c} \cdot (\vec{a} \times \vec{b}) \quad \text{--- (3)}$$

Since dot product is commutative, So

$$\Delta = \vec{b} \times \vec{c} \cdot \vec{a} \quad \text{--- (4)}$$

$$\Delta = \vec{c} \times \vec{a} \cdot \vec{b} \quad \text{--- (5)}$$

$$\Delta = \vec{a} \times \vec{b} \cdot \vec{c} \quad \text{--- (6)}$$

Since vectors product is anti Commutative

$$\Delta = -\vec{a} \cdot \vec{c} \times \vec{b} \quad \text{--- (7)}$$

$$\Delta = -\vec{b} \cdot \vec{a} \times \vec{c} \quad \text{--- (8)}$$

$$\Delta = -\vec{c} \cdot \vec{b} \times \vec{a} \quad \text{--- (9)}$$

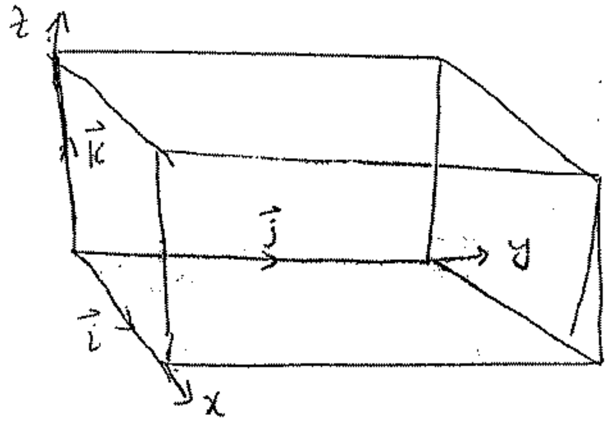
$$\Delta = -\vec{c} \times \vec{b} \cdot \vec{a} \quad \text{--- (10)}$$

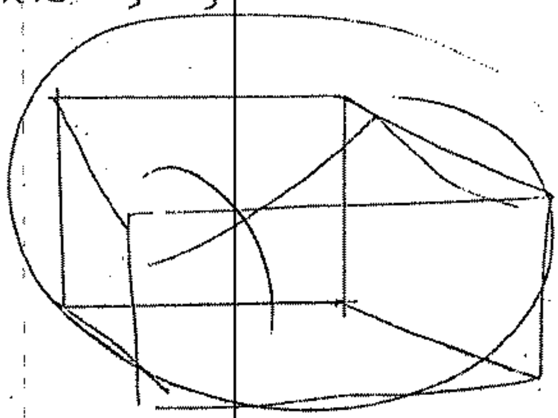
$$\Delta = -\vec{a} \times \vec{c} \cdot \vec{b} \quad \text{--- (11)}$$

$$\Delta = -\vec{b} \times \vec{a} \cdot \vec{c} \quad \text{--- (12)}$$

VOLUME OF PARALLELOPIPED

$V = \vec{a} \cdot \vec{b} \times \vec{c}$, where $\vec{a}$, $\vec{b}$, $\vec{c}$ are its edges





($\frac{Q\,4}{19}$) Find the volume of a parallelopiped whose edges are represented by

$\vec{a} = 3\vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{c} = \vec{i} - 3\vec{j} - 4\vec{k}$

Sol:- $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - 3\vec{j} + \vec{k}$

$\vec{c} = \vec{i} - 3\vec{j} - 4\vec{k}$

Volume of llelpiped is

$V = \vec{a} \cdot \vec{b} \cdot \times \vec{c}$

$$V = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -3 & -4 \end{vmatrix}$$

Expanding by R_1

$V = 3(12+3) + (1+8) - 1(-6+3) = 45 + 9 + 3$

$$\boxed{V = 57 \text{ cubic units}}$$

v.v. imp

Prove that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.$$

This Questio has a trick So Becareful when solving this Question.

Proof:- $\vec{a} = [a_1, a_2, a_3]$

$\vec{b} = [b_1, b_2, b_3]$

$\vec{c} = [c_1, c_2, c_3]$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1 \end{vmatrix}$$ Expanding by R_1

$$= \hat{i}[a_2(b_1c_2-b_2c_1) - a_3(b_3c_1-b_1c_3)]$$
$$+\hat{j}[a_3(b_2c_3-b_3c_2) - a_1(b_1c_2-b_2c_1)]$$
$$+\hat{k}[a_1(b_3c_1-b_1c_3) - a_2(b_2c_3-b_3c_2)]$$

$$= \hat{i}[a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 + \underbrace{a_1b_1c_1 - a_1b_1c_1}_{\text{introduction}}]$$

$$+\hat{j}[a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 + \underbrace{a_2b_2c_2 - a_2b_2c_2}_{\text{introduction}}]$$

$$+\hat{k}[a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 + \underbrace{a_3b_3c_3 - a_3b_3c_3}_{\text{introduction}}]$$

P.T.O

$$= \hat{i}\left[a_2b_1c_2 + a_3b_1c_3 + a_1b_1c_1 - (a_2b_2c_1 + a_3b_3c_1 + a_1b_1c_1)\right]$$

$$+\hat{j}\left[a_3b_2c_3 + a_1b_2c_1 + a_2b_2c_2 - (a_3b_3c_2 + a_1b_1c_2 + a_2b_2c_2)\right]$$

$$+\hat{k}\left[a_1b_3c_1 + a_2b_2c_2 + a_3b_3c_3 - (a_1b_1c_3 + a_2b_2c_3 + a_3b_3c_3)\right]$$

$$= b_1(a_2c_2 + a_3c_3 + a_1c_1)\hat{i} - c_1(a_2b_2 + a_3b_3 + a_1b_1)\hat{i}$$

$$+ b_2(a_3c_3 + a_1c_1 + a_2c_2)\hat{j} - c_2(a_3b_3 + a_1b_1 + a_2b_2)\hat{j}$$

$$+ b_3(a_1c_1 + a_2c_2 + a_3c_3)\hat{k} - c_3(a_1b_1 + a_2b_2 + a_3b_3)\hat{k}$$

$$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$- (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$\vec{a} \times \vec{b} \times \vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

proved.

NOTE THAT

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.$$

$$\mathrm{I} \times (\mathrm{II} \times \mathrm{III}) = (\mathrm{I}\cdot\mathrm{III})\,\mathrm{II} - (\mathrm{I}\cdot\mathrm{II})\,\mathrm{III}.$$

Q6/19 Prove that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

Proof:- To prove that

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$$

Since

$$\vec{a}\times(\vec{b}\times\vec{c}) = \cancel{(\vec{a}\cdot\vec{c})\vec{b}} - \cancel{(\vec{a}\cdot\vec{b})\vec{c}}$$

$$\vec{b}\times(\vec{c}\times\vec{a}) = \cancel{(\vec{b}\cdot\vec{a})\vec{c}} - \cancel{(\vec{b}\cdot\vec{c})\vec{a}}$$

$$\vec{c}\times(\vec{a}\times\vec{b}) = \cancel{(\vec{c}\cdot\vec{b})\vec{a}} - \cancel{(\vec{c}\cdot\vec{a})\vec{b}}$$ adding

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$$

proved.

*

v.imp Q5/19 Show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{b}\times\vec{d})\vec{c}-(\vec{a}\cdot\vec{b}\times\vec{c})\vec{d}$

$$= (\vec{a}\cdot\vec{c}\times\vec{d})\vec{b}-(\vec{b}\cdot\vec{c}\times\vec{d})\vec{a}$$

proof:- To show

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b}\cdot\vec{d})\vec{c}-(\vec{a}\times\vec{b}\cdot\vec{c})\vec{d}$$

$$= (\vec{a}\cdot\vec{b}\times\vec{d})\vec{c}-(\vec{a}\cdot\vec{b}\times\vec{c})\vec{d}$$

let $\vec{a}\times\vec{b}=\vec{e}$

So. P.T.O

$\Rightarrow (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{e} \times (\vec{c} \times \vec{d})$

$= (\vec{e} \cdot \vec{d})\vec{c} - (\vec{e} \cdot \vec{c})\vec{d}$

But $\vec{e} = \vec{a} \times \vec{b}$

$= (\vec{a} \times \vec{b} \cdot \vec{d})\vec{c} - (\vec{a} \times \vec{b} \cdot \vec{c})\vec{d}$

$= (\vec{a} \cdot \vec{b} \times \vec{d})\vec{c} - (\vec{a} \cdot \vec{b} \times \vec{c})\vec{d}$ — (1)

$(\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c})$

Let $\vec{c} \times \vec{d} = \vec{f}$

$\Rightarrow (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{f}$

$= -\vec{f} \times (\vec{a} \times \vec{b})$

$= -\{(\vec{f} \cdot \vec{b})\vec{a} - (\vec{f} \cdot \vec{a})\vec{b}\}$

$= (\vec{f} \cdot \vec{a})\vec{b} - (\vec{f} \cdot \vec{b})\vec{a}$

$= (\vec{a} \cdot \vec{f})\vec{b} - (\vec{b} \cdot \vec{f})\vec{a}$

But $\vec{f} = \vec{c} \times \vec{d}$

$= (\vec{a} \cdot \vec{c} \times \vec{d})\vec{b} - (\vec{b} \cdot \vec{c} \times \vec{d})\vec{a}$ — (2)

From (1) and (2)

we have proved.

Q 7 / 19 Show that $\vec{a}\cdot(\vec{b}\times\vec{c})\times\vec{d} = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{c}\cdot\vec{d} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{d} \end{vmatrix}$

Sol:- $\vec{a}\cdot(\vec{b}\times\vec{c})\times\vec{d} = \vec{a}\cdot\left[(\vec{b}\times\vec{c})\times\vec{d}\right]$

As we know that vector product is anti-commutative.

$$= \vec{a}\cdot\left[-\vec{d}\times(\vec{b}\times\vec{c})\right] = -\vec{a}\cdot\left[\vec{d}\times(\vec{b}\times\vec{c})\right]$$

$$= -\vec{a}\cdot\left[(\vec{d}\cdot\vec{c})\vec{b} - (\vec{d}\cdot\vec{b})\vec{c}\right]$$

$$= -(\vec{a}\cdot\vec{b})(\vec{d}\cdot\vec{c}) + (\vec{a}\cdot\vec{c})(\vec{d}\cdot\vec{b})$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})$$

$$\vec{a}\cdot(\vec{b}\times\vec{c})\times\vec{d} = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{c}\cdot\vec{d} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{d} \end{vmatrix}$$ proved

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Q 8 / 19 Show that $(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b}) = (\vec{a}\cdot\vec{b}\times\vec{c})^2$.

Sol:-

$$(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c})\cdot\left[(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\right]$$

$$= (\vec{b}\times\vec{c})\cdot\left[(\vec{c}\times\vec{a}\cdot\vec{b})\vec{a} - (\vec{c}\times\vec{a}\cdot\vec{a})\vec{b}\right]$$

$$= (\vec{b}\times\vec{c})\cdot\left[(\vec{c}\times\vec{a}\cdot\vec{b})\vec{a} - (\vec{c}\times\vec{a}\times\vec{a})\vec{b}\right]$$

$$= (\vec{b}\times\vec{c})\cdot\left[(\right.$$

$$= (\vec{b} \times \vec{c}) \cdot [(\vec{c} \cdot \vec{a} \times \vec{b})\vec{a} - 0] \quad (\because \vec{a} \times \vec{a} = 0)$$

$$= (\vec{b} \times \vec{c} \cdot \vec{a})(\vec{c} \cdot \vec{a} \times \vec{b})$$

$$= (\vec{b} \cdot \vec{c} \times \vec{a})(\vec{c} \cdot \vec{a} \times \vec{b})$$

$$= (\vec{a} \cdot \vec{b} \times \vec{c})(\vec{a} \cdot \vec{b} \times \vec{c})$$

$$= (\vec{a} \cdot \vec{b} \times \vec{c})^2$$

Thus

$$(\vec{b} \times \vec{c}) \cdot (\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{b} \times \vec{c})^2.$$

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vimp Ex 18) State and prove Lagrange's identity & deduce

$$(\vec{a} \times \vec{b})^2 = a^2 b^2 - (a \cdot b)^2$$

write it in component form.

Sol:- (1) To prove

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

interchange 'x' & '.'

$$\vec{a} \times \vec{b} \cdot (\vec{c} \times \vec{d}) = \vec{a} \cdot [\vec{b} \times (\vec{c} \times \vec{d})]$$

P.T.O

$$= \vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$

$$= \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c}-(\vec{b}\cdot\vec{c})\vec{d}]$$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d}) \quad (1)$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix} \quad \text{proved}$$

Second part

putting $\vec{c}=\vec{a}$, $\vec{d}=\vec{b}$ in (1)

$$(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{b}\cdot\vec{a})(\vec{a}\cdot\vec{b})$$

$$(\vec{a}\times\vec{b})^2 = a^2b^2-(\vec{a}\cdot\vec{b})^2 \quad (2)$$

3rd part

In the components form.

let $\hat{a}=[a_1, a_2, a_3]$, $\hat{b}=[b_1, b_2, b_3]$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \hat{i}(a_2b_3-a_3b_2)+\hat{j}(a_3b_1-a_1b_3)+\hat{k}(a_1b_2-a_2b_1)$$

$$(\vec{a}\times\vec{b})^2 = |\vec{a}\times\vec{b}|^2 = (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2 + (a_1b_2-a_2b_1)^2$$

$$a^2 = |\vec{a}|^2 = a_1^2+a_2^2+a_3^2$$

$$b^2 = |\vec{b}|^2 = b_1^2+b_2^2+b_3^2$$

$$\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$$

$(2) \Rightarrow$

$$(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2 = (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$$

which is the required component form