Metric Spaces

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Real Valued Function

Let $f: A \to R$ be a function. Clearly domain of f is A, in other words f is defined on A. Since co-domain of f is R, we can say that f is real valued function.

Metric

Let X be a non-empty set and R be a set of real numbers.

Let $d: X \times X \to R$ be a function

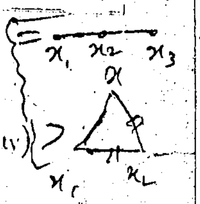
Then "d" is called "metric" on X, if "d" satisfies each of the following four conditions;

(M$_1$) $d(x_1, x_2) \geq 0 \quad \forall\, x_1, x_2 \in X$

(M$_2$) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2 \quad \forall\, x_1, x_2 \in X$

(M$_3$) $d(x_1, x_2) = d(x_2, x_1) \quad \forall\, x_1, x_2 \in X$ (Symmetric Property)

(M$_4$) $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3) \quad \forall\, x_1, x_2, x_3 \in X$ (Triangular Inequality)



If "d" is a "metric" on X then the pair (X, d) is called metric space.

Note

The non-negative real number $d(x_1, x_2)$ is called distance between points x_1 and x_2 in the metric "d".

Usual Metric on R

Let $d: R \times R \to R$ be a metric on R given by $d(x_1, x_2) = |x_1 - x_2|$

Then "d" is called a usual metric on R and (R, d) is called usual metric space.

Usual Metric on R^2

Let $d: R^2 \times R^2 \to R$ be a metric on R^2 given by

$d(P_1,P_2) \Leftarrow d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Then "d" is called a usual metric on R^2 and (R^2, d) is usual metric space.

Usual Metric on R^3

Let $d: R^3 \times R^3 \to R$ be a metric on R^3 given by

$$d[(x_1, y_1, z_1), (x_2, y_2, z_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Then "d" is called a usual metric on R^3 and (R^3, d) is usual metric space.

Note

When we say that R is a metric space without giving a metric on R then it is assumed that metric on R is "usual metric".
Similarly we take the case of R^2 and R^3.

Example

Let X be the set of all towns marked on a plane geographically map and let $d(x_1, x_2)$ be the length of the shortest rout from town x_1 to x_2. Show that "d" is a metric on X.

Solution

Here function $d: X \times X \to R$ is defined as

$d(x_1, x_2)$ =Length of shortest route from town x_1 to x_2.

(M_1) Since (Length of shortest route from town x_1 to x_2) ≥ 0

$\therefore \quad d(x_1, x_2) \geq 0$

(M_2) Let $d(x_1, x_2) = 0 \Rightarrow$ Length of shortest route from town x_1 to $x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

Let $\quad x_1 = x_2 \quad \Rightarrow \quad$ Length of shortest route from town x_1 to $x_2 = 0$

$\Rightarrow \quad d(x_1, x_2) = 0$

(M_3) Since $d(x_1, x_2)$ = Length of shortest route from town x_1 to x_2

= Length of shortest route from town x_2 to x_1

$= d(x_2, x_1)$

(M_4) Let $\quad x_1, x_2, x_3 \in X$

Then x_1, x_2, x_3 are non-collinear or collinear

If x_1, x_2, x_3 are non-collinear, then they form a triangle and we know that sum of length of two sides of a triangle is always greater than the third side.

$\therefore \quad d(x_1, x_2) + d(x_2, x_3) > d(x_1, x_3)$ ------------ --- (i)

Let x_1, x_2, x_3 are collinear.

Then $\quad d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3)$ -------- --- (ii)

From (i) and (ii), we get

$d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$

Hence "d" is a metric on X.

Example

Let $X = R$ be the set of all real numbers and let $d: R \times R \to R$ be defined by $d(x_1, x_2) = |x_1 - x_2|$ denotes the absolute value of the number $x_1 - x_2$. Show that (R, d) is a metric space.

Solution

Here function $d: R \times R \to R$ is defined as

$d(x_1, x_2) = |x_1 - x_2|$

(M_1) Since $|x_1 - x_2| \geq 0$

$\therefore \quad d(x_1, x_2) \geq 0$

(M_2) Let $d(x_1, x_2) = 0 \quad \Rightarrow \quad |x_1 - x_2| = 0$

$\Rightarrow \quad x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

Let $\quad x_1 = x_2 \quad \Rightarrow \quad x_1 - x_2 = 0$

$\Rightarrow \quad |x_1 - x_2| = 0$

$\Rightarrow \quad d(x_1, x_2) = 0$

Thus $d(x_1, x_2) = 0 \iff x_1 = x_2$

(M_3) Since $d(x_1, x_2) = |x_1 - x_2|$

$$= |-(x_2 - x_1)|$$
$$= |x_2 - x_1|$$
$$= d(x_2, x_1)$$

(M_4) Since $d(x_1, x_2) = |x_1 - x_2|$

$$d(x_2, x_3) = |x_2 - x_3|$$
$$d(x_1, x_3) = |x_1 - x_3|$$

Now $d(x_1, x_3) = |x_1 - x_3|$

$$= |x_1 - x_2 + x_2 - x_3|$$
$$\leq |x_1 - x_2| + |x_2 - x_3|$$
$$= d(x_1, x_2) + d(x_2, x_3)$$

Thus $(R; d)$ is a metric space.

Example

Let $X = R^2$ be a set of all ordered pairs (x, y); $x, y \in R$. Let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued function "d" defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$ is a metric on R^2

Solution

Here function $d: R^2 \times R^2 \to R$ is defined as

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

(M_1) Since $|x_1 - x_2| + |y_1 - y_2| \geq 0$

$$\therefore d(P_1, P_2) \geq 0$$

(M_2) Let $d(P_1, P_2) = 0 \Rightarrow |x_1 - x_2| + |y_1 - y_2| = 0$

$$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$$
$$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$$
$$\Rightarrow x_1 = x_2, \quad y_1 = y_2$$
$$\Rightarrow (x_1, y_1) = (x_2, y_2)$$
$$\Rightarrow P_1 = P_2$$

Let $P_1 = P_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

$$\Rightarrow x_1 = x_2, \quad y_1 = y_2$$
$$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$$
$$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$$
$$\Rightarrow |x_1 - x_2| + |y_1 - y_2| = 0$$
$$\Rightarrow d(P_1, P_2) = 0$$

Thus $d(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$

(M_3) Since $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$

$$= |-(x_2 - x_1)| + |-(y_2 - y_1)|$$
$$= |x_2 - x_1| + |y_2 - y_1|$$
$$= d(P_2, P_1)$$

(M_4) Since $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$

$$d(P_2, P_3) = |x_2 - x_3| + |y_2 - y_3|$$
$$d(P_1, P_3) = |x_1 - x_3| + |y_1 - y_3|$$

Since $d(P_1, P_3) = |x_1 - x_3| + |y_1 - y_3|$

$$= |x_1 - x_2 + x_2 - x_3| + |y_1 - y_2 + y_2 - y_3|$$
$$\leq |x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3|$$
$$= |x_1 - x_2| + |y_1 - y_2| + |x_2 - x_3| + |y_2 - y_3|$$
$$= d(P_1, P_2) + d(P_2, P_3)$$

Hence "d" is metric on R^2.

Example

Let $X = R^2$ be a set of all ordered pairs(x, y); $x, y \in R$. Let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued function"d" defined by $d(P_1, P_2) = max\,(|x_1 - x_2|, |y_1 - y_2|)$ is a metric on R^2.

Solution

Here function $d: R^2 \times R^2 \to R$ is defined as

$$d(P_1, P_2) = max\,(|x_1 - x_2| + |y_1 - y_2|)$$

(M$_1$) Since $max\,(|x_1 - x_2| + |y_1 - y_2|) \geq 0$

$(\because |x_1 - x_2| \geq 0\ \&\ |y_1 - y_2| \geq 0)$

$\therefore d(P_1, P_2) \geq 0$

(M$_2$) Let $d(P_1, P_2) = 0 \Rightarrow max\,(|x_1 - x_2|, |y_1 - y_2|) = 0$

$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow P_1 = P_2$

Let $P_1 = P_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0$

$\Rightarrow max\,(|x_1 - x_2|, |y_1 - y_2|) = 0$

$\Rightarrow d(P_1, P_2) = 0$

Thus $d(P_1, P_2) = 0 \Leftrightarrow P_1 = P_2$

(M$_3$) Since $d(P_1, P_2) = max\,(|x_1 - x_2|, |y_1 - y_2|)$

$$= max(|-(x_2 - x_1)|, |-(y_2 - y_1)|)$$
$$= max(|x_2 - x_1|, |y_2 - y_1|)$$
$$= d(P_2, P_1)$$

(M₄) Since $d(P_1, P_2) = max\,(|x_1 - x_2|, |y_1 - y_2|) = |x_1 - x_2|$ (Say)

$d(P_2, P_3) = max\,(|x_2 - x_3|, |y_2 - y_3|) = |x_2 - x_3|$ (Say)

$d(P_1, P_3) = max\,(|x_1 - x_3|, |y_1 - y_3|) = |x_1 - x_3|$ (Say)

Now $d(P_1, P_3) = |x_1 - x_3|$

$$= |x_1 - x_2 + x_2 - x_3|$$
$$\leq |x_1 - x_2| + |x_2 - x_3|$$
$$= d(P_1, P_2) + d(P_2, P_3)$$

(We can get the same results in the remaining cases.)

Hence "d" is metric on R^2.

Example

Let $X = R^2$ be a set of all ordered pairs(x, y); $x, y \in R$. Let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued function "d" defined by $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$ is a metric on R^2.

Solution

Here function $d: R^2 \times R^2 \to R$ is defined as

$$d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$$

(M₁) Since $[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}} \geq 0$

$\therefore \quad d(P_1, P_2) \geq 0$

(M₂) Let $d(P_1, P_2) = 0 \Rightarrow [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}} = 0$

$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$

$\Rightarrow (x_1 - x_2)^2 = 0, \quad (y_1 - y_2)^2 = 0$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow P_1 = P_2$

Let $P_1 = P_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow x_1 = x_2, \quad y_1 = y_2$

$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0$

$\Rightarrow (x_1 - x_2)^2 = 0, \quad (y_1 - y_2)^2 = 0$

$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$

$\Rightarrow [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}} = 0$

$\Rightarrow d(P_1, P_2) = 0$

(M$_3$) Since $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$

$$= [\{-(x_2 - x_1)\}^2 + \{-(y_2 - y_1)\}^2]^{\frac{1}{2}}$$

$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{1}{2}}$$

$$= d(P_2, P_1)$$

(M$_4$) Let $P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3) \in R^2$ then P_1, P_2, P_3 are collinear or non-collinear.

If P_1, P_2, P_3 are collinear, then

$$d(P_1, P_2) + d(P_2, P_3) = d(P_1, P_3) \text{ ------------- (1)}$$

If P_1, P_2, P_3 are non-collinear, then they form a triangle and we know that, we know that sum of length of two sides of a triangle is always greater than the third side.

$$\therefore \quad d(P_1, P_2) + d(P_2, P_3) > d(P_1, P_3) \text{ ------------- (2)}$$

From (1) & (2) we get,

$$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$$

Hence "d" is metric on R^2.

Example

Let $X = R^3$ be a set of all ordered pairs(x, y); $x, y \in R$. Let $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2) \in R^3$. Show that the non-negative real valued function "d" defined by $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$ is a metric on R^3.

$+(z_1 - z_2)^2$

Solution

Here function $d: R^3 \times R^3 \to R$ is defined as

$$d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}}$$

(M$_1$) Since $[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}} \geq 0$

$$\therefore \quad d(P_1, P_2) \geq 0$$

(M$_2$) Let $d(P_1, P_2) = 0 \Rightarrow [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}} = 0$

$$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 0$$

$$\Rightarrow (x_1 - x_2)^2 = 0, (y_1 - y_2)^2 = 0, (z_1 - z_2)^2 = 0$$

$$\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0, \quad z_1 - z_2 = 0$$

$$\Rightarrow x_1 = x_2, \quad y_1 = y_2 \quad z_1 = z_2$$

$$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$$

$$\Rightarrow P_1 = P_2$$

Let $P_1 = P_2 \Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$

$\Rightarrow \quad x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2$

$\Rightarrow \quad x_1 - x_2 = 0, \quad y_1 - y_2 = 0, \quad z_1 - z_2 = 0$

$\Rightarrow \quad (x_1 - x_2)^2 = 0, (y_1 - y_2)^2 = 0, (z_1 - z_2)^2 = 0$

$\Rightarrow \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 0$

$\Rightarrow \quad [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}} = 0$

$\Rightarrow \quad d(P_1, P_2) = 0$

(M₃) Since $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}}$

$$= [\{-(x_2 - x_1)\}^2 + \{-(y_2 - y_1)\}^2 + \{-(z_2 - z_1)\}^2]^{\frac{1}{2}}$$

$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{\frac{1}{2}}$$

$$= d(P_2, P_1)$$

(M₄) Let $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2), P_3(x_3, y_3, z_3) \in R^3$ then P_1, P_2, P_3 are collinear or non-collinear.

If P_1, P_2, P_3 are collinear, then

$d(P_1, P_2) + d(P_2, P_3) = d(P_1, P_3)$ -------------- (1)

If P_1, P_2, P_3 are non-collinear, then they form a triangle and we know that, we know that sum of length of two sides of a triangle is always greater than the third side.

$\therefore \quad d(P_1, P_2) + d(P_2, P_3) > d(P_1, P_3)$ -------------- (2)

From (1) & (2) we get,

$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$

Hence "d" is metric on R^3.

Imp

Example

Show that every non-empty set can be given a metric and hence can be converted into metric space.

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Solution

Let X be any non-empty set.

Let $d_\circ : X \times X \to R$ be defined by

$$d_\circ(x_1, x_2) = \begin{cases} 0 & if\ x_1 = x_2 \\ 1 & if\ x_1 \neq x_2 \end{cases}$$

We shall prove that $d_\circ$ is a metric on X.

(M₁) Here $d_\circ(x_1, x_2) \geq 0 \quad (\because d_\circ(x_1, x_2) = 0$ or $d_\circ(x_1, x_2) = 1)$

(M₂) Let $d_\circ(x_1, x_2) = 0 \Rightarrow x_1 = x_2$ (By definition)

Let $x_1 = x_2 \Rightarrow d_\circ(x_1, x_2) = 0$ (By definition)

Thus $d_\circ(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$

(M_3) (i) Let $d_\circ(x_1,x_2)=0 \Rightarrow x_1=x_2$ (By definition)

$\Rightarrow x_2=x_1$

$\Rightarrow d_\circ(x_2,x_1)=0$

(ii) Let $d_\circ(x_1,x_2)=1 \Rightarrow x_1\neq x_2$ (By definition)

$\Rightarrow x_2\neq x_1$ ~~(By definition)~~

$\Rightarrow d_\circ(x_2,x_1)=1$

Hence in both the cases $d_\circ(x_1,x_2)=d_\circ(x_2,x_1)$

(M_4) Let $x_1,x_2,x_3\in X$

(i) Let $x_1=x_2=x_3$ then $d_\circ(x_1,x_2)=0$

& $d_\circ(x_2,x_3)=0$

also $d_\circ(x_1,x_3)=0$

$\therefore \quad d(x_1,x_2)+d(x_2,x_3)=d(x_1,x_3)$

(ii) Let $x_1\neq x_2\neq x_3$ then $d_\circ(x_1,x_2)=1$

& $d_\circ(x_2,x_3)=1$

also $d_\circ(x_1,x_3)=1$

$\therefore \quad \underbrace{d(x_1,x_2)+d(x_2,x_3)}_{1+1}>d(x_1,x_3)$

Similar type of verification in all remaining cases leads us to the conclusion that $\quad d(x_1,x_2)+d(x_2,x_3)\geq d(x_1,x_3) \quad \forall\, x_1,x_2,x_3\in X$

Hence $(X,d_\circ)$ is a metric space.

Note

Let X be any non-empty set. Let $d_\circ : X\times X\rightarrow R$ be defined by

$$d_\circ(x_1,x_2)=\begin{cases}0 & if\ x_1=x_2\\ 1 & if\ x_1\neq x_2\end{cases}$$

Then $d_\circ$ is called discrete metric on X.

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Question

Let C be the set of all complex numbers and let $d: C\times C\rightarrow R$ be defined by $d(z_1,z_2)=|z_1-z_2|$ d is a metric on C

Solution

Here function $d: C\times C\rightarrow R$ is defined as

$$d(z_1,z_2)=|z_1-z_2|$$

(M_1) Since $|z_1-z_2|\geq 0$

$\therefore \quad d(z_1,z_2)\geq 0$

(M_2) Let $d(z_1,z_2)=0 \Rightarrow |z_1-z_2|=0$

$\Rightarrow z_1-z_2=0$

$\Rightarrow z_1=z_2$

Let $z_1 = z_2 \quad \Rightarrow \quad z_1 - z_2 = 0$

$\Rightarrow \quad |z_1 - z_2| = 0$

$\Rightarrow \quad d(z_1, z_2) = 0$

Thus $d(z_1, z_2) = 0 \iff z_1 = z_2$

(M₃) Since $d(z_1, z_2) = |z_1 - z_2|$

$= |-(z_2 - z_1)|$

$= |z_2 - z_1|$

$= d(z_2, z_1)$

(M₄) Since $d(z_1, z_2) = |z_1 - z_2|$

$d(z_2, z_3) = |z_2 - z_3|$

$d(z_1, z_3) = |z_1 - z_3|$

Now $d(z_1, z_3) = |z_1 - z_3|$

$= |z_1 - z_2 + z_2 - z_3|$

$\leq |z_1 - z_2| + |z_2 - z_3|$

$= d(z_1, z_2) + d(z_2, z_3)$

Thus (C, d) is a metric space.

Question

Let d be a metric on X and let $d': X \times X \to R$ be given by $d'(x_1, x_2) = min\,(1, d(x_1, x_2))$. Is d' a metric on X?

Solution

Here function $d': X \times X \to R$ be given by

$$d'(x_1, x_2) = min\,(1, d(x_1, x_2))$$

(M₁) Since $min\,(1, d(x_1, x_2)) \geq 0$

$\therefore \quad d'(x_1, x_2) \geq 0$

(M₂) Let $d'(x_1, x_2) = 0 \quad \Rightarrow \quad min\,(1, d(x_1, x_2)) = 0$

$\Rightarrow \quad d(x_1, x_2) = 0 \qquad \because 1 \neq 0$

$\Rightarrow \quad x_1 = x_2 \qquad \because d$ is metric on X.

Let $x_1 = x_2 \quad \Rightarrow \quad d(x_1, x_2) = 0 \qquad \because d$ is metric on X.

$\Rightarrow \quad min\,(1, d(x_1, x_2)) = 0$

$\Rightarrow \quad d'(x_1, x_2) = 0$

Thus $d'(x_1, x_2) = 0 \iff x_1 = x_2$

(M₃) Since $d'(x_1, x_2) = min\,(1, d(x_1, x_2))$

$= min\,(1, d(x_2, x_1)) \qquad \because d$ is metric on X.

$= d'(x_2, x_1)$

(M₄) Since $d'(x_1, x_2) = min\,(1, d(x_1, x_2)) = d(x_1, x_2)$ (Say)

$d'(x_2, x_3) = min\,(1, d(x_2, x_3)) = d(x_2, x_3)$ (Say)

$d'(x_1, x_3) = min\,(1, d(x_1, x_3)) = d(x_1, x_3)$ (Say)

Since d is a metric on X.

$\therefore \quad d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$

$\Rightarrow \quad d'(x_1, x_2) + d'(x_2, x_3) \geq d'(x_1, x_3)$

We get the same result in the remaining cases.

$\therefore \quad d'$ is a metric on X.

Question

Let (X_1, d_1) and (X_2, d_2) be two metric spaces

Define $d'[(x_1, x_2), (y_1, y_2)] = \sum_{i=1}^{2} d_i(x_i, y_i)$. Is d' a metric on $X_1 \times X_2$.

Solution

Here function $d' : X_1 \times X_2 \to R$ is defined as

$$d'[(x_1, x_2), (y_1, y_2)] = \sum_{i=1}^{2} d_i(x_i, y_i)$$
$$= d_1(x_1, y_1) + d_2(x_2, y_2)$$

(M$_1$) Since $d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0$

$\because$ $d_1(x_1, y_1) \geq 0$, $d_2(x_2, y_2) \geq 0$

$\because$ d_1, d_2 are metrics on X_1 and X_2 respectively.

$\therefore$ $d'\big((x_1, x_2), (y_1, y_2)\big) \geq 0$

(M$_2$) Let $d'\big((x_1, x_2), (y_1, y_2)\big) = 0$ $\Rightarrow$ $d_1(x_1, y_1) + d_2(x_2, y_2) = 0$

$\Rightarrow$ $d_1(x_1, y_1) = 0$, $d_2(x_2, y_2) = 0$

$\Rightarrow$ $x_1 = y_1$, $x_2 = y_2$

$\because$ d_1, d_2 are metrics on $X_1 \times X_2$

$\Rightarrow$ $(x_1, x_2) = (x_2, y_2)$

Let $(x_1, x_2) = (y_1, y_2)$ $\Rightarrow$ $x_1 = y_1$, $x_2 = y_2$

$\Rightarrow$ $d_1(x_1, y_1) = 0$ $d_2(x_2, y_2) = 0$

($\because$ d_1, d_2 are metrics on X_1 & X_2 respectively)

$\Rightarrow$ $d_1(x_1, y_1) + d_2(x_2, y_2) = 0$

$\Rightarrow$ $d'\big((x_1, x_2), (y_1, y_2)\big) = 0$

Thus $d'\big((x_1, x_2), (y_1, y_2)\big) = 0 \Leftrightarrow (x_1, x_2) = (y_1, y_2)$

(M$_3$) Since $d'\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$

$$= d_1(y_1, x_1) + d_2(y_2, x_2)$$

$$= d'\big((y_1, y_2), (x_1, x_2)\big)$$

(M$_4$) Since $d'\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$

$d'\big((y_1, y_2), (z_1, z_2)\big) = d_1(y_1, z_1) + d_2(y_2, z_2)$

$d'\big((x_1, x_2), (z_1, z_2)\big) = d_1(x_1, z_1) + d_2(x_2, z_2)$

Now $d'\big((x_1, x_2), (y_1, y_2)\big) + d'\big((y_1, y_2), (z_1, z_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$

$$+ d_1(y_1, z_1) + d_2(y_2, z_2)$$
$$= d_1(x_1, y_1) + d_1(y_1, z_1)$$
$$+ d_2(x_2, y_2) + d_2(y_2, z_2)$$
$$\geq d_1(x_1, z_1) + d_2(x_2, z_2)$$

$$\left(\begin{array}{l} \because d_1, d_2 \text{ are metrics on } X_1 \ \& \ X_2 \text{ resp.} \\ \therefore d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1) \\ \& \ d_2(x_2, y_2) + d_1(y_2, z_2) \geq d_2(x_2, z_2) \end{array}\right)$$

$$= d'\big((x_1, x_2), (z_1, z_2)\big)$$

$\therefore$ d' is a metric on $X_1 \times X_2$.

Question

Let (X_1, d_1) and (X_2, d_2) be two metric spaces.

Let $d''[(x_1, x_2), (y_1, y_2)] = max\,(d_1(x_1, y_1), d_2(x_2, y_2))$.

Is d'' a metric on $X_1 \times X_2$.

Solution

Here function $d'': X_1 \times X_2 \to R$ is defined as

$$d''[(x_1, x_2), (y_1, y_2)] = max\,(d_1(x_1, y_1), d_2(x_2, y_2))$$

(M$_1$) Since $max(d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0)$

$\because$ $d_1(x_1, y_1) \geq 0$, $d_2(x_2, y_2) \geq 0$

$\because$ d_1, d_2 are metrics on X_1 and X_2 respectively

$\therefore$ $d''((x_1, x_2), (y_1, y_2)) \geq 0$.

(M$_2$) Let $d''((x_1, x_2), (y_1, y_2)) = 0 \quad \Rightarrow \quad max\,(d_1(x_1, y_1), d_2(x_2, y_2)) = 0$

$\Rightarrow \quad d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0$

$\Rightarrow \quad x_1 = y_1, \qquad x_2 = y_2$

($\because$ d_1, d_2 are metrics on X_1, X_2 respectively)

$\Rightarrow \quad (x_1, x_2) = (y_1, y_2)$

Let $(x_1, x_2) = (y_1, y_2) \quad \Rightarrow \quad x_1 = y_1, \quad x_2 = y_2$

$\Rightarrow \quad d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0$

($\because$ d_1, d_2 are metrics on X_1, X_2 respectively)

$\Rightarrow \quad max\,(d_1(x_1, y_1), d_2(x_2, y_2)) = 0$

$\Rightarrow \quad d''((x_1, x_2), (y_1, y_2)) = 0$

$$d''((x_1, x_2), (y_1, y_2)) = 0 \Leftrightarrow (x_1, x_2) = (y_1, y_2).$$

(M$_3$) Since $d''[(x_1, x_2), (y_1, y_2)] = max\,(d_1(x_1, y_1), d_2(x_2, y_2))$

$= max\,(d_1(y_1, x_1), d_2(y_2, x_2))$

($\because$ d_1, d_2 are metrics on X_1, X_2 respectively)

$= d''((y_1, y_2), (x_1, x_2))$

(M$_4$) Let. $d''[(x_1, x_2), (y_1, y_2)] = max\,(d_1(x_1, y_1), d_2(x_2, y_2)) = d_1(x_1, y_1)$ (Say)

$d''[(y_1, y_2), (z_1, z_2)] = max\,(d_1(y_1, z_1), d_2(y_2, z_2)) = d_1(y_1, z_1)$ (Say)

$d''[(x_1, x_2), (z_1, z_2)] = max\,(d_1(x_1, z_1), d_2(x_2, z_2)) = d_1(x_1, z_1)$ (Say)

Since d_1 is a metric on X_1.

$\therefore$ $d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1)$

$\Rightarrow d''[(x_1, x_2), (y_1, y_2)] + d''[(y_1, y_2), (z_1, z_2)] \geq d''[(x_1, x_2), (z_1, z_2)]$

(We get the same result in the remaining cases.)

$\therefore$ d'' is a metric on $X_1 \times X_2$.

Question

Let (X, d) be a metric space and let $d': X \times X \to R$ be given by $d'(x_1, x_2) = \frac{d(x_1,x_2)}{1 + d(x_1,x_2)}$. Prove that d' is metric on X.

Solution

Here function $d': X \times X \to R$ be defined by

$$d'(x_1, x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}$$

(M$_1$) Since $\frac{d(x_1,x_2)}{1+d(x_1,x_2)} \geq 0$

$\because$ $d(x_1, x_2) \geq 0$

$\because$ d is a metric on X.

$\therefore$ $d'(x_1, x_2) \geq 0$

(M$_2$) Let $d'(x_1, x_2) = 0 \Rightarrow \frac{d(x_1,x_2)}{1+d(x_1,x_2)} = 0$

$\Rightarrow d(x_1, x_2) = 0$

$\Rightarrow x_1 = x_2$ $\quad$ ($\because$ d is a metric on X.)

Let $x_1 = x_2$ $\quad \Rightarrow d(x_1, x_2) = 0$ $\quad$ ($\because$ d is a metric on X.)

$\Rightarrow \frac{d(x_1,x_2)}{1+d(x_1,x_2)} = 0$

$\Rightarrow d'(x_1, x_2) = 0$

Thus $d'(x_1, x_2) = 0 \quad \Leftrightarrow \quad x_1 = x_2$

(M$_3$) Since $d'(x_1, x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}$ $\quad$ $\therefore$ d is metric on x

$$= \frac{d(x_2,x_1)}{1+d(x_2,x_1)}$$

$$= d'(x_2, x_1)$$

(M$_4$) Since $d'(x_1, x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}$

$d'(x_2, x_3) = \frac{d(x_2,x_3)}{1+d(x_2,x_3)}$

$d'(x_1, x_3) = \frac{d(x_1,x_3)}{1+d(x_1,x_3)}$

Now $d'(x_1, x_2) + d'(x_2, x_3) = \frac{d(x_1,x_2)}{1 + d(x_1,x_2)} + \frac{d(x_2,x_3)}{1 + d(x_2,x_3)}$

$$\geq \frac{d(x_1,x_2)}{1 + d(x_1,x_2) + d(x_2,x_3)} + \frac{d(x_2,x_3)}{1 + d(x_1,x_2) + d(x_2,x_3)}$$

$$= \frac{d(x_1,x_2) + d(x_2,x_3)}{1 + d(x_1,x_2) + d(x_2,x_3)}$$

$\therefore$ $d'(x_1, x_2) + d'(x_2, x_3) \geq \frac{d(x_1,x_3)}{1 + d(x_1,x_3)}$ $\quad$ ($\because$ $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$ $\because$ d is a metric on X.)

$$= d'(x_1, x_3)$$

$\therefore$ d' is a metric on X.

Question

Let $X = R$ and $d(x_1, x_2) = |x_1| + |x_2|$. Show that d is not a metric on R.

Solution

$$
\begin{aligned}
\text{Let } d(x_1, x_2) = 0 \quad &\Rightarrow \quad |x_1| + |x_2| = 0 \\
&\Rightarrow \quad |x_1| = 0, \quad |x_2| = 0 \\
&\Rightarrow \quad x_1 = 0, \quad x_2 = 0 \\
&\Rightarrow \quad x_1 = x_2
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } x_1 = x_2 \quad &\Rightarrow \quad |x_1| = |x_2| \\
&\Rightarrow \quad |x_1| + |x_2| = |x_2| + |x_2| \quad (\text{Adding } |x_2| \text{ both sides}) \\
&\Rightarrow \quad d(x_1, x_2) = 2|x_2| \\
&\Rightarrow \quad d(x_1, x_2) = 0 \quad \text{if} \quad |x_2| = 0
\end{aligned}
$$

i.e. $d(x_1, x_2)$ is not always zero.

$\because$ d is not a metric on X.

Question

Let $X = R$ and $d(x_1, x_2) = max\,(|x_1|, |x_2|)$. Show that d is not a metric on R.

Solution

$$
\begin{aligned}
\text{Let } d(x_1, x_2) = 0 \quad &\Rightarrow \quad max\,(|x_1|, |x_2|) = 0 \\
&\Rightarrow \quad |x_1| = 0, \quad |x_2| = 0 \\
&\Rightarrow \quad x_1 = 0, \quad x_2 = 0 \\
&\Rightarrow \quad x_1 = x_2
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } x_1 = x_2 \quad &\Rightarrow \quad |x_1| = |x_2| \\
&\Rightarrow \quad max\,(|x_1|, |x_2|) = |x_2| \\
&\Rightarrow \quad d(x_1, x_2) = 0 \quad \text{if} \quad |x_2| = 0
\end{aligned}
$$

i.e. $d(x_1, x_2)$ is not always zero.

Thus d is not a metric on X.

Question

Let (X, d) be a metric space and let $d'' : X \times X \to R$ be given by $d''(x_1, x_2) = \frac{1 - d(x_1, x_2)}{1 + d(x_1, x_2)}$. Prove that d'' is not a metric on X.

Solution

$$
\begin{aligned}
\text{Let } \quad d''(x_1, x_2) = 0 \quad &\Rightarrow \quad \frac{1 - d(x_1, x_2)}{1 + d(x_1, x_2)} = 0 \\
&\Rightarrow \quad 1 - d(x_1, x_2) = 0 \\
&\Rightarrow \quad d(x_1, x_2) = 1 \\
&\Rightarrow \quad x_1 \neq x_2
\end{aligned}
$$

$\because$ d is a metric on X and $d(x_1, x_2) = 0 \;\Leftrightarrow\; x_1 = x_2$

Thus $d''(x_1, x_2) = 0 \;\not\Rightarrow\; x_1 = x_2$

Thus d'' is not a metric on X.

S

OPEN SPHERE

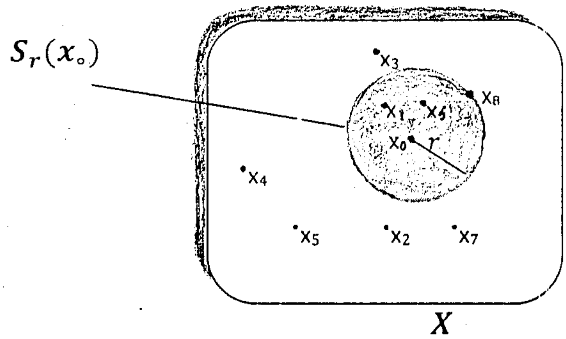
Open sphere

Let (X, d) be a metric space. Let $x_\circ \in X$ and $r > 0$, then open sphere with centre at $x_\circ$ and radius equal to r is denoted by $S_r(x_\circ)$ and is defined as $\quad S_r(x_\circ) = \{\, x | x \in X,\ d(x, x_\circ) < r \,\}$

Note

(i) Let $X = \{\, x_\circ, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \,\}$ and $r > 0$

$S_r(x_\circ) = ?$



Then by definition of open sphere $S_r(x_\circ) = \{\, x_\circ, x_1, x_6 \,\}$

(ii) $S_r(x_\circ) \subseteq X$

(iii) $S_r(x_\circ) \neq \phi$

(iv) Here we shall study the open spheres of the following shapes.

(a) Open interval (b) Open disc (c) Open ball

The shape of an open sphere depends upon the metric space (X, d).

Example

Let R be the metric space. Let $x_0 = 1,\ r = \frac{1}{2}$. Find $S_{\frac{1}{2}}(1)$.

Solution

Here metric space is (R, d), where metric $d: R \times R \to R$ is defined as $d(x_1, x_2) = |x_1 - x_2|$

We know that

$S_r(x_\circ) = \{\, x | x \in X,\ d(x, x_\circ) < r \,\}$

Put $X = R, \quad x_\circ = 1, \quad r = \frac{1}{2}$

$$\therefore\ S_{\frac{1}{2}}(1) = \left\{ x | x \in R,\ d(x, 1) < \tfrac{1}{2} \right\}$$

$$= \left\{ x | x \in R,\ |x - 1| < \tfrac{1}{2} \right\}$$

$|x-1| < 1/2$

$$= \left\{ x | x \in R,\ x - 1 < \tfrac{1}{2},\ x - 1 > -\tfrac{1}{2} \right\}$$

$\pm(x-1) < \frac{1}{2}$

$$= \left\{ x | x \in R,\ x < 1 + \tfrac{1}{2},\ x > 1 - \tfrac{1}{2} \right\}$$

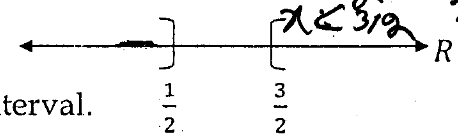
$x - 1 < \frac{1}{2}$ or $-(x-1) < \frac{1}{2}$

$x < 1 + 1/2$ or $x - 1 > -1/2$

$$= \left\{ x | x \in R,\ \tfrac{1}{2} < x < \tfrac{3}{2} \right\}$$

$x < 3/2$ $\quad x > 1/2$

$$= \left] \tfrac{1}{2}, \tfrac{3}{2} \right[$$



Open sphere in this case is an open interval.

K.S

Note

An open sphere in a usual metric space R is always an "open interval".

Example

Let the metric space be R^2 and let $P_\circ = (a, b)$ and $r = 1$. Find $S_r(P_\circ)$.

Solution

Here metric space is (R, d), where metric $d: R \times R \to R$ is defined as $d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

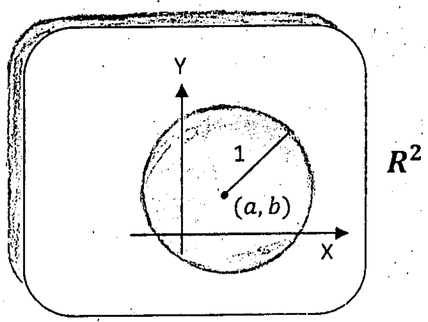
We know that

$$S_r(P_\circ) = \{ P | P \in X,\ d(P, P_\circ) < r \}$$ k.S

Put $X = R^2,\quad P_\circ = (a, b),\quad P = (x, y),\quad r = 1$

$$\therefore\ S_1(a, b) = \{ (x, y) | (x, y) \in R^2,\ d((x, y), (a, b)) < 1 \}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ \sqrt{(x - a)^2 + (y - b)^2} < 1 \right\}$$
$$= \{ (x, y) | (x, y) \in R^2,\ (x - a)^2 + (y - b)^2 < 1 \}$$

This is an open disc with centre at (a, b) and radius 1.



Note

An open sphere in a usual metric space R^2 is always an "open disc".

Example

Let the metric space be R^2 and d_1 be the metric on R^2 defined by
$$d_1(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|.$$
Let $P_\circ = (0, 0)$ and $r = \frac{1}{\sqrt{2}}$. Find $S_r(x_\circ)$.

Solution

Here metric space is (R^2, d_1), where metric $d_1: R^2 \times R^2 \to R$ is defined as $d_1[(x_1, y_1), (x_2, y_2)] = |x_1 - x_2| + |x_1 - x_2|$

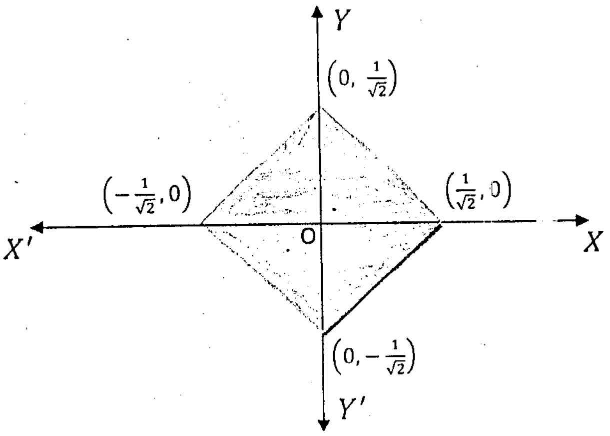
We know that

$$S_r(P_\circ) = \{ P | P \in X,\ d(P, P_\circ) < r \}$$ k.S

Put $X = R^2,\quad P_\circ = (0,0),\quad P = (x, y),\quad r = \frac{1}{\sqrt{2}}$

$$\therefore\ S_{\frac{1}{\sqrt{2}}}(0,0) = \left\{ (x, y) | (x, y) \in R^2,\ d_1((x, y), (0,0)) < \tfrac{1}{\sqrt{2}} \right\}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ |x - 0| + |y - 0| < \tfrac{1}{\sqrt{2}} \right\}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ |x| + |y| < \tfrac{1}{\sqrt{2}} \right\}$$ — End which is required open sphere.
$$= \left\{ (x, y) | (x, y) \in R^2,\ \pm x \pm y < \tfrac{1}{\sqrt{2}} \right\}$$
$$= \left\{ (x, y) | (x, y) \in R^2,\ \frac{x}{\pm\frac{1}{\sqrt{2}}} \pm \frac{y}{\pm\frac{1}{\sqrt{2}}} < 1 \right\}$$

This is an open square with x-intercepts $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ and y-intercepts $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$



Example

Let $(X, d_\circ)$ be a discrete metric space. Let $x_\circ \in X$ and $r > 0$

Find $S_r(x_\circ)$, when (i) $r \leq 1$ (ii) $r > 1$

Solution

Here metric space is $(X, d_\circ)$, where $d_\circ : X \times X \to R$ is defined by

$$d_\circ(x_1, x_2) = \begin{cases} 0 & if\ x_1 = x_2 \\ 1 & if\ x_1 \neq x_2 \end{cases}$$

We know that

$$S_r(x_\circ) = \{ x | x \in X,\ d(x, x_\circ) < r \} \quad \text{---------} \quad (1)$$

1, 0

When $r \leq 1$

If $x \neq x_\circ$ then from equation (1) we get $1 < r$ (False)

If $x = x_\circ$ then from equation (1) we get $0 < r$ (True)

Thus $S_r(x_\circ) = \{ x | x \in X,\ x = x_\circ \} = \{ x_\circ \}$

When $r > 1$

If $x \neq x_\circ$ then from equation (1) we get $1 < r$ (True)

If $x = x_\circ$ then from equation (1) we get $0 < r$ (True)

Thus $S_r(x_\circ) = \{ x | x \in X,\ x = x_\circ\ or\ x \neq x_\circ \}$

$= \{ x | x \in X,\ x = x_\circ \} \cup \{ x | x \in X,\ x \neq x_\circ \}$

$= \{ x_\circ \} \cup X - \{ x_\circ \}$

$= X$

Note

From above example we conclude that

(i) An open sphere with radius less than or equal to 1 in a discrete metric space is always singleton.

(ii) An open sphere with radius greater than 1 in a discrete metric space is always the full space X.

Question

Let C be the set of all complex numbers and let $d: C \times C \rightarrow R$ be defined by $d(z_1, z_2) = |z_1 - z_2|$. Find $S_r(x_\circ)$ when $x_\circ = 1,\ r = 0.01$

Solution

The given metric space is (C, d), where $d: C \times C \rightarrow R$ be defined by

$$d(z_1, z_2) = |z_1 - z_2|$$

Now $\quad S_r(x_\circ) = \{\, x | x \in X,\ \ d(x, x_\circ) < r \,\}$

Put $X = C, \quad x_\circ = 1 \quad r = 0.01$

$\therefore\ S_{0.01}(1) = \{\, x | x \in C,\ \ d(x, 1) < 0.01\}$

$= \{\, x | x \in C,\ \ |x - 1| < 0.01\}$ ——— (1) $|x-1| < 0.01$

$\pm(x-1) < 0.01$

Since $x \in C \quad \therefore\ x = a + ib$ $\quad$ 27 $\quad$ $x-1 < 0.01;\ -(x-1) < 0.01$

$\Rightarrow \quad x - 1 = a + ib - 1$ $\quad$ $x < 1+0.01;\ x-1 > -0.01$

$\Rightarrow \quad x - 1 = (a - 1) + ib$ $\quad$ $x < 1.01;\ x > 0.99$

$\Rightarrow \quad |x - 1| = \sqrt{(a-1)^2 + b^2}$ $\quad$ $]0.99, 1.01[$

S_0

$\therefore (1) \Rightarrow S_{0.01}(1) = \left\{\, (a + ib) | (a + ib) \in C,\ \sqrt{(a-1)^2 + b^2} < 0.01 \right\}$

$= \{(a + ib) | (a + ib) \in C, (a - 1)^2 + (b - 0)^2 < (.01)^2\}$

This is an open disc with centre at (1,0) and radius equal to 0.01.

Question

Let d be a metric on X and let $d': X \times X \rightarrow R$ be given by $d'(x_1, x_2) = min\,(1, d(x_1, x_2))$. Describe $S_r(x_\circ)$.

Solution

Here given metric space is (X, d'), where $d': X \times X \rightarrow R$ be given by

$$d'(x_1, x_2) = min\,(1, d(x_1, x_2))$$

Now $\quad S_r(x_\circ) = \{\, x | x \in X,\ \ d'(x, x_\circ) < r \}$

$= \{\, x | x \in X,\ \ min\,(1, d(x_1, x_2)) < r \}$

This is the required open sphere.

Question

Let (X, d) be a metric space and let $d': X \times X \rightarrow R$ be given by $d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$. Describe $S_r(x_\circ)$.

Solution

Here given metric space is (X, d'), where $d': X \times X \rightarrow R$ be given by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$$

Now $\quad S_r(x_\circ) = \{\, x | x \in X,\ \ d'(x, x_\circ) < r \,\}$

$= \left\{\, x | x \in X,\ \ \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} < r \right\}$

This is the required open sphere.

A11

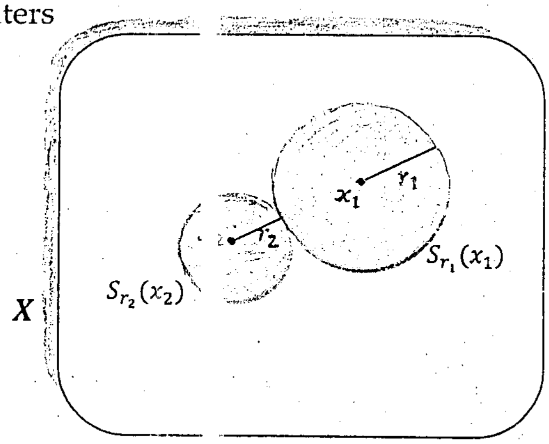
Theorem

Let x_1, x_2 be any two distinct points of a metric space X. Prove that there exist two open spheres $S_{r_1}(x_1)$ and $S_{r_2}(x_2)$ in X such that

$$S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$$

Proof

Let $S_{r_1}(x_1)$ and $S_{r_2}(x_2)$ be two open spheres with centers x_1 and x_2 and radii r_1 and r_2 respectively.



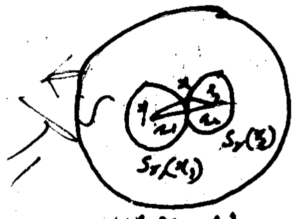
Let $d(x_1, x_2) = r_1 + r_2$

We are to prove that

$$S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$$

We shall prove it by contradiction method.

Suppose $\quad S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \phi$



Let $\quad x \in S_{r_1}(x_1) \cap S_{r_2}(x_2)$

$\Rightarrow \quad x \in S_{r_1}(x_1) \quad$ and $\quad x \in S_{r_2}(x_2)$

$\Rightarrow \quad d(x, x_1) < r_1 \quad$ and $\quad d(x, x_2) < r_2 \quad$ ———— (1) (X, d)

Since $\quad r_1 + r_2 = d(x_1, x_2) \leq d(x_1, x) + d(x, x_2)$

$\because \ d$ is a metric on X.

$\Rightarrow \quad r_1 + r_2 \leq d(x_1, x) + d(x, x_2)$

$\Rightarrow \quad r_1 + r_2 < r_1 + r_2 \qquad$ [By (1)]

It is impossible.

Thus our supposition $S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \phi$ is wrong.

Hence $\ S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$

Let x_1 and x_2 be any two pts of a metric space X
Let $S_r(x_1)$ and $S_r(x_2)$ be any open spheres in X
To prove $S_r(x_1) \cap S_r(x_2) = \phi$
Let $x \in S_r(x_1) \cap S_r(x_2)$
Thus by triangle inequality
Then $d(x_1, x_2) \leq d(x_1, x) + d(x, x_2)$
$r_1 + r_2 \leq r_1 + r_2$
$\because d(x_1, x_2) = r_1 + r_2$ — (1)
It is impossible.
Thus our supposition $S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \phi$ is wrong

$S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$

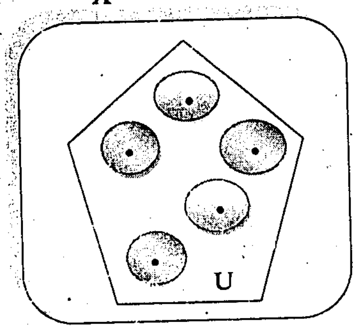
A subset U of a metric space (X, d) is called an open set if for every $x \in U$ there exists a real number $r > 0$ such that $x \in S_r(x) \subseteq U$

OPEN SET

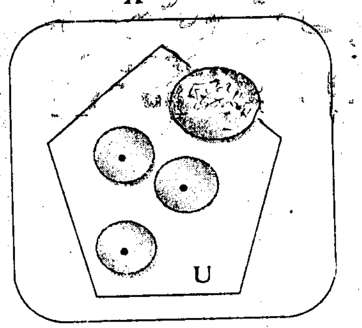
Open Set

Let (X, d) be a metric space. Let $U \subseteq X$. The U is called an open set, if for each $x \in U, \exists\, r > 0$, such that $S_r(x) \subseteq U$.

i.e. U is called an open set, if each point of U is the centre of some open sphere, which is contained in U.



U is an open set.



U is not an open set.

Example

Let R be a usual metric space (The ordinary real number line) and let U=$]0,1[$, then show that U is open.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Let $x_\circ \in U$, Let $r > 0$

Then $S_r(x_\circ) = \{x \mid x \in R,\ d(x, x_\circ) < r\}$

$= \{x \mid x \in R,\ |x - x_\circ| < r\}$

$= \{x \mid x \in R,\ x - x_\circ < r,\ x - x_\circ > -r\}$

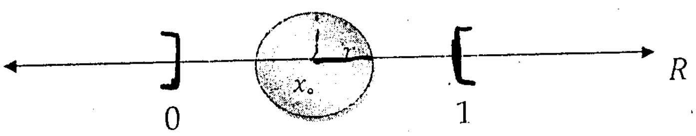
$= \{x \mid x \in R,\ x < x_\circ + r,\ x > x_\circ - r\}$

$= \{x \mid x \in R,\ x_\circ - r < x < x_\circ + r\}$

$=]x_\circ - r,\ x_\circ + r[$

We can find a value of r for which $S_r(x_\circ) =]x_\circ - r, x_\circ + r[\subseteq U =]0,1[$

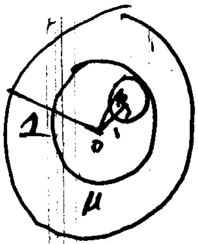
Thus U=$]0,1[$ is an open set.



Note

In the above example if we take $x_\circ = 0.99$. Let $r = 0.001$

Then $S_{0.001}(0.99) =]0.99 - 0.001, 0.99 + 0.001[=]0.981, 0.991[\subseteq]0,1[$



Example

Let R^2 be a usual metric space (The ordinary real plane)

Let U= $\{(x,y)|(x,y) \in R^2, x^2+y^2 < 1\}$. Show that U is an open set.

(X, d)

Solution

Let (X,d) be a metric space

Let $U = \{P|P \in R^2, x^2+y^2<1\}$

To prove U is openset

Let $P_0 \in U$

construction open sphere centre at P_0 having radius equal to r.

Take $P \in S_r(P_0)$ such that $d(P,P_0)<r$ from the pt 0 at P

This in this way we get a vector triangle now from triangle inequality

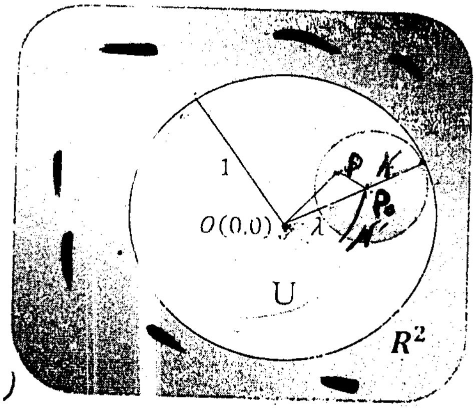
Here metric space is (R^2, d), where $d: R^2 \times R^2 \to R$ is given by

$$d[(x_1,y_1),(x_2,y_2)] = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Let $P_\circ \in$ U.

Let $d(O,P_\circ) = \lambda$

Let $r = 1-\lambda$, then $r>0$



We shall prove that $S_r(P_\circ) \subseteq$ U

Let $P \in S_r(P_\circ) \Rightarrow d(P,P_\circ) < r$

Since (R^2, d) is a metric space,

$\therefore \quad d(P,P_\circ) + d(P_\circ,O) \geq d(P,O)$

$\Rightarrow \quad r+\lambda > d(P,O) \qquad \because r > d(P,P_0)$

$\Rightarrow \quad 1-\lambda+\lambda > d(P,O) \qquad \because r = 1-\lambda$

$\Rightarrow \quad d(P,O) < 1$

$\Rightarrow \quad P \in$ U

Since $P \in S_r(P_\circ) \Rightarrow P \in$ U

$\therefore \quad S_r(P_\circ) \subseteq$ U

Hence U is an open set.

$K=S$ P_3

Example

Let R be a usual metric space (The ordinary real number line) and let U= $\{x|x \in R,\ 0 \leq x < 1\}$, then show that U is not open.

Solution

Here metric space is (R,d), where $d: R \times R \to R$ is given by

$d(x_1,x_2) = |x_1 - x_2|$

$d(0,P) \leq d(0,P_0) + d(P_0,P)$

$d(0,P_0) = \lambda$ — ①

$d(P_0,P) < r$ — ②

$r+\lambda = 1$

Here U= $\{x|x \in R,\ 0 \leq x < 1\}$

$= [0,1[$

$d(0,P) \leq r+\lambda$

Let $x_\circ = 0 \in$ U, Let $r>0$

Then $S_r(0) = \{x|x \in R,\ d(x,0) < r\}$

$= \{x|x \in R,\ |x-0| < r\}$

$= \{x|x \in R,\ |x| < r\}$

$= \{x|x \in R,\ x < r,\ x > -r\}$

$= \{x|x \in R,\ -r < x < r\}$

$=]-r, +r[$

$d(0,P) \leq 1$

$d(0,P) < 1$

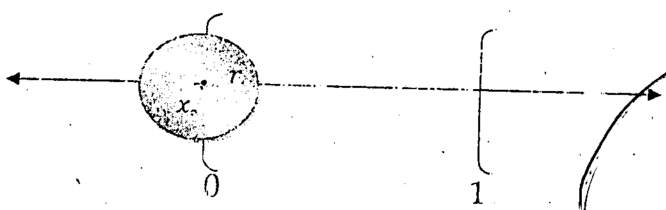
$P \in U$

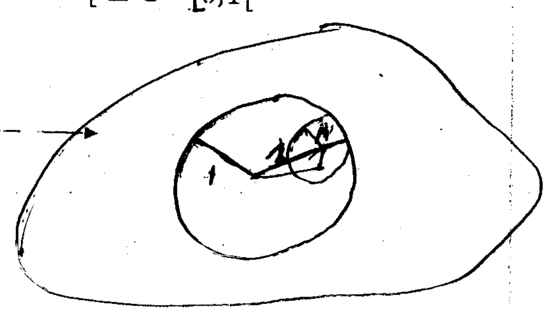
We can find a value of r for which $S_r(0) =]-r\ \ +r[\not\subseteq$ U$=[0,1[$

Thus U=$[0,1[$ is not an open set.

$P \in S_r(P_0)$

$S_r(P_0) \subseteq U$





$= 1-\lambda$

Theorem

Every non-empty subset of a discrete metric space is open.

Proof

Let $(X, d_\circ)$ be a discrete metric space.

Let $U \subseteq X$ such that $U \neq \phi$

We shall prove that U is an open set.

Let $x_\circ \in U$.

Let $0 < r < 1$

Then $S_r(x_\circ) = \{x | x \in X,\ d(x, x_\circ) < r\}$

$= \{x_\circ\}$.

$\therefore$ The open sphere in a discrete metric space, whose radius is less then 1, is always singleton. take centre

Since $S_r(x_\circ) = \{x_\circ\} \subseteq U$ Centre point

$\Rightarrow$ U is an open set.

Example

Let R be a usual metric space (The ordinary real number line) and let U= { 0 }, then show that U is not open.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ is given by

$$d(x_1 . x_2) = |x_1 - x_2|$$

Here $U = \{0\}$

Let $x_\circ = 0 \in U$, Let $r > 0$

Then $S_r(0) = \{x | x \in R,\ d(x, 0) < r\}$

$= \{x | x \in R,\ |x - 0| < r\}$

$= \{x | x \in R,\ |x| < r\}$

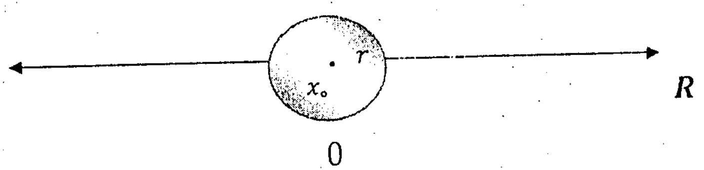
$= \{x | x \in R,\ x < r,\ x > -r\}$

$= \{x | x \in R,\ -r < x < r\}$

$=]-r, +r[$

We can find a value of r for which $S_r(0) =]-r, +r[\not\subseteq U = \{0\}$

Thus U= { 0 } is not an open set.



Theorem

Let (X, d) be a metric space, then

(i) Union of any collection $\{ U_\alpha : \alpha \in I \}$ of open sets is open.

(ii) Intersection of finite number of open sets is open.

(iii) The Whole space X and the empty set ϕ are both open.

Proof

(i) Let $\{ U_\alpha : \alpha \in I \}$ be any collection of open sets in (X, d).

We are to prove that, $\bigcup_{\alpha \in I} U_\alpha$ is an open set.

Let $x \in \bigcup_{\alpha \in I} U_\alpha$

Then $x \in U_\alpha$ for some $\alpha \in I$

Since each U_α is an open set therefore there exist $r > 0$

Such that $S_r(x) \subseteq U_\alpha$ for some $\alpha \in I$

$\Rightarrow S_r(x) \subseteq \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is an open set.

(ii) Let $\{ U_\alpha : \alpha = 1, 2, \ldots \ldots n \}$ be finite collection of open sets in (X, d).

We are to prove that $\bigcap_{\alpha = 1}^{n} U_\alpha$ is an open set.

Let $x \in \bigcap_{\alpha = 1}^{n} U_\alpha$

$\Rightarrow x \in U_\alpha \quad \forall \alpha = 1, 2, \ldots \ldots n$

Since each U_α is an open set therefore there exist $r > 0$

Such that $S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall \alpha = 1, 2, \ldots \ldots n$

Let $r = min \{ r_1, r_2, r_3, \ldots \ldots r_n \}$

Then $S_r(x) \subseteq S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall \alpha = 1, 2, \ldots \ldots n$

$\Rightarrow S_r(x) \subseteq U_\alpha \quad \forall \alpha = 1, 2, \ldots \ldots n$

$\Rightarrow S_r(x) \subseteq \bigcap_{\alpha = 1}^{n} U_\alpha$

$\Rightarrow \bigcap_{\alpha = 1}^{n} U_\alpha$ is an open set.

(iii) To show that empty set ϕ is an open set, we have to show that each point in ϕ is the centre of some open sphere which is contained in ϕ. But since there is no point in ϕ, the condition is automatically satisfied.

Hence ϕ is an open set.

Since every open sphere centered at a point of X is contained in X.

$\therefore$ X is an open set.

S11

Theorem M

An open sphere in a metric space (X, d) is an open set.

Proof

Let $S_r(x_\circ)$ be an open sphere in (X, d).

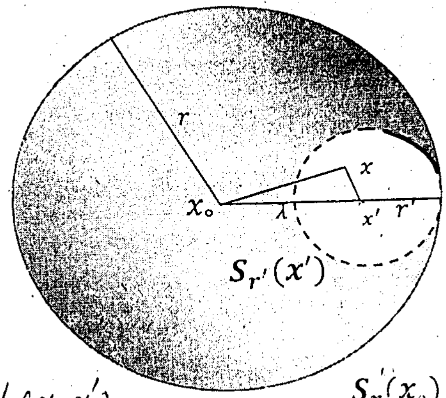
Let $x' \in S_r(x_\circ) \Rightarrow d(x', x_\circ) < r$

Let $d(x', x_\circ) = \lambda$

Let $r' = r - \lambda$, then $r' > 0$

We shall prove that $S_{r'}(x') \subseteq S_r(x_\circ)$

Let $x \in S_{r'}(x') \Rightarrow d(x, x') < r'$



$S_r(x_\circ)$

Since (X, d) is a metric space,

$\therefore \quad d(x, x') + d(x', x_\circ) \geq d(x, x_\circ)$

$\Rightarrow \quad r' + \lambda > d(x, x_\circ) \qquad \because \lambda' > d(x, x')$

$\Rightarrow \quad r - \lambda + \lambda > d(x, x_\circ) \qquad \because r' = r - \lambda$

$\Rightarrow \quad d(x, x_\circ) < r$

$\Rightarrow \quad x \in S_r(x_\circ)$

33 K 8

Since $x \in S_{r'}(x') \Rightarrow x \in S_r(x_\circ)$

$\therefore \quad S_{r'}(x') \subseteq S_r(x_\circ)$

Thus $S_r(x_\circ)$ is an open set.

Hence open sphere in a metric space is an open set.

S13

Theorem

A subset U of a metric space X is open if and only if U is union of open spheres.

Proof

Let (X, d) be a metric space. Let $U \subseteq X$. We have to prove that

U is an open set $\Leftrightarrow$ U is the union of open spheres.

We suppose that U is an open set. Since U is open therefore each point of U is the centre of some open sphere which is contained in U. Thus U is the union of open spheres.

Conversely suppose that U is the union of open spheres. Thus U is the union of open sets. ($\because$ Open spheres in metric space are open sets.)

Since the union of any number of open sets in a metric space is an open set. Thus U is an open set.

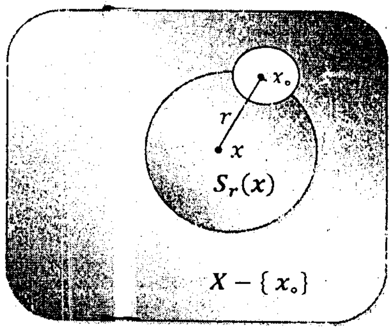
Theorem

Let X be a metric space and let $\{x_\circ\}$ be a singleton subset of X. Then $X - \{x_\circ\}$ is open.

Proof

Let $x \in X - \{x_\circ\}$

Let $d(x, x_\circ) = r$ --------- (1)



We shall prove that

$S_r(x) \subseteq X - \{x_\circ\}$

Let $x' \in S_r(x)$

$\Rightarrow d(x', x) < r$ ---------- (2)

From (1) and (2) we get

$\Rightarrow d(x', x) \neq d(x, x_\circ)$

$\Rightarrow d(x, x') \neq d(x, x_\circ)$ [$\because$ d is a metric on X. So $d(x', x) = d(x, x')$]

$\Rightarrow x' \neq x_\circ$

$\Rightarrow x' \notin \{x_\circ\}$

$\Rightarrow x' \in X - \{x_\circ\}$

Since $x' \in S_r(x) \Rightarrow x' \in X - \{x_\circ\}$

$\therefore S_r(x) \subseteq X - \{x_\circ\}$

Since every point x of $X - \{x_\circ\}$ is the centre of some open sphere contained in $X - \{x_\circ\}$.

Hence $X - \{x_\circ\}$ is an open set.

Question

Can a finite subset of a metric space be open?

Solution

We know that

(i) If $(X, d_\circ)$ is a discrete metric space, then every subset of X is open. Therefore a finite subset of a metric space is open.

(ii) If (R, d) is a usual metric space then $\{0\} \subseteq R$, is not open. Therefore a finite subset $\{0\}$ of R is not open.

Thus in general, we can say that, finite subset of a metric space may or may not open.

Metric Topology

The topology determined by a metric is called "metric topology".

Check in Past Paper

Theorem

If T is a collection of all open sets in a metric space (X, d), then T is a topology on X.

OR

A "metric space" is a topological space.

Proof

Let T be the collection of all open sets in a metric space (X, d).
We are to prove that, T is a topology on X.

(T_1) Let $U_\alpha \in T \quad \forall\ \alpha \in I$

$\Rightarrow \quad U_\alpha$ is an open set. $\quad \forall\ \alpha \in I$

$\Rightarrow \quad \bigcup_{\alpha \in I} U_\alpha$ is open.

($\because$ Union of any number of open sets is open.)

$\Rightarrow \quad \bigcup_{\alpha \in I} U_\alpha \in T$

(T_2) Let $U_\alpha \in T \quad \forall\ \alpha = 1, 2, \ldots n$

$\Rightarrow \quad \bigcap_{\alpha = 1}^{n} U_\alpha$ is an open set.

($\because$ Intersection of finite number of open sets is open.)

$\Rightarrow \quad \bigcap_{\alpha = 1}^{n} U_\alpha \in T \qquad$ (By definition of T)

(T_3) Since ϕ, X both are open.

$\therefore \quad \phi, X \in T \qquad$ (By definition of T)

Thus T is a topology on X.

i.e. (X, T) is a topological space.

This shows that a "metric space" is a "topological space" whose topology is "metric topology".

Theorem

Every non-empty set can be given a metric topology.

Proof

We know that

(i) Every non-empty set can be given a metric and can be converted into metric space.

~~Therefore a finite subset of a metric space is open.~~

(ii) Every "metric space" is a "topological space" whose topology is a "metric topology".

Thus from (i) and (ii) we conclude that every non-empty set can be given a metric topology.

CLOSED SET

Closed Set

Let (X, d) be a metric space. Let $F \subseteq X$.

Then F is closed $\Leftrightarrow$ $F' = X - F$ is open

complement

Example

Let $X = R$ be the metric space and let $A = [a, b]$, where $a, b \in R$, & $a < b$. Show that A is closed set.

Solution

The given metric space is (R, d), where $d: R \times R \to R$ is given by

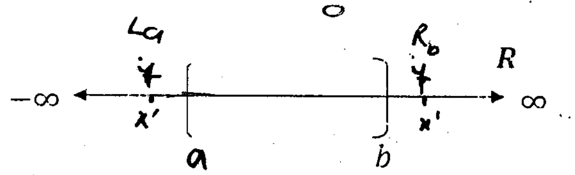
$$d(x_1, x_2) = |x_1 - x_2|$$

Since $A = [a, b]$

$A' = R - [a, b]$

$=]-\infty, a[\cup]b, \infty[$

$= L_a \cup R_b$



In order to prove A is closed; we will have to prove that A' is open.

Let $x' \in A'$ $\Rightarrow$ $x' \in L_a \cup R_b$

$\Rightarrow$ $x' \in L_a$ or $x' \in R_b$

Case-I If $x' \in L_a$ then $x' < a$

Let $d(x', a) = r$

$\Rightarrow$ $|x' - a| = r$

$\Rightarrow$ $x' - a = -r$ $\because$ $x' < a$

$\Rightarrow$ $x' + r = a$ ------------------------------ (1)

Now $S_r(x') = \{ x | x \in R,\ d(x, x') < r \}$

$= \{ x | x \in R,\ |x - x'| < r \}$

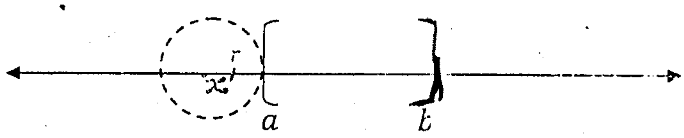
$= \{ x | x \in R,\ x - x' < r,\ x - x' > -r \}$

$= \{ x | x \in R,\ x < x' + r,\ x > x' - r \}$

$= \{ x | x \in R,\ x' - r < x < x' + r \}$

$=]x' - r,\ x' + r[$

$=]x' - r,\ a[$ [By (1)]



Thus $S_r(x') =]x' - r,\ a[\subseteq L_a \subseteq L_a \cup R_b = A'$

i.e. $S_r(x') \subseteq A'$

Hence in this case A' is open.

Case-II If $x' \in R_b$ then $x' > b$

Let $d(x', b) = r$

$\Rightarrow \quad |x' - b| = r$

$\Rightarrow \quad x' - b = r \quad \because x' > b$

$\Rightarrow \quad x' - r = b$ ---------------------------------- (2)

Now $\quad S_r(x') = \{ x | x \in R,\ d(x, x') < r \}$

$= \{ x | x \in R,\ |x - x'| < r \}$

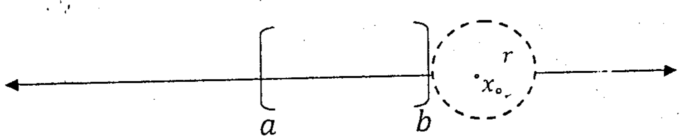
$= \{ x | x \in R,\ x - x' < r,\ x - x' > -r \}$

$= \{ x | x \in R,\ x < x' + r,\ x > x' - r \}$

$= \{ x | x \in R,\ x' - r < x < x' + r \}$

$=]\, x' - r,\ x' + r\, [$

$=]\, b,\ x' + r\, [\qquad$ [By (2)]



Thus $\quad S_r(x') =]\, b,\ x' + r\, [\subseteq R_b \subseteq L_a \cup R_b = A'$

i.e. $\quad S_r(x') \subseteq A'$.

Hence in this case A' is also open.

Since in both the cases A' is open. Therefore A is closed set.

Example

Let R^2 be the metric space.

Let $F = \{ (x, y) | (x, y) \in R^2,\ (x - a)^2 + (y - b)^2 \leq 1 \}$.

Show that F is closed set.

Solution

Here given metric space is (R^2, d) where $d : R^2 \times R^2 \to R$ is given by $\quad d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Here $\quad F = \{ (x, y) | (x, y) \in R^2,\ (x - a)^2 + (y - b)^2 \leq 1 \}$

Thus $\quad F' = \{ (x, y) | (x, y) \in R^2,\ (x - a)^2 + (y - b)^2 > 1 \}$

In order to prove that F is closed, we will show that F' is open.

Let $P' \in F'$. Let $d(P', P_0) = \lambda$

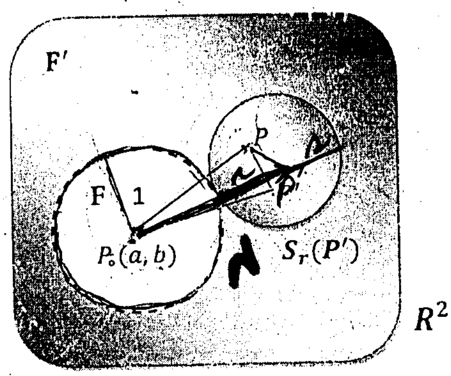
Let $r = \lambda - 1$, clearly $r > 0$

We shall prove that $S_r(P') \subseteq F'$

Let $P \in S_r(P') \Rightarrow d(P, P') < r$

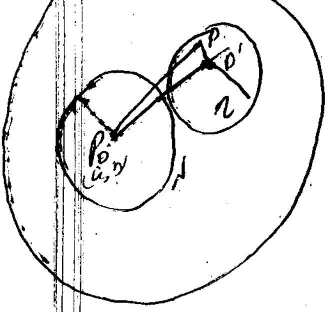
Since d is a metric on R^2

$\therefore\ d(P', P) + d(P, P_0) \geq d(P', P_0)$



let $P \in F'$

$\Rightarrow \quad r + d(P,P_\circ) > \lambda$ $\quad \because \lambda > d(P,P')$ construct an open sphere centred at P and having radius r such that $d(P_0,P') = \lambda$



$\Rightarrow \quad d(P,P_\circ) > \lambda - r$

$\Rightarrow \quad d(P,P_\circ) > \underline{\lambda - (\lambda - 1) = 1}$ $\quad \lambda-(\lambda-1)$, $\lambda-\lambda+1=1$

$\Rightarrow \quad d(P,P_\circ) > 1$

$\Rightarrow \quad P \in F'$

Since $P \in S_r(P') \Rightarrow P \in F'$ $\quad$ take a pt $P' \in S_r(P)$ such that $d(P,P') < r$

$\therefore \quad S_r(P') \subseteq F'$

$\Rightarrow \quad F'$ is an open set. $\quad$ Join the point P_0, with P & P'

$\Rightarrow \quad F$ is closed set. $\quad$ In this way we get vector triangle P_0PP'

Example

Let R^2 be the metric space. $\quad$ From triangle inequality

Let $A = \{ (x,y) | (x,y) \in R^2,\ x^2 + y^2 \leq 1 \}$ be a subset of R^2.

Is A a closed set in R^2? $\quad d(P_0,P) \leq d(P_0,P') + d(P',P)$

Solution

$d(P_0,P) \leq \lambda + r$

Here given metric space is (R^2, d) where $d: R^2 \times R^2 \to R$ is given by $\quad d[(x_1,y_1),(x_2,y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Here $\quad A = \{ (x,y) | (x,y) \in R^2,\ x^2 + y^2 \leq 1 \}$ $\quad$ As $r = \lambda - 1$

Thus $\quad A' = \{ (x,y) | (x,y) \in R^2,\ x^2 + y^2 > 1 \}$ $\quad d(P_0,P) \leq \lambda + (\lambda - 1)$

In order to prove that A is closed, we will show that A is open.

$d(P_0,P) \leq 1$

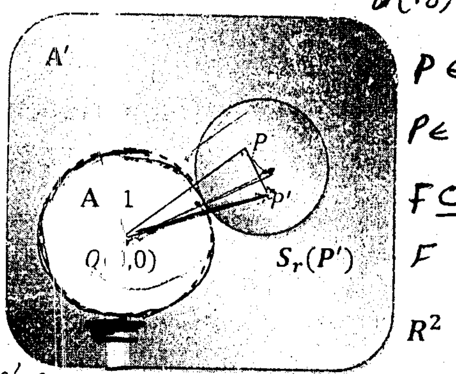
Let $P' \in A'$. Let $d(P',O) = \lambda$

Let $r = \lambda - 1$, clearly $r > 0$ $\quad P \in F'$

We shall prove that $S_r(P') \subseteq A'$ $\quad P \in S_r(P')$

Let $P \in S_r(P') \Rightarrow d(P,P') < r$ $\quad F \subseteq S_r(P')$

Since d is a metric on R^2 $\quad F$ is closed



$\therefore \ d(P',P) + d(P,O) \geq d(P',O)$

$\Rightarrow \quad r + d(P,O) > \lambda$ $\quad \because \lambda > d(P',P)$

$\Rightarrow \quad d(P,O) > \lambda - r$

$\Rightarrow \quad d(P,O) > \lambda - (\lambda - 1) = 1$

$\Rightarrow \quad d(P,O) > 1$

$\Rightarrow \quad P \in A'$

Since $P \in S_r(P') \Rightarrow P \in A'$

$\therefore \quad S_r(P') \subseteq A'$

$\Rightarrow \quad A'$ is an open set.

$\Rightarrow \quad A$ is closed set.

Example

Let R be the real line and let $A = \{ x | x \in R,\ 0 \le x < 1 \}$, be a subset of R. Show that A is not closed.

Solution

The given metric space is (R, d), where $d: R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $\quad A = \{ x | x \in R,\ 0 \le x < 1 \}$

$\quad\quad = [0,1[$

$\therefore \quad A' = R - A$

$\quad\quad =]-\infty, 0 [\cup [1, \infty [$

Note that, $1 \in A'$. We take $x_\circ = 1$, and $r > 0$

Then $S_r(x_\circ) = \{ x | x \in X,\ d(x, x_\circ) < r \}$

Put $x_\circ = 1$ and $X = R$

$S_r(1) = \{ x | x \in R,\ d(x, 1) < r \}$

$= \{ x | x \in R,\ |x - 1| < r \}$

$= \{ x | x \in R,\ x - 1 < r,\ x - 1 > -r \}$

$= \{ x | x \in R,\ x < 1 + r,\ x > 1 - r \}$

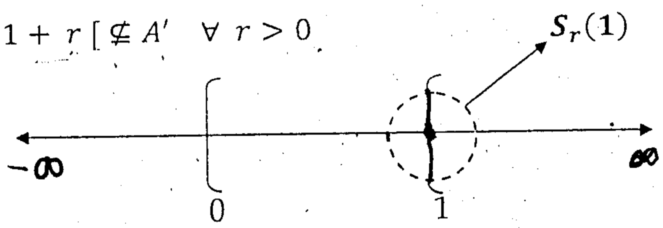
$= \{ x | x \in R,\ 1 - r < x < 1 + r \}$

$=] 1 - r, 1 + r [$

But $S_r(1) =] 1 - r, 1 + r [\not\subseteq A' \quad \forall\ r > 0$

Thus A' is not open.

$\Rightarrow \quad A$ is not closed.



Theorem Imp

A subset U of a metric space is open if and only if $X - U$ is closed.
A subset F ″ ″ ″ ″ ″ ″ F′ closed

Proof

Let (X, d) be a metric space. We have to prove that

U is open $\Leftrightarrow$ $X - U$ is closed.

Suppose U is an open set.

Then $\quad (X - U)' = (U')' \quad\quad \because X - U = U'$

$\quad\quad = U \quad\quad$ (Open set)

Since $(X - U)'$ is an open set.

$\therefore \quad X - U$ is a closed set.

Conversely suppose that $X - U$ is a closed set.

Then $(X - U)'$ is an open set.

$\Rightarrow \quad (U')'$ is an Open set. $\quad \because \quad X - U = U'$

$\Rightarrow \quad U$ is an Open set.

Theorem

Let X be a metric space.

(i) Intersection of any collection $\{F_\alpha : \alpha \in I\}$ of closed sets is closed.

(ii) Union of finite collection $\{F_1, F_2, \ldots\ldots, F_n\}$ of closed set is closed.

(iii) X and ϕ are closed.

Proof

(i) Let $\{F_\alpha : \alpha \in I\}$ be any collection of closed sets in (X, d).

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

Then $F_\alpha{}'$ is open. $\forall\ \alpha \in I$

$\Rightarrow \bigcup_{\alpha \in I} F_\alpha{}'$ is open. ($\because$ Union of any number of open sets is open)

$\Rightarrow \left(\bigcap_{\alpha \in I} F_\alpha\right)'$ is open. $\because \bigcup_{\alpha \in I} F_\alpha{}' = \left(\bigcap_{\alpha \in I} F_\alpha\right)'$

$\Rightarrow \bigcap_{\alpha \in I} F_\alpha$ is closed. $\bigcap_{\alpha \in I} F_\alpha = F_1 \cap F_2 \cap F_3 \cdots\cdots$

(ii) Let $\{F_\alpha : \alpha = 1, 2, \ldots, n\}$ be any finite collection of closed sets in (X, d).

Then $F_\alpha{}'$ is open. $\forall\ \alpha = 1, 2, \ldots, n$

$\Rightarrow \bigcap_{\alpha=1}^{n} F_\alpha{}'$ is open. ($\because$ Intersection of finite number of open sets is open)

$\Rightarrow \left(\bigcup_{\alpha=1}^{n} F_\alpha\right)'$ is open. $\because \bigcap_{\alpha=1}^{n} F_\alpha{}' = \left(\bigcup_{\alpha=1}^{n} F_\alpha\right)'$

$\Rightarrow \bigcup_{\alpha=1}^{n} F_\alpha$ is closed.

(iii) Since $\phi' = X - \phi = X$ which is open.

$\Rightarrow \phi$ is closed.

And $X' = X - X = \phi$ which is open.

$\Rightarrow X$ is closed.

Question

Is N closed in R?

Solution

Here $N = \{1, 2, 3, \ldots\ldots\}$

$N' = R - N$

$=]-\infty, 1[\ \cup\ ]1, 2[\ \cup\ ]2, 3[\ \cup \ldots\ldots$

$=$ Union of open intervals in R

$=$ Union of open sets ($\because$ An open intervals in R is an open set)

$=$ Open set ($\because$ Union of any number of open sets is an open set)

Since N' is an open set.

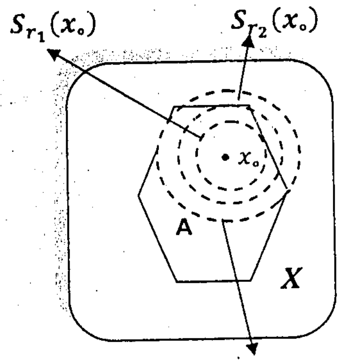
$\Rightarrow N$ is a closed set.

فرض کیا (X,d) ایک میٹرک سپیس ہے۔ فرض کیا A سب سیٹ ہے X کا اور x_0 رکن ہے X کا
x_0 لمٹنگ پوائنٹ ہوگا A کا اگر ہر اوپن سفیئر جس کا سنٹر x_0 ہے وہ x_0 کے علاوہ
A کا کسی دوسرے ایلیمنٹ پر مشتمل ہو

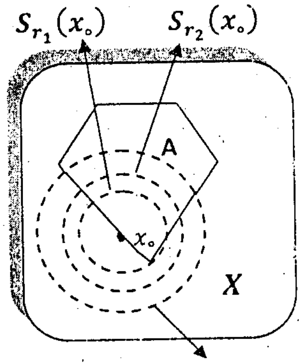
LIMIT POINT

Limit Point

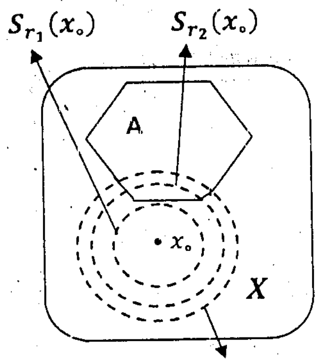
Let (X,d) be a metric space. Let $A \subseteq X$ and $x_\circ \in X$. Then $x_\circ$ is called limit point of A if each open sphere centered at $x_\circ$ contains at least one point of A different from $x_\circ$.



(Fig – 1) $S_{r_3}(x_\circ)$



(Fig – 2) $S_{r_3}(x_\circ)$



(Fig – 3) $S_{r_3}(x_\circ)$

In Fig–1, $x_\circ$ is a limit point of A.

In Fig–2, $x_\circ$ is also a limit point of A.

In Fig–3, $x_\circ$ is not a limit point of A.

Theorem

Let (X,d) be a discrete metric space. Let $A \subseteq X$. Then A has no limit point.

Proof

Consider the discrete metric space $(X, d_\circ)$.

Here $d_\circ : X \times X \to R$ is defined by

$$d_\circ(x_1, x_2) = \begin{cases} 0 & if\ x_1 = x_2 \\ 1 & if\ x_1 \neq x_2 \end{cases}$$

We have to prove that, $A \subseteq X$ has no limit point.

We shall prove it by contradiction method.

Suppose $x_\circ \in X$ such that $x_\circ$ is a limit point of A.

Let $0 < r < 1$ then $S_r(x_\circ) = \{x_\circ\}$ ---------- (1)

$\because$ In a discrete metric space the open sphere with radius less than 1 is always singleton.

Here (1) shows that $S_r(x_\circ)$ contains no point of A different from $x_\circ$.

Thus $x_\circ$ is not a limit point of A.

Hence A has no limit point.

Question

Let R be the metric space. Let $A = \left\{x | x \in R,\ x = \frac{1}{n}, n \in N\right\}$ be a subset of R. Show that "0" is a limit point of A.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\dots\right\}$

Then $S_r(0) = \{x | x \in R,\ d(x, 0) < r\}; \qquad r > 0$

$$= \{x | x \in R,\ |x - 0| < r\}$$
$$= \{x | x \in R,\ |x| < r\}$$
$$= \{x | x \in R,\ x < r,\ x > -r\}$$
$$= \{x | x \in R,\ -r < x < r\}$$
$$=]-r, +r[$$

Clearly for every $r > 0$, $S_r(0) =]-r, +r[$ contains a point of A different from "0".

Thus "0" is the limit point of A.

Question

Let R be the metric space. Let $A = \left\{x | x \in R, x = 1 \text{ or } x = 1 + \frac{1}{n}, n \in N\right\}$ be a subset of R. Show that "1" is a limit point of A.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \left\{x | x \in R, x = 1 \text{ or } x = 1 + \frac{1}{n}, n \in N\right\}$

$$= \{x | x \in R, x = 1\} \cup \{x | x \in R, x = 1 + 1/n, n \in N\}$$
$$= \{1\} \cup \left\{2, \frac{3}{2}, \frac{4}{3}, \dots\dots\right\}$$
$$= \left\{1, 2, \frac{3}{2}, \frac{4}{3}, \dots\dots\right\}$$

Now $S_r(1) = \{x | x \in R,\ d(x, 1) < r\}$

$$= \{x | x \in R,\ |x - 1| < r\}$$
$$= \{x | x \in R,\ x - 1 < r,\ x - 1 > -r\}$$
$$= \{x | x \in R,\ x < 1 + r,\ x > 1 - r\}$$
$$= \{x | x \in R,\ 1 - r < x < 1 + r\}$$
$$=]1 - r, 1 + r[$$

Clearly for every $r > 0$, $S_r(1) =]1 - r, 1 + r[$ contains a point of A different from "1".

Thus "1" is a limit point of A.

Question

Let R be the metric space. Let $A = \{ x | x \in R,\ 0 < x < 1 \}$ be a subset of R. Show that "0" and "1" are the limit point of A.

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \{ x | x \in R,\ 0 < x < 1 \}$

$=]\,0,\ 1\,[$

(i) First we shall prove that "0" is the limit point of A.

Now $S_r(0) = \{x | x \in R,\ d(x, 0) < r\};\quad r > 0$

$= \{x | x \in R,\ |x - 0| < r\}$

$= \{x | x \in R,\ |x| < r\}$

$= \{x | x \in R,\ x < r,\ x > -r\}$

$= \{x | x \in R,\ -r < x < r\}$

$=]-r,\ +r[$

Clearly for every $r > 0$, $S_r(0) =]-r,\ +r\,[$ contains a point of A different from "0".

Thus "0" is a limit point of A.

(ii) Now we shall prove that "1" is the limit point of A.

Now $S_r(1) = \{ x | x \in R,\ d(x, 1) < r \}$

$= \{ x | x \in R,\ |x - 1| < r \}$

$= \{ x | x \in R,\ x - 1 < r,\ x - 1 > -r \}$

$= \{ x | x \in R,\ x < 1 + r,\ x > 1 - r \}$

$= \{ x | x \in R,\ 1 - r < x < 1 + r \}$

$=]\,1 - r,\ 1 + r\,[$

Clearly for every $r > 0$, $S_r(1) =]\,1 - r,\ 1 + r\,[$ contains a point of A different from "1".

Thus "1" is a limit point of A.

Question

Let R be the metric space. Describe the limit points of the followings.
(a) N (b) Z

Solution

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

(a) Here $N = \{1, 2, 3, \ldots\}$

Let $a \in R$ be a limit point of N.

Then $a \in N$ or $a \notin N$

Case - I When $a \in N$

Then $S_r(a) = \{x | x \in R,\ d(x, a) < r\}; \quad r > 0$

$= \{x | x \in R,\ |x - a| < r\}$

$= \{x | x \in R,\ x - a < r,\ x - a > -r\}$

$= \{x | x \in R,\ x < a + r,\ x > a - r\}$

$= \{x | x \in R,\ a - r < x < a + r\}$

$=]\, a - r,\ a + r\, [$

Clearly for every $r > 0$, $S_r(1) =]\, a - r,\ a + r\, [$ contains no point of N different from "a".

Thus "a" is not the limit point of N.

Case - II When $a \notin N$, we can also prove that "a" is not a limit point of N.

Thus N has no limit point.

(b) Here $Z = \{\ldots \ldots -3, -2, -1, 0, 1, 2, \ldots \ldots \ldots\}$

Let $a \in R$ be a limit point of Z.

Then $a \in Z$ or $a \notin Z$

Case - I When $a \in Z$

Then $S_r(a) = \{x | x \in R,\ d(x, a) < r\}; \quad r > 0$

$= \{x | x \in R,\ |x - a| < r\}$

$= \{x | x \in R,\ x - a < r,\ x - a > -r\}$

$= \{x | x \in R,\ x < a + r,\ x > a - r\}$

$= \{x | x \in R,\ a - r < x < a + r\}$

$=]\, a - r,\ a + r\, [$

Clearly for every $r > 0$, $S_r(1) =]\, a - r,\ a + r\, [$ contains no point of Z different from "a".

Thus "a" is not the limit point of Z.

Case - II When $a \notin Z$, we can also prove that "a" is not a limit point of Z.

Thus Z has no limit point.

فرض (X, d) ایک میٹرک سپیس ہے اور x_0 رکھتا ہے X کا
فرض کیا N سب سیٹ ہے X کا پھر N نیبرہوڈ کہلائے گا x_0 کا

NEIGHBOURHOOD

اگر ایک اوپن سفیر ہے جس کا سنٹر x_0 ہو جو پورا $x_0 \in S_r(x_0) \subseteq N$

Neighbourhood

Let (X,d) be a metric space. Let $x_\circ \in X$. Let $N \subseteq X$. Then N is called a neighbourhood of $x_\circ$, if $\exists$ an open sphere $S_r(x_\circ)$ such that $x_\circ \in S_r(x_\circ) \subseteq N$.

Example

Let R be the usual metric space. Let $x_\circ = 0 \in R$. Show that $]-r,\ r[$, $]-r,\ r]$, $[-r,\ r[$, and $[-r,\ r]$, $(r > 0)$ is a neighbourhood of 0.

Solution

We know that,

In a usual metric space R, the open sphere is an open interval.

(i) Now $0 \in]-r,\ r[\subseteq]-r,\ r[$ Where $]-r,\ r[$ is an open sphere in R

$\Rightarrow$ $]-r,\ r[$ is a neighbourhood of "0".

(ii) Now $0 \in]-r,\ r[\subseteq]-r,\ r]$ Where $]-r,\ r[$ is an open sphere in R

$\Rightarrow$ $]-r,\ r]$ is a neighbourhood of "0".

(iii) Now $0 \in]-r,\ r[\subseteq [-r,\ r[$ Where $]-r,\ r[$ is an open sphere in R

$\Rightarrow$ $[-r,\ r[$ is a neighbourhood of "0".

(iv) Now $0 \in]-r,\ r[\subseteq [-r,\ r]$ Where $]-r,\ r[$ is an open sphere in R

$\Rightarrow$ $[-r,\ r]$ is a neighbourhood of "0".

A 10

Theorem

Let (X,d) be a metric space. Let $A \subseteq X$. Let $x_\circ$ be a limit point of A. Then every neighbourhood of $x_\circ$ contains infinitely many points of A.

Proof

Let N be a neighbourhood of $x_\circ$, then $\exists$ an open sphere $S_r(x_\circ)$ (where $r > 0$) such that

$$x_\circ \in S_r(x_\circ) \subseteq N \text{ ---------------- } (1)$$

We are to prove that N contains infinite points of A.

We prove it by contradiction method.

Suppose N contains finite points of A.

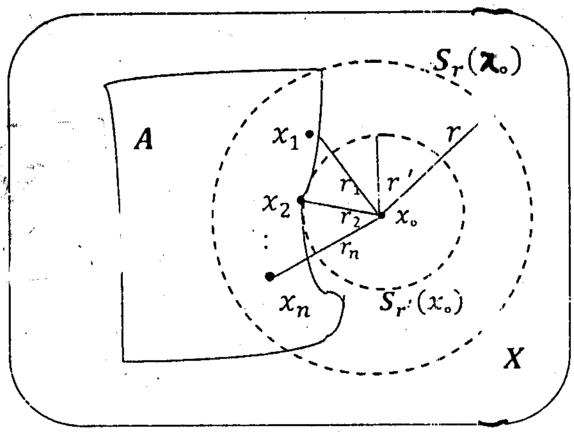
Then by (1) $S_r(x_\circ)$ also contains finite points of A.

Suppose $S_r(x_\circ)$ contains n points $x_1, x_2, x_3, \ldots\ldots, x_n$ of A.

Then $\quad A \cap S_r(x_\circ) = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$

Let $\quad d(x_\circ, x_i) = r_i, \quad i = 1,2,3,\ldots\ldots,n$

Let $\quad r' = min\,(r_1, r_2, r_3, \ldots\ldots, r_n)$



Clearly $S_{r'}(x_\circ)$ contains no point of A different from $x_\circ$.

This shows that, $x_\circ$ is not a limit point of A. This is a contradiction.

Hence N contains infinitely many points of A