Metric Spaces

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Real Valued Function

Let \( f: A \rightarrow \mathbb{R} \) be a function. Clearly domain of \( f \) is \( A \), in other words \( f \) is defined on \( A \). Since co-domain of \( f \) is \( \mathbb{R} \), we can say that \( f \) is real valued function.

Metric

Let \( X \) be a non-empty set and \( \mathbb{R} \) be a real numbers.

Let \( d: X \times X \rightarrow \mathbb{R} \) be a function

Then "\( d \)" is called "metric" on \( X \), if "\( d \)" satisfies each of the following four conditions:

\[
\begin{align*}
(M_1) & \quad d(x_1, x_2) \geq 0 \quad \forall x_1, x_2 \in X \\
(M_2) & \quad d(x_1, x_2) = 0 \iff x_1 = x_2 \quad \forall x_1, x_2 \in X \\
(M_3) & \quad d(x_1, x_2) = d(x_2, x_1) \quad \forall x_1, x_2 \in X \quad \text{(Symmetric Property)} \\
(M_4) & \quad d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3) \quad \forall x_1, x_2, x_3 \in X \quad \text{(Triangular Inequality)}
\end{align*}
\]

If "\( d \)" is a "metric" on \( X \) then the pair \((X, d)\) is called metric space.

Note

The non-negative real number \( d(x_1, x_2) \) is called distance between points \( x_1 \) and \( x_2 \) in the metric "\( d \)".

Usual Metric on \( \mathbb{R} \)

Let \( d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be a metric on \( \mathbb{R} \) given by \( d(x_1, x_2) = |x_1 - x_2| \)

Then "\( d \)" is called a usual metric on \( \mathbb{R} \) and \((\mathbb{R}, d)\) is called usual metric space.

Usual Metric on \( \mathbb{R}^2 \)

Let \( d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be a metric on \( \mathbb{R}^2 \) given by

\[
d([P_1, P_2] \triangle [P_1', P_2']) = \sqrt{(x_1' - x_1)^2 + (y_1' - y_1)^2}
\]

Then "\( d \)" is called a usual metric on \( \mathbb{R}^2 \) and \((\mathbb{R}^2, d)\) is usual metric space.

Usual Metric on \( \mathbb{R}^3 \)

Let \( d: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} \) be a metric on \( \mathbb{R}^3 \) given by

\[
d([P_1, P_2] \triangle [P_1', P_2']) = \sqrt{(x_1' - x_1)^2 + (y_1' - y_1)^2 + (z_1' - z_1)^2}
\]

Then "\( d \)" is called a usual metric on \( \mathbb{R}^3 \) and \((\mathbb{R}^3, d)\) is usual metric space.

Note

When we say that \( \mathbb{R} \) is a metric space without giving a metric on \( \mathbb{R} \) then it is assumed that metric on \( \mathbb{R} \) is "usual metric". Similarly we take the case of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).
**Example**
Let $X$ be the set of all towns marked on a plane geographically map and let $d(x_1, x_2)$ be the length of the shortest route from town $x_1$ to $x_2$. Show that "d" is a metric on $X$.

**Solution**
Here function $d: X \times X \to R$ is defined as

$$d(x_1, x_2) = \text{Length of shortest route from town } x_1 \text{ to } x_2.$$  

$(M_1)$ Since (Length of shortest route from town $x_1$ to $x_2$) $\geq 0$  
\[d(x_1, x_2) \geq 0\]  

$(M_2)$ Let $d(x_1, x_2) = 0 \implies$ Length of shortest route from town $x_1$ to $x_2 = 0$  
\[x_1 = x_2\]  

$(M_3)$ Let $x_1, x_2 \in X$  
Then $x_1, x_2, x_3$ are non-collinear or collinear  
If $x_1, x_2, x_3$ are non-collinear, then they form a triangle and we know that sum of length of two sides of a triangle is always greater than the third side.
\[d(x_1, x_2) + d(x_2, x_3) > d(x_1, x_3) \quad \text{--- (i)} \]

$(M_4)$ Let $x_1, x_2, x_3 \in X$  
Then $x_1, x_2, x_3$ are collinear.  
From (i) and (ii), we get
\[d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)\]

Hence "d" is a metric on $X$.

**Example**
Let $X = R$ be the set of all real numbers and let $d: R \times R \to R$ be defined by $d(x_1, x_2) = |x_1 - x_2|$ denotes the absolute value of the number $x_1 - x_2$. Show that $(R, d)$ is a metric space.

**Solution**
Here function $d: R \times R \to R$ is defined as

$$d(x_1, x_2) = |x_1 - x_2|$$  

$(M_1)$ Since $|x_1 - x_2| \geq 0$  
\[d(x_1, x_2) \geq 0\]  

$(M_2)$ Let $d(x_1, x_2) = 0 \implies |x_1 - x_2| = 0$.  
\[x_1 - x_2 = 0\]  
\[x_1 = x_2\]

Thus $d(x_1, x_2) = 0 \iff x_1 = x_2$
(M₃) Since \( d(x_1, x_2) = |x_1 - x_2| \)
\[ = |-(x_2 - x_1)| \]
\[ = |x_2 - x_1| \]
\[ = d(x_2, x_1) \]

(M₄) Since \( d(x_1, x_2) = |x_1 - x_2| \)
\[ d(x_2, x_3) = |x_2 - x_3| \]
\[ d(x_1, x_3) = |x_1 - x_3| \]
Now \( d(x_1, x_3) = |x_1 - x_3| \)
\[ = |x_1 - x_2 + x_2 - x_3| \]
\[ \leq |x_1 - x_2| + |x_2 - x_3| \]
\[ = d(x_1, x_2) + d(x_2, x_3) \]

Thus \((R, d)\) is a metric space.

Example
Let \( X = R^2 \) be a set of all ordered pairs \((x, y); x, y \in R\). Let \( P_1(x_1, y_1), P_2(x_2, y_2) \in R^2 \). Show that the non-negative real valued function "\( d \)" defined by \( d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| \) is a metric on \( R^2 \).

Solution
Here function \( d: R^2 \times R^2 \rightarrow R \) is defined as
\[ d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| \]
(M₁) Since \( |x_1 - x_2| + |y_1 - y_2| \geq 0 \)
\[ \Rightarrow d(P_1, P_2) \geq 0 \]

(M₂) Let \( d(P_1, P_2) = 0 \)
\[ \Rightarrow |x_1 - x_2| + |y_1 - y_2| = 0 \]
\[ \Rightarrow |x_1 - x_2| = 0, |y_1 - y_2| = 0 \]
\[ \Rightarrow x_1 = x_2, y_1 = y_2 \]
\[ \Rightarrow (x_1, y_1) = (x_2, y_2) \]
\[ \Rightarrow P_1 = P_2 \]
Let \( P_1 = P_2 \)
\[ \Rightarrow (x_1, y_1) = (x_2, y_2) \]
\[ \Rightarrow x_1 = x_2, y_1 = y_2 \]
\[ \Rightarrow x_1 - x_2 = 0, y_1 - y_2 = 0 \]
\[ \Rightarrow |x_1 - x_2| = 0, |y_1 - y_2| = 0 \]
\[ \Rightarrow |x_1 - x_2| + |y_1 - y_2| = 0 \]
\[ \Rightarrow d(P_1, P_2) = 0 \]
Thus \( d(P_1, P_2) = 0 \iff P_1 = P_2 \)

(M₃) Since \( d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| \)
\[ = |-(x_2 - x_1)| + |-(y_2 - y_1)| \]
\[ = |x_2 - x_1| + |y_2 - y_1| \]
\[ = d(P_2, P_1) \]

(M₄) Since \( d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| \)
\[ d(P_2, P_3) = |x_2 - x_3| + |y_2 - y_3| \]
\[ d(P_1, P_3) = |x_1 - x_3| + |y_1 - y_3| \]
Since \( d(P_1, P_3) = |x_1 - x_3| + |y_1 - y_3| \)
\[ = |x_1 - x_2 + x_2 - x_3| + |y_1 - y_2 + y_2 - y_3| \]
\[ \leq |x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3| \]
\[ = |x_1 - x_2| + |y_1 - y_2| + |x_2 - x_3| + |y_2 - y_3| \]
\[ = d(P_1, P_2) + d(P_2, P_3) \]

Hence "d" is metric on \( R^2 \).

**Example**

Let \( X = R^2 \) be a set of all ordered pairs \((x, y)\); \( x, y \in \mathbb{R} \). Let \( P_1(x_1, y_1), P_2(x_2, y_2) \in R^2 \). Show that the non-negative real valued function "d" defined by \( d(P_1, P_2) = \max(|x_1 - x|, |y_1 - y|) \) is a metric on \( R^2 \).

**Solution**

Here function \( d: R^2 \times R^2 \to \mathbb{R} \) is defined as
\[
\begin{align*}
   d(P_1, P_2) &= \max(|x_1 - x_2|, |y_1 - y_2|) \\
\end{align*}
\]

\((M_1)\) Since \( \max(|x_1 - x_2|, |y_1 - y_2|) \geq 0 \)
\[
(\because |x_1 - x_2| \geq 0 \& |y_1 - y_2| \geq 0 ) \\
\therefore d(P_1, P_2) \geq 0
\]

\((M_2)\) Let \( d(P_1, P_2) = 0 \) \( \Rightarrow \max(|x_1 - x_2|, |y_1 - y_2|) = 0 \)
\[
\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0 \\
\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0 \\
\Rightarrow x_1 = x_2, \quad y_1 = y_2 \\
\Rightarrow (x_1, y_1) = (x_2, y_2) \\
\Rightarrow P_1 = P_2
\]

Let \( P_1 = P_2 \) \( \Rightarrow (x_1, y_1) = (x_2, y_2) \)
\[
\Rightarrow x_1 = x_2, \quad y_1 = y_2 \\
\Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0 \\
\Rightarrow |x_1 - x_2| = 0, \quad |y_1 - y_2| = 0 \\
\Rightarrow \max(|x_1 - x_2|, |y_1 - y_2|) = 0 \\
\Rightarrow d(P_1, P_2) = 0
\]

Thus \( d(P_1, P_2) = 0 \) \( \iff P_1 = P_2 \)

\((M_3)\) Since \( d(P_1, P_2) = \max(|x_1 - x|, |y_1 - y|) \)
\[
\begin{align*}
   &= \max(|-(x_2 - x_1)|, |y_2 - y_1|) \\
   &= \max(|x_2 - x_1|, |y_2 - y_1|) \\
   &= d(P_2, P_1)
\end{align*}
\]
Since \( d(P_1, P_2) = \max \{ |x_1 - x_2|, |y_1 - y_2| \} = |x_1 - x_2| \) (Say)
\( d(P_2, P_3) = \max \{ |x_2 - x_3|, |y_2 - y_3| \} = |x_2 - x_3| \) (Say)
\( d(P_1, P_3) = \max \{ |x_1 - x_3|, |y_1 - y_3| \} = |x_1 - x_3| \) (Say)

Now \( d(P_1, P_3) = |x_1 - x_3| \)
\( = |x_1 - x_2 + x_2 - x_3| \)
\( \leq |x_1 - x_2| + |x_2 - x_3| \)
\( = d(P_1, P_2) + d(P_2, P_3) \)

(We can get the same results in the remaining cases.)

Hence "d" is metric on \( R^2 \).

**Example**

Let \( X = R^2 \) be a set of all ordered pairs \((x, y)\); \( x, y \in R \). Let \( P_1(x_1, y_1), P_2(x_2, y_2) \in R^2 \). Show that the non-negative real valued function "d" defined by \( d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} \) is a metric on \( R^2 \).

**Solution**

Here function \( d: R^2 \times R^2 \to R \) is defined as
\[
d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}
\]

\((M_1)\) Since \([(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} \geq 0 \)
\[
\therefore \quad d(P_1, P_2) \geq 0
\]

\((M_2)\) Let \( d(P_1, P_2) = 0 \)  \( \Rightarrow \) \([(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} = 0 \)
\[
\Rightarrow \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0
\]
\[
\Rightarrow \quad (x_1 - x_2)^2 = 0, \quad (y_1 - y_2)^2 = 0
\]
\[
\Rightarrow \quad x_1 - x_2 = 0, \quad y_1 - y_2 = 0
\]
\[
\Rightarrow \quad x_1 = x_2, \quad y_1 = y_2
\]
\[
\Rightarrow \quad (x_1, y_1) = (x_2, y_2)
\]
\[
\Rightarrow \quad P_1 = P_2
\]

Let \( P_1 = P_2 \)  \( \Rightarrow \) \((x_1, y_1) = (x_2, y_2)\)
\[
\Rightarrow \quad x_1 = x_2, \quad y_1 = y_2
\]
\[
\Rightarrow \quad x_1 - x_2 = 0, \quad y_1 - y_2 = 0
\]
\[
\Rightarrow \quad (x_1 - x_2)^2 = 0, \quad (y_1 - y_2)^2 = 0
\]
\[
\Rightarrow \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0
\]
\[
\Rightarrow \quad [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} = 0
\]
\[
\Rightarrow \quad d(P_1, P_2) = 0
\]
(M₃) Since \( d(P₁, P₂) = [(x₁ - x₂)^2 + (y₁ - y₂)^2]^{\frac{1}{2}} \)
\[ = [(-(x₂ - x₁))^2 + (y₂ - y₁)^2]^{\frac{1}{2}} \]
\[ = [(x₂ - x₁)^2 + (y₂ - y₁)^2]^{\frac{1}{2}} \]
\[ = d(P₂, P₁) \]

(M₄) Let \( P₁(x₁, y₁), P₂(x₂, y₂), P₃(x₃, y₃) \in R^2 \) then \( P₁, P₂, P₃ \) are collinear or non-collinear.
If \( P₁, P₂, P₃ \) are collinear, then
\[ d(P₁, P₂) + d(P₂, P₃) = d(P₁, P₃) \quad -----(1) \]
If \( P₁, P₂, P₃ \) are non-collinear, then they form a triangle and we know that, we know that sum of length of two sides of a triangle is always greater than the third side.
\[ \therefore d(P₁, P₂) + d(P₂, P₃) > d(P₁, P₃) \quad -----(2) \]
From (1) & (2) we get,
\[ d(P₁, P₂) + d(P₂, P₃) ≥ d(P₁, P₃) \]
Hence "\( d \)" is metric on \( R^2 \).

Example
Let \( X \) be a set of all ordered pairs \((x, y)\); \( x, y \in R \). Let \( P₁(x₁, y₁), P₂(x₂, y₂) \in X \). Show that the non-negative real valued function "\( d \)" defined by \( d(P₁, P₂) = [(x₁ - x₂)^2 + (y₁ - y₂)^2 + (z₁ - z₂)^2]^{\frac{1}{2}} \) is a metric on \( R^3 \).

Solution
Here function \( d : R^3 \times R^3 \rightarrow R \) is defined as
\[ d(P₁, P₂) = [(x₁ - x₂)^2 + (y₁ - y₂)^2 + (z₁ - z₂)^2]^{\frac{1}{2}} \]

(M₁) Since \([x₁ - x₂)^2 + (y₁ - y₂)^2 + (z₁ - z₂)^2]^{\frac{1}{2}} \geq 0 \)
\[ \therefore d(P₁, P₂) \geq 0 \]

(M₂) Let \( d(P₁, P₂) = 0 \) \( \Rightarrow [(x₁ - x₂)^2 + (y₁ - y₂)^2 + (z₁ - z₂)^2]^{\frac{1}{2}} = 0 \)
\[ \Rightarrow (x₁ - x₂)^2 + (y₁ - y₂)^2 + (z₁ - z₂)^2 = 0 \]
\[ \Rightarrow (x₁ - x₂)^2 = 0, (y₁ - y₂)^2 = 0, (z₁ - z₂)^2 = 0 \]
\[ \Rightarrow x₁ = x₂, \quad y₁ = y₂, \quad z₁ = z₂ \]
\[ \Rightarrow (x₁, y₁, z₁) = (x₂, y₂, z₂) \]
\[ \Rightarrow P₁ = P₂ \]
Let \( P_1 = P_2 \Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2) \)
\[ \Rightarrow x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2 \]
\[ \Rightarrow x_1 - x_2 = 0, \quad y_1 - y_2 = 0, \quad z_1 - z_2 = 0 \]
\[ \Rightarrow (x_1 - x_2)^2 = 0, \quad (y_1 - y_2)^2 = 0, \quad (z_1 - z_2)^2 = 0 \]
\[ \Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 0 \]
\[ \Rightarrow d(P_1, P_2) = 0 \]

\((M_3)\) Since \( d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2} \)
\[ = [(-(x_2 - x_1))^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \]
\[ = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \]
\[ = d(P_2, P_1) \]

\((M_4)\) Let \( P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2), P_3(x_3, y_3, z_3) \in \mathbb{R}^3 \) then \( P_1, P_2, P_3 \) are collinear or non-collinear.
If \( P_1, P_2, P_3 \) are collinear, then
\[ d(P_1, P_3) + d(P_2, P_3) = d(P_1, P_3) \] \((1)\)
If \( P_1, P_2, P_3 \) are non-collinear, then they form a triangle and we know that, we know that sum of length of two sides of a triangle is always greater than the third side.
\[ \therefore d(P_1, P_2) + d(P_2, P_3) > d(P_1, P_3) \] \((2)\)
From \((1)\) & \((2)\) we get,
\[ d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3) \]
Hence "\( d \)" is metric on \( \mathbb{R}^3 \).

**Example**
Show that every non-empty set can be given a metric and hence can be converted into metric space.

**Solution**
Let \( X \) be any non-empty set.
Let \( d_+: X \times X \rightarrow \mathbb{R} \) be defined by
\[ d_+(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases} \]
We shall prove that \( d_+ \) is a metric on \( X \).

\((M_1)\) Here \( d_+(x_1, x_2) \geq 0 \) \( (\because d_+(x_1, x_2) = 0 \text{ or } d_+(x_1, x_2) = 1) \)

\((M_2)\) Let \( d_+(x_1, x_2) = 0 \Rightarrow x_1 = x_2 \) \( (\text{By definition}) \)
Let \( x_1 = x_2 \Rightarrow d_+(x_1, x_2) = 0 \) \( (\text{By definition}) \)
Thus \( d_+(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2 \)
(M3) (i) Let \( d_s(x_1, x_2) = 0 \) \( \Rightarrow \) \( x_1 = x_2 \) (By definition) 
\[ x_2 = x_1 \]
\[ d_s(x_2, x_1) = 0 \]

(ii) Let \( d_s(x_1, x_2) = 1 \) \( \Rightarrow \) \( x_1 \neq x_2 \) (By definition) 
\[ x_2 \neq x_1 \]
\[ d_s(x_2, x_1) = 1 \]

Hence in both the cases \( d_s(x_1, x_2) = d_s(x_2, x_1) \)

(M4) Let \( x_1, x_2, x_3 \in X \)

(i) Let \( x_1 = x_2 = x_3 \) then \( d_s(x_1, x_2) = 0 \) 
\& \( d_s(x_2, x_3) = 0 \)
also \( d_s(x_1, x_3) = 0 \)
\[ d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3) \]

(ii) Let \( x_1 \neq x_2 \neq x_3 \) then \( d_s(x_1, x_2) = 1 \) 
\& \( d_s(x_2, x_3) = 1 \)
also \( d_s(x_1, x_3) = 1 \)
\[ d(x_1, x_2) + d(x_2, x_3) > d(x_1, x_3) \]

Similar type of verification in all remaining cases leads us to the conclusion that 
\[ d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3) \quad \forall x_1, x_2, x_3 \in X \]

Hence \( (X, d_s) \) is a metric space.

Note
Let \( X \) be any non-empty set. Let \( d_s: X \times X \to \mathbb{R} \) be defined by 
\[ d_s(x_1, x_2) = \begin{cases} 
0 & \text{if } x_1 = x_2 \\
1 & \text{if } x_1 \neq x_2 
\end{cases} \]

Then \( d_s \) is called discrete metric on \( X \).

Question
Let \( C \) be the set of all complex numbers and let \( d: C \times C \to \mathbb{R} \) be defined by \( d(z_1, z_2) = |z_1 - z_2| \) \( d \) is a metric on \( C \)

Solution
Here function \( d: C \times C \to \mathbb{R} \) is defined as 
\[ d(z_1, z_2) = |z_1 - z_2| \]

(M1) Since \( |z_1 - z_2| \geq 0 \)
\[ \therefore d(z_1, z_2) \geq 0 \]

(M2) Let \( d(z_1, z_2) = 0 \) \( \Rightarrow \) \( |z_1 - z_2| = 0 \)
\[ \Rightarrow z_1 - z_2 = 0 \]
\[ \Rightarrow z_1 = z_2 \]
Let \( z_1 = z_2 \) \( \Rightarrow \) \( z_1 - z_2 = 0 \)
\( \Rightarrow \) \( |z_1 - z_2| = 0 \)
\( \Rightarrow \) \( d(z_1, z_2) = 0 \)
Thus \( d(z_1, z_2) = 0 \iff z_1 = z_2 \)

\((M_3)\) Since \( d(z_1, z_2) = |z_1 - z_2| = -(z_2 - z_1) \)
\( = |z_2 - z_1| \)
\( = d(z_2, z_1) \)

\((M_4)\) Since \( d(z_1, z_2) = |z_1 - z_2| \)
\( d(z_2, z_3) = |z_2 - z_3| \)
\( d(z_1, z_3) = |z_1 - z_3| \)
Now \( d(z_1, z_3) = |z_1 - z_3| \)
\( = |z_1 - z_2 + z_2 - z_3| \)
\( \leq |z_1 - z_2| + |z_2 - z_3| \)
\( = d(z_1, z_2) + d(z_2, z_3) \)

Thus \((C, d)\) is a metric space.

**Question**
Let \( d \) be a metric on \( X \) and let \( d': X \times X \to R \) be given by
\( d'(x_1, x_2) = \min \{1, d(x_1, x_2)\} \). Is \( d' \) a metric on \( X \)?

**Solution**
Here function \( d': X \times X \to R \) be given by
\( d'(x_1, x_2) = \min \{1, d(x_1, x_2)\} \).

\((M_1)\) Since \( \min \{1, d(x_1, x_2)\} \geq 0 \)
\( \therefore d'(x_1, x_2) \geq 0 \)

\((M_2)\) Let \( d'(x_1, x_2) = 0 \Rightarrow \min \{1, d(x_1, x_2)\} = 0 \)
\( \Rightarrow d(x_1, x_2) = 0 \forall 1 \neq 0 \)
\( \Rightarrow x_1 = x_2 \forall d \) is metric on \( X \).
Let \( x_1 = x_2 \)
\( \Rightarrow d(x_1, x_2) = 0 \forall d \) is metric on \( X \).
\( \Rightarrow \min \{1, d(x_1, x_2)\} = 0 \)
\( \Rightarrow d'(x_1, x_2) = 0 \)
Thus \( d'(x_1, x_2) = 0 \iff x_1 = x_2 \)

\((M_3)\) Since \( d'(x_1, x_2) = \min \{1, d(x_1, x_2)\} \)
\( = \min \{1, d(x_2, x_1)\} \forall d \) is metric on \( X \).
\( = d'(x_2, x_1) \)

\((M_4)\) Since \( d'(x_1, x_2) = \min \{1, d(x_1, x_2)\} = d(x_1, x_2) \) (Say)
\( d'(x_2, x_3) = \min \{1, d(x_2, x_3)\} = d(x_2, x_3) \) (Say)
\( d'(x_1, x_3) = \min \{1, d(x_1, x_3)\} = d(x_1, x_3) \) (Say)

Since \( d \) is a metric on \( X \).
\( \therefore d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3) \)
\( \Rightarrow d'(x_1, x_2) + d'(x_2, x_3) \geq d'(x_1, x_3) \)
We get the same result in the remaining cases.
\( \therefore d' \) is a metric on \( X \).
Question

Let \((X_1, d_1)\) and \((X_2, d_2)\) be two metric spaces.

Define \(d'(x_1, x_2, y_1, y_2) = \sum_{i=1}^{2} d_i(x_i, y_i)\). Is \(d'\) a metric on \(X_1 \times X_2\).

Solution

Here function \(d': X_1 \times X_2 \rightarrow R\) is defined as

\[
d'(x_1, x_2, y_1, y_2) = \sum_{i=1}^{2} d_i(x_i, y_i)
\]

(Since \(d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0\))

\[
\therefore d_1(x_1, y_1) \geq 0, \quad d_2(x_2, y_2) \geq 0
\]

\[
\therefore d_1, d_2 \text{ are metrics on } X_1 \text{ and } X_2 \text{ respectively.}
\]

\[
\therefore d'(x_1, x_2, y_1, y_2) \geq 0
\]

\(\text{(M_2)}\) Let \(d'(x_1, x_2, y_1, y_2) = 0\) \(\Rightarrow d_1(x_1, y_1) + d_2(x_2, y_2) = 0\)

\[
\Rightarrow d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0
\]

\[
\Rightarrow x_1 = y_1, \quad x_2 = y_2
\]

\[
\therefore d_1, d_2 \text{ are metrics on } X_1 \times X_2
\]

Let \((x_1, x_2) = (y_1, y_2)\) \(\Rightarrow x_1 = y_1, \quad x_2 = y_2\)

\[
\Rightarrow d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0
\]

(\(\because d_1, d_2 \text{ are metrics on } X_1 \text{ and } X_2 \text{ respectively})

\[
\Rightarrow d_1(x_1, y_1) + d_2(x_2, y_2) = 0
\]

\[
\Rightarrow d'(x_1, x_2, (y_1, y_2)) = 0
\]

Thus \(d'(x_1, x_2, (y_1, y_2)) = 0 \iff (x_1, x_2) = (y_1, y_2)\)

\(\text{(M_3)}\) Since \(d'(x_1, x_2, (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)\)

\[
= d_1(y_1, x_1) + d_2(y_2, x_2)
\]

\[
= d'(y_1, x_2, (x_1, x_2))
\]

\(\text{(M_4)}\) Since \(d'(x_1, x_2, (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)\)

\[
d'(y_1, y_2, (z_1, z_2)) = d_1(y_1, z_1) + d_2(y_2, z_2)
\]

\[
d'(x_1, x_2, (z_1, z_2)) = d_1(x_1, z_1) + d_2(x_2, z_2)
\]

Now \(d'(x_1, x_2, (y_1, y_2)) + d'(y_1, y_2, (z_1, z_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)\)

\[
+ d_1(y_1, z_1) + d_2(y_2, z_2)
\]

\[
= d_1(x_1, y_1) + d_1(y_1, z_1)
\]

\[
+ d_2(x_2, y_2) + d_2(y_2, z_2)
\]

\[
\geq d_1(x_1, z_1) + d_2(x_2, z_2)
\]

(\(\because d_1, d_2 \text{ are metrics on } X_1 \text{ and } X_2 \text{ respectively})

\[
\therefore d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1)
\]

\[
\& d_2(x_2, y_2) + d_2(y_2, z_2) \geq d_2(x_2, z_2)
\]

\[
= d'(x_1, x_2, (z_1, z_2))
\]

\(\therefore d'\) is a metric on \(X_1 \times X_2\).
Question
Let \((X_1, d_1)\) and \((X_2, d_2)\) be two metric spaces.
Let \(d''([x_1, x_2], (y_1, y_2)] = \max (d_1(x_1, y_1), d_2(x_2, y_2))\).
Is \(d''\) a metric on \(X_1 \times X_2\).

Solution
Here function \(d'': X_1 \times X_2 \rightarrow R\) is defined as
\[d''([x_1, x_2], (y_1, y_2)] = \max (d_1(x_1, y_1), d_2(x_2, y_2))\]
(M1) Since \(\max (d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0)\)
\[\Rightarrow d_1(x_1, y_1) \geq 0, \quad d_2(x_2, y_2) \geq 0\]
\[\Rightarrow d_1, d_2 \text{ are metrics on } X_1 \text{ and } X_2 \text{ respectively}\]
\[\Rightarrow d''((x_1, x_2), (y_1, y_2)) \geq 0\]
(M2) Let \(d''((x_1, x_2), (y_1, y_2)) = 0\) \(\Rightarrow \max (d_1(x_1, y_1), d_2(x_2, y_2)) = 0\)
\[\Rightarrow d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0\]
\[\Rightarrow x_1 = y_1, \quad x_2 = y_2\]
\[\Rightarrow (x_1, x_2) = (y_1, y_2)\]
(M3) Since \(d''([x_1, x_2], (y_1, y_2)] = \max (d_1(x_1, y_1), d_2(x_2, y_2))\)
\[= \max (d_1(y_1, x_1), d_2(y_2, x_2))\]
\[\Rightarrow d''((x_1, x_2), (y_1, y_2)) = 0\]
(M4) Let \(d''([x_1, x_2], (y_1, y_2)] = \max (d_1(x_1, y_1), d_2(x_2, y_2)) = d_1(x_3, y_1)\) (Say)
\[d''([y_1, y_2], (z_1, z_2)] = \max (d_1(y_1, z_1), d_2(y_2, z_2)) = d_1(y_1, z_1)\] (Say)
\[d''([x_1, x_2], (z_1, z_2)] = \max (d_1(x_1, z_1), d_2(x_2, z_2)) = d_1(x_1, z_1)\] (Say)
Since \(d_1\) is a metric on \(X_1\).
\[\Rightarrow d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1)\]
\[\Rightarrow d''([x_1, x_2], (y_1, y_2)] + d''([y_1, y_2], (z_1, z_2)] \geq d''([x_1, z_2], (z_1, z_2)]\]
(We get the same result in the remaining cases.)
\[\Rightarrow d''\] is a metric on \(X_1 \times X_2\).
Question
Let \((X, d)\) be a metric space and let \(d' : X \times X \to \mathbb{R}\) be given by
\[
d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}\]
Prove that \(d'\) is metric on \(X\).

Solution
Here function \(d' : X \times X \to \mathbb{R}\) be defined by
\[
d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}\]

(M1) Since \(\frac{d(x_1, x_2)}{1 + d(x_1, x_2)} \geq 0\)
\[
\Rightarrow d(x_1, x_2) \geq 0
\]
\[
\Rightarrow d'\) is a metric on \(X\).
\[
\Rightarrow d'(x_1, x_2) \geq 0
\]

(M2) Let \(d'(x_1, x_2) = 0 \Rightarrow \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} = 0\)
\[
\Rightarrow d(x_1, x_2) = 0
\]
\[
\Rightarrow x_1 = x_2 \quad (\because d'\) is a metric on \(X\).)

(M3) Since \(d'(x_1, x_2) = \frac{d(x_2, x_1)}{1 + d(x_2, x_1)}\)
\[
= \frac{d(x_2, x_1)}{1 + d(x_2, x_1)}
\]
\[
d'(x_2, x_1)\]

(M4) Since \(d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}\)
\[
d'(x_2, x_3) = \frac{d(x_2, x_3)}{1 + d(x_2, x_3)}
\]
\[
d'(x_1, x_3) = \frac{d(x_1, x_3)}{1 + d(x_1, x_3)}
\]

Now \(d'(x_1, x_2) + d'(x_2, x_3) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} + \frac{d(x_2, x_3)}{1 + d(x_2, x_3)}\)
\[
\geq \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} + \frac{d(x_2, x_3)}{1 + d(x_2, x_3)}
\]
\[
= \frac{d(x_1, x_2) + d(x_2, x_3)}{1 + d(x_1, x_2) + d(x_2, x_3)}
\]
\[
\Rightarrow d'(x_1, x_2) + d'(x_2, x_3) \geq d'(x_1, x_3) \quad (\because d'(x_1, x_2) + d'(x_2, x_3) \geq d'(x_1, x_3))
\]
\[
\Rightarrow d'\) is a metric on \(X\).
**Question**

Let $X = R$ and $d(x_1, x_2) = |x_1| + |x_2|$. Show that $d$ is not a metric on $R$.

**Solution**

Let $d(x_1, x_2) = 0 \implies |x_1| + |x_2| = 0$

$\implies |x_1| = 0, \quad |x_2| = 0$

$\implies x_1 = 0, \quad x_2 = 0$

$\implies x_1 = x_2$

Let $x_1 = x_2 \implies |x_1| = |x_2|$

$\implies |x_1| + |x_2| = |x_2| + |x_2|$ (Adding $|x_2|$ both sides)

$\implies d(x_1, x_2) = 2|x_2|$

$\implies d(x_1, x_2) = 0$ if $|x_2| = 0$

i.e. $d(x_1, x_2)$ is not always zero.

$\therefore d$ is not a metric on $X$.}

**Question**

Let $X = R$ and $d(x_1, x_2) = max(|x_1|, |x_2|)$. Show that $d$ is not a metric on $R$.

**Solution**

Let $d(x_1, x_2) = 0 \implies max(|x_1|, |x_2|) = 0$

$\implies |x_1| = 0, \quad |x_2| = 0$

$\implies x_1 = 0, \quad x_2 = 0$

$\implies x_1 = x_2$

Let $x_1 = x_2 \implies |x_1| = |x_2|$

$\implies max(|x_1|, |x_2|) = |x_2|$

$\implies d(x_1, x_2) = 0$ if $|x_2| = 0$

i.e. $d(x_1, x_2)$ is not always zero.

Thus $d$ is not a metric on $X$.

**Question**

Let $(X, d)$ be a metric space and let $d'' : X \times X \to R$ be given by

$$d''(x_1, x_2) = \frac{1 - d(x_1, x_2)}{1 + d(x_1, x_2)}$$

Prove that $d''$ is metric on $X$.

**Solution**

Let $d''(x_1, x_2) = 0 \implies \frac{1 - d(x_1, x_2)}{1 + d(x_1, x_2)} = 0$

$\implies 1 - d(x_1, x_2) = 0$

$\implies d(x_1, x_2) = 1$

$\implies x_1 \neq x_2$

$\therefore d$ is a metric on $X$ and $d(x_1, x_2) = 0 \iff x_1 = x_2$

Thus $d''(x_1, x_2) = 0 \iff x_1 = x_2$

Thus $d''$ is not a metric on $X$. 
**OPEN SPHERE**

Open sphere

Let \( (X, d) \) be a metric space. Let \( x_\ast \in X \) and \( r > 0 \), then open sphere with centre at \( x_\ast \) and radius equal to \( r \) is denoted by \( S_r(x_\ast) \) and is defined as

\[
S_r(x_\ast) = \{ x | x \in X, \ d(x, x_\ast) < r \}
\]

**Note**

(i) Let \( X = \{ x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \} \) and \( r > 0 \)

\[
S_r(x_\ast) = ?
\]

(ii) \( S_r(x_\ast) \subseteq X \)

(iii) \( S_r(x_\ast) \neq \emptyset \)

(iv) Here we shall study the open spheres of the following shapes.

(a) Open interval  
(b) Open disc  
(c) Open ball

The shape of an open sphere depends upon the metric space \((X, d)\).

**Example**

Let \( R \) be the metric space. Let \( x_0 = 1, \ r = \frac{1}{2} \). Find \( S_{\frac{1}{2}}(1) \).

**Solution**

Here metric space is \((R, d)\), where metric \(d : R \times R \rightarrow R\) is defined as

\[
d(x_1, x_2) = |x_1 - x_2|
\]

We know that

\[
S_r(x_\ast) = \{ x | x \in X, \ d(x, x_\ast) < r \}
\]

Put \( X = R \), \( x_\ast = 1 \), \( r = \frac{1}{2} \)

\[
S_{\frac{1}{2}}(1) = \{ x | x \in R, \ d(x, 1) < \frac{1}{2} \}
\]

\[
= \{ x | x \in R, \ |x - 1| < \frac{1}{2} \}
\]

\[
= \{ x | x \in R, \ x - 1 < \frac{1}{2}, \ x - 1 > -\frac{1}{2} \}
\]

\[
= \{ x | x \in R, \ x < 1 + \frac{1}{2}, \ x > 1 - \frac{1}{2} \}
\]

\[
= \{ x | x \in R, \ \frac{1}{2} < x < \frac{3}{2} \}
\]

Open sphere in this case is an open interval.
Note

An open sphere in a usual metric space $R$ is always an "open interval".

Example

Let the metric space be $R^2$ and let $P_a = (a, b)$ and $r = 1$. Find $S_r(P_a)$.

Solution

Here metric space is $(R, d)$, where metric $d: R \times R \to R$ is defined as

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We know that

$$S_r(P_a) = \{ P | P \in X, \ d(P, P_a) < r \}$$

Put $X = R^2$, $P_a = (a, b)$, $P = (x, y)$, $r = 1$

$$S_1(a, b) = \{ (x, y) | (x, y) \in R^2, \ d((x, y), (a, b)) < 1 \}$$

$$= \{ (x, y) | (x, y) \in R^2, \ \sqrt{(x - a)^2 + (y - b)^2} < 1 \}$$

This is an open disc with centre at $(a, b)$ and radius 1.

Note

An open sphere in a usual metric space $R^2$ is always an "open disc".

Example

Let the metric space be $R^2$ and $d_1$ be the metric on $R^2$ defined by

$$d_1(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|.$$  

Let $P_a = (0, 0)$ and $r = \frac{1}{\sqrt{2}}$. Find $S_r(P_a)$.

Solution

Here metric space is $(R^2, d_1)$, where metric $d_1: R^2 \times R^2 \to R$ is defined as

$$d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

We know that

$$S_r(P_a) = \{ P | P \in X, \ d(P, P_a) < r \}$$

Put $X = R^2$, $P_a = (0, 0)$, $P = (x, y)$, $r = \frac{1}{\sqrt{2}}$

$$S_{\frac{1}{\sqrt{2}}}(0, 0) = \left\{ (x, y) | (x, y) \in R^2, \ d_1((x, y), (0, 0)) < \frac{1}{\sqrt{2}} \right\}$$

$$= \left\{ (x, y) | (x, y) \in R^2, \ |x - 0| + |y - 0| < \frac{1}{\sqrt{2}} \right\}$$

$$= \left\{ (x, y) | (x, y) \in R^2, \ |x| + |y| < \frac{1}{\sqrt{2}} \right\}$$

$$= \left\{ (x, y) | (x, y) \in R^2, \ \pm x \pm y < \frac{1}{\sqrt{2}} \right\}$$

This is an open square with $x$-intercepts $\pm \frac{1}{\sqrt{2}}$ and $y$-intercepts $\pm \frac{1}{\sqrt{2}}$.
Let \((X, d_0)\) be a discrete metric space. Let \(x_0 \in X\) and \(r > 0\).

Find \(S_r(x_0)\), when (i) \(r \leq 1\) (ii) \(r > 1\)

**Solution**

Here metric space is \((X, d_0)\), where \(d_0: X \times X \to \mathbb{R}\) is defined by

\[
    d_0(x_1, x_2) = \begin{cases} 
    0 & \text{if } x_1 = x_2 \\
    1 & \text{if } x_1 \neq x_2 
    \end{cases}
\]

We know that

\[
    S_r(x_0) = \{ x | x \in X, \quad d_0(x, x_0) < r \}
\]

When \(r \leq 1\)

- If \(x \neq x_0\) then from equation (1) we get \(1 < r\) (False)
- If \(x = x_0\) then from equation (1) we get \(0 < r\) (True)

Thus \(S_r(x_0) = \{ x | x \in X, \quad x = x_0 \} = \{ x_0 \}\)

When \(r > 1\)

- If \(x \neq x_0\) then from equation (1) we get \(1 < r\) (True)
- If \(x = x_0\) then from equation (1) we get \(0 < r\) (True)

Thus \(S_r(x_0) = \{ x | x \in X, \quad x = x_0 \quad \text{or} \quad x \neq x_0 \}
\]

\[
= \{ x | x \in X, \quad x = x_0 \} \cup \{ x | x \in X, \quad x \neq x_0 \}
\]

\[
= \{ x_0 \} \cup X - \{ x_0 \}
\]

\[
= X
\]

**Note**

From above example we conclude that

(i) An open sphere with radius less than or equal to 1 in a discrete metric space is always singleton.

(ii) An open sphere with radius greater than 1 in a discrete metric space is always the full space \(X\).
**Question**

Let $C$ be the set of all complex numbers and let $d: C \times C \to \mathbb{R}$ be defined by $d(z_1, z_2) = |z_1 - z_2|$. Find $S_r(x_0)$ when $x_0 = 1$, $r = 0.01$

**Solution**

The given metric space is $(C, d)$, where $d: C \times C \to \mathbb{R}$ be defined by

$$d(z_1, z_2) = |z_1 - z_2|$$

Now $S_r(x_0) = \{ x | x \in X, \ d(x, x_0) < r \}$

Put $X = C$, $x_0 = 1$  $r = 0.01$

$$S_{0.01}(1) = \{ x | x \in C, \ d(x, 1) < 0.01 \}$$

$$= \{ x | x \in C, \ |x - 1| < 0.01 \}$$

Since $x \in C$  $\Rightarrow$  $x = a + ib$

$$\Rightarrow \quad x - 1 = a + ib - 1$$

$$\Rightarrow \quad x - 1 = (a - 1) + ib$$

$$\Rightarrow \quad |x - 1| = \sqrt{(a - 1)^2 + b^2}$$

$$\therefore S_{0.01}(1) = \{ (a + ib)|(a + ib) \in C, \ \sqrt{(a - 1)^2 + b^2} < 0.01 \}$$

$$= \{ (a + ib)|(a + ib) \in C, (a - 1)^2 + (b - 0)^2 < (.01)^2 \}$$

This is an open disc with centre at $(1, 0)$ and radius equal to 0.01.

**Question**

Let $d$ be a metric on $X$ and let $d': X \times X \to \mathbb{R}$ be given by

$$d'(x_1, x_2) = \min (1, d(x_1, x_2))$$. Describe $S_r(x_0)$.

**Solution**

Here given metric space is $(X, d')$, where $d': X \times X \to \mathbb{R}$ be given by

$$d'(x_1, x_2) = \min (1, d(x_1, x_2))$$

Now $S_r(x_0) = \{ x | x \in X, \ d'(x, x_0) < r \}$

$$= \{ x | x \in X, \ \min (1, d(x_1, x_2)) < r \}$$

This is the required open sphere.

**Question**

Let $(X, d)$ be a metric space and let $d': X \times X \to \mathbb{R}$ be given by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$$. Describe $S_r(x_0)$.

**Solution**

Here given metric space is $(X, d')$, where $d': X \times X \to \mathbb{R}$ be given by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$$

Now $S_r(x_0) = \{ x | x \in X, \ d'(x, x_0) < r \}$

$$= \{ x | x \in X, \ \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} < r \}$$

This is the required open sphere.
Theorem

Let \( x_1, x_2 \) be any two distinct points of a metric space \( X \). Prove that there exist two open spheres \( S_{r_1}(x_1) \) and \( S_{r_2}(x_2) \) in \( X \) such that

\[
S_{r_1}(x_1) \cap S_{r_2}(x_2) = \emptyset
\]

Proof

Let \( S_{r_1}(x_1) \) and \( S_{r_2}(x_2) \) be two open spheres with centers \( x_1 \) and \( x_2 \) and radii \( r_1 \) and \( r_2 \) respectively.

Let \( d(x_1, x_2) = r_1 + r_2 \)

We are to prove that

\[
S_{r_1}(x_1) \cap S_{r_2}(x_2) = \emptyset
\]

We shall prove it by contradiction method.

Suppose \( S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \emptyset \)

Let \( x \in S_{r_1}(x_1) \cap S_{r_2}(x_2) \)

\[
\Rightarrow x \in S_{r_1}(x_1) \quad \text{and} \quad x \in S_{r_2}(x_2)
\]

\[
\Rightarrow d(x, x_1) < r_1 \quad \text{and} \quad d(x, x_2) < r_2
\]

Since \( r_1 + r_2 = d(x_1, x_2) \leq d(x_1, x) + d(x, x_2) \)

\[
\therefore d \text{ is a metric on } X.
\]

\[
\Rightarrow r_1 + r_2 \leq d(x_1, x) + d(x, x_2)
\]

\[
\Rightarrow r_1 + r_2 < r_1 + r_2 \quad \text{[By (1)]}
\]

It is impossible.

Thus our supposition \( S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \emptyset \) is wrong.

Hence \( S_{r_1}(x_1) \cap S_{r_2}(x_2) = \emptyset \)

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A subset $U$ of a metric space $(X, d)$ is called an open set if for every $x \in U$ there exists a real number $r > 0$ such that $x \in S_r(x) \subseteq U$.

**Open Set**

Let $(X, d)$ be a metric space. Let $U \subseteq X$. The $U$ is called an open set, if for each $x \in U$, $\exists \ r > 0$, such that $S_r(x) \subseteq U$.

i.e. $U$ is called an open set, if each point of $U$ is the centre of some open sphere, which is contained in $U$.

![Diagram of open and not open sets](image)

**Example**

Let $R$ be a usual metric space (The ordinary real number line) and let $U=]0,1[\,$, then show that $U$ is open.

**Solution**

Here metric space is $(R, d)$, where $d: R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Let $x_o \in U, r$. Let $r > 0$

Then $S_r(x_o) = \{x \in R, \ d(x_o, x) < r\}$

$$= \{x \in R, \ |x - x_o| < r\}$$

$$= \{x \in R, \ x - x_o < r, \ x - x_o > -r\}$$

$$= \{x \in R, \ x < x_o + r, \ x > x_o - r\}$$

$$= \{x \in R, \ x_o - r < x < x_o + r\}$$

$$= ]x_o - r, x_o + r[$$

We can find a value of $r$ for which $S_r(x_o) = ]x_o - r, x_o + r[ \subseteq U=]0,1[$

Thus $U=]0, 1[$ is an open set.

![Number line with open interval](image)

**Note**

In the above example if we take $x_o = 0.99$. Let $r = 0.001$

Then $S_{0.001}(0.99) = ]0.99 - 0.001, 0.99 + 0.001[ = ]0.981, 0.991[ \subseteq ]0, 1[$
Example

Let $R^2$ be a usual metric space (The ordinary real plane)

Let $U= \{(x,y)| (x,y) \in R^2, x^2+y^2 < 1\}$. Show that $U$ is an open set.

Solution

Here metric space is $(R^2, d)$, where $d: R^2 \times R^2 \to R$ is given by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Let $P_0 \in U$.

Let $d(O, P_0) = \lambda$

Let $r = 1 - \lambda$, then $r > 0$

We shall prove that $S_r(P_0) \subseteq U$.

Let $P \in S_r(P_0) \Rightarrow d(P, P_0) < r$

Then $S_r(P_0) \subseteq U$

Hence $U$ is an open set.

Example

Let $R$ be a usual metric space (The ordinary real number line) and let $U = \{x | x \in R, 0 \leq x < 1\}$, then show that $U$ is open.

Solution

Here metric space is $(R, d)$, where $d: R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Let $x, 0 \in U$, Let $r > 0$

Then $S_r(0) = \{x | x \in R, d(x, 0) < r\}$

Let $x, 0 \in U$

Then $S_r(0) = \{x | x \in R, 0 < x < 1\}$

We can find a value of $r$ for which $S_r(0) = \{x | x \in R, 0 < x < 1\}$ is not an open set.

Thus $U = [0, 1]$ is not an open set.
Theorem
Every non-empty subset of a discrete metric space is open.

Proof
Let \((X, d, \Phi)\) be a discrete metric space.
Let \(U \subseteq X\) such that \(U \neq \emptyset\).
We shall prove that \(U\) is an open set.
Let \(x_0 \in U\).
Let \(0 < r < 1\).
Then \(S_r(x_0) = \{ x \mid x \in X, \quad d(x, x_0) < r \}\)
\[ = \{ x_0 \} \]
\(\because\) The open sphere in a discrete metric space, whose radius is less then \(1\), is always singleton.
Since \(S_r(x_0) = \{ x_0 \} \subseteq U\) \(\quad\text{Centre point}\)
\(\Rightarrow\) \(U\) is an open set.

Example
Let \(R\) be a usual metric space (The ordinary real number line) and let \(U = \{ 0 \}\), then show that \(U\) is open.

Solution
Here metric space is \((R, d)\), where \(d: R \times R \to R\) is given by \(d(x_1, x_2) = |x_1 - x_2|\).
Here \(U = \{ 0 \}\).
Let \(x_0 = 0 \in U\), Let \(r > 0\).
Then \(S_r(0) = \{ x \mid x \in R, \quad d(x, 0) < r \}\)
\[ = \{ x \mid x \in R, \quad |x - 0| < r \}\]
\[ = \{ x \mid x \in R, \quad |x| < r \}\]
\[ = \{ x \mid x \in R, \quad x < r, \quad x > -r \}\]
\[ = \{ x \mid x \in R, \quad -r < x < r \}\]
\[ = ] -r, +r[ \]
We can find a value of \(r\) for which \(S_r(0) = ] -r, +r[ \not\subseteq U = \{ 0 \}\)
Thus \(U = \{ 0 \}\) is not an open set.
Theorem

Let $(X, d)$ be a metric space, then

(i) Union of any collection $\{ U_\alpha : \alpha \in I \}$ of open sets is open.

(ii) Intersection of finite number of open sets is open.

(iii) The whole space $X$ and the empty set $\phi$ are both open.

Proof

(i) Let $\{ U_\alpha : \alpha \in I \}$ be any collection of open sets in $(X, d)$.

We are to prove that, $\bigcup_{\alpha \in I} U_\alpha$ is an open set.

Let $x \in \bigcup_{\alpha \in I} U_\alpha$

Then $x \in U_\alpha$ for some $\alpha \in I$

Since each $U_\alpha$ is an open set therefore there exist $r > 0$

Such that $S_r(x) \subseteq U_\alpha$ for some $\alpha \in I$

$\Rightarrow S_r(x) \subseteq \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is an open set.

(ii) Let $\{ U_\alpha : \alpha = 1, 2, \ldots, n \}$ be finite collection of open sets in $(X, d)$.

We are to prove that $\bigcap_{\alpha = 1}^n U_\alpha$ is an open set.

Let $x \in \bigcap_{\alpha = 1}^n U_\alpha$

$\Rightarrow x \in U_\alpha \quad \forall \alpha = 1, 2, \ldots, n$

Since each $U_\alpha$ is an open set therefore there exist $r > 0$

Such that $S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall \alpha = 1, 2, \ldots, n$

Let $r = \min \{ r_1, r_2, r_3, \ldots, r_n \}$

Then $S_r(x) \subseteq S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall \alpha = 1, 2, \ldots, n$

$\Rightarrow S_r(x) \subseteq U_\alpha \quad \forall \alpha = 1, 2, \ldots, n$

$\Rightarrow S_r(x) \subseteq \bigcap_{\alpha = 1}^n U_\alpha$

$\Rightarrow \bigcap_{\alpha = 1}^n U_\alpha$ is an open set.

(iii) To show that empty set $\phi$ is an open set, we have to show that each point in $\phi$ is the centre of some open sphere which is contained in $\phi$.

But since there is no point in $\phi$, the condition is automatically satisfied.

Hence $\phi$ is an open set.

Since every open sphere centered at a point of $X$ is contained in $X$.

$\therefore X$ is an open set.
Theorem

An open sphere in a metric space \((X, d)\) is an open set.

Proof

Let \( S_r(x_o) \) be an open sphere in \((X, d)\).

Let \( x' \in S_r(x_o) \Rightarrow d(x', x_o) < r \)

Let \( d(x', x_o) = \lambda \)

Let \( r' = r - \lambda \), then \( r' > 0 \)

We shall prove that \( S_r'(x') \subseteq S_r(x_o) \)

Let \( x \in S_r'(x') \Rightarrow d(x, x') < r' \)

Since \((X, d)\) is a metric space,
\[
\begin{align*}
\therefore \quad d(x, x') + d(x', x_o) & \geq d(x, x_o) \\
& \Rightarrow r' + \lambda > d(x, x_o) \\
& \Rightarrow r - \lambda + \lambda > d(x, x_o) \quad \therefore \ r' = r - \lambda \\
& \Rightarrow d(x, x_o) < r \\
& \Rightarrow x \in S_r(x_o)
\end{align*}
\]

Since \( x \in S_r'(x') \Rightarrow x \in S_r(x_o) \)

\[ S_r'(x') \subseteq S_r(x_o) \]

Thus \( S_r(x_o) \) is an open set.

Hence open sphere in a metric space is an open set.

Theorem

A subset \( U \) of a metric space \( X \) is open if and only if \( U \) is union of open spheres.

Proof

Let \((X, d)\) be a metric space. Let \( U \subseteq X \). We have to prove that \( U \) is an open set \( \iff \) \( U \) is the union of open spheres.

We suppose that \( U \) is an open set. Since \( U \) is open therefore each point of \( U \) is the centre of some open sphere which is contained in \( U \).

Thus \( U \) is the union of open spheres.

Conversely suppose that \( U \) is the union of open spheres. Thus \( U \) is the union of open sets. ( \( \because \) Open spheres in metric space are open sets.)

Since the union of any number of open sets in a metric space is an open set. Thus \( U \) is an open set.
**Theorem**

Let \( X \) be a metric space and let \( \{ x_0 \} \) be a singleton subset of \( X \). Then \( X - \{ x_0 \} \) is open.

**Proof**

Let \( x \in X - \{ x_0 \} \)
Let \( d(x, x_0) = r \). \( \quad \cdots (1) \)

We shall prove that
\[
S_r(x) \subseteq X - \{ x_0 \}
\]

Let \( x' \in S_r(x) \)
\[
d(x', x) < r \quad \cdots (2)
\]

From (1) and (2) we get
\[
d(x', x) \neq d(x, x_0)
\]
\[
d(x, x') \neq d(x, x_0) \quad \because \text{d is a metric on } X. \text{ So } d(x', x) = d(x, x')
\]
\[
x' \neq x_0
\]
\[
x' \notin \{ x_0 \}
\]
\[
x' \in X - \{ x_0 \}
\]

Since \( x' \in S_r(x) \) \( \implies x' \in X - \{ x_0 \} \)
\[
S_r(x) \subseteq X - \{ x_0 \}
\]

Since every point \( x \) of \( X - \{ x_0 \} \) is the center of some open sphere contained in \( X - \{ x_0 \} \).

Hence \( X - \{ x_0 \} \) is an open set.

**Question**

Can a finite subset of a metric space be open?

**Solution**

We know that

(i) If \((X, d)\) is a discrete metric space, then every subset of \(X\) is open.

Therefore a finite subset of a metric space is open.

(ii) If \((R, d)\) is a usual metric space then \(\{0\} \subseteq R, \) is not open.

Therefore a finite subset \(\{0\}\) of \(R\) is not open.

Thus in general, we can say that, finite subset of a metric space may or may not open.

**Metric Topology**

The topology determined by a metric is called "metric topology".
Theorem

If T is a collection of all open sets in a metric space \((X, d)\), then T is a topology on X.

OR

A "metric space" is a topological space.

Proof

Let T be the collection of all open sets in a metric space \((X, d)\).

We are to prove that, T is a topology on X.

(T1) Let \( U_\alpha \in T \quad \forall \ \alpha \in I \)

\( \Rightarrow \) \( U_\alpha \) is an open set. \( \forall \ \alpha \in I \)

\( \Rightarrow \) \( \bigcup_{\alpha \in I} U_\alpha \) is open.

( \( \therefore \) Union of any number of open sets is open. )

\( \Rightarrow \) \( \bigcup_{\alpha \in I} U_\alpha \in T \)

(T2) Let \( U_\alpha \in T \quad \forall \ \alpha = 1,2,...n \)

\( \Rightarrow \) \( \bigcap_{\alpha = 1}^n U_\alpha \) is an open set.

( \( \therefore \) Intersection of finite number of open sets is open. )

\( \Rightarrow \) \( \bigcap_{\alpha = 1}^n U_\alpha \in T \) (By definition of T)

(T3) Since \( \phi, X \) both are open.

\( \therefore \) \( \phi, X \in T \) (By definition of T)

Thus T is a topology on X.

i.e. \((X, T)\) is a topological space.

This shows that a "metric space" is a "topological space" whose topology is "metric topology".

Theorem

Every non-empty set can be given a metric topology.

Proof

We know that

(i) Every non-empty set can be given a metric and can be converted into metric space.

Therefore, a finite subset of a metric space is open.

(ii) Every "metric space" is a "topological space" whose topology is a "metric topology".

Thus from (i) and (ii) we conclude that every non-empty set can be given a metric topology.
CLOSED SET

Closed Set
Let \((X, d)\) be a metric space. Let \(F \subseteq X\).
Then \(F\) is closed \(\iff F' = X - F\) is open.

Example
Let \(X = R\) be the metric space and let \(A = [a, b]\), where \(a, b \in R, \; a < b\). Show that \(A\) is closed set.

Solution
The given metric space is \((R, d)\), where \(d: R \times R \to R\) is given by
\[
d(x_1, x_2) = |x_1 - x_2|
\]
Since \(A = [a, b]\)
\[
A' = R - [a, b]
\]
\[
= ] - \infty, a[ \cup ]b, \infty[
= L_a \cup R_b
\]
In order to prove \(A\) is closed; we will have to prove that \(A'\) is open.
Let \(x' \in A' \implies x' \in L_a \cup R_b \implies x' \in L_a \text{ or } x' \in R_b
Case-I
If \(x' \in L_a\) then \(x' < a\)
Let \(d(x', a) = r\)
\[
\implies |x' - a| = r
\]
\[
\implies x' - a = -r \implies x' < a
\]
\[
\implies x' + r = a \quad \text{--------------------------- (1)}
\]
Now \(S_r(x') = \{ x | x \in R, \; d(x, x') < r \}\)
\[
= \{ x | x \in R, \; |x - x'| < r \}
= \{ x | x \in R, \; x - x' < r, \; x - x' > -r \}
= \{ x | x \in R, \; x < x' + r, \; x > x' - r \}
= \{ x | x \in R, \; x' - r < x < x' + r \}
= ] x' - r, x' + r [\]
\[
= ] x' - r, a [ \quad \text{[ By (1)]}
\]
Thus \(S_r(x') = ] x' - r, a [ \subseteq L_a \subseteq L_a \cup R_b = A'\)
i.e. \(S_r(x') \subseteq A'\)
Hence in this case \(A'\) is open.
Case-II If \( x' \in R_b \) then \( x' > b \)

Let \( d(x', b) = r \)

\[
\Rightarrow |x' - b| = r
\]

\[
\Rightarrow x' - b = r \quad \therefore x' > b
\]

\[
\Rightarrow x' - r = b \quad \text{----------------------------- (2)}
\]

Now \( S_r(x') = \{ x | x \in R, d(x, x') < r \} \)

\[
= \{ x | x \in R, |x - x'| < r \}
\]

\[
= \{ x | x \in R, x - x' < r, x - x' > -r \}
\]

\[
= \{ x | x \in R, x < x' + r, x > x' - r \}
\]

\[
= \{ x | x \in R, x' - r < x < x' + r \}
\]

\[
= [x' - r, x' + r [ \quad \text{[By (2)]}
\]

Thus \( S_r(x') = [b, x' + r [ \subseteq R_b \subseteq \bar{R}_a \cup R_b = A' \)

i.e. \( S_r(x') \subseteq A' \)

Hence in this case \( A' \) is also open.

Since in both the cases \( A' \) is open. Therefore \( A \) is closed set.

Example

Let \( R^2 \) be the metric space.

Let \( F = \{(x, y) | (x, y) \in R^2, (x - a)^2 + (y - b)^2 \leq 1 \} \)

Show that \( F \) is closed set.

Solution

Here given metric space is \((R^2, d)\) where \(d: R^2 \times R^2 \rightarrow R\) is given by \(d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\)

Here \( F = \{(x, y) | (x, y) \in R^2, (x - a)^2 + (y - b)^2 \leq 1 \} \)

Thus \( F' = \{(x, y) | (x, y) \in R^2, (x - a)^2 + (y - b)^2 > 1 \} \)

In order to prove that \( F \) is closed, we will show that \( F' \) is open.

Let \( P' \in F' \). Let \( d(P', P_0) = \lambda \)

Let \( r = \lambda - 1 \), clearly \( r > 0 \)

We shall prove that \( S_r(P') \subseteq F' \)

Let \( P \in S_r(P') \Rightarrow d(P, P') < r \)

Since \( d \) is a metric on \( R^2 \)

\[
\therefore d(P', P) + d(P, P_0) \geq d(P', P_0)
\]

\[
\Rightarrow d(P', P) + d(P, P_0) > d(P', P_0)
\]

\[
\Rightarrow d(P', P) > d(P', P_0) - d(P, P_0)
\]

Thus \( P \in S_r(P') \subseteq F' \).
let \( P \in F \)

\[ \Rightarrow \quad r + d(P, P_0) > \lambda \quad \therefore \lambda \geq d(P, P_0) \]

\[ \Rightarrow \quad d(P, P_0) > \lambda - r \]

\[ \Rightarrow \quad d(P, P_0) > \lambda - (\lambda - 1) = 1 \]

\[ \Rightarrow \quad d(P, P_0) > 1 \]

\[ \Rightarrow \quad P \in F' \]

Since \( P \in S_r(P_0) \Rightarrow P \in F' \)

\[ \therefore \quad S_r(P_0) \subseteq F' \]

\[ \Rightarrow \quad F' \text{ is an open set.} \]

\[ \Rightarrow \quad F \text{ is closed set.} \]

**Example**

Let \( R^2 \) be the metric space.

Let \( A = \{(x, y) | (x, y) \in R^2, \ x^2 + y^2 \leq 1 \} \) be a subset of \( R^2 \).

Is \( A \) a closed set in \( R^2 \)?

**Solution**

Here given metric space is \((R^2, d)\) where \( d: R^2 \times R^2 \rightarrow R \) is given by

\[ d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \]

Here \( A = \{(x, y) | (x, y) \in R^2, \ x^2 + y^2 \leq 1 \} \)

Thus \( A' = \{(x, y) | (x, y) \in R^2, \ x^2 + y^2 > 1 \} \)

In order to prove that \( A \) is closed, we will show that \( A \) is open.

Let \( P' \in A' \). Let \( d(P', O) = \lambda \)

Let \( r = \lambda - 1 \), clearly \( r > 0 \)

We shall prove that \( S_r(P') \subseteq A' \)

Let \( P \in S_r(P') \Rightarrow d(P, P') < r \)

Since \( d \) is a metric on \( R^2 \)

\[ \Rightarrow \quad d(P', P) + d(P, O) \geq d(P', O) \]

\[ \Rightarrow \quad r + d(P, O) > \lambda \quad \therefore \lambda > d(P, P') \]

\[ \Rightarrow \quad d(P, O) > \lambda - r \]

\[ \Rightarrow \quad d(P, O) > \lambda - (\lambda - 1) = 1 \]

\[ \Rightarrow \quad d(P, O) > 1 \]

\[ \Rightarrow \quad P \in A' \]

Since \( P \in S_r(P') \Rightarrow P \in A' \)

\[ \therefore \quad S_r(P') \subseteq A' \]

\[ \Rightarrow \quad A' \text{ is an open set.} \]

\[ \Rightarrow \quad A \text{ is closed set.} \]
Example
Let \( R \) be the real line and let \( A = \{ x \in R, 0 \leq x < 1 \} \), be a subset of \( R \). Show that \( A \) is not closed.

Solution
The given metric space is \((R, d)\), where \( d: R \times R \to R \) is given by
\[
d(x_1, x_2) = |x_1 - x_2|
\]
Here
\[
A = \{ x \in R, 0 \leq x < 1 \}
\]
= \([0,1]\)
\[
A' = R - A
\]
= \([-\infty, 0 ) \cup [1, \infty]\)

Note that, \( 1 \in A' \). We take \( x_0 = 1 \), and \( r > 0 \)

Then \( S_r(x_0) = \{ x \in X, d(x, x_0) < r \} \)

Put \( x_0 = 1 \) and \( X = R \)
\[
S_r(1) = \{ x \in R, d(x, 1) < r \}
\]
= \{ x \in R, |x - 1| < r \}
= \{ x \in R, x - 1 < r, x - 1 > -r \}
= \{ x \in R, x < 1 + r, x > 1 - r \}
= \{ x \in R, 1 - r < x < 1 + r \}
= \] 1 - r, 1 + r [

But \( S_r(1) = ] 1 - r, 1 + r [ \not\in A' \) \( \forall r > 0 \)

Thus \( A' \) is not open.
\[
\Rightarrow A \text{ is not closed.}
\]

Theorem
A subset \( U \) of a metric space is open if and only if \( X - U \) is closed.

Proof
Let \((X, d)\) be a metric space. We have to prove that
\[
U \text{ is open } \iff X - U \text{ is closed.}
\]
Suppose \( U \) is an open set.
Then \( (X - U)' = (U')' \) \( \Rightarrow X - U = U' \) (Open set)

Since \( (X - U)' \) is an open set.
\[
\Rightarrow X - U \text{ is a closed set.}
\]

Conversely suppose that \( X - U \) is a closed set.
Then \( (X - U)' \) is an open set.
\[
\Rightarrow (U')' \text{ is an Open set.} \Rightarrow X - U = U'
\]
\[
\Rightarrow U \text{ is an Open set.}
\]
Theorem
Let $X$ be a metric space.

(i) Intersection of any collection $\{ F_\alpha : \alpha \in I \}$ of closed sets is closed.

(ii) Union of finite collection $\{ F_1, F_2, \ldots, F_n \}$ of closed set is closed.

(iii) $X$ and $\phi$ are closed.

Proof
(i) Let $\{ F_\alpha : \alpha \in I \}$ be any collection of closed sets in $(X, d)$.
Then $F'_\alpha$ is open. $\forall \alpha \in I$

\[ \Rightarrow \bigcup_{\alpha \in I} F'_\alpha \text{ is open.} \quad (\because \text{Union of any number of open sets is open}) \]

\[ \Rightarrow \left( \bigcap_{\alpha \in I} F_\alpha \right)' \text{ is open.} \quad \therefore \bigcup_{\alpha \in I} F'_\alpha = \left( \bigcap_{\alpha \in I} F_\alpha \right)' \]

\[ \Rightarrow \bigcap_{\alpha \in I} F_\alpha \text{ is closed.} \]

(ii) Let $\{ F_\alpha : \alpha = 1, 2, \ldots, n \}$ be any finite collection of closed sets in $(X, d)$.
Then $F'_\alpha$ is open. $\forall \alpha = 1, 2, \ldots, n$

\[ \Rightarrow \bigcap_{\alpha = 1}^{n} F'_\alpha \text{ is open.} \quad (\because \text{Intersection of finitely many open sets is open}) \]

\[ \Rightarrow \left( \bigcup_{\alpha = 1}^{n} F_\alpha \right)' \text{ is open.} \quad \therefore \bigcap_{\alpha = 1}^{n} F'_\alpha = \left( \bigcup_{\alpha = 1}^{n} F_\alpha \right)' \]

\[ \Rightarrow \bigcup_{\alpha = 1}^{n} F_\alpha \text{ is closed.} \]

(iii) Since $\phi' = X - \phi = X$ which is open.

\[ \Rightarrow \phi \text{ is closed.} \]

And $X' = X - X = \phi$ which is open.

\[ \Rightarrow X \text{ is closed.} \]

Question
Is $N$ closed in $R$?

Solution
Here $N = \{ 1, 2, 3, \ldots \}$

\[ N' = R - N \]

\[ = \left( -\infty, 1 \right] \cup \left( 1, 2 \right] \cup \left( 2, 3 \right] \cup \ldots \]

= Union of open intervals in $R$

= Union of open sets $\quad (\because \text{An open intervals in } R \text{ is an open set})$

= Open set $\quad (\because \text{Union of any number of open sets is an open set})$

Since $N'$ is an open set.

\[ \Rightarrow N \text{ is a closed set.} \]
Limit Point

Let \((X, d)\) be a metric space. Let \(A \subseteq X\) and \(x_0 \in X\). Then \(x_0\) is called a limit point of \(A\) if each open sphere centered at \(x_0\) contains at least one point of \(A\) different from \(x_0\).

\[
S_{r_1}(x_0) \quad S_{r_2}(x_0) \quad S_{r_3}(x_0)
\]

(Fig - 1) \(S_{r_3}(x_0)\) (Fig - 2) \(S_{r_2}(x_0)\) (Fig - 3) \(S_{r_3}(x_0)\)

In Fig - 1, \(x_0\) is a limit point of \(A\).

In Fig - 2, \(x_0\) is also a limit point of \(A\).

In Fig - 3, \(x_0\) is not a limit point of \(A\).

Theorem

Let \((X, d)\) be a discrete metric space. Let \(A \subseteq X\). Then \(A\) has no limit point.

Proof

Consider the discrete metric space \((X, d_0)\).

Here \(d_0: X \times X \rightarrow \mathbb{R}\) is defined by

\[
d_0(x_1, x_2) = \begin{cases} 
0 & \text{if } x_1 = x_2 \\
1 & \text{if } x_1 \neq x_2
\end{cases}
\]

We have to prove that, \(A \subseteq X\) has no limit point.

We shall prove it by contradiction method.

Suppose \(x_0 \in X\) such that \(x_0\) is a limit point of \(A\).

Let \(0 < r < 1\) then \(S_r(x_0) = \{x_0\}\) ---(1)

\(\because\) In a discrete metric space the open sphere with radius less than 1 is always singleton.

Here (1) shows that \(S_r(x_0)\) contains no point of \(A\) different from \(x_0\).

Thus \(x_0\) is not a limit point of \(A\).

Hence \(A\) has no limit point.
Question

Let $R$ be the metric space. Let $A = \{x|x \in R, x = \frac{1}{n}, n \in N\}$ be a subset of $R$. Show that “0” is a limit point of $A$.

Solution

Here metric space is $(R, d)$, where $d: R \times R \to R$ be defined by

$d(x_1, x_2) = |x_1 - x_2|$

Here $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \}$

Then $S_r(0) = \{x|x \in R, d(x, 0) < r\}; r > 0$

= $\{x|x \in R, |x - 0| < r\}$
= $\{x|x \in R, |x| < r\}$
= $\{x|x \in R, x < r, x > -r\}$
= $\{x|x \in R, -r < x < r\}$
= $[-r, +r[$

Clearly for every $r > 0$, $S_r(0) = ] - r, +r [$ contains a point of $A$ different from “0”.

Thus “0” is the limit point of $A$.

Question

Let $R$ be the metric space. Let $A = \{x|x \in R, x = 1$ or $x = 1 + \frac{1}{n}, n \in N\}$ be a subset of $R$. Show that “1” is a limit point of $A$.

Solution

Here metric space is $(R, d)$, where $d: R \times R \to R$ be defined by

$d(x_1, x_2) = |x_1 - x_2|$

Here $A = \{x|x \in R, x = 1$ or $x = 1 + \frac{1}{n}, n \in N\}$

= $\{x|x \in R, x = 1\} \cup \{x|x \in R, x = 1 + 1/n, n \in N\}$

= $\{1\} \cup \{2, \frac{3}{2}, \frac{4}{3}, \ldots, \}$

= $\{1, 2, \frac{3}{2}, \frac{4}{3}, \ldots, \}$

Now $S_r(1) = \{x|x \in R, d(x, 1) < r\}$

= $\{x|x \in R, |x - 1| < r\}$
= $\{x|x \in R, x - 1 < r, x - 1 > -r\}$
= $\{x|x \in R, x < 1 + r, x > 1 - r\}$
= $\{x|x \in R, 1 - r < x < 1 + r\}$
= $[1 - r, 1 + r[$

Clearly for every $r > 0$, $S_r(1) = ]1 - r, 1 + r [$ contains a point of $A$ different from “1”.

Thus “1” is a limit point of $A$. 
Question
Let $R$ be the metric space. Let $A = \{ x | x \in R, \ 0 < x < 1 \}$ be a subset of $R$. Show that "0" and "1" are the limit point of $A$.

Solution
Here metric space is $(R, d)$, where $d: R \times R \rightarrow R$ be defined by
$$d(x_1, x_2) = |x_1 - x_2|$$
Here $A = \{ x | x \in R, \ 0 < x < 1 \}$

First we shall prove that "0", is the limit point of $A$.

Now $S_r(0) = \{ x | x \in R, \ d(x, 0) < r \}$; $r > 0$

$= \{ x | x \in R, \ |x - 0| < r \}$
$= \{ x | x \in R, \ |x| < r \}$
$= \{ x | x \in R, \ x < r, \ x > -r \}$
$= \{ x | x \in R, \ -r < x < r \}$
$= ] - r, + r [$
Clearly for every $r > 0$, $S_r(0) = ] - r, + r [$ contains a point of $A$ different from "0".
Thus "0" is a limit point of $A$.

(ii) Now we shall prove that "1" is the limit point of $A$.

Now $S_r(1) = \{ x | x \in R, \ d(x, 1) < r \}$

$= \{ x | x \in R, \ |x - 1| < r \}$
$= \{ x | x \in R, \ x - 1 < r, \ x - 1 > -r \}$
$= \{ x | x \in R, \ x < 1 + r, \ x > 1 - r \}$
$= \{ x | x \in R, \ 1 - r < x < 1 + r \}$
$= ] 1 - r, 1 + r [$
Clearly for every $r > 0$, $S_r(1) = ] 1 - r, 1 + r [$ contains a point of $A$ different from "1".
Thus "1" is a limit point of $A$.

Question
Let $R$ be the metric space. Describe the limit points of the followings.
(a) $N$ (b) $Z$

Solution
Here metric space is $(R, d)$, where $d: R \times R \rightarrow R$ be defined by
$$d(x_1, x_2) = |x_1 - x_2|$$
(a) Here $N = \{1, 2, 3, \ldots\}$

Let $a \in \mathbb{R}$ be a limit point of $N$.

Then $a \in N$ or $a \notin N$

**Case - I** When $a \in N$

Then $S_r(a) = \{x | x \in \mathbb{R}, d(x, a) < r \}; \quad r > 0$

\[
= \{x | x \in \mathbb{R}, |x - a| < r \}
= \{x | x \in \mathbb{R}, x - a < r, x - a > -r \}
= \{x | x \in \mathbb{R}, x < a + r, x > a - r \}
= \{x | x \in \mathbb{R}, a - r < x < a + r \}
= ]a - r, a + r[
\]

Clearly for every $r > 0$, $S_r(1) = ]a - r, a + r[ \text{ contains no point of } N \text{ different from } "a".$

Thus "a" is not the limit point of $N$.

**Case - II** When $a \notin N$, we can also prove that "a" is not a limit point of $N$.

Thus $N$ has no limit point.

(b) Here $Z = \{..., -3, -2, -1, 0, 1, 2, ..., \ldots\}$

Let $a \in \mathbb{R}$ be a limit point of $Z$.

Then $a \in Z$ or $a \notin Z$

**Case - I** When $a \in Z$

Then $S_r(a) = \{x | x \in \mathbb{R}, d(x, a) < r \}; \quad r > 0$

\[
= \{x | x \in \mathbb{R}, |x - a| < r \}
= \{x | x \in \mathbb{R}, x - a < r, x - a > -r \}
= \{x | x \in \mathbb{R}, x < a + r, x > a - r \}
= \{x | x \in \mathbb{R}, a - r < x < a + r \}
= ]a - r, a + r[
\]

Clearly for every $r > 0$, $S_r(1) = ]a - r, a + r[ \text{ contains no point of } Z \text{ different from } "a".$

Thus "a" is not the limit point of $Z$.

**Case - II** When $a \notin Z$, we can also prove that "a" is not a limit point of $Z$.

Thus $Z$ has no limit point.
Neighbourhood
Let \((X, d)\) be a metric space. Let \(x_0 \in X\). Let \(N \subseteq X\). Then \(N\) is called a neighbourhood of \(x_0\), if \(\exists\) an open sphere \(S_r(x_0)\) such that \(x_0 \in S_r(x_0) \subseteq N\).

Example
Let \(R\) be the usual metric space. Let \(x_0 = 0 \in R\). Show that \([-r, r], [-r, r], [r, -r], [-r, r], (r > 0)\) is a neighbourhood of 0.

Solution
We know that,

In a usual metric space \(R\), the open sphere is an open interval.

(i) Now \(0 \in \] -r, r \[ \subseteq \] -r, r \[\) Where \(\] -r, r \[\) is an open sphere in \(R\) 
\(\Rightarrow\) \(\] -r, r \[\) is a neighbourhood of “0”.
(ii) Now \(0 \in \] -r, r \[ \subseteq \] -r, -r \[\) Where \(\] -r, r \[\) is an open sphere in \(R\) 
\(\Rightarrow\) \(\] -r, r \[\) is a neighbourhood of “0”.
(iii) Now \(0 \in \] -r, r \[ \subseteq \] -r, r \[\) Where \(\] -r, r \[\) is an open sphere in \(R\) 
\(\Rightarrow\) \(\] -r, r \[\) is a neighbourhood of “0”.
(iv) Now \(0 \in \] -r, r \[ \subseteq \] -r, r \[\) Where \(\] -r, r \[\) is an open sphere in \(R\) 
\(\Rightarrow\) \(\] -r, r \[\) is a neighbourhood of “0”.

Theorem
Let \((X, d)\) be a metric space. Let \(A \subseteq X\): Let \(x_0\) be a limit point of \(A\). Then every neighbourhood of \(x_0\) contains infinitely many points of \(A\).

Proof
Let \(N\) be a neighbourhood of \(x_0\), then \(\exists\) an open sphere \(S_r(x_0)\) (where \(r > 0\)) such that

\(x_0 \in S_r(x_0) \subseteq N\)  --- (1)

We are to prove that \(N\) contains infinite points of \(A\).

We prove it by contradiction method.

Suppose \(N\) contains finite points of \(A\).
Then by (1) \(S_r(x_0)\) also contains finite points of \(A\).
Suppose \(S_r(x_0)\) contains \(n\) points \(x_1, x_2, x_3, \ldots, x_n\) of \(A\).
Then \(A \cap S_r(x_0) = \{x_1, x_2, x_3, \ldots, x_n\}\)
Let \(d(x_0, x_i) = r_i, \ i = 1, 2, 3, \ldots, n\)
Let \(r' = \min (r_1, r_2, r_3, \ldots, r_n)\)
Clearly $S_r(x_*)$ contains no point of $A$ different from $x_*$.

This shows that, $x_*$ is not a limit point of $A$. This is a contradiction.

Hence $N$ contains infinitely many points of $A$. 

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