Transforming a Process into a Markov Chain

Transforming Process into a Markov Chain

Suppose that whether or not it rains today depends on previous weather conditions through the last two days.

Specifically, suppose that

- if it has rained for the past two days, then it will rain tomorrow with probability 0.7;
- if it rained today but not yesterday, then it will rain tomorrow with probability 0.5;
- if it rained yesterday but not today, then it will rain tomorrow with probability 0.4;
- if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

Simply knowing if it's **raining today** is **not enough** to predict tomorrow's rain in this model. Because the **rain from yesterday** is also needed, the process **fails the Markov Property**.

However, we can transform this model into a Markov chain by saying that the state at any time is determined by the weather conditions during both that day and the previous day. In other words, we can say that the process is in

- state 0: if it rained both today and yesterday, [or simply (R,R)]
- state 1: if it rained today but not yesterday, [or simply (N,R)]
- state 2: if it rained yesterday but not today, [or simply (R,N)]
- state 3: if it did not rain either yesterday or today. [or simply (N,N)]

Here N representing not rain and R representing as rain.

Yesterday	Today	Tomorrow	Probability
R	R	R N	0.7 ?
N	R	R N	0.5 ?
R	N	R N	0.4 ?
N	N	R N	0.2 ?

={ CALCULATION }=

$$(R,R) \rightarrow \begin{cases} (R, R): 0.7 \\ (R, N): 0.3 \end{cases}$$

$$(N,R) \rightarrow \begin{cases} (R, R): 0.5 \\ (R, N): 0.5 \end{cases}$$

$$(R,N) \rightarrow \begin{cases} (N, R): 0.4 \\ (N, N): 0.6 \end{cases}$$

$$(N,N) \rightarrow \begin{cases} (N, R): 0.2 \\ (N, N): 0.8 \end{cases}$$

Now this a four-state Markov chain and the following questions can be asked:

(i) Write the mathematical model of the above stochastic process.

(ii) Draw the state transition diagram.

(iii) Write the transition probabilities P_{ij}.

(iv) Write the transition probability matrix P.

Second-Order Markov Property

The process $\{X_n\}$ is said to be a **second-order Markov chain** if, for all $n \geq 1$and for all states $i_0, i_1, \dots, i_{n+1} \in S$,

$$P(X_{n+1} = i_{n+1} \mid X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$
$$= P(X_{n+1} = i_{n+1} \mid X_n = i_n, X_{n-1} = i_{n-1}).$$

A second-order Markov chain can be **converted into a first-order chain** by defining an *augmented state variable*:

$$Y_n = (X_n, X_{n-1}),$$

whose state space is $S \times S = \{(i,j): i, j \in S\}$.

Then:

$$P_{ij} = P\big(Y_{n+1} = j \coloneqq (i_{n+1}, i_n) \mid Y_n = i \coloneqq (i_n, i_{n-1})\big)$$
$$= P(X_{n+1} = i_{n+1} \mid X_n = i_n, X_{n-1} = i_{n-1})$$

Thus $\{Y_n\}$forms a **Markov chain** on $S \times S$ (it is also known as first order Markov chain).

Question

A student's studying habit each night is classified into one of three states:

State	Habit Description
I	Intense Study ($>$ 3 hours)
M	Moderate Study (1 to 3 hours)
N	No Study ($<$ 1 hour)

Suppose the probability of a student choosing to Intensely Study (I) tomorrow is determined by the last two nights' habits as follows:

1. If the habit for the past two nights was (Intense, Intense), the probability is 0.8.
2. If the habit was (No Study, No Study), the probability is 0.1.
3. If the habit was (Moderate, Intense), the probability is 0.5.
4. In all other combinations, the probability of Intense Study tomorrow is 0.3.

- ***Write the state space of the Markov Model related to the above problem.***
- ***Draw the state transition diagram taking each state as vertex of graph.***
- ***Write the transition probability matrix.***

Thank you so much