

# **Stationary Distribution in Markov Chain & Initial State Vector Method**

## Motivation

Consider two state Markov model  $\{X_n, n \geq 0\}$  as follows:

A student moves between two places on campus every hour:

- State 0: Football Ground
- State 1: Canteen

with

$$P(X_{n+1} = 1 | X_n = 0) = 0.3,$$

$$P(X_{n+1} = 0 | X_n = 1) = 0.4.$$

Then the transition probability matrix is as follows:

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

One can note

$$P^8 \approx \begin{bmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{bmatrix}$$

Now what  $P^9, P^{10}, \dots$

Let  $\pi = (\pi_0, \pi_1)$ .

Solve:

$$\pi P = \pi, \quad \pi_0 + \pi_1 = 1$$

Equations:

$$\pi_0 = 0.7\pi_0 + 0.4\pi_1 \dots (1)$$

$$\pi_1 = 0.3\pi_0 + 0.6\pi_1 \dots (2)$$

Substitute  $\pi_1 = 1 - \pi_0$  in (1):

$$\pi_0 = 0.7\pi_0 + 0.4(1 - \pi_0)$$

$$\pi_0 = 0.7\pi_0 + 0.4 - 0.4\pi_0$$

$$\pi_0 = 0.3\pi_0 + 0.4$$

$$0.7\pi_0 = 0.4$$

$$\pi_0 = \frac{4}{7} \approx 0.5714$$

Similarly, one can get  $\pi_1 = \frac{3}{7} \approx 0.4286$ .

## Interpretation

In the long run:

- The student spends 57.14% of the time in the football ground
- The student spends 42.86% of the time in the canteen

Even if the student starts in one place, the proportions eventually settle to these stationary probabilities.

## Stationary Probabilities

Let  $\{X_n, n \geq 0\}$  be a Markov chain with transition probability matrix  $P$ . A probability vector  $\pi$  is stationary if:

$$\pi P = \pi$$

and

$$\sum_i \pi_i = 1, \pi_i \geq 0.$$

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**Note:** A discrete-time Markov chain has a stationary distribution if and only if it is irreducible and positive recurrent.

- ✚ A Markov chain is **irreducible** if for every pair of states  $i, j, i \rightarrow j$ , that is,  $\exists n \geq 1$  such that  $P^n(i, j) > 0$ .
- ✚ A state  $i$  is **positive recurrent** if its expected return time  $\mathbb{E}_i[T_i]$  is **finite**, where  $T_i = \inf \{n \geq 1: X_n = i\}$ .
- ✚ A chain is **positive recurrent** if *all* of its states are positive recurrent.

## The Initial State Vector Method

We have the transition probability matrix:

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

What is the probability of change in state from 1 to 2 after five steps (initial state is 0)?

Since we start from state 1:

$$\pi^{(0)} = [0, 1, 0]$$

This means:  $P(X_0 = 0) = 0$ ,  $P(X_0 = 1) = 1$  and  $P(X_0 = 2) = 0$ .

**After 1 Step**

$$\begin{aligned} \pi^{(1)} &= \pi^{(0)} \times P = [0, 1, 0] \times \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \\ &= [0.3, 0.4, 0.3] \end{aligned}$$

**After 2 Step**

$$\begin{aligned}\pi^{(2)} &= \pi^{(1)} \times P = [0.3, 0.4, 0.3] \times \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \\ &= [0.33, 0.37, 0.30]\end{aligned}$$

**After 3 Step**

$$\begin{aligned}\pi^{(3)} &= \pi^{(2)} \times P = [0.33, 0.37, 0.30] \times P \\ &= [0.336, 0.370, 0.294]\end{aligned}$$

**After 4 Step**

$$\begin{aligned}\pi^{(4)} &= \pi^{(3)} \times P = [0.336, 0.370, 0.294] \times P \\ &= [0.3378, 0.3706, 0.2916]\end{aligned}$$

**After 5 Step**

$$\begin{aligned}\pi^{(5)} &= \pi^{(4)} \times P = [0.3378, 0.3706, 0.2916] \times P \\ &= [0.3384, 0.3708, 0.2908]\end{aligned}$$

Finally, the probability of going from state 1 to state 2 in exactly 5 steps is **0.2908**.

**Problem:**

A mobile phone market has three companies: **A**, **B**, and **C**. Each year, customers may switch between companies according to the following patterns:

- **Company A** retains 80% of its customers, loses 10% to B, and 10% to C
- **Company B** retains 70% of its customers, loses 20% to A, and 10% to C
- **Company C** retains 60% of its customers, loses 30% to A, and 10% to B

**Current market share:** A = 40%, B = 35%, C = 25%

**Question:** What will be the market share of each company after 3 years?

**Solution:**

Let's order the states as [A, B, C]. Then the transition probability matrix is as follows:

$$P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$$

The current market share gives us our initial state vector:

$$\pi^{(0)} = [0.40, 0.35, 0.25]$$

**Market Share After 1 Year:**

$$\begin{aligned}\pi^{(1)} &= \pi^{(0)} \times P = [0.40, 0.35, 0.25] \times \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix} \\ &= [0.465, 0.310, 0.225]\end{aligned}$$

**Market Share After 2 Years:**

$$\begin{aligned}\pi^{(2)} &= \pi^{(1)} \times P = [0.465, 0.310, 0.225] \times P \\ &= [0.5015, 0.2860, 0.2125]\end{aligned}$$

**Market Share After 3 Years:**

$$\begin{aligned}\pi^{(3)} &= \pi^{(2)} \times P = [0.5015, 0.2860, 0.2125] \times P \\ &= [0.522, 0.272, 0.206]\end{aligned}$$

***Final Answer:***

After 3 years, the projected market shares are:

- Company A: 52.2%
- Company B: 27.2%
- Company C: 20.6%

***Thank you so much***