# Chapter 5 – Differentiation

Course Title: Real Analysis 1 Course Code: MTH321
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Differentiation allows us to find rates of change. For example, it allows us to find the rate of change of velocity with respect to time (which is acceleration). Calculus courses succeed in conveying an idea of what a derivative is, and the students develop many technical skills in computations of derivatives or applications of them. We shall return to the subject of derivatives but with a different objective.
Now we wish to see a little deeper and to understand the basis on which that theory develops.
Let $f$ be defined and real valued on $(a,b)$. For any point $c\in(a,b)$, form the quotient

$$\frac{f(x)-f(c)}{x-c}.$$

We fix point $c$ and study the behaviour of this quotient as $x\to c$.

## ❖ DERIVATIVE OF A FUNCTION

***Definition:*** Let $f$ be defined on an open interval $(a,b)$, and assume that $c\in(a,b)$. Then $f$ is said to be differentiable at $c$ whenever the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists. This limit is denoted by $f'(c)$ and is called the derivative of $f$ at point $c$.

***Definition*:** If $f$ is differentiable at each point of $(a,b)$, then we say $f$ is differentiable on $(a,b)$.

## ❖ *Remarks*

- There are so many notations to represents the derivative of the function in the literature.
- If $x-c=h$, then we have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}.$$

## ❖ *Example*

($i$) A function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ;\ x\neq 0\\ 0 & ;\ x=0\end{cases}$$

This function is differentiable at $x=0$ because

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}$$
$$=\lim_{x\to 0}x\sin\frac{1}{x}=0.$$

($ii$) Let $f(x) = x^n$ ; $n \geq 0$ ($n$ is integer), $x \in \mathbb{R}$.

Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to c}\frac{x^n - c^n}{x-c}\\ &= \lim_{x\to c}\frac{(x-c)(x^{n-1}+cx^{n-2}+\ldots+c^{n-2}x+c^{n-1})}{x-c}\\ &= \lim_{x\to c}(x^{n-1}+cx^{n-2}+\ldots+c^{n-2}x+c^{n-1})\\ &= nc^{n-1}.\end{aligned}$$

This implies that $f$ is differentiable every where and $f'(x) = nx^{n-1}$.

❖ ***Theorem*** (Differentiability implies continuity)

Let $f$ be defined on $(a,b)$, if $f$ is differentiable at a point $x \in (a,b)$, then $f$ is continuous at $x$.

*Proof*

We know that

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x} = f'(x)\text{, where } t \neq x \quad \text{and} \quad a < t < b.$$

Now

$$\begin{aligned}\lim_{t\to x}\big(f(t)-f(x)\big) &= \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}(t-x)\\ &= f'(x)\cdot 0\\ &= 0\end{aligned}$$

$$\Rightarrow \lim_{t\to x} f(t) = f(x).$$

This show that $f$ is continuous at $x$. ❑

❖ ***Remarks***

($i$) The converse of the above theorem does not hold.

Consider $f(x) = |x| = \begin{cases} x & if \quad x \geq 0, \\ -x & if \quad x < 0. \end{cases}$

Then $f'(0)$ does not exists but $f(x)$ is continuous at $x = 0$.

($ii$) If $f$ is discontinuous at some point $c$ of the domain of the function then $f'(c)$ does not exist. e.g.

$$f(x) = \begin{cases} 1 & if \quad x \geq 0, \\ 0 & if \quad x < 0. \end{cases}$$

A function $f$ is discontinuous at $x = 0$ therefore it is not differentiable at $x = 0$.

❖ ***Question***

Prove that a differentiable function is continuous, but the converse is not true.

❖ ***Theorem***

Suppose $f$ and $g$ are defined on $(a,b)$ and are differentiable at a point $x \in (a,b)$, then $f+g$, $fg$ and $\dfrac{f}{g}$ are differentiable at $x$ and

(*i*) $(f+g)'(x) = f'(x) + g'(x)$,

(*ii*) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$,

(*iii*) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$, proved $g(x) \neq 0$.

The proof of this theorem can be get from any F.Sc or B.Sc textbook.

❖ ***Remark***

As we know $\lim\limits_{t\to x}\dfrac{f(t)-f(x)}{t-x} = f'(x)$, this gives

$$\frac{f(t)-f(x)}{t-x} = f'(x) + u(t),$$

where $u(t)$ is a function such that $u(t) \to 0$ as $t \to x$.

This gives us $f(t) - f(x) = (t-x)\left[f'(x) + u(t)\right]$, where $u(t) \to 0$ as $t \to x$, as an alternative definition of derivative.

❖ ***Theorem (Chain Rule)***

Suppose $f$ is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in (a,b)$. A function $g$ is defined on an interval $I$ which contains the range of $f$, and $g$ is differentiable at the point $f(x)$. If $h(t) = g\left(f(t)\right)$ ; $a \le t \le b$, then $h$ is differentiable at $x$ and

$$h'(x) = g'\left(f(x)\right) \cdot f'(x).$$

***Proof***

Let $y = f(x)$.

By the definition of the derivative, we have

$$f(t) - f(x) = (t-x)\left[f'(x) + u(t)\right] \quad \ldots\ldots\ldots\ldots (i)$$

and $$g(s) - g(y) = (s-y)\left[g'(y) + v(s)\right] \quad \ldots\ldots\ldots\ldots (ii)$$

where $t \in [a,b]$, $s \in I$ and $u(t) \to 0$ as $t \to x$ and $v(s) \to 0$ as $s \to y$.

Let us suppose $s = f(t)$. Then

$$\begin{aligned} h(t) - h(x) &= g\left(f(t)\right) - g\left(f(x)\right) = g(s) - g(y) \\ &= \left[s-y\right]\left[g'(y)+v(s)\right] && \text{by } (ii) \\ &= \left[f(t)-f(x)\right]\left[g'(y)+v(s)\right] \\ &= (t-x)\left[f'(x)+u(t)\right]\left[g'(y)+v(s)\right] && \text{by } (i) \end{aligned}$$

or if $t \neq x$

$$\frac{h(t)-h(x)}{t-x} = \left[f'(x)+u(t)\right]\left[g'(y)+v(s)\right],$$

taking the limit as $t \to x$ we have

$$h'(x) = [f'(x)+0][g'(y)+0]$$
$$= g'(f(x))\cdot f'(x), \qquad \because\ y = f(x)$$

This is the required result. ❑

### ❖ *Example*

Let us find the derivative of $\sin(2x)$, One way to do that is through some trigonometric identities. Indeed, we have

$$\sin(2x) = 2\sin(x)\cos(x)\cdot$$

So we will use the product formula to get

$$\left(\sin(2x)\right)' = 2\left(\sin'(x)\cos(x) + \sin(x)\cos'(x)\right)$$

which implies

$$\left(\sin(2x)\right)' = 2\left(\cos^2(x) - \sin^2(x)\right)\cdot$$

Using the trigonometric formula $\cos(2x) = \cos^2(x) - \sin^2(x)$, we have

$$\left(\sin(2x)\right)' = 2\cos(2x)\cdot$$

Once this is done, you may ask about the derivative of $\sin(5x)$? The answer can be found using similar trigonometric identities, but the calculations are not as easy as before. We will see how the Chain Rule formula will answer this question in an elegant way.

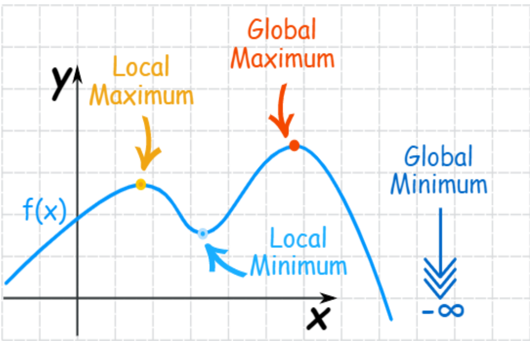
Let us find the derivative of $\sin(5x)\cdot$

We have $h(x) = f(g(x))$, where $g(x) = 5x$ and $f(x) = \sin x$. Then the Chain rule implies that $h'(x)$ exists and

$$h'(x) = 5\cdot[\cos(5x)] = 5\cos(5x)\cdot$$

### ❖ *Maxima and Minima of Functions*

Maxima and minima of a function are the largest and smallest value of the function respectively either within a given range or on the entire domain. Collectively they are also known as extrema of the function. The maxima and minima are the respective plurals of maximum and minimum of a function. Before understanding maxima and minima in detail, let's understand the local maximum and minimum value of the function first.




### ❖ LOCAL MAXIMUM

***Definition:*** Let $f$ be a real valued function defined on a set $E \subseteq \mathbb{R}$, we say that $f$ has a local maximum at a point $p \in E$ if there exist $\delta > 0$ such that $f(x) \le f(p)$ for all $x \in E$ with $|x - p| < \delta$.

Local minimum is defined likewise.

### ❖ GLOBAL (OR ABSOLUTE) MAXIMUM AND MINIMUM

***Definition:*** The maximum or minimum over the entire domain of the function is called an "global" or "absolute" maximum or minimum.

❖ ***Remark:*** There might be only one global maximum (and one global minimum) but there can be more than one local maximum or minimum.

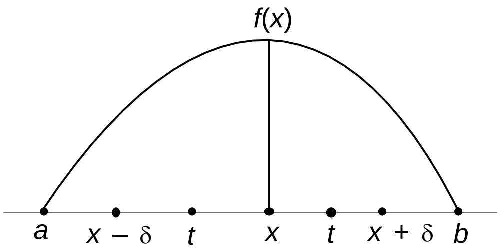
### ❖ *Theorem*

Let $f$ be defined on $[a,b]$ and it is differentiable on $(a,b)$. If $f$ has a local maximum at a point $x \in (a,b)$ and if $f'(x)$ exist, then $f'(x)=0$.

***Proof***

Choose a $\delta > 0$ such that

$$a < x-\delta < x < x+\delta < b$$




Now if $x-\delta < t < x$ then

$$\frac{f(t)-f(x)}{t-x} \geq 0.$$

Taking limit as $t \to x$ we get

$$f'(x) \geq 0 \quad \ldots\ldots\ldots\ldots (i)$$

If $x < t < x+\delta$, then

$$\frac{f(t)-f(x)}{t-x} \leq 0$$

Again, taking limit when $t \to x$ we get

$$f'(x) \leq 0 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Combining ($i$) and ($ii$) we have $f'(x)=0$. ❑

### ❖ *Theorem*

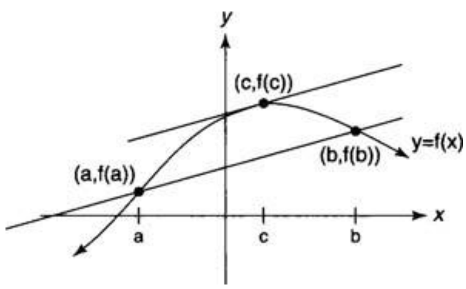
Let $f$ be defined on $[a,b]$ and it is differentiable on $(a,b)$. If $f$ has a local minimum at a point $x \in (a,b)$ and if $f'(x)$ exist then $f'(x)=0$.

The proof of this theorem is like the proof of above theorem.

### ❖ *Lagrange's Mean Value Theorem.*

Let $f$ be continuous on $[a,b]$ and differentiable on $(a,b)$. Then there exists a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$




***Proof.***

Let us design a new function

$$h(t) = \left[f(b)-f(a)\right]t-(b-a)f(t) \qquad ,(a \leq t \leq b)$$

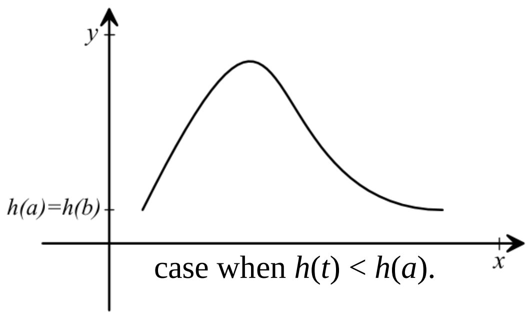
then clearly $h(a)=h(b)$.

Since $h(t)$ depends upon $t$ and $f(t)$ therefore it possesses all the properties of $f$.

Now there are two cases:

i) $h$ is a constant.
implies that $h'(x) = 0 \ \ \forall \ \ x \in (a,b)$.

ii) $h$ is not a constant, then
if $h(t) > h(a) = h(b)$ for some $t \in (a,b)$,
then there exists a point $c \in (a,b)$ at which $h$ attains its maximum implies that $h'(c) = 0$.
and if $h(t) < h(a) = h(b)$
then there exists a point $c \in (a,b)$ at which $h$ attain its minimum implies that $h'(c) = 0$.



case when $h(t) < h(a)$.

Since $h(t) = [f(b) - f(a)]t - (b-a)f(t)$,
therefore $h'(c) = [f(b) - f(a)] - (b-a)f'(c)$.

This gives that $\dfrac{f(b) - f(a)}{b-a} = f'(c)$ as desired. ❑

### ❖ *Generalized Mean Value Theorem*

If $f$ and $g$ are continuous real valued functions on closed interval $[a,b]$ and $f$ and $g$ are differentiable on $(a,b)$, then there is a point $c \in (a,b)$ at which

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c),$$

***Proof.***

Let

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \qquad (a \le t \le b)$$

Since $h$ involves $f$ and $g$ therefore $h$ is

i) continuous on close interval $[a,b]$.
ii) differentiable on open interval $(a,b)$.
iii) and $h(a) = h(b)$.

To prove the theorem, we have to show that $h'(c) = 0$ for some $c \in (a,b)$.

There are two cases to be discussed:

($i$) If $h$ is constant function, then $h'(x) = 0 \ \ \forall \ \ x \in (a,b)$.

($ii$) If $h$ is not constant, then
if $h(t) > h(a) = h(b)$ for some $t \in (a,b)$,
then there exists a point $c \in (a,b)$ at which $h$ attains its maximum,
this implies that $h'(c) = 0$,
and if $h(t) < h(a) = h(b)$ for some $t \in (a,b)$,
then there exists a point $c \in (a,b)$ at which $h$ attain its minimum,
this implies that $h'(c) = 0$.

Hence

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$$

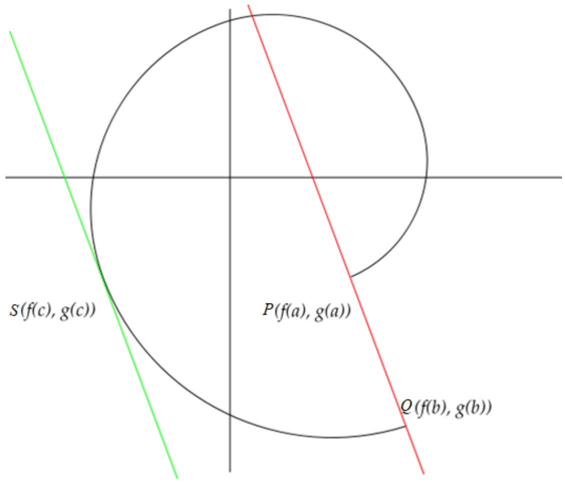
This gives the desire result. ❑

### ❖ *Geometric interpretation of generalized MVT*

Consider a plane curve $C$ represented by

$$x = f(t), \ y = g(t).$$

Then generalized mean value theorem (MVT) states that there is a point $S$ on $C$ between two points $P\big(f(a), g(a)\big)$ and $Q\big(f(b), g(b)\big)$ of $C$ such that the tangent at $S$ to the curve $C$ is parallel to the chord $PQ$.




### ❖ *Theorem (Darboux's Theorem)*

Suppose $f$ is a real differentiable function on some interval $I$ with $a, b \in I, a < b$ and suppose $\lambda$ is a number between $f'(a)$ and $f'(b)$ then there exist a point $x \in (a,b)$ such that $f'(x) = \lambda$.

***Proof***

Without loss of generality assume that $f'(a) < \lambda < f'(b)$.

Also assume that $g(t) = f(t) - \lambda t$ for $t \in I$.

Then $g'(t) = f'(t) - \lambda$

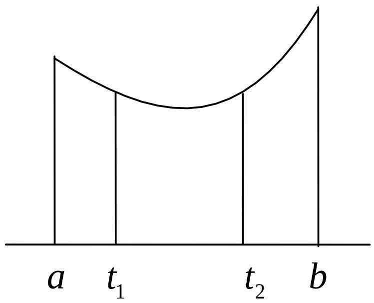
If $t = a$ we have

$$g'(a) = f'(a) - \lambda.$$

Since $f'(a) - \lambda < 0$, therefore $g'(a) < 0$.

This implies that $g$ is monotonically decreasing at $a$.

So there exists a point $t_1 \in (a,b)$ such that $g(a) > g(t_1)$.




Similarly,

$$g'(b) = f'(b) - \lambda$$

Since $f'(b) - \lambda > 0$, therefore $g'(b) > 0$.

This implies that $g$ is monotonically increasing at $b$.

So there exists a point $t_2 \in (a,b)$ such that $g(t_2) < g(b)$

This implies the function attain its minimum on $(a,b)$ at a point $x$ (say) such that $g'(x) = 0 \quad \Rightarrow \quad f'(x) - \lambda = 0$

$$\Rightarrow \quad f'(x) = \lambda.$$

❑

### ❖ *Question*

Let $f$ be defined for all real $x$ and suppose that $|f(x) - f(y)| \le (x-y)^2$ for all real $x$ and $y$. Then prove that $f$ is constant.

***Solution***

Since $\quad |f(x) - f(y)| \le (x-y)^2,$

Therefore

$$-(x-y)^2 \le f(x) - f(y) \le (x-y)^2.$$

Dividing throughout by $x-y$ for $x \neq y$, we get

$$-(x-y) \leq \frac{f(x)-f(y)}{x-y} \leq (x-y) \quad \text{when} \quad x>y$$

and

$$-(x-y) \geq \frac{f(x)-f(y)}{x-y} \geq (x-y) \quad \text{when} \quad x<y$$

Taking limit as $x \to y$, we get

$$\left.\begin{array}{l} 0 \leq f'(y) \leq 0 \\ 0 \geq f'(y) \geq 0 \end{array}\right] \quad \Rightarrow \; f'(y)=0$$

This shows that function is constant.

❖ ***Question (L'Hospital Rule)***

Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$ and $f(x)=g(x)=0$.

Prove that $\lim\limits_{t \to x} \dfrac{f(t)}{g(t)} = \dfrac{f'(x)}{g'(x)}$ .

*Proof*

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t)-0}{g(t)-0} = \lim_{t \to x} \frac{f(t)-f(x)}{g(t)-(x)} \qquad \because \; f(x)=g(x)=0$$

$$= \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \cdot \frac{t-x}{g(t)-(x)}$$

$$= \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \cdot \lim_{t \to x} \frac{1}{\dfrac{g(t)-(x)}{t-x}}$$

$$= \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \cdot \frac{1}{\lim\limits_{t \to x} \dfrac{g(t)-(x)}{t-x}} = f'(x) \cdot \frac{1}{g'(x)} = \frac{f'(x)}{g'(x)}.$$

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***References:***

(1) *Principles of Mathematical Analysis*
*Walter Rudin (McGraw-Hill, Inc.)*

(2) *Introduction to Real Analysis*
*R.G.Bartle, and D.R. Sherbert (John Wiley & Sons, Inc.)*

(3) *Mathematical Analysis,*
*Tom M. Apostol, (Pearson; 2nd edition.)*

(4) *Elementary Real Analysis*
*B.S. Thomson, J.B. Brickner, A.M. Bruckner*
*(ClassicalRealAnalysis.com; 2nd Edition)*

(5) *Paul's Online Notes*
*http://tutorial.math.lamar.edu/*