Chapter 4 – Limit & Continuity

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In this chapter we will introduce the important notion of the limit of a function. The intuitive idea of the function *f* having a limit *L* at the point *a* is that the values *f*(*x*) are close to *L* when *x* is close to (but different from) *a*. But it is necessary to have a technical way of working with the idea of "close to" and this is accomplished in the $\varepsilon-\delta$ definition given below.
In order for the idea of the limit of a function *f* at a point *a* to be meaningful, it is necessary that *f* be defined at points near *a*. It need not be defined at the point *a*, but it should be defined at enough points close to *a* to make the study interesting. This is the reason for the following definition.

❖ LIMIT OF THE FUNCTION

Definition: Suppose $E\subseteq\mathbb{R}$ and $f:E\to\mathbb{R}$ be a function. A number L is called the limit of f when x approaches to *a* if for all $\varepsilon>0$, there exists $\delta>0$ (depending upon ε) such that

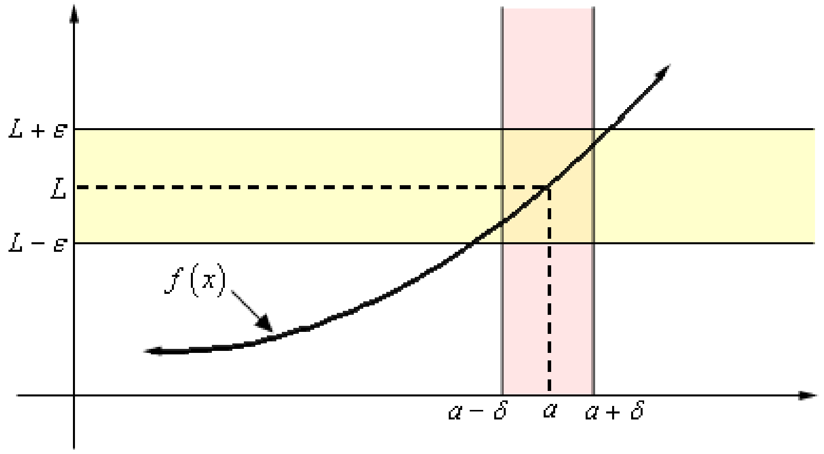
$$|f(x)-L|<\varepsilon \text{ whenever } 0<|x-a|<\delta.$$

Notation: It is written as $\lim\limits_{x\to a} f(x)=L$.

Note: *i*) It is to be noted that $a\in\mathbb{R}$ but that *a* need not a point of E in the above definition (*a* is a limit point of E which may or may not belong to E.)
ii) Even if $a\in E$, we may have $f(a)\neq\lim\limits_{x\to a} f(x)$.

Example:

In the following diagram we have illustrated $\lim\limits_{x\to a} f(x)=L$.



What the definition is telling us is that for any number $\varepsilon>0$ that we pick we can go to our graph and sketch two horizontal lines at $L+\varepsilon$ and $L-\varepsilon$ as shown on the graph above. Then somewhere out there in the world is another number $\delta>0$, which we will need to determine, that will allow us to add in two vertical lines to our graph at $a+\delta$ and $a-\delta$.

❖ ***Example***

(i) Consider the function $f(x)=\dfrac{x^2-1}{x-1}$, $x\neq 1$.

It is to be noted that f is not defined at $x=1$ but if $x\neq 1$ and is very close to 1, then $f(x)$ is close to 2.

To check limit of $f(x)\to 2$ as $x\to 1$, let's start off by letting $\varepsilon>0$ be any number then we need to find a number $\delta>0$ so that the following will be true.

$$\left|\frac{x^2-1}{x-1}-2\right|<\varepsilon \text{ whenever } 0<|x-1|<\delta.$$

We'll start by simplifying the left inequality in an attempt to get a guess for δ. Doing this gives,

$$\left|\frac{x^2-1}{x-1}-2\right|=|x+1-2|=|x-1|<\varepsilon \quad \text{implies } 0<|x-1|<\delta=\varepsilon. \qquad ❑$$

(ii) Lets see by definition: $\lim\limits_{x\to 2}(5x-4)=6$.

Let's start off by letting $\varepsilon>0$ be any number then we need to find a number $\delta>0$ so that the following will be true.

$$|(5x-4)-6|<\varepsilon \text{ whenever } 0<|x-2|<\delta.$$

We'll start by simplifying the left inequality in an attempt to get a guess for δ. Doing this gives,

$$|(5x-4)-6|=|5x-4-6|=|5x-10|=5|x-2|<\varepsilon \quad \text{implies } 0<|x-2|<\delta=\frac{\varepsilon}{5}. \qquad ❑$$

Note: Today, we have developed lot of tools to find the limit of functions without using the definition (even without knowing the limit). Here our aim is to understand the limit by definition.

If the definition of limit is violated or leads to something absurd even by choosing one value of ε, then we say limit doesn't exist.

❖ ***Example***

$$\lim_{x\to 0}\sin\frac{1}{x} \quad \text{does not exist.}$$

Suppose that $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ exists and take it to be l, then there exist a positive real number δ such that

$$\left|\sin\frac{1}{x}-l\right|<1 \quad \text{when} \quad 0<|x-0|<\delta \quad (\text{we take here } \varepsilon=1>0)$$

We can find a positive integer n such that

$$\frac{2}{n\pi}<\delta \quad \text{then} \quad \frac{2}{(4n+1)\pi}<\delta \quad \text{and} \quad \frac{2}{(4n+3)\pi}<\delta.$$

It thus follows

$$\left|\sin\frac{(4n+1)\pi}{2}-l\right| < 1 \quad \Rightarrow \left|1-l\right| < 1$$

and $$\left|\sin\frac{(4n+3)\pi}{2}-l\right| < 1 \quad \Rightarrow \left|-1-l\right| < 1 \quad \text{or} \quad \left|1+l\right| < 1.$$

So that

$$2=\left|1+l+1-l\right| \le \left|1+l\right|+\left|1-l\right| < 1+1$$

$$\Rightarrow\ 2 < 2.$$

This is impossible; hence limit of the function does not exist. ❑

❖ ***Example***

Consider the function $f:[0,1]\to\mathbb{R}$ defined as

$$f(x)=\begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irratioanl} \end{cases}$$

Show that $\lim\limits_{x\to p} f(x)$, where $p\in[0,1]$ does not exist.

Solution

On the contrary, suppose that $\lim\limits_{x\to p} f(x)=q$.

Then for given $\varepsilon>0$ we can find $\delta>0$ such that

$$\left|f(x)-q\right|<\varepsilon \quad \text{whenever} \quad 0<\left|x-p\right|<\delta.$$

Consider two points r and s from interval $\left(p-\delta, p+\delta\right)\subset[0,1]$ such that r is rational and s is irrational.

Then $f(r)=0$ & $f(s)=1$.

Now

$$\begin{aligned} 1&=\left|f(s)\right|=\left|f(s)-q+q\right| \\ &=\left|(f(s)-q+q-0\right| \\ &=\left|f(s)-q+q-f(r)\right| \qquad (\text{since } 0=f(r)). \\ &\le\left|f(s)-q\right|+\left|f(r)-q\right|<\varepsilon+\varepsilon. \end{aligned}$$

i.e. $1<2\varepsilon$

In particular, if we take $\varepsilon=\dfrac{1}{4}$, then $1<\dfrac{1}{2}$.

This is absurd.

Hence the limit of the function does not exist. ❑

❖ ***Theorem***

If $\lim\limits_{x\to c} f(x)$ exists, then it is unique.

Proof

Suppose $\lim\limits_{x\to c} f(x)$ is not unique.

Take $\lim\limits_{x\to c} f(x)=l_1$ and $\lim\limits_{x\to c} f(x)=l_2$, where $l_1\neq l_2$.

So for $\varepsilon>0$, there exists real numbers δ_1 and δ_2 such that

$$|f(x)-l_1|<\frac{\varepsilon}{2} \quad \text{whenever} \quad |x-c|<\delta_1$$

$$\& \quad |f(x)-l_2|<\frac{\varepsilon}{2} \quad \text{whenever} \quad |x-c|<\delta_2.$$

Now
$$\begin{aligned} |l_1-l_2| &= \left|\left(f(x)-l_1\right)-\left(f(x)-l_2\right)\right| \\ &\le |f(x)-l_1|+|f(x)-l_2| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}, \quad \text{whenever} \quad |x-c|<\min(\delta_1,\delta_2). \end{aligned}$$

That is, $0\le |l_1-l_2|<\varepsilon$ for all $\varepsilon>0$.

$\Rightarrow l_1-l_2=0$ or $l_1=l_2$. ❑

❖ RIGHT HAND LIMIT OF THE FUNCTION

Definition: Suppose $E\subseteq\mathbb{R}$ and $f:E\to\mathbb{R}$ be a function. If for all $\varepsilon>0$, there exists $\delta>0$ (depending upon ε) such that

$$|f(x)-L|<\varepsilon \text{ whenever } a<x<a+\delta,$$

Then L is called right hand limit of function *f* at *a*.

Notation: It is written as $\lim\limits_{x\to a+} f(x)=L$.

❖ LEFT HAND LIMIT OF THE FUNCTION

Definition: Suppose $E\subseteq\mathbb{R}$ and $f:E\to\mathbb{R}$ be a function. If for all $\varepsilon>0$, there exists $\delta>0$ (depending upon ε) such that

$$|f(x)-L|<\varepsilon \text{ whenever } a-\delta<x<a,$$

Then L is called left hand limit of function *f* at *a*.

Notation: It is written as $\lim\limits_{x\to a-} f(x)=L$.

Remark: One can easily prove that if the right hand limit or left hand limit of the function exists then it is unique.

Examples:

(i) Consider a function $f(x)=\dfrac{|\sin x|}{\sin x}$ for $x\in\mathbb{R}$.

It is easy to see that $\lim\limits_{x\to 0+}\dfrac{|\sin x|}{\sin x}=1$, but $\lim\limits_{x\to 0-}\dfrac{|\sin x|}{\sin x}=-1$.

(ii) Suppose

$$f(x)=\begin{cases} 2x+1, & x<1; \\ 5 & x=1; \\ 7x^2-4 & x>1. \end{cases}$$

To compute $\lim\limits_{x\to 1+} f(x)$, we use the part of the definition for *f* which applies to $x>1$, so

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+}(2x+1) = 3.$$

To compute $\lim_{x\to 1-} f(x)$, we use the part of the definition for f which applies to $x<1$, so

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}(7x^2-4) = 3.$$

Note that $\lim_{x\to 1+} f(x) = \lim_{x\to 1-} f(x) = 3,$ but $f(1)=5$. ❑

The proof of the following theorem can be seen in FSc or BSc mathematics book.

❖ ***Theorem***

Suppose f is a function define on E may not containing point a. Then

$$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = \lim_{x\to a} f(x).$$

❖ **LIMIT AS A INFINITY**

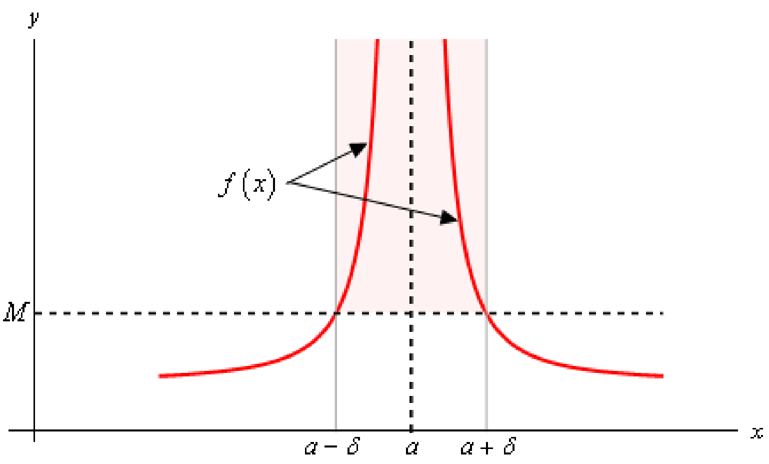
Definition: Let $f(x)$ be a function defined on an interval that contains $x=a$, except possibly at $x=a$. Then we say that

$$\lim_{x\to a} f(x) = \infty$$

if for every number $M>0$, there is some number $\delta>0$ such that

$$f(x) > M \text{ whenever } 0<|x-a|<\delta.$$

Above definitions is telling us that no matter how large we choose M to be we can always find an interval around $x=a$, given by $0<|x-a|<\delta$ for some number $\delta>0$, so that as long as we stay within that interval the graph of the function will be above the line $y=M$ as shown in the graph. Similarly, one can define limit as negative infinity.



❖ **LIMIT AS NEGATIVE INFINITY:**

Definition: Let $f(x)$ be a function defined on an interval that contains $x=a$, except possibly at $x=a$. Then we say that

$$\lim_{x\to a} f(x) = -\infty$$

if for every number $N<0$, there is some number $\delta>0$ such that

$$f(x) < N \text{ whenever } 0<|x-a|<\delta.$$

❖ ***Example***

Use the definition of the limit to prove the following limit.

$$\lim_{x\to 0}\frac{1}{x^2}=\infty .$$

Solution:

Let $M>0$ be any number and we'll need to choose a δ so that,

$$\frac{1}{x^2}>M \quad \text{whenever} \quad 0<|x-0|=|x|<\delta .$$

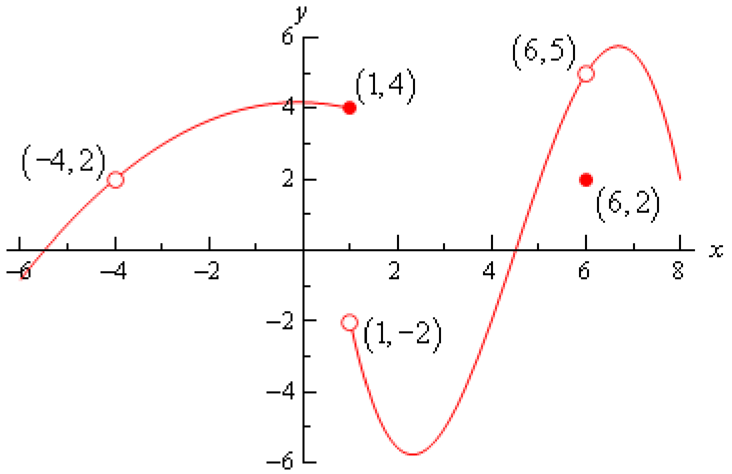
We take

$$\frac{1}{x^2}>M \quad \Rightarrow \quad x^2<\frac{1}{M}$$

$$\Rightarrow \quad |x|<\frac{1}{\sqrt{M}}=\delta .$$

❑

Exercise: Given the following graph of function f:



(a) $f(-4)$ (b) $\lim_{x\to -4-} f(x)$ (c) $\lim_{x\to -4+} f(x)$ (d) $\lim_{x\to -4} f(x)$

(e) $f(1)$ (f) $\lim_{x\to 1-} f(x)$ (g) $\lim_{x\to 1+} f(x)$ (h) $\lim_{x\to 1} f(x)$

(i) $f(6)$ (j) $\lim_{x\to 6-} f(x)$ (k) $\lim_{x\to 6+} f(x)$ (l) $\lim_{x\to 6} f(x)$

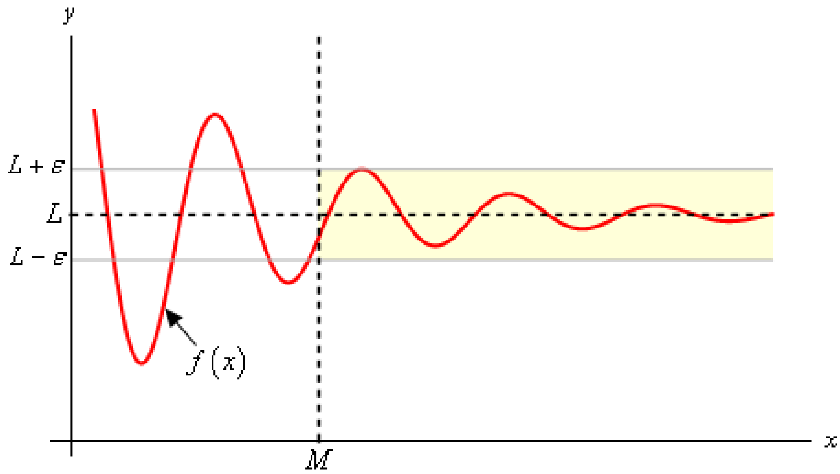
❖ **LIMIT AT INFINITY**

Definition: Let X and Y be subsets of $\mathbb{R}$. A function $f:X\to Y$ is said to tend to limit L as $x\to\infty$, if for a real number $\varepsilon>0$ however small, there exists a positive number M which depends upon ε such that distance

$$|f(x)-L|<\varepsilon \quad \text{when} \quad x>M .$$

Notation: This is written as $\lim_{x\to\infty} f(x)=L$.

Above definition tells us that no matter how close to L we want to get, mathematically this is given by $|f(x)-L|<\varepsilon$ for any chosen $\varepsilon>0$, we can find another number M such that provided we take any x bigger than M, then the graph of the function for that x will be closer to L than $L-\varepsilon$ and $L+\varepsilon$.



Similarly, one can define limit at negative infinity.

❖ **LIMIT AT NEGATIVE INFINITY**

Definition: Let X and Y be subsets of $\mathbb{R}$. A function $f: X \to Y$ is said to tend to limit L as $x \to -\infty$, if for a real number $\varepsilon > 0$ however small, there exists a positive number N which depends upon ε such that distance

$$|f(x) - L| < \varepsilon \quad \text{when } x < N.$$

Notation: This is written as $\lim\limits_{x\to\infty} f(x) = L$.

❖ ***Example***

By definition, prove that $\lim\limits_{x\to\infty} \dfrac{2x}{1+x} = 2$.

We have $\left|\dfrac{2x}{1+x} - 2\right| = \left|\dfrac{2x-2-2x}{1+x}\right| = \left|\dfrac{-2}{1+x}\right| < \dfrac{2}{x}$.

Now if $\varepsilon > 0$ is given we can find $M = \dfrac{2}{\varepsilon}$ so that

$$\left|\frac{2x}{1+x} - 2\right| < \varepsilon \quad \text{whenever} \quad x > M = \frac{2}{\varepsilon}.$$

❑

The following theorem is very useful to find the limit of different function. Here we are not giving the proof as one can found it in the mathematics book of FSc.

❖ ***Theorem***

Let $f: E \to \mathbb{R}$ and $g: E \to \mathbb{R}$ be real valued functions. If $\lim\limits_{x\to p} f(x) = A$ and $\lim\limits_{x\to p} g(x) = B$ then i- $\lim\limits_{x\to p}\left(f(x)\pm g(x)\right)=A\pm B$,

ii- $\lim\limits_{x\to p}(fg)(x)=AB$,

iii- $\lim\limits_{x\to p}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{A}{B}$, provided $B\neq 0$.

❖ CONTINUITY

Definition: Suppose $E\subseteq\mathbb{R}$ and $f:E\to\mathbb{R}$ be a function. Then f is said to be continuous at p if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(x)-f(p)|<\varepsilon \text{ for all points } x\in E \text{ for which } 0<|x-p|<\delta.$$

Definition: If f is continuous at every point of E, then f is said to be continuous on E.

Note: Comparing the definition of continuity with the definition of the limit, It is to be noted that f has to be continuous at p iff $\lim\limits_{x\to p}f(x)=f(p)$.

❖ *Examples*

A function $f(x)=x^2$ is continuous for all $x\in\mathbb{R}$.

Here $f(x)=x^2$. Take $p\in\mathbb{R}$ and $\varepsilon>0$.

Then we have to show

$$|f(x)-f(p)|<\varepsilon \;\Rightarrow\; \left|x^2-p^2\right|<\varepsilon \text{ whenever } |x-p|<\delta.$$

Now
$$\begin{aligned}\left|x^2-p^2\right| &=|(x-p)(x+p)|\\ &=|(x-p)(x-p+2p)|\\ &\le|x-p|\left(|x-p|+2|p|\right)\end{aligned}$$

Now if $|x-p|<\delta$, then we have

$$\begin{aligned}\left|x^2-p^2\right| &\le |x-p|\left(|x-p|+2|p|\right)\\ &<\delta\left(\delta+2|p|\right)=\varepsilon.\end{aligned}$$

Since p is arbitrary real number,

therefore, the function $f(x)$ is continuous for all real numbers. ❑

❖ *Example*

A function $f(x)=\sqrt{x}$ is continuous on $[0,\infty[$.

Let c be an arbitrary point such that $0<c<\infty$

For $\varepsilon>0$, we have

$$\begin{aligned}|f(x)-f(c)| &= \left|\sqrt{x}-\sqrt{c}\right|\\ &=\frac{|x-c|}{\sqrt{x}+\sqrt{c}}<\frac{|x-c|}{\sqrt{c}}\end{aligned}$$

$$\Rightarrow \left| f(x) - f(c) \right| < \varepsilon \quad \text{whenever} \quad \frac{|x-c|}{\sqrt{c}} < \varepsilon$$

i.e. $|x-c| < \sqrt{c}\,\varepsilon = \delta$

$\Rightarrow$ f is continuous for $x = c$.

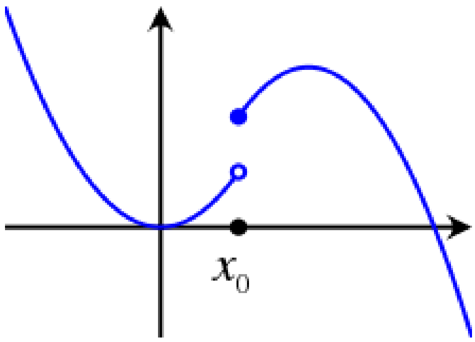
$\because$ c is an arbitrary point lying in $[0,\infty[$

$\therefore$ $f(x) = \sqrt{x}$ is continuous on $[0,\infty[$ ❑

❖ **RIGHT CONTINUOUS AND LEFT CONTINUOUS**

Definition**:** Let f be a real valued function. It is said to be right continuous at point *a* if $\lim\limits_{x\to a+} f(x) = f(a)$ and it is said to be left continuous at point *a* if $\lim\limits_{x\to a-} f(x) = f(a)$.

❖ ***Example***



Consider a function given in above graph. We see f is not continuous at point x_0. It is right continuous at point x_0 but not left continuous at point x_0.

❖ ***Example***

Let

$$f(x) = \begin{cases} x^2 + 1 & if \;\; x \le 2, \\ \dfrac{x^2-4}{x-2} & if \;\; x > 2. \end{cases}$$

Then f is left continuous at 2 but it is not right continuous at 2.

❖ **RIGHT CONTINUOUS AND LEFT CONTINUOUS**

Definition**:** A function $f : [a,b] \to \mathbb{R}$ is said to be continuous on closed interval $[a,b]$ if

f is continuous on (a,b)

f is right continuous at a.

f is left continuous at b.

❖ ***Theorem (The intermediate value theorem)***

Suppose f is continuous on $[a,b]$ and $f(a) \ne f(b)$, then given a number λ that lies between $f(a)$ and $f(b)$, there exist a point $c \in (a,b)$ with $f(c) = \lambda$.

Proof

Without loss of generality, we can consider $f(a) < f(b)$ and $f(a) < \lambda < f(b)$. Also let $S = \{x \in [a,b] \mid f(x) < \lambda\}$. Then S is non-empty as $a \in S$ and b is an upper bound of S.

Since we are dealing with the set of real numbers, therefore supremum of S exist in $\mathbb{R}$, say $c = \sup S$.

Since f is continuous on $[a,b]$, in particular at $x = c$, therefore for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \text{ whenever } 0 < |x - c| < \delta.$$

This means that

$$f(x) - \varepsilon < f(c) < f(x) + \varepsilon \quad \text{for all } x \text{ between } c - \delta \text{ and } c + \delta.$$

By the properties of the supremum, there exist x_1 between $c - \delta$ and c that is contained in S, so that

$$f(c) < f(x_1) + \varepsilon < \lambda + \varepsilon. \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Choose x_2 between c and $c + \delta$. Then $x_2 \notin S$, so we have

$$f(c) > f(x_2) - \varepsilon \geq \lambda - \varepsilon. \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii), we have for all $\varepsilon > 0$,

$$\lambda - \varepsilon < f(c) < \lambda + \varepsilon.$$

$$\Rightarrow \quad |f(c) - \lambda| < \varepsilon$$

So ultimately, we have $\quad f(c) = \lambda.$ ❑

❖ UNIFORM CONTINUITY

Definition: Suppose $f : E \to \mathbb{R}$ is a real valued function. We say that f is uniformly continuous on E if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(p) - f(q)| < \varepsilon \quad \forall \quad p, q \in E \quad \text{for which} \quad |p - q| < \delta.$$

The uniform continuity is a property of a function on a set, that is, it is a global property but continuity can be defined at a single point i.e. it is a local property.

Uniform continuity of a function at a point has no meaning.

It is evident that every uniformly continuous function is continuous.

To emphasize a difference between continuity and uniform continuity on set S, we consider the following examples.

❖ *Example*

Let S be a half open interval $0 < x \leq 1$ and let $f(x)$ be defined for each x in S by the formula $f(x) = x^2$. It is uniformly continuous on S. To prove this, assume $x, y \in (0,1]$ and take

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x - y||x + y| \\ &< 2|x - y| \end{aligned}$$

If $|x - y| < \delta$ then $|f(x) - f(y)| < 2\delta = \varepsilon$

Hence if ε is given we need only to take $\delta = \dfrac{\varepsilon}{2}$ to guarantee that

$$|f(x) - f(y)| < \varepsilon \text{ for every pair } x, y \text{ with } |x - y| < \delta$$

Thus f is uniformly continuous on the set S. ❑

❖ ***Example***

Let S be the half open interval $0 < x \le 1$ and let a function f be defined for each x in S by the formula $f(x) = \dfrac{1}{x}$. This function is continuous on the set S, however we shall prove that this function is not uniformly continuous on S.

Solution

Let suppose $\varepsilon = 10$ and suppose we can find a δ , $0 < \delta < 1$, to satisfy the condition of the definition.

Taking $x = \delta$, $y = \dfrac{\delta}{11}$, we obtain

$$|x - y| = \frac{10\delta}{11} < \delta$$

and

$$|f(x) - f(y)| = \left| \frac{1}{\delta} - \frac{11}{\delta} \right| = \frac{10}{\delta} > 10$$

Hence for these two points we have $|f(x) - f(y)| > 10$.

This contradict the definition of uniform continuity.

Hence the given function being continuous on a set S is not uniformly continuous on S. ❑

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