Chapter 3 – Series

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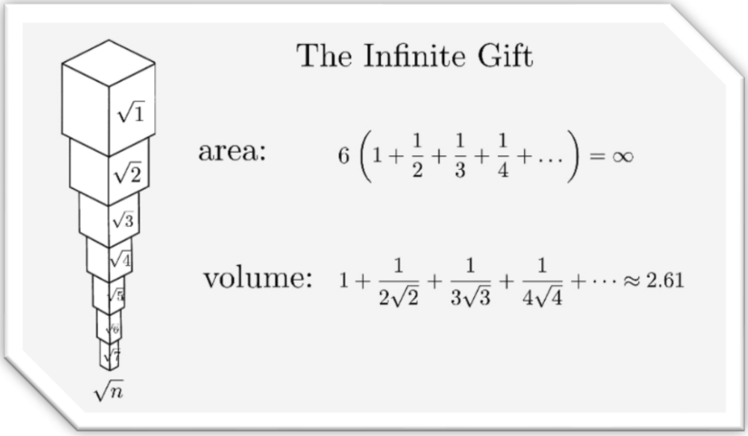

I'm standing 5 m from a wall. I jump half the distance (2.5 m) towards the wall. I halve the distance again (1.25 m) and continue getting closer to the wall by stepping half the remaining distance each time. Do I ever reach the wall? Zeno, the 5th century BCE Greek philosopher, proposed a similar question in his famous Paradoxes (search for Zeno's paradox).

The first known example of an infinite sum was when Greek mathematician Archimedes showed in the 3rd century BCE that the area of a segment of a parabola is 4/3 the area of a triangle with the same base. The notation he used was different, of course, and some of the approach was more geometric than algebraic, but his approach of summing infinitely small quantities was quite remarkable for the time.

Mathematicians Madhava from Kerala, India studied infinite series around 1350 CE. Among his many contributions, he discovered the infinite series for the trigonometric functions of sine, cosine, tangent and arctangent, and many methods for calculating the circumference of a circle

In the 17th century, James Gregory (1638-1675) worked in the new decimal system on infinite series and published several Maclaurin series. In 1715, a general method for constructing the Taylor series for all functions for which they exist was provided by Brook Taylor (1685-1731). Leonhard Euler (1707-1783) derived series for sine, cosine, exp, log, etc., and he also discovered relationships between them. He also introduced sigma notation (Σ) for sums of series.

Infinite Series

Let $\{a_n\}$ be a given sequence. Then a sum of the form

$$a_1+a_2+a_3+\dots$$

is called an infinite series.

Another way of writing this infinite series is $\sum\limits_{n=1}^{\infty}a_n$ or $\sum_{n=1}^{\infty}a_n$ or simply $\sum a_n$.

Convergence and divergence of the series

A series $\sum\limits_{n=1}^{\infty}a_n$ is said to be convergent if the sequence $\{s_n\}$, where $s_n=\sum\limits_{k=1}^{n}a_k$, is convergent.

If the sequence $\{s_n\}$ diverges then the series is said to be diverge.

Remarks:

For a series $\sum_{n=1}^{\infty} a_n$, the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_k$, is called the sequence of partial sum of the series. The numbers a_n are called terms and s_n are called partial sums. One can note that

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3 \text{ and}$$
$$s_n = a_1 + a_2 + \dots + a_n \text{ or } s_n = s_{n-1} + a_n .$$

If the sequence $\{s_n\}$ converges to s, we say that the series converges and write $\sum_{n=1}^{\infty} a_n = s$, the number s is called the sum or value of the series but it should be clearly understood that the 's' is the limit of the sequence of sums and is not obtained simply by addition.

Also note that the behaviors of the series remain unchanged by addition or deletion of the first finite terms. Just as a sequence may be indexed such that its first element is not a_n, but is a_0, or a_5 or a_{99}, we will denote the series having these numbers as their first element by the symbols

$$\sum_{n=0}^{\infty} a_n \text{ or } \sum_{n=5}^{\infty} a_n \text{ or } \sum_{n=99}^{\infty} a_n .$$

Review:

- Let $\{a_n\}$ be a convergent sequence, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1}$.

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof

Assume that $s_n = a_1 + a_2 + a_3 + \dots + a_n$.

As $\sum_{n=1}^{\infty} a_n$ is convergent, therefore $\{s_n\}$ is convergent.

Suppose $\lim_{n\to\infty} s_n = s$, then we have $\lim_{n\to\infty} s_{n-1} = s$.

Now we have $s_n = s_{n-1} + a_n$ for $n > 1$,

or $a_n = s_n - s_{n-1}$ for $n > 1$.

Therefore
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(s_n - s_{n-1}\right)$$
$$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$$
$$= s - s = 0 .$$
❑

Remark:

(i) The converse of the above theorem is false. For example, consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$. We know that the sequence $\{s_n\}$, where $s_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$, is divergent therefore $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent series, although $\lim_{n\to\infty} a_n = 0$.

(ii) The above theorem shows that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent (This is called basic divergent test).

Examples:

(i) Is the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is convergent or divergent?

Solution.

Assume $a_n = 1+\frac{1}{n}$.

Now we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$.

Hence $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is divergent (by basic divergent test)

(ii) Show that the series $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+...$ is divergent.

Solution.

The above series can be written as $\sum_{n=1}^{\infty}\sqrt{\frac{n}{2(n+1)}}$.

Then take

$$a_n = \sqrt{\frac{n}{2(n+1)}},$$

As we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}} \neq 0$.

Hence the given series is divergent by basic comparison test.

(iii) Is the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is convergent or divergent?

Solution.

Assume that $a_n = \frac{n^n}{n!}$.

As $$a_n = \frac{n^n}{n\cdot(n-1)\cdot(n-2)\cdot...\cdot 3\cdot 2\cdot 1}$$

$$= \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \ldots \cdot \frac{n}{2} \cdot \frac{n}{1} \geq 1 \text{ for all } n \geq 1.$$

We conclude $\lim_{n\to\infty} a_n$ cannot be zero.

Hence $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent (by basic divergent test).

Questions:

(i) Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$ but converse is not true.

(ii) Prove that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.

Review:

- A series $\sum a_n$ is convergent if and only if it sequence of partial sum $\left\{s_n := \sum_{k=1}^{n} a_k\right\}$ is convergent.
- A sequence in $\mathbb{R}$ is convergent iff it is Cauchy sequence.
- A sequence $\{s_n\}$ is Cauchy sequence if and only if for all $\varepsilon > 0$ there exists positive integer n_0 such $|s_n - s_m| < \varepsilon$ for all $n, m > n_0$ (or $n \geq m > n_0$).

Theorem (General Principle of Convergence or Cauchy Criterion for Series)

A series $\sum a_n$ is convergent if and only if for any real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left| \sum_{i=m+1}^{n} a_i \right| < \varepsilon \qquad \forall\ n \geq m > n_0$$

Proof

Assume that $s_n = a_1 + a_2 + a_3 + \ldots + a_n$.

Then $\sum a_n$ is convergent if and only if $\{s_n\}$ is convergent.

Now $\{s_n\}$ is convergent if and only $\{s_n\}$ is Cauchy sequence,

that is, for all real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$|\ s_n - s_m| < \varepsilon \qquad \forall\ n \geq m > n_0 \ \ldots\ldots\ldots (i)$$

As $n > m$, therefore

$$s_n = s_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

$$\Rightarrow\ s_n - s_m = a_{m+1} + a_{m+2} + \ldots + a_n.$$

So by using (i), we have

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \quad \forall\ n \geq m > n_0.$$

This gives

$$\left| \sum_{i=m+1}^{n} a_i \right| < \varepsilon \quad \forall\ n \geq m > n_0.$$

❑

Review:

- A bounded and monotone sequence, then it is convergent.
- An unbounded sequence is divergent.

Theorem

Let $\sum a_n$ be an infinite series of non-negative terms and let $\{s_n\}$ be a sequence of its partial sums. Then $\sum a_n$ is convergent if $\{s_n\}$ is bounded and it diverges if $\{s_n\}$ is unbounded.

Proof

We have $s_n = a_1 + a_2 + a_3 + ... + a_n$, this give $s_{n+1} = s_n + a_{n+1}$.

As we have given $a_n \geq 0$ for all $n \geq 1$ and $s_{n+1} = s_n + a_{n+1} \geq s_n$ for all $n \geq 1$.

Therefore, the sequence $\{s_n\}$ is monotonic increasing.

Now if $\{s_n\}$ is bounded then we concluded that $\{s_n\}$ is convergent.

Now if $\{s_n\}$ is unbounded, then it is divergent.

Hence we conclude that $\sum a_n$ is convergent if $\{s_n\}$ is bounded and it divergent if $\{s_n\}$ is unbounded. ❑

Review:

- A series $\sum a_n$ is divergent if and only if there exists real number $\varepsilon > 0$, such that for all positive integer n_0,

$$\left| \sum_{i=m+1}^{n} a_i \right| > \varepsilon \quad \text{whenever } n > m > n_0$$

Theorem (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0$ for all n. Also suppose that for a fixed positive number λ and positive integer k,

$$a_n < \lambda b_n \quad \forall\, n \geq k.$$

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Proof

(i) Suppose $\sum b_n$ is convergent and

$$a_n < \lambda b_n \quad \forall\, n \geq k,\ \lambda > 0, \ \ldots\ldots\ldots\ldots (i)$$

By Cauchy criterion; for any positive number $\varepsilon > 0$ there exists n_0 such that

$$\sum_{i=m+1}^{n} b_i < \frac{\varepsilon}{\lambda} \qquad n > m > n_0$$

from (i)

$$\sum_{i=m+1}^{n} a_i < \lambda \sum_{i=m+1}^{n} b_i < \varepsilon, \quad n > m > n_0 \quad \Rightarrow \sum a_n \text{ is convergent.}$$

(ii) Now suppose $\sum a_n$ is divergent then there exists a real number $\beta > 0$, such that

$$\sum_{i=m+1}^{n} a_i > \lambda\beta \ , \qquad n > m .$$

From (i)

$$\sum_{i=m+1}^{n} b_i > \frac{1}{\lambda}\sum_{i=m+1}^{n} a_i > \beta, \qquad n > m$$

$\Rightarrow \ \sum b_n$ is divergent. ❑

Example

Prove that $\sum \frac{1}{\sqrt{n}}$ is divergent.

Since $n \geq \sqrt{n} > 0 \quad \forall \ \ n \geq 1$.

$\Rightarrow \ \frac{1}{n} \leq \frac{1}{\sqrt{n}}$

$\Rightarrow \ \sum \frac{1}{\sqrt{n}}$ is divergent as $\sum \frac{1}{n}$ is divergent. ❑

Example

The series $\sum \frac{1}{n^\alpha}$ is convergent if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Let $\quad s_n = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + ... + \frac{1}{n^\alpha}$.

If $\alpha > 1$ then

$$s_n < s_{2n} \quad \text{and} \quad \frac{1}{n^\alpha} < \frac{1}{(n-1)^\alpha} .$$

Now $\quad s_{2n} = \left[1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \frac{1}{4^\alpha} + ... + \frac{1}{(2n)^\alpha}\right]$

$$= \left[1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + \ldots + \frac{1}{(2n-1)^\alpha}\right] + \left[\frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \frac{1}{6^\alpha} + \ldots + \frac{1}{(2n)^\alpha}\right]$$

$$= \left[1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + \ldots + \frac{1}{(2n-1)^\alpha}\right] + \frac{1}{2^\alpha}\left[1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{(n)^\alpha}\right]$$

$$< \left[1 + \frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \ldots + \frac{1}{(2n-2)^\alpha}\right] + \frac{1}{2^\alpha}s_n$$ (replacing 3 by 2, 5 by 4 and so on.)

$$= 1 + \frac{1}{2^\alpha}\left[1 + \frac{1}{2^\alpha} + \ldots + \frac{1}{(n-1)^\alpha}\right] + \frac{1}{2^\alpha}s_n$$

$$=1+\frac{1}{2^{\alpha}}s_{n-1}+\frac{1}{2^{\alpha}}s_{n} \quad <1+\frac{1}{2^{\alpha}}s_{2n}+\frac{1}{2^{\alpha}}s_{2n} \qquad \because\ s_{n-1}<s_n<s_{2n}$$

$$=1+\frac{2}{2^{\alpha}}s_{2n}$$

$$\Rightarrow\ s_{2n}<1+\frac{1}{2^{\alpha-1}}s_{2n}.$$

$$\Rightarrow\ \left(1-\frac{1}{2^{\alpha-1}}\right)s_{2n}<1 \quad \Rightarrow\ \left(\frac{2^{\alpha-1}-1}{2^{\alpha-1}}\right)s_{2n}<1 \quad \Rightarrow\ s_{2n}<\frac{2^{\alpha-1}}{2^{\alpha-1}-1},$$

i.e. $s_n<s_{2n}<\dfrac{2^{\alpha-1}}{2^{\alpha-1}-1}$

$\Rightarrow$ $\{s_n\}$ is bounded and also monotonic. Hence, we conclude that $\sum\frac{1}{n^{\alpha}}$ is convergent when $\alpha>1$.

If $\alpha\le 1$ then

$$n^{\alpha}\le n \quad \forall\ n\ge 1$$

$$\Rightarrow\ \frac{1}{n^{\alpha}}\ge\frac{1}{n} \quad \forall\ n\ge 1$$

Since $\sum\frac{1}{n}$ is divergent therefore $\sum\frac{1}{n^{\alpha}}$ is divergent when $\alpha\le 1$. ❑

Theorem (Limit Comparison Test)

Let $a_n>0$, $b_n>0$ and $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lambda$, where $\lambda\ge 0$.

(i) If $\lambda\ne 0$, then the series $\sum a_n$ and $\sum b_n$ behave alike.

(ii) If $\lambda=0$ and if $\sum b_n$ is convergent, then $\sum a_n$ is convergent. If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

Proof

Since $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lambda$, therefore for $\varepsilon>0$, there exists positive integer n_0 such that

$$\left|\frac{a_n}{b_n}-\lambda\right|<\varepsilon \quad \forall\ n\ge n_0.\ \ldots\ldots\ldots\ldots\ (*)$$

(i) If $\lambda\ne 0$, then take $\varepsilon=\frac{\lambda}{2}$ (as λ will be positive)

$$\Rightarrow\ \left|\frac{a_n}{b_n}-\lambda\right|<\frac{\lambda}{2} \quad \forall\ n\ge n_0.$$

$$\Rightarrow\ -\frac{\lambda}{2}<\frac{a_n}{b_n}-\lambda<\frac{\lambda}{2} \qquad \forall\ n\ge n_0.$$

$$\Rightarrow\ \lambda-\frac{\lambda}{2}<\frac{a_n}{b_n}<\lambda+\frac{\lambda}{2}\quad \forall\ n\geq n_0.$$

$$\Rightarrow\ \frac{\lambda}{2}<\frac{a_n}{b_n}<\frac{3\lambda}{2}\quad \forall\ n\geq n_0.$$

Then we got

$$a_n<\frac{3\lambda}{2}b_n \quad \text{and} \quad b_n<\frac{2}{\lambda}a_n \text{ for } n\geq n_0.$$

Hence by comparison test we conclude that $\sum a_n$ and $\sum b_n$ converge or diverge together.

(ii) If $\lambda=0$, then (*) implies $a_n<\varepsilon b_n$

Hence by comparison test we conclude that $\sum a_n$ is convergent if $\sum b_n$ converges. Also $\sum b_n$ is divergent if $\sum a_n$ diverges. ❑

Example

Is the series $\sum \frac{1}{n}\sin^2\frac{x}{n}$ is convergent or divergent for real x?

Consider $a_n=\frac{1}{n}\sin^2\frac{x}{n}$ and take $b_n=\frac{1}{n^3}$.

Then $$\frac{a_n}{b_n}=n^2\sin^2\frac{x}{n}=\frac{\sin^2\frac{x}{n}}{\frac{1}{n^2}}$$

$$=x^2\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2$$

Applying limit as $n\to\infty$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}x^2\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2=x^2\left(\lim_{n\to\infty}\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2=x^2(1)=x^2.$$

$\Rightarrow$ $\sum a_n$ and $\sum b_n$ have the similar behavior for all finite values of x except x = 0.

Since $\sum\frac{1}{n^3}$ is convergent series therefore the given series is also convergent for finite values of x except x = 0.

If $x=0$, then the given series is also convergent because it is just zero. ❑

Theorem (Cauchy Condensation Test)

Let $a_n \geq 0$, $a_n > a_{n+1}$ for all $n \geq 1$ (i.e. $\{a_n\}$ is positive term decreasing sequence). Then the series $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

Proof

The condensation test follows from noting that if we collect the terms of the series into groups of lengths 2^n, each of these groups will be less than $2^n a_{2^n}$ by monotonicity. Observe,

$$\sum_{n=1}^{\infty} a_n = a_1 + \underbrace{a_2 + a_3}_{\leq a_2 + a_2} + \underbrace{a_4 + a_5 + a_6 + a_7}_{\leq a_4 + a_4 + a_4 + a_4} + \cdots + \underbrace{a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1}}_{\leq a_{2^n} + a_{2^n} + \cdots + a_{2^n}} + \cdots$$

$$\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} + \cdots = \sum_{n=0}^{\infty} 2^n a_{2^n}.$$

We have use the fact that a_n is decreasing sequence. The convergence of the original series now follows from direct comparison to this "condensed" series. To see that convergence of the original series implies the convergence of this last series, we similarly put,

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \underbrace{a_1 + a_2}_{\leq a_1 + a_1} + \underbrace{a_2 + a_4 + a_4 + a_4}_{\leq a_2 + a_2 + a_3 + a_3} + \cdots + \underbrace{a_{2^n} + a_{2^{n+1}} + \cdots + a_{2^{n+1}}}_{\leq a_{2^n} + a_{2^n} + a_{(2^n+1)} + a_{(2^n+1)} + \cdots + a_{(2^{n+1}-1)}} + \cdots$$

$$\leq a_1 + a_1 + a_2 + a_2 + a_3 + a_3 + \cdots + a_n + a_n + \cdots = 2\sum_{n=1}^{\infty} a_n.$$

And we have convergence, again by direct comparison. And we are done. Note that we have obtained the estimate

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=0}^{\infty} 2^n a_{2^n} \leq 2\sum_{n=1}^{\infty} a_n.$$ ❑

Example

Find value of *p* for which $\sum \frac{1}{n^p}$ is convergent or divergent.

If $p \leq 0$ then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, therefore the series diverges when $p \leq 0$.

If $p > 0$ then the condensation test is applicable, and we are lead to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{(p-1)k}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{(p-1)}}\right)^k$$

$$= \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

Now $2^{1-p} < 1$ iff $1-p < 0$ i.e. when $p > 1$.

And the result follows by comparing this series with the geometric series having common ratio less than one.

The series diverges when $2^{1-p} = 1$ (i.e. when $p = 1$).

The series is also divergent if $0 < p < 1$. ❑

Example

Prove that if $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges and if $p \le 1$ the series is divergent.

Since $\{\ln n\}$ is increasing, therefore $\left\{\frac{1}{n \ln n}\right\}$ decreases

and we can use the condensation test to the above series.

We have $a_n = \frac{1}{n(\ln n)^p}$

$$\Rightarrow \quad a_{2^n} = \frac{1}{2^n\left(\ln 2^n\right)^p} \qquad \Rightarrow \quad 2^n a_{2^n} = \frac{1}{(n \ln 2)^p}$$

Now $\sum 2^n a_{2^n} = \sum \frac{1}{(n \ln 2)^p} = \frac{1}{(\ln 2)^p} \sum \frac{1}{n^p}$.

This converges when $p > 1$ and diverges when $p \le 1$. ❑

Example

Prove that $\sum \frac{1}{\ln n}$ is divergent.

Since $\{\ln n\}$ is increasing there $\left\{\frac{1}{\ln n}\right\}$ decreases.

We can apply the condensation test to check the behavior of the series.

Take $a_n = \frac{1}{\ln n}$, then $a_{2^n} = \frac{1}{\ln 2^n}$.

So $2^n a_{2^n} = \frac{2^n}{\ln 2^n} \quad \Rightarrow \quad 2^n a_{2^n} = \frac{2^n}{n \ln 2}$.

Since $\frac{2^n}{n} > \frac{1}{n} \qquad \forall \; n \ge 1$

and $\sum \frac{1}{n}$ is diverges therefore the given series is also diverges. ❑

Alternating Series

A series in which successive terms have opposite signs is called an alternating series.

Example:

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \text{ is an alternating series.}$$

Review:

- If $\{s_n\}$ is convergent to s, then every subsequence of $\{s_n\}$ converges to s.
- If $\sum a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.
- If a sequence is decreasing and bounded below then it is convergent.

Theorem (Alternating Series Test or Leibniz Test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim\limits_{n\to\infty} a_n = 0$ then the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ converges.

Proof

Looking at the odd numbered partial sums of this series we find that

$$s_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \ldots + (a_{2n-1} - a_{2n}) + a_{2n+1}.$$

Since $\{a_n\}$ is decreasing therefore all the terms in the parenthesis are non-negative

$$\Rightarrow s_{2n+1} > 0 \quad \forall\ n.$$

Moreover

$$s_{2n+3} = s_{2n+1} - a_{2n+2} + a_{2n+3}$$
$$= s_{2n+1} - \left(a_{2n+2} - a_{2n+3}\right)$$

Since $a_{2n+2} - a_{2n+3} \geq 0$ therefore $s_{2n+3} \leq s_{2n+1}$.

Hence the sequence of odd numbered partial sum is decreasing and is bounded below by zero. (as it has +ive terms)

It is therefore convergent.

Thus s_{2n+1} converges to some limit l (say).

Now consider the even numbered partial sum. We find that

$$s_{2n+2} = s_{2n+1} - a_{2n+2}$$

and

$$\lim_{n\to\infty} s_{2n+2} = \lim_{n\to\infty}\left(s_{2n+1} - a_{2n+2}\right)$$
$$= \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} a_{2n+2} = l - 0 = l \qquad \because \lim_{n\to\infty} a_n = 0.$$

so that the even partial sum is also convergent to l.

$\Rightarrow$ both sequences of odd and even partial sums converge to the same limit.

Hence, we conclude that the corresponding series is convergent. ❑

Absolute Convergence

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

Review:

- A series $\sum a_n$ is convergent if and only if for any real number $\varepsilon > 0$, there exists a positive integer n_0 such that $\left|\sum_{i=m+1}^{n} a_i\right| < \varepsilon$ for all $n > m > n_0$.
- For all $a_i \in \mathbb{R}$, $i = 1, 2, ..., n$; $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$.

Theorem

An absolutely convergent series is convergent.

Proof:

If $\sum |a_n|$ is convergent then by Cauchy criterion for convergence; for a real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left|\sum_{i=m+1}^{n} |a_i|\right| = \sum_{i=m+1}^{n} |a_i| < \varepsilon \quad \forall\ n, m > n_0 . \quad \ldots\ldots\ldots\ldots.. \text{(i)}$$

Also, we have

$$\left|\sum_{i=m+1}^{n} a_i\right| < \sum_{i=m+1}^{n} |a_i| \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \text{(ii)}$$

By using (i) and (ii), one has

$$\left|\sum_{i=m+1}^{n} a_i\right| < \varepsilon \quad \forall\ n, m > n_0 .$$

This implies the series $\sum a_n$ is convergent.

Note:

The converse of the above theorem does not hold.

e.g. $\sum \frac{(-1)^{n+1}}{n}$ is convergent but $\sum \frac{1}{n}$ is divergent. ❑

Question:

- Prove that every absolute convergent series is convergent, but convers is not true in general.

Review

- Let $x, y \in \mathbb{R}$ and $x < y$. Then there exist number r such that $x < r < y$.
- If $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n < l$, $l > 0$, then there exist positive integer n_0 such that $a_n < l$ for $n \ge n_0$.

Theorem (The Root Test)

Let $\lim_{n\to\infty} |a_n|^{1/n} = p$.

Then $\sum a_n$ converges absolutely if $p < 1$ and it diverges if $p > 1$.

Proof

Let $p<1$ then there exist real number r such that $p<r<1$.

As we have $\lim_{n\to\infty}|a_n|^{1/n}=p$, there is some n_0 so that

$$|a_n|^{1/n} < r \quad \forall\ n>n_0$$

$$\Rightarrow\ |a_n| < r^n < 1 \qquad \forall\ n>n_0.$$

Since $\sum r^n$ is convergent because it is a geometric series with $|r|<1$, therefore $\sum|a_n|$ is convergent.

$\Rightarrow\ \sum a_n$ converges absolutely.

Now let $p>1$. Also we have $\lim_{n\to\infty}|a_n|^{1/n}=p$, there is some n_0 so that

$\Rightarrow\ |a_n|^{1/n} > 1$ for $n\geq n_0$.

$\Rightarrow\ |a_n| > 1$ for $n\geq n_0$.

$\Rightarrow\ \lim_{n\to\infty}|a_n|\neq 0 \quad \Rightarrow\ \lim_{n\to\infty}a_n\neq 0.$

$\Rightarrow\ \sum a_n$ is divergent. ❑

Note:

The above test gives no information when $p=1$.

e.g. Consider the series $\sum\frac{1}{n}$ and $\sum\frac{1}{n^2}$.

For each of these series; $p=1$, but $\sum\frac{1}{n}$ is divergent and $\sum\frac{1}{n^2}$ is convergent.

Theorem (Ratio Test)

The series $\sum a_n$

(i) Converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$. (ii) Diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$.

Proof

If (i) holds we can find $\beta<1$ and integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for } n\geq N$$

In particular

$$\left|\frac{a_{N+1}}{a_N}\right| < \beta$$

$$\Rightarrow\ |a_{N+1}| < \beta|a_N|$$

$$\Rightarrow\ |a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$$

$$\Rightarrow\ |a_{N+3}| < \beta^3|a_N|$$

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$$\Rightarrow \left|a_{N+p}\right| < \beta^{p}\left|a_N\right|$$

$$\Rightarrow \left|a_n\right| < \beta^{n-N}\left|a_N\right| \quad \text{we put } N+p=n.$$

i.e. $\left|a_n\right| < \left|a_N\right|\beta^{-N}\beta^{n}$ for $n \geq N$.

Sine $\sum \beta^n$ is convergent because it is geometric series with common ratio less than 1, therefore $\sum a_n$ is convergent (by comparison test).

If (ii) holds, then we can find integer n_0 such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{for } n \geq n_0.$$

This gives

$$\left|a_{n+1}\right| \geq \left|a_n\right| \quad \text{for } n \geq n_0,$$

that is, the terms are getting larger and guaranteed to not be negative, therefore $\lim_{n\to\infty}\left|a_n\right| \neq 0$. This provide us $\lim_{n\to\infty} a_n \neq 0$.

$$\Rightarrow \sum a_n \text{ is divergent.}$$ ❑

Note:

The knowledge $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ implies nothing about the convergent or divergent of series.

Example

Prove that series $\sum a_n$ with $a_n = \left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^{n+1}\right]^{-n}$ is divergent.

Since $\frac{n}{n+1} < 1$, therefore $a_n > 0 \quad \forall \; n$.

Also
$$(a_n)^{\frac{1}{n}} = \left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^{n+1}\right]^{-1}$$

$$= \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n}{n+1}\right)^{n}\right]^{-1} = \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n+1}{n}\right)^{-n}\right]^{-1}$$

$$= \left(1 + \frac{1}{n}\right)\left[1 - \left(1 + \frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left[1-\left(1+\frac{1}{n}\right)^{-n}\right]^{-1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\ \lim_{n\to\infty}\left[1-\left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= 1\cdot\left[1-e^{-1}\right]^{-1} = \left[1-\frac{1}{e}\right]^{-1} = \left[\frac{e-1}{e}\right]^{-1} = \frac{e}{e-1} > 1.$$

This implies the given series is divergent. ❑

Dirichlet's Theorem

Suppose that

(i) $\{s_n\}$, $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ is bounded and

(ii) $\{b_n\}$ be positive term decreasing sequence such that $\lim\limits_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ is convergent.

Proof

Since $\{s_n\}$ is bounded, therefore, there exists a positive number λ such that

$$|s_n| \le \lambda \qquad \forall\ n \ge 1.$$

Then
$$a_i b_i = (s_i - s_{i-1}) b_i \qquad \text{for } i \ge 2$$
$$= s_i b_i - s_{i-1} b_i$$
$$= s_i b_i - s_{i-1} b_i + s_i b_{i+1} - s_i b_{i+1}$$
$$= s_i (b_i - b_{i+1}) - s_{i-1} b_i + s_i b_{i+1}$$

$$\Rightarrow \sum_{i=m+1}^{n} a_i b_i = \sum_{i=m+1}^{n} s_i (b_i - b_{i+1}) - (s_m b_{m+1} - s_n b_{n+1})$$

Since $\{b_n\}$ is positive term decreasing,

therefore
$$\left|\sum_{i=m+1}^{n} a_i b_i\right| = \left|\sum_{i=m+1}^{n} s_i (b_i - b_{i+1}) - s_m b_{m+1} + s_n b_{n+1}\right|$$
$$\le \sum_{i=m+1}^{n} \{|s_i| (b_i - b_{i+1})\} + |s_m| b_{m+1} + |s_n| b_{n+1}$$
$$\le \sum_{i=m+1}^{n} \{\lambda (b_i - b_{i+1})\} + \lambda b_{m+1} + \lambda b_{n+1} \qquad \because\ |s_i| \le \lambda$$
$$= \lambda\left(\sum_{i=m+1}^{n} (b_i - b_{i+1}) + b_{m+1} + b_{n+1}\right)$$
$$= \lambda\left((b_{m+1} - b_{n+1}) + b_{m+1} + b_{n+1}\right) = 2\lambda b_{m+1} < 2\lambda b_{m+1} + 1.$$

$$\Rightarrow \left|\sum_{i=m+1}^{n} a_i b_i\right| < \varepsilon, \qquad \text{where } \varepsilon = 2\lambda b_{m+1} + 1 \text{ a certain number}$$

$\Rightarrow$ The $\sum a_n b_n$ is convergent. (We have use Cauchy criterion here.) ❑

Theorem

Suppose that

(i) $\sum a_n$ is convergent and

(ii) $\{b_n\}$ is monotonic convergent sequence,

then $\sum a_n b_n$ is also convergent.

Proof

Suppose $\{b_n\}$ is decreasing and it converges to b.

Put $c_n = b_n - b$ for all n.

$\Rightarrow$ $c_n \geq 0$ and $\lim\limits_{n\to\infty} c_n = 0$.

Since $\sum a_n$ is convergent,

therefore $\{s_n\}$, $s_n = a_1 + a_2 + \ldots + a_n$ is convergent, that is, $\{s_n\}$ is bounded.

By Dirichlet's theorem, we have $\sum a_n c_n$ is convergent.

Since $a_n b_n = a_n c_n + a_n b$ and $\sum a_n c_n$ and $\sum a_n b$ are convergent,

therefore $\sum a_n b_n$ is convergent.

Now if $\{b_n\}$ is increasing and converges to b then we shall put $c_n = b - b_n$. ❑

Example

A series $\sum \dfrac{1}{(n\ln n)^\alpha}$ is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

To see this we proceed as follows

$$a_n = \frac{1}{(n\ln n)^\alpha}$$

Take
$$b_n = 2^n a_{2^n} = \frac{2^n}{\left(2^n \ln 2^n\right)^\alpha} = \frac{2^n}{\left(2^n\ n\ln 2\right)^\alpha}$$
$$= \frac{2^n}{2^{n\alpha} n^\alpha \left(\ln 2\right)^\alpha} = \frac{1}{2^{n\alpha-n} n^\alpha \left(\ln 2\right)^\alpha}$$
$$= \frac{1}{\left(\ln 2\right)^\alpha} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n^\alpha}$$

Since $\sum \dfrac{1}{n^\alpha}$ is convergent when $\alpha > 1$ and $\left(\dfrac{1}{2}\right)^{(\alpha-1)n}$ is decreasing for $\alpha > 1$ and it converges to 0. Therefore $\sum b_n$ is convergent

$\Rightarrow$ $\sum a_n$ is also convergent.

Now $\sum b_n$ is divergent for $\alpha \leq 1$ therefore $\sum a_n$ diverges for $\alpha \leq 1$. ❑

Example

To check $\sum \frac{1}{n^\alpha \ln n}$ is convergent or divergent.

We have $a_n = \frac{1}{n^\alpha \ln n}$

Take $b_n = 2^n a_{2^n} = \frac{2^n}{(2^n)^\alpha (\ln 2^n)} = \frac{2^n}{2^{n\alpha}(n \ln 2)}$

$$= \frac{1}{\ln 2} \cdot \frac{2^{(1-\alpha)n}}{n} = \frac{1}{\ln 2} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n}$$

$\because \sum \frac{1}{n}$ is divergent although $\left\{\left(\frac{1}{2}\right)^{n(\alpha-1)}\right\}$ is decreasing, tending to zero for $\alpha > 1$

therefore $\sum b_n$ is divergent.

$\Rightarrow \sum a_n$ is divergent.

The series also divergent if $\alpha \leq 1$.
i.e. it is always divergent. ❑

References:

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[2] Introduction to Real Analysis by R.G.Bartle, and D.R. Sherbert (John Wiley & Sons, Inc.)
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[4] Real Analysis by Dipak Chatterjee (PHI Learning, 2nd edition.)