# Chapter 2 – Sequences

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Sequences form an important component of Mathematical Analysis and arise in many situations. The first rigorous treatment of sequences was made by A. Cauchy (1789-1857) and George Cantor (1845-1918). A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence (some authors considered it finite sequence); a sequence must continue without interruption. Formally it is defined as follows:

**Sequence**

A function whose domain is the set of natural numbers and range is a subset of real numbers is called *real sequence*.

Since in this chapter, we shall be concerned with *real sequences* only, we shall refer to them as just *sequences*.

**Notation:**

A sequence is usually denoted as

$$\{s_n\}_{n=1}^{\infty} \text{ or } \{s_n : n\in\mathbb{N}\} \text{ or } \{s_1, s_2, s_3, ...\} \text{ or simply as } \{s_n\} \text{ or by } (s_n).$$

But it is not limited to above notations only.

The values $s_n$ are called the *terms* or the *elements* of the sequence $\{s_n\}$.

e.g. i) $\{n\} = \{1, 2, 3, \ldots\}$.

ii) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

iii) $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, \ldots\}$.

iv) $\{2, 3, 5, 7, 11, ...\}$, a sequence of positive prime numbers.

v) $\{s_n\}$ such that $s_1 = 1$, $s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n$.

**Range of a sequence**

The set of all distinct terms of a sequence is called its range.

**Remark:**

In a sequence $\{s_n\}$, since $n\in\mathbb{N}$ and $\mathbb{N}$ is an infinite set, the number of the terms of a sequence is always infinite. However, the range of the sequence may be finite.

**Subsequence**

It is a sequence whose terms are contained in given sequence.

A subsequence of $\{s_n\}$ is usually written as $\{s_{n_k}\}$.

**Examples:**

1. $\{2,4,6,...\}$ is subsequence of $\{1,2,3,...\}$
2. $\left\{\dfrac{1}{2n}\right\}$ and $\left\{\dfrac{1}{n+1}\right\}$ is subsequence of $\left\{\dfrac{1}{n}\right\}$.

**Increasing sequence**

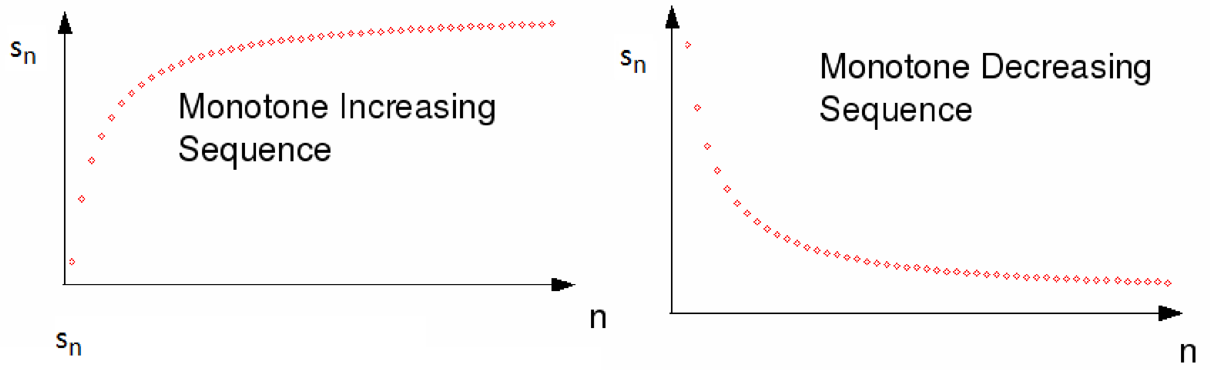
A sequence $\{s_n\}$ is said to be an increasing sequence if $s_{n+1} \geq s_n \quad \forall \; n \geq 1$.

**Decreasing sequence**

A sequence $\{s_n\}$ is said to be a decreasing sequence if $s_{n+1} \leq s_n \quad \forall \; n \geq 1$.

**Monotonic sequence**

A sequence $\{s_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.




**Remarks:**

- A sequence $\{s_n\}$ is monotonically increasing if $s_{n+1} - s_n \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically increasing if $\dfrac{s_{n+1}}{s_n} \geq 1, \quad \forall \; n \geq 1$.
- A sequence $\{s_n\}$ is monotonically decreasing if $s_n - s_{n+1} \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically decreasing if $\dfrac{s_n}{s_{n+1}} \geq 1, \quad \forall \; n \geq 1$.

**Strictly Increasing or Decreasing**

A sequence $\{s_n\}$ is called strictly increasing or decreasing according as

$$s_{n+1} > s_n \quad \text{or} \quad s_{n+1} < s_n \qquad \forall \; n \geq 1.$$

**Examples:**

- $\{n\} = \{1,2,3,...\}$ is an increasing sequence (also it is strictly increasing).
- $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence. (also it is strictly decreasing).
- $\{1,1,2,2,3,3,...\}$ is increasing sequence but it is not strictly increasing.
- $\{\cos n\pi\} = \{-1,1,-1,1,...\}$ is neither increasing nor decreasing.

**Questions:**

1) Prove that $\left\{1+\frac{1}{n}\right\}$ is a decreasing sequence.

2) Is $\left\{\frac{n+1}{n+2}\right\}$ is increasing or decreasing sequence?

**Bounded Sequence**

A sequence $\{s_n\}$ is said to be bounded if there is a positive number $\lambda$ such that

$$|s_n| \le \lambda \quad \forall \ n \in \mathbb{N}.$$

For such a sequence, every term belongs to the interval $[-\lambda, \lambda]$. Also inequality in the above definition can be replaced with strict inequality. Alternatively, a sequence is bounded if its range is a bounded set.

It can be noted that if the sequence is bounded then its supremum and infimum exist.

If $S$ and $s$ are the supremum and infimum of the bounded sequence $\{s_n\}$, then we write $S = \sup s_n$ and $s = \inf s_n$.

**Remarks:**

It is easy to conclude that if $\{s_n\}$ is bounded sequence and $n_0$ is positive integer then there exists $\lambda > 0$ such that

$$|s_n| \le \lambda \text{ whenever } n \ge n_0.$$

**Examples:**

(i) $\{u_n\} = \left\{\frac{(-1)^n}{n}\right\}$ is a bounded sequence

(ii) $\{v_n\} = \{\sin n\}$ is also bounded sequence. Its supremum is 1 and infimum is $-1$.

(iii) The geometric sequence $\{ar^{n-1}\}$, $r > 1$ is an unbounded above sequence. It is bounded below by $a$.

(iv) $\{\exp(n)\}$ is an unbounded sequence.

**Convergence of the sequence**

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

is getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

*A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L.*

However, this is too vague and too weak to serve as definition but a rough guide for the intuition, this is misleading in other respects. What about the sequence

0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, 0.00001, 0.00002, ...?

Surely this should converge to 0 but the terms do not get steadily "closer and closer" but back off a bit at each second step.

The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of "arbitrarily close" using inequalities.

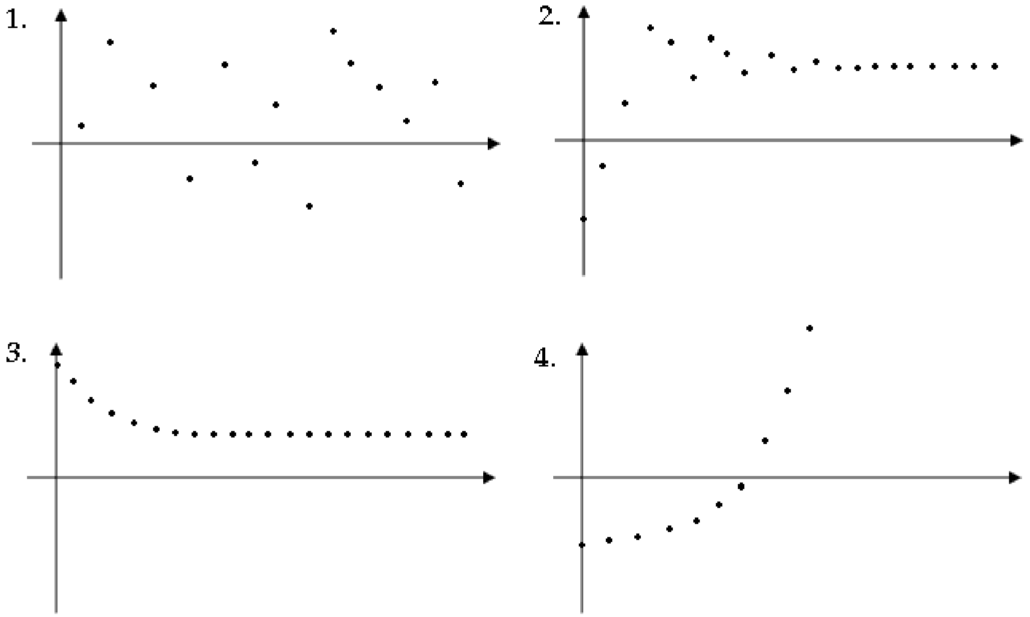
**Definition**

A sequence $\{s_n\}$ of real numbers is said to convergent to limit '$s$' as $n \to \infty$, if for every real number $\varepsilon > 0$, there exists a positive integer $n_0$, depending on $\varepsilon$, so that

$$|s_n - s| < \varepsilon \quad \text{whenever } n > n_0.$$

A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to *divergent* (it will be discussed later).

We will try to understand it by graph of some sequence. Graphs of any four sequences is drawn in the picture below.




**Examples**

*a*) Prove that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ (or $\left\{\frac{1}{n}\right\}$ converges to $0$).

*Solution:* Let $\varepsilon > 0$ be given. By the Archimedean Property, there is a positive integer $n_0 = n_0(\varepsilon)$ such that $n_0 \cdot \varepsilon > 1$, that is, $\frac{1}{n_0} < \varepsilon$. Then, if $n > n_0$, we have

$$\frac{1}{n} < \frac{1}{n_0} < \varepsilon.$$

Thus we proved that for all $\varepsilon > 0$, there exists $n_0$, depending upon $\varepsilon$, such that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \quad \text{whenever} \quad n > n_0.$$

Hence $\left\{\frac{1}{n}\right\}$ converges to point '0'.

*b*) Prove that $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ (by definition).

*Solution*: Let $\varepsilon > 0$ be given. Now consider

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} < \frac{1}{n}. \quad (\text{Since } n^2+1 > n^2 > 0)$$

Now if we choose $n_0$ such that $\frac{1}{n_0} < \varepsilon$ (or $n_0 > \frac{1}{\varepsilon}$), then the above expression gives us

$$\left|\frac{1}{n^2+1} - 0\right| < \frac{1}{n} \le \frac{1}{n_0} < \varepsilon \quad \text{whenever} \quad n \ge n_0 > \frac{1}{\varepsilon}.$$

Hence, we conclude that, $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$.

*c*) Prove that $\lim_{n\to\infty} \frac{3n+2}{n+1} = 3$ (by definition).

*Solution*: Let $\varepsilon > 0$ be given. Now consider

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right|$$

$$= \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} \qquad (\because\ n+1 > n > 0)$$

Now if we take $n_0$ such that $\frac{1}{n_0} < \varepsilon$ (or $n_0 > \frac{1}{\varepsilon}$), then the above expression gives us

$$\left|\frac{3n+2}{n+1} - 3\right| < \varepsilon \quad \text{whenever} \quad n \ge n_0.$$

Hence, we conclude that $\lim_{n\to\infty} \frac{3n+2}{n+1} = 3$.

**Questions:**

Use definition of the limits to prove the followings:

a) $\lim_{n\to\infty} \frac{2n}{n+1} = 2$. b) $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$ c) $\lim_{n\to\infty} \frac{1}{3^n} = 0$

**Definitions**

i. A bounded sequence which does not converge is said to *oscillate finitely*.

ii. A sequence $\{s_n\}$ is said to be *divergent to* $\infty$, if to each given positive number $\Delta$, there correspond an integer $m$ such that

$$s_n > \Delta \text{ for all } n \geq m.$$

iii. A sequence $\{s_n\}$ is said to be *divergent to* $-\infty$, if to each given positive number $\Delta$, there correspond an integer $m$ such that

$$s_n < -\Delta \text{ for all } n \geq m.$$

iv. A sequence $\{s_n\}$ is said to *oscillate infinitely*, if it is unbounded and is divergent neither to $\infty$ nor to $-\infty$.

**Examples**

a. $\{1+(-1)^n\}$ oscillates finitely.

b. $\{(-1)^n n\}$ oscillates infinitely.

c. $\{2^n\}$ diverges to $\infty$.

d. $\{-2n\}$ diverges to $-\infty$.

**Question**

Prove that $\{-e^n\}$ diverges to $-\infty$ (by definition)

***Solution.***

Suppose $\Delta > 0$ be given and $s_n = -e^n$ .

Take $s_n < -\Delta$, i.e. $-e^n < -\Delta \quad \Rightarrow \; e^n > \Delta \quad \Rightarrow \; n > \log \Delta$.

Now if $m$ is positive integer such that $m > \log \Delta$, then

$s_n < -\Delta$ for all $n > m$.

This implies $\{-e^n\}$ is diverges to $-\infty$.

**Question**

Prove that $\{5^n\}$ diverges to $\infty$ (by definition).

Prove that $\{n^2\}$ diverges to $\infty$ (by definition).

**Review**

- Triangular inequality: If $a, b \in \mathbb{R}$, then $\big| |a| - |b| \big| \leq \big| a \pm b \big| \leq |a| + |b|$.
- If $0 \leq a < \varepsilon$ for all $\varepsilon > 0$, then $a = 0$.

**Theorem**

A convergent sequence of real number has one and only one limit (i.e. limit of the sequence is unique.)

*Proof:*

Suppose $\{s_n\}$ converges to two limits $s$ and $t$, where $s \neq t$.

Then for all $\varepsilon > 0$, there exists two positive integers $n_1$ and $n_2$ such that

$$|s_n - s| < \frac{\varepsilon}{2} \qquad \forall \; n > n_1 \; \dots\dots\dots\dots\dots\dots \; (1)$$

and $$|s_n - t| < \frac{\varepsilon}{2} \qquad \forall \; n > n_2. \; \dots\dots\dots\dots\dots\dots \; (2)$$

As (1) and (2) hold simultaneously for all $n > \max\{n_1, n_2\}$.

Thus, for all $n > \max\{n_1, n_2\}$ we have

$$\begin{aligned}0 \le |s-t| &= |s - s_n + s_n - t| \\ &\le |s_n - s| + |s_n - t| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .\end{aligned}$$

As $\varepsilon$ is arbitrary, we get $|s-t| = 0$, this gives $s = t$, that is, the limit of the sequence is unique. ❑

**Theorem**

If the sequence $\{s_n\}$ converges to *s*, where $s \neq 0$, then there exists a positive integer $n_1$ such that $|s_n| > \frac{1}{2}|s|$ for all $n > n_1$.

*Proof:*

Since $\{s_n\}$ converges to *s*, therefore for all real $\varepsilon > 0$, there exists positive integer $n_1$ such that

$$|s_n - s| < \varepsilon \quad \text{for } n > n_1 .$$

We fix $\varepsilon = \frac{1}{2}|s| > 0$ to get

$$|s_n - s| < \frac{1}{2}|s| \quad \text{for } n > n_1 ,$$

that is,

$$-\frac{1}{2}|s| > -|s_n - s| \quad \text{for } n > n_1 . \quad \ldots\ldots\ldots\ldots (1)$$

Now

$$\begin{aligned}\frac{1}{2}|s| &= |s| - \frac{1}{2}|s| \\ &< |s| - |s_n - s| \quad \text{for } n > n_1 \qquad \text{(by using (1))} \\ &\le |s + (s_n - s)| \quad \text{for } n > n_1\end{aligned}$$

This ultimately gives us

$$|s_n| > \frac{1}{2}|s| \text{ for all } n > n_1 .$$ ❑

**Theorem**

Let *a* and *b* be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to *s* and *t* respectively, then

(*i*) $\{as_n + bt_n\}$ converges to *as* + *bt*.

(*ii*) $\{s_n t_n\}$ converges to *st*.

(*iii*) $\left\{\frac{s_n}{t_n}\right\}$ converges to $\frac{s}{t}$, provided $t_n \neq 0$ for all $n$ and $t \neq 0$.

*Proof:*

Since $\{s_n\}$ and $\{t_n\}$ converge to *s* and *t* respectively, therefore

$$|s_n - s| < \varepsilon \qquad \forall\ n > n_1 \in \mathbb{N}$$
$$|t_n - t| < \varepsilon \qquad \forall\ n > n_2 \in \mathbb{N}$$

Also $\exists\ \lambda > 0$ such that $|s_n| < \lambda \quad \forall\ n > 1$ $\qquad$ ( $\because$ $\{s_n\}$ is bounded )

(*i*) We have

$$\begin{aligned}\left|(as_n + bt_n) - (as + bt)\right| &= \left|a(s_n - s) + b(t_n - t)\right| \\ &\le \left|a(s_n - s)\right| + \left|b(t_n - t)\right| \\ &< |a|\varepsilon + |b|\varepsilon \qquad \forall\ n > \max\{n_1, n_2\} \\ &= \varepsilon_1,\end{aligned}$$

where $\varepsilon_1 = |a|\varepsilon + |b|\varepsilon$ a certain number.

This implies $\{as_n + bt_n\}$ converges to *as* + *bt*.

(*ii*)
$$\begin{aligned}|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &= |s_n(t_n - t) + t(s_n - s)| \\ &\le |s_n|\cdot|(t_n - t)| + |t|\cdot|(s_n - s)| \\ &< \lambda\varepsilon + |t|\varepsilon \qquad \forall\ n > \max\{n_1, n_2\} \\ &= \varepsilon_2, \qquad \text{where } \varepsilon_2 = \lambda\varepsilon + |t|\varepsilon \text{ a certain number.}\end{aligned}$$

This implies $\{s_n t_n\}$ converges to *st*.

(*iii*)
$$\begin{aligned}\left|\frac{1}{t_n} - \frac{1}{t}\right| &= \left|\frac{t - t_n}{t_n t}\right| \\ &= \frac{|t_n - t|}{|t_n||t|} < \frac{\varepsilon}{\frac{1}{2}|t||t|} \qquad \forall\ n > \max\{n_1, n_2\} \\ &\because\ |t_n| > \frac{1}{2}|t| \\ &= \frac{\varepsilon}{\frac{1}{2}|t|^2} = \varepsilon_3, \qquad \text{where } \varepsilon_3 = \frac{\varepsilon}{\frac{1}{2}|t|^2} \text{ a certain number.}\end{aligned}$$

This implies $\left\{\frac{1}{t_n}\right\}$ converges to $\frac{1}{t}$.

Hence $\left\{\frac{s_n}{t_n}\right\} = \left\{s_n \cdot \frac{1}{t_n}\right\}$ converges to $s \cdot \frac{1}{t} = \frac{s}{t}$. ( from (ii) ) ❑

**Question**

Prove that if $\lim\limits_{n\to\infty} s_n = t$, then $\lim\limits_{n\to\infty} |s_n| = |t|$ but converse is not true in general.

**Question**

Prove that every convergent sequence is bounded.

**Solution:**

Consider a sequence $\{s_n\}$ converges to limit $l$, that is, for all $\varepsilon > 0$, there exists positive integer $n_0$ such that

$$|s_n - l| < \varepsilon \quad \text{for all } n > n_0.$$

For $\varepsilon = 1$, we have

$$|s_n - l| < 1 \quad \text{for all } n > n_0 \text{ …….. } (i)$$

Now $$|s_n| < |s_n - l + l| \leq |s_n - l| + |l|$$

Using (*i*), in above expression, we get

$$|s_n| < 1 + |l| \quad \text{for all } n > n_0.$$

Now take $\lambda = \max\{|s_1|, |s_2|, \ldots, |s_{n_0}|, 1 + |l|\}$, then we have

$$|s_n| \leq \lambda \quad \text{for all } n \in \mathbb{N}.$$

This implies $\{s_n\}$ is bounded. ❑

**Review:**

- For all $a, b, c \in \mathbb{R}$, $|a - b| < c \Leftrightarrow b - c < a < b + c$ or $a - c < b < a + c$.

**Theorem (Sandwich Theorem or Squeeze Theorem)**

Suppose that $\{s_n\}$ and $\{t_n\}$ be two convergent sequences such that $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = s$. If $s_n < u_n < t_n$ for all $n \geq n_0$, then the sequence $\{u_n\}$ also converges to *s*.

*Proof:*

Since the sequence $\{s_n\}$ and $\{t_n\}$ converge to the same limit *s* (say), therefore for given $\varepsilon > 0$ there exists two positive integers $n_1$ and $n_2$ such that

$$|s_n - s| < \varepsilon \qquad \forall\ n > n_1,$$

$$|t_n - s| < \varepsilon \qquad \forall\ n > n_2.$$

i.e. $$s - \varepsilon < s_n < s + \varepsilon \qquad \forall\ n > n_1,$$

$$s - \varepsilon < t_n < s + \varepsilon \qquad \forall\ n > n_2.$$

Also, we have given

$$s_n < u_n < t_n \qquad \forall\ n > n_0.$$

Consider $n_3 = \max\{n_0, n_1, n_2\}$, then we have

$$s - \varepsilon < s_n < u_n < t_n < s + \varepsilon \qquad \forall\ n > n_3$$

$$\Rightarrow\ s - \varepsilon < u_n < s + \varepsilon \qquad \forall\ n > n_3$$

i.e. $|u_n - s| < \varepsilon \quad \forall\ n > n_3$

i.e. $\lim\limits_{n\to\infty} u_n = s$. ❑

**Example**

Show that $\lim\limits_{n\to\infty}\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}\right)=0$.

***Solution.***

Consider

$$s_n=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}$$

As $\underbrace{\frac{1}{(2n)^2}+\frac{1}{(2n)^2}+\ldots+\frac{1}{(2n)^2}}_{n\ times}\le s_n<\underbrace{\frac{1}{n^2}+\frac{1}{n^2}+\ldots+\frac{1}{n^2}}_{n\ times}$ ,

that is,

$$n\cdot\frac{1}{(2n)^2}\ \le\ s_n\ <\ n\cdot\frac{1}{n^2}\qquad\Rightarrow\ \frac{1}{4n}\ \le\ s_n\ <\frac{1}{n}$$

$$\Rightarrow\ \lim_{n\to\infty}\frac{1}{4n}\ \le\ \lim_{n\to\infty}s_n\ <\ \lim_{n\to\infty}\frac{1}{n}\qquad\Rightarrow\ 0\ \le\ \lim_{n\to\infty}s_n\ <\ 0$$

$$\Rightarrow\ \lim_{n\to\infty}s_n=0.$$

❑

**Theorem**

For each irrational number *x*, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim\limits_{n\to\infty}r_n=x$.

*Proof:*

Since *x* and *x* + 1 are two different real numbers, so there exist a rational number $r_1$ such that

$$x<r_1<x+1$$

Similarly there exists a rational number $r_2\ne r_1$ such that

$$x<r_2<\min\left\{r_1,x+\frac{1}{2}\right\}<x+1$$

Continuing in this manner we have

$$x<r_3<\min\left\{r_2,x+\frac{1}{3}\right\}<x+1$$

$$x<r_4<\min\left\{r_3,x+\frac{1}{4}\right\}<x+1$$

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$$x<r_n<\min\left\{r_{n-1},x+\frac{1}{n}\right\}<x+1$$

This implies that there is a sequence $\{r_n\}$ of the distinct rational number such that

$$x<r_n<x+\frac{1}{n}.$$

Since $\lim_{n\to\infty}(x)=\lim_{n\to\infty}\left(x+\frac{1}{n}\right)=x$.

Therefore

$$\lim_{n\to\infty} r_n = x. \qquad \square$$

**Theorem**

Let a sequence $\{s_n\}$ be a bounded sequence.

($i$) If $\{s_n\}$ is monotonically increasing then it converges to its supremum.

($ii$) If $\{s_n\}$ is monotonically decreasing then it converges to its infimum.

**Proof**

($i$) Let $S=\sup s_n$ and take $\varepsilon>0$.

Since there exists $s_{n_0}$ such that $S-\varepsilon<s_{n_0}$

Since $\{s_n\}$ is monotonically increasing,

therefore

$$S-\varepsilon<s_{n_0}<s_n<S<S+\varepsilon \qquad \text{for } n>n_0$$
$$\Rightarrow\ S-\varepsilon<s_n<S+\varepsilon \qquad \text{for } n>n_0$$
$$\Rightarrow\ |s_n-S|<\varepsilon \qquad \text{for } n>n_0$$
$$\Rightarrow\ \lim_{n\to\infty} s_n = S$$

($ii$) Let $s=\inf s_n$ and take $\varepsilon>0$.

Since there exists $s_{n_1}$ such that $s_{n_1}<s+\varepsilon$

Since $\{s_n\}$ is monotonically decreasing,

therefore

$$s-\varepsilon<s<s_n<s_{n_1}<s+\varepsilon \qquad \text{for } n>n_1$$
$$\Rightarrow\ s-\varepsilon<s_n<s+\varepsilon \qquad \text{for } n>n_1$$
$$\Rightarrow\ |s_n-s|<\varepsilon \qquad \text{for } n>n_1$$

Thus $\lim_{n\to\infty} s_n = s$ $\square$

**Questions:**

1. Let $\{s_n\}$ be a sequence and $\lim_{n\to\infty} s_n = s$. Then prove that $\lim_{n\to\infty} s_{n+1} = s$.
2. Prove that a bounded increasing sequence converges to its supremum.
3. Prove that a bounded decreasing sequence converges to its infimum.
4. Prove that if a sequence $\{s_n\}$ converges to $l$, then every subsequence of $\{s_n\}$ converges to $l$.
5. If the subsequence $\{s_{2n}\}$ and $\{s_{2n-1}\}$ of sequence $\{s_n\}$ converges to the same limit $l$ then $\{s_n\}$ converges to $l$.

**Recurrence Relation**

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

**Example:**

Let $t_1 > 1$ and let $\{t_n\}$ be defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$ for $n \geq 1$.

(i) Show that $\{t_n\}$ is decreasing sequence.
(ii) It is bounded below.
(iii) Find the limit of the sequence.

Since $t_1 > 1$ and $\{t_n\}$ is defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$ ; $n \geq 1$

$$\Rightarrow t_n > 0 \quad \forall \; n \geq 1$$

Also $t_n - t_{n+1} = t_n - 2 + \dfrac{1}{t_n}$

$$= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n - 1)^2}{t_n} > 0.$$

$$\Rightarrow t_n > t_{n+1} \quad \forall \; n \geq 1.$$

This implies that $t_n$ is monotonically decreasing.

Since $t_n > 1 \quad \forall \; n \geq 1$,

$\Rightarrow t_n$ is bounded below.

Since $t_n$ is decreasing and bounded below therefore $t_n$ is convergent.

Let us suppose $\lim\limits_{n\to\infty} t_n = t$.

Then $\lim\limits_{n\to\infty} t_{n+1} = \lim\limits_{n\to\infty} t_n \quad \Rightarrow \lim\limits_{n\to\infty}\left(2 - \dfrac{1}{t_n}\right) = \lim\limits_{n\to\infty} t_n$

$$\Rightarrow 2 - \frac{1}{t} = t \quad \Rightarrow \frac{2t-1}{t} = t \quad \Rightarrow 2t - 1 = t^2 \quad \Rightarrow t^2 - 2t + 1 = 0$$

$$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow t = 1.$$ ❑

**Question:**

- Let $\{t_n\}$ be a positive term sequence. Find the limit of the sequence if $4t_{n+1} = \dfrac{2}{5} - 3t_n$ for all $n \geq 1$.
- Let $\{u_n\}$ be a sequence of positive numbers. Then find the limit of the sequence if $u_{n+1} = \dfrac{1}{u_n} + \dfrac{1}{4}u_{n-1}$ for $n \geq 1$.

- The Fibonacci numbers are: $F_1 = F_2 = 1$ , and for every $n \geq 3$, $F_n$ is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Find the $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ (this limit is known as golden number)

**Cauchy Sequence**

A sequence $\{s_n\}$ of real number is said to be a *Cauchy sequence* if for given number $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$|s_n - s_m| < \varepsilon \qquad \forall\ m,n > n_0$$

**Example**

The sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

Suppose $s_n = \frac{1}{n}$ and $\varepsilon > 0$ be given. We choose a positive integer $n_0 = n_0(\varepsilon)$ such that $n_0 > \frac{2}{\varepsilon}$.

Then if $m,n > n_0$, we have $\frac{1}{n} < \frac{1}{n_0} < \frac{\varepsilon}{2}$ and similarly $\frac{1}{m} < \frac{1}{n_0} < \frac{\varepsilon}{2}$. Therefore, it follows that if $m,n > n_0$, then

$$|s_n - s_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\left\{\frac{1}{n}\right\}$ is Cauchy sequence.

**Theorem**

A Cauchy sequence of real numbers is bounded.

*Proof:*

Let $\{s_n\}$ be a Cauchy sequence. Then for given number $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$|s_n - s_m| < \varepsilon \qquad \forall\ m,n > n_0 .$$

Take $\varepsilon = 1$, then we have

$$|s_n - s_m| < 1 \qquad \forall\ m,n > n_0 .$$

Fix $m = n_0 + 1$ then

$$\begin{aligned} |s_n| &= |s_n - s_{n_0+1} + s_{n_0+1}| \\ &\leq |s_n - s_{n_0+1}| + |s_{n_0+1}| \\ &< 1 + |s_{n_0+1}| \qquad \forall\ n > n_0 . \end{aligned}$$

Now take $\lambda = \max\left\{|s_1|,|s_2|,...,|s_{n_0}|,1+\left|s_{n_0+1}\right|\right\}$, then we have

$$|s_n| \le \lambda \quad \text{for all } n \in \mathbb{N}.$$

Hence we conclude that $\{s_n\}$ is a Cauchy sequence, which is bounded one. ❑

***Remarks:***

The converse of the above theorem does not hold, that is, every bounded sequence is not Cauchy.

Consider the sequence $\{s_n\}$, where $s_n = (-1)^n$, $n \ge 1$. It is bounded sequence because

$$\left|(-1)^n\right| = 1 < 2 \qquad \forall \ n \ge 1.$$

But it is not a Cauchy sequence if it is then for $\varepsilon = 1$ we should be able to find a positive integer $n_0$ such that $\left|s_n - s_m\right| < 1$ for all $m, n > n_0$.

But with $m = 2k+1$, $n = 2k+2$ when $2k+1 > n_0$, we arrive at

$$\left|s_n - s_m\right| = \left|(-1)^{2n+2} - (-1)^{2k+1}\right|$$

$$= \left|1+1\right| = 2 < 1 \quad \text{is absurd.}$$

Hence $\{s_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence.

**Questions:**

a. Prove that every Cauchy sequence of real number is bounded but converse is not true.
b. Prove that every convergent sequence is bounded but converse is not true.

**Theorem**

Every Cauchy sequence of real numbers has a convergent subsequence.

*Proof:*

Suppose $\{s_n\}$ is a Cauchy sequence, therefore it is bounded.

First, we assume that $\{s_n : n \in \mathbb{N}\}$ has maximum value, then set

$$s_{n_1} = \max\{s_n : n \ge 1\}$$
$$s_{n_2} = \max\{s_n : n > n_1\}$$
$$s_{n_3} = \max\{s_n : n > n_2\} \text{ and so on}$$

Then clearly $\{s_{n_k}\}$ is subsequence of $\{s_n\}$ and it is decreasing and bounded.

Hence it is convergent.

On the other hand, if $\{s_n : n \in \mathbb{N}\}$ has no maximum value, then there exist some positive integer $N$ such that $\{s_n : n > N\}$ has no maximum value.

Now for $m > N$, we can find some $s_m$ such that $s_m > s_N$, otherwise one of the $s_{N+1}, s_{N+2}, ..., s_m$ will be the maximum value of .$\{s_n : n > N\}$.

So assume $s_{n_1} = s_{N+1}$.

Now $s_{n_2}$ can be the first term after $s_{n_1}$ such that $s_{n_2} > s_{n_1}$.

Then $s_{n_3}$ can be the first term after $s_{n_2}$ such that $s_{n_3} > s_{n_2}$.

Continuing in this way, we get $\{s_{n_k}\}$ be a subsequence of $\{s_n\}$ such that it is increasing and bounded. Thus it is convergent. ❑

**Question:**

Prove that every bounded sequence has convergent subsequence.

**Theorem (Cauchy's General Principle for Convergence)**

A sequence of real number is convergent if and only if it is a Cauchy sequence.

*Proof:*

Let $\{s_n\}$ be a convergent sequence, which converges to $s$.

Then for given $\varepsilon > 0$ $\exists$ a positive integer $n_0$, such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \forall \; n > n_0$$

Now for $n > m > n_0$

$$\begin{aligned} |s_n - s_m| &= |s_n - s + s - s_m| \\ &\leq |s_n - s| + |s - s_m| = |s_n - s| + |s_m - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \end{aligned}$$

This shows that $\{s_n\}$ is a Cauchy sequence.

Conversely, suppose that $\{s_n\}$ is a Cauchy sequence then for $\varepsilon > 0$, there exists a positive integer $m_1$ such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \;\; \forall \; n, m > m_1 \;\; \ldots\ldots\ldots.. (i)$$

Since $\{s_n\}$ is a Cauchy sequence,

therefore it has a subsequence $\{s_{n_k}\}$ converging to *s* (say).

This implies there exists a positive integer $m_2$ such that

$$|s_{n_k} - s| < \frac{\varepsilon}{2} \qquad \forall \; n_k > m_2 \;\; \ldots\ldots\ldots.. (ii)$$

Now

$$\begin{aligned} |s_n - s| &= |s_n - s_{n_k} + s_{n_k} - s| \\ &\leq |s_n - s_{n_k}| + |s_{n_k} - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall \; n > \max\{m_1, m_2\}, \end{aligned}$$

this shows that $\{s_n\}$ is a convergent sequence. ❑

**Example**

Prove that $\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right\}$ is divergent sequence.

Let $\{t_n\}$ be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

For $m, n \in \mathbb{N}$, $n > m$ we have

$$\begin{aligned} |t_n - t_m| &= \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} \\ &> \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \quad (n-m \text{ times}) \\ &= (n-m)\frac{1}{n} = 1 - \frac{m}{n}. \end{aligned}$$

In particular if $n = 2m$ then

$$|t_n - t_m| > \frac{1}{2}.$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent. ❑

**Theorem (Nested intervals)**

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n$ $\forall\ n \geq 1$, and $(b_n - a_n) \to 0$ as $n \to \infty$ then $\bigcap I_n$ contains one and only one point.

*Proof:*

Since $I_{n+1} \subset I_n$, therefore

$$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n < b_n < b_{n-1} < \ldots < b_3 < b_2 < b_1.$$

Note that $\{a_n\}$ is increasing sequence, bounded above by $b_1$ and bounded below by $a_1$.
Also note that $\{b_n\}$ is decreasing sequence bounded below by $a_1$ and bounded above by $b_1$.
This implies both $\{a_n\}$ and $\{b_n\}$ are monotone and bounded sequences and hence convergent.
Suppose $\{a_n\}$ converges to $a$ and $\{b_n\}$ converges to $b$.

But $|a - b| = |a - a_n + a_n - b_n + b_n - b|$

$$\leq |a_n - a| + |a_n - b_n| + |b_n - b| \to 0 \quad \text{as} \quad n \to \infty.$$

$\Rightarrow\ a = b$

and $\quad a_n < a < b_n \quad \forall\ n \geq 1.$

This given $\bigcap I_n = \{a\}$, that is, $\bigcap I_n$ contains only one point. ❑

**Limit inferior of the sequence**

Suppose $\{s_n\}$ is bounded below then we define limit inferior of $\{s_n\}$ as follow

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} u_n, \quad \text{where} \quad u_n = \inf\{s_k : k \geq n\}.$$

If $s_n$ is not bounded below then we define

$$\liminf_{n\to\infty} s_n = -\infty.$$

**Limit superior of the sequence**

Suppose $\{s_n\}$ is bounded above then we define limit superior of $\{s_n\}$ as follow

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n, \text{ where } v_n = \sup\{s_k : k \ge n\}$$

If $s_n$ is not bounded above then we define

$$\limsup_{n\to\infty} s_n = +\infty.$$

**Remarks:**

i. *Limit inferior* is also known as *lower limit* and *limit superior* is also known sas *upper limit* of the sequence in the literature with the notations $\underline{\lim}$ and $\overline{\lim}$ respectively.

ii. A bounded sequence has unique limit inferior and superior.

iii. It is easy to prove that limit inferior is less than or equal to limit superior.

**Examples**

(i) Let $s_n = (-1)^n$, then limit superior of $\{s_n\}$ is 1 and limit inferior of $\{s_n\}$ is $-1$.

(*ii*) Let $s_n = (-1)^n\left(1+\dfrac{1}{n}\right)$

then limit superior of $\{s_n\}$ is 1 and limit inferior of $\{s_n\}$ is $-1$.

(*iii*) Let $s_n = \left(1+\dfrac{1}{n}\right)\cos n\pi$.

Then $u_n = \inf\{s_k : k \ge n\}$

$$= \inf\left\{\left(1+\frac{1}{n}\right)\cos n\pi, \left(1+\frac{1}{n+1}\right)\cos(n+1)\pi, \left(1+\frac{1}{n+2}\right)\cos(n+2)\pi, \ldots\right\}$$

$$= \begin{cases} \left(1+\dfrac{1}{n}\right)\cos n\pi & \text{if } n \text{ is odd} \\ \left(1+\dfrac{1}{n+1}\right)\cos(n+1)\pi & \text{if } n \text{is even} \end{cases}$$

$$\Rightarrow \liminf_{n\to\infty} s_n s_n = \lim_{n\to\infty} u_n = -1.$$

Also $v_n = \sup\{s_k : k \ge n\}$

$$= \begin{cases} \left(1+\dfrac{1}{n+1}\right)\cos(n+1)\pi & \text{if } n \text{ is odd} \\ \left(1+\dfrac{1}{n}\right)\cos n\pi & \text{if } n \text{is even} \end{cases}$$

$$\Rightarrow \limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n = 1.$$ ❑

**Theorem**

If $\{s_n\}$ is a convergent sequence, then

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$$

*Proof:*

Let $\lim_{n\to\infty} s_n = s$ then for a real number $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$|s_n - s| < \varepsilon \quad \text{whenever } n \geq n_0.$$

i.e. $\quad s - \varepsilon < s_n < s + \varepsilon \quad \text{whenever } n \geq n_0.$ ............... (i)

If we take $u_n = \inf\{s_k : k \geq n\}$ and $v_n = \sup\{s_k : k \geq n\}$, then (i) gives us

$$s - \varepsilon < u_n \leq v_n < s + \varepsilon \quad \text{whenever } n \geq n_0.$$

This gives $\lim_{n\to\infty} u_n = s$ and $\lim_{n\to\infty} v_n = s$

that is, $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$. ❑

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