# Lecture 29: Discrete Mathematics

**Course Title:** Discrete Mathematics **Course Code:** MTH211 **Class:** BSM-II

## Objectives

The main aim of the lecture is to define the notion of

- *factorial function.*
- *permutations.*
- *permutations with repetition.*

## References:


- S. Lipschutz and M. Lipson, Schaum's Outlines Discrete Mathematics, Third Edition, McGraw-Hil, 2007.
- K.H. Rosen, Discrete Mathematics and its Application, MeGraw-Hill, 6th edition. 2007.
- K.A. Ross, C.R.B. Wright, Discrete Mathematics, Prentice Hall. New Jersey, 2003.
- https://www.freepik.com/ (for background image)

**Factorial Function**

Let $n$ be a non-negative integer. For positive integer $n$, the product of the positive integers less that equal to $n$ is called $n$ factorial denoted by $n!$. Also, conventionally we define $0! = 1$.

Namely:

$$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-2)(n-1)n = n(n-1)(n-2)\cdot \ldots \cdot 3\cdot 2\cdot 1\,.$$

Accordingly, $1! = 1$ and $n! = n(n - l)!$.

**Permutations**

Any arrangement of a set of $n$ objects in a given order is called a *permutation* of the object (taken all at a time). Any arrangement of any $r \leq n$ of these objects in a given order is called an "$r$-permutation" or "a permutation of the $n$ objects taken $r$ at a time."

Consider, for example, the set of letters $A$, $B$, $C$, $D$. Then:

(i) *BDCA*, *DCBA*, and *ACDB* are permutations of the four letters (taken all at a time).

(ii) *BAD*, *ACB*, *DBC* are permutations of the four letters taken three at a time.

(iii) *AD*, *BC*, *CA* are permutations of the four letters taken two at a time.

We usually are interested in the number of such permutations without listing them.

The number of permutations of $n$ objects taken $r$ at a time will be denoted by $P(n, r)$ (other texts may use ${}_nP^r$, $P_{n,r}$ , or $(n)_r$ ).

The following theorem applies.

**Theorem:** $$P(n, r) = n(n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

We emphasize that there are $r$ factors in $n(n-1)(n-2) \cdots (n-r+1)$.

**Example**

Find the number *m* of permutations of six objects, say, *A*, *B*, *C*, *D*, *E*, *F*, taken three at a time.
In other words, find the number of “three-letter words” using only the given six letters without repetition. Let us represent the general three-letter word by the following three positions:

$$\_\_\_\_, \_\_\_\_, \_\_\_\_$$

The first letter can be chosen in 6 ways; following this the second letter can be chosen in 5 ways; and, finally, the third letter can be chosen in 4 ways.
Write each number in its appropriate position as follows:

$$6\,,\,5\,,\,4$$

By the Product Rule there are $m = 6 \cdot 5 \cdot 4$ 120 possible three-letter words without repetition from the six letters.
Namely, there are 120 permutations of 6 objects taken 3 at a time. This agrees with the formula in Theorem given above:

$$P(6,3) = 6 \cdot 5 \cdot 4 = 120.$$

Consider now the special case of $P(n, r)$ when $r = n$. We get the following result.

**Corollary**

There are $n!$ permutations of $n$ objects (taken all at a time).

For example, there are 3! = 6 permutations of the three letters A, B, C. These are:

ABC, ACB, BAC, BCA, CAB, CBA.

**Permutations with Repetitions**

Frequently we want to know the number of permutations of a multiset, that is, a set of objects some of which are alike. We will let

$$P(n; n_1, n_2, ..., n_r)$$

denote the number of permutations of $n$ objects of which $n_1$ are alike, $n_2$ are alike, . . ., $n_r$ are alike. The general formula follows:

**Theorem:** $P(n; n_1, n_2, ..., n_r) = \dfrac{n!}{n_1!\, n_2! \cdots n_r!}$.

We indicate the proof of the above theorem by a particular example. Suppose we want to form all possible five-letter “words” using the letters from the word “*BABBY*.” Now there are 5! = 120 permutations of the objects $B_1$, $A$, $B_2$, $B_3$, $Y$, where the three $B$’s are distinguished. Observe that the following six permutations

$$B_1B_2\, B_3AY,\ B_2B_1\, B_3AY,\ B_3B_1\, B_2AY,\ B_1B_3\, B_2AY,\ B_2B_3\, B_1AY,\ B_3B_2\, B_1AY$$

produce the same word when the subscripts are removed.

The 6 comes from the fact that there are 3! = 3·2·1 = 6 different ways of placing the three *B*'s in the first three positions in the permutation. This is true for each set of three positions in which the *B*'s can appear. Accordingly, the number of different five-letter words that can be formed using the letters from the word "*BABBY*" is:

$$P(5;3) = \frac{5!}{3!} = 20 .$$

**Example:**

Find the number *m* of seven-letter words that can be formed using the letters of the word "*BENZENE*."

We seek the number of permutations of 7 objects of which 3 are alike (the three *E*'s), and 2 are alike (the two *N*'s). Thus

$$m = P(7;3,2) = \frac{7!}{3!\,2!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1}$$

$$= 420.$$

THANKS FOR YOUR ATTENTION