Lecture 23: Discrete Mathematics

Course Title: Discrete Mathematics **Course Code:** MTH211 **Class:** BSM-II

Objectives

The main aim of the lecture is to

- *define connected & connected components of graph,*
- *define distance & diameter in the graphs,*
- *define cutpoints & bridges,*
- *discuss Königsberg bridge problem*

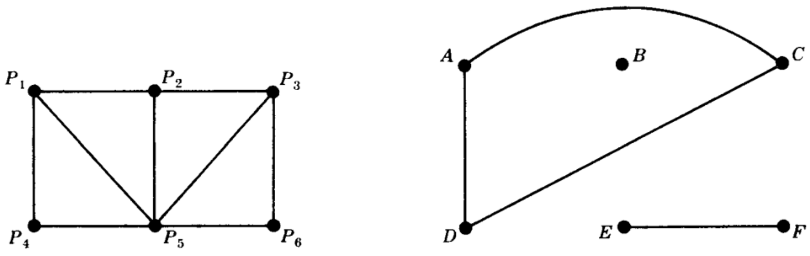
References:

- S. Lipschutz and M. Lipson, Schaum's Outlines Discrete Mathematics, Third Edition, McGraw-Hil, 2007.
- K.H. Rosen, Discrete Mathematics and its Application, MeGraw-Hill, 6th edition. 2007.
- K.A. Ross, C.R.B. Wright, Discrete Mathematics, Prentice Hall. New Jersey, 2003.

Connectivity, Connected Components

A graph G is ***connected*** if there is a path between any two of its vertices.The graph in Fig. (a) is connected, but the graph in Fig. (b) is not connected since, for example, there is no path between vertices D and E.

Suppose G is a graph. A connected subgraph H of G is called a ***connected component*** of G if H is not contained in any larger connected subgraph of G. It is intuitively clear that any graph G can be partitioned into its connected components. For example, the graph G in Fig. (b) has three connected components, the subgraphs induced by the vertex sets $\{A, C, D\}$, $\{E, F\}$, and $\{B\}$.



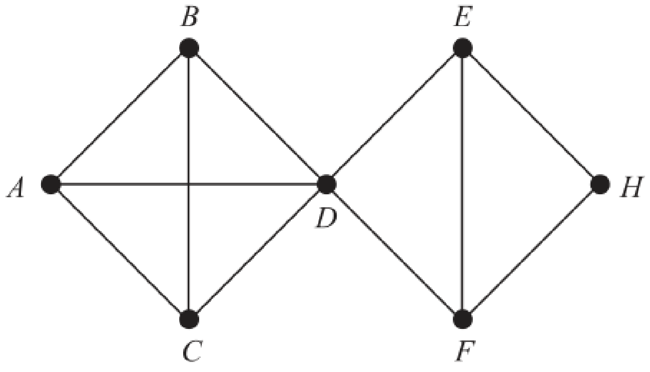
(a) (b)

Distance and Diameter

Consider a connected graph G. The distance between vertices u and v in G, written $d(u, v)$, is the length of the shortest path between u and v. The diameter of G, written diam(G), is the maximum distance between any two points in G.

For example, in Fig. (a),

$$\mathrm{d}(A, F) = 2 \text{ and } \mathrm{diam}(G) = 3,$$

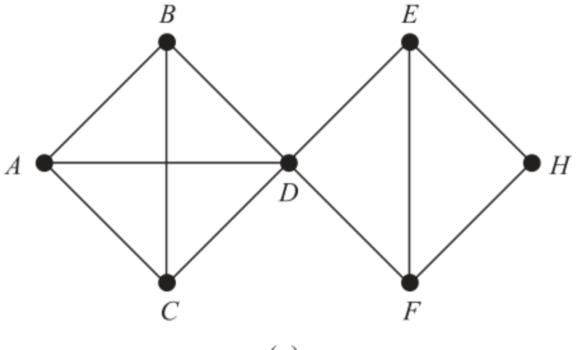


(a)

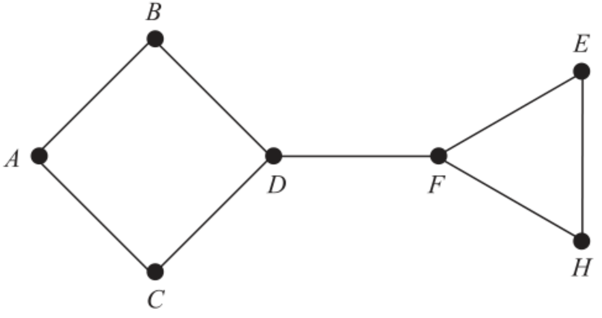
Cutpoints and Bridges

Let G be a connected graph. A vertex v in G is called a ***cutpoint*** if $G - v$ is disconnected. (Recall that $G - v$ is the graph obtained from G by deleting v and all edges containing v.) An edge e of G is called a ***bridge*** if $G - e$ is disconnected. (Recall that $G - e$ is the graph obtained from G by simply deleting the edge e).

In Fig. (a), the vertex D is a cut point and there are no bridges. In Fig. (b), the edge = $\{D, F\}$ is a bridge. (Its endpoints D and F are necessarily cutpoints.)



(a)

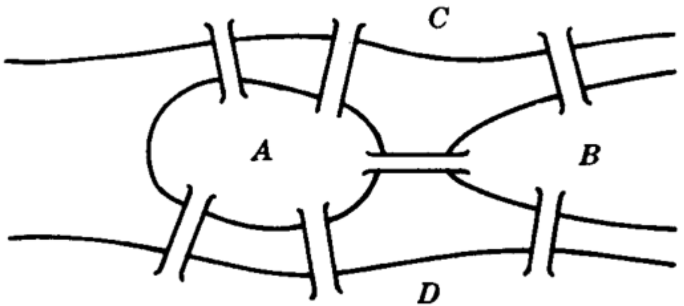


(b)

Königsberg Bridge Problem

The eighteenth-century East Prussian town of Königsberg included two islands and seven bridges as shown in Figure.

Question: Beginning anywhere and ending anywhere, can a person walk through town crossing all seven bridges but not crossing any bridge twice?

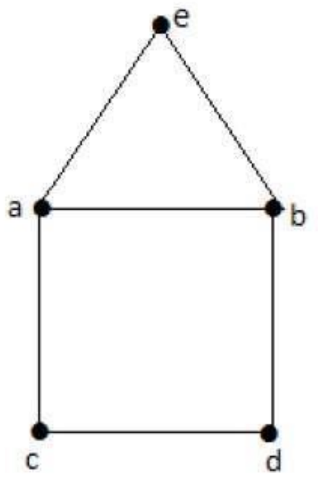


(*a*) **Königsberg in 1736**

Before going to the solution of Königsberg bridge problem, we need to understand the following definitions:

Traversable & Traversable Trail

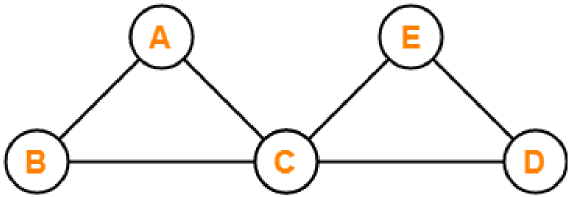
A multigraph is said to be ***traversable*** if "it can be drawn without any breaks in the curve and without repeating any edges," that is, if there is a path which includes all vertices and uses each edge exactly once. Such a path must be a trail (since no edge is used twice) and will be called a ***traversable trail***. Clearly a traversable multigraph must be finite and connected.



(A path is ***simple path*** if all the vertices are distinct and path is said to be ***trail*** if all the edges are distinct)

Eulerian graph & Eulerian trail

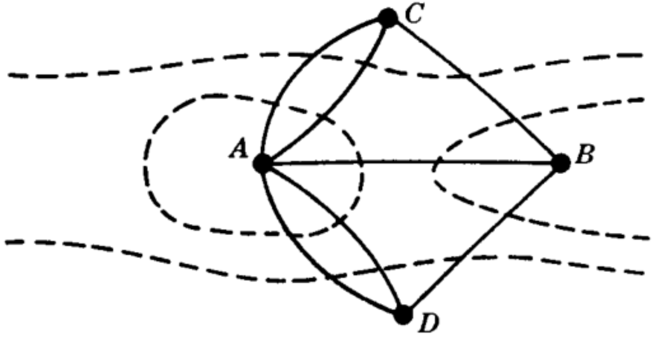
A graph G is called an ***Eulerian graph*** if there exists a closed traversable trail, called an ***Eulerian trail***.



Example of Euler Graph

Towards Solutions of Königsberg Bridge Problem

The people of Königsberg wrote to the celebrated Swiss mathematician L. Euler about this question. Euler proved in 1736 that such a walk is impossible. He replaced the islands and the two sides of the river by points and the bridges by curves, obtaining graph shown in figure.
Euler actually proved the converse of the above statement, which is contained in the following theorem and corollary.



(b) Euler's graphical representation

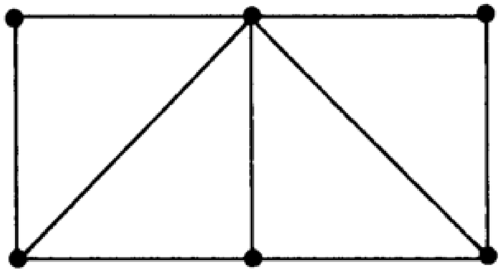
Theorem (Euler): A finite connected graph is Eulerian if and only if each vertex has even degree.

Corollary: Any finite connected graph with two odd vertices is traversable. A traversable trail may begin at either odd vertex and will end at the other odd vertex.

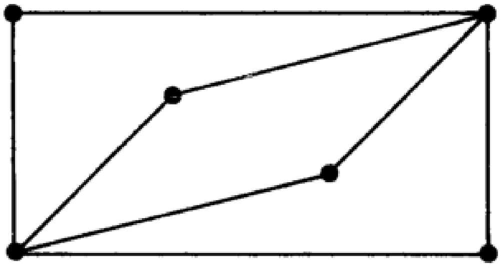
Hamiltonian Graphs*

A Hamiltonian circuit in a graph *G* is a closed path that visits every vertex in *G* exactly once. (Such a closed path must be a cycle.) If *G* does admit a Hamiltonian circuit, then *G* is called a Hamiltonian graph.

Note that an Eulerian circuit traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once but may repeat edges. Figure below gives an example of a graph which is Hamiltonian but not Eulerian, and vice versa.



(*a*) Hamiltonian and non-Eulerian



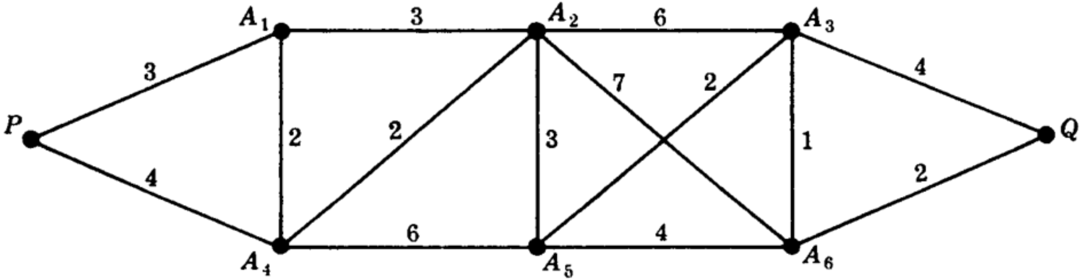
(*b*) Eulerian and non-Hamiltonian

*This was named after the nineteenth-century Irish mathematician William Hamilton (1803–1865).

Labeled and Weighted Graphs

A graph G is called a ***labeled graph*** if its edges and/or vertices are assigned data of one kind or another. In particular, G is called a ***weighted graph*** if each edge e of G is assigned a nonnegative number $w(e)$ called the ***weight*** or ***length*** of v.

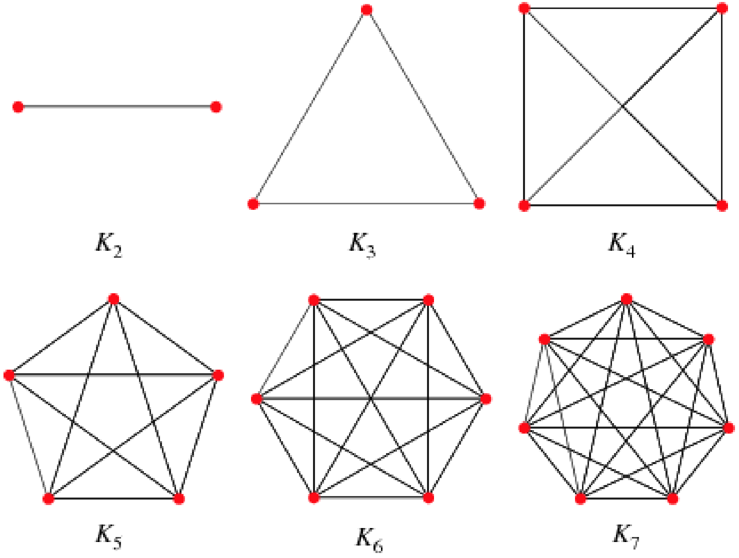
The ***weight* (**or ***length*) *of a path*** in such a weighted graph G is defined to be the sum of the weights of the edges in the path.



Complete Graphs

A graph G is said to be ***complete*** if every vertex in G is connected to every other vertex in G.

Thus, a complete graph G must be connected. The complete graph with n vertices is denoted by K_n.

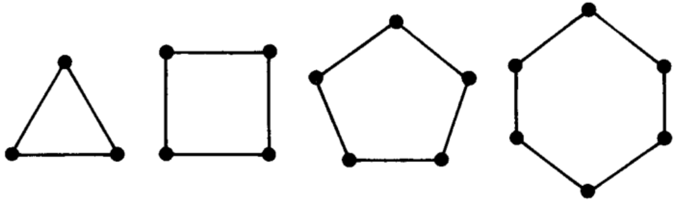


Regular Graphs

A graph G is ***regular of degree k*** or ***k-regular*** if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree.

(i) 0-regular

(ii) 1-regular



(iii) 2-regular

Thanks for your attention.