Lecture 16: Discrete Mathematics

Course Title: Discrete Mathematics **Course Code:** MTH211 **Class:** BSM-II

Objectives

The main aim of the lecture is to

- *give examples related to Big-O notation,*
  little-o notation and
  little-omega notation.

References:

- S. Lipschutz and M. Lipson, Schaum's Outlines Discrete Mathematics, Third Edition, McGraw-Hil, 2007.
- K.H. Rosen, Discrete Mathematics and its Application, MeGraw-Hill, 6th edition. 2007.
- K.A. Ross, C.R.B. Wright, Discrete Mathematics, Prentice Hall. New Jersey, 2003.
- https://en.wikipedia.org/wiki/Big_O_notation

Examples of Big O Notation:

Example 1:

$$x^3 + 3x + 1 = O(x^3).$$

Sol. $f(x) = x^3 + 3x + 1$, $g(x) = \underline{x^3}$

$|f(x)| = |x^3 + 3x + 1| = |x^3|\left|1 + \frac{3}{x^2} + \frac{1}{x^3}\right|$

$\leq |x^3|\left(1 + \left|\frac{3}{x^2}\right| + \left|\frac{1}{x^3}\right|\right)$

$= |x^3|\left(1 + 3\cdot\left|\frac{1}{x^2}\right| + \left|\frac{1}{x^3}\right|\right) \leftarrow (1)$

If $x > 1$, then $\frac{1}{x} < 1$, $\frac{1}{x^2} < 1$, $\frac{1}{x^3} < 1$

Using it in (1)

$|f(x)| < |x^3|(1 + 3 + 1)$

$\Rightarrow |f(x)| < 5|g(x)|$ for $x > 1$

$\Rightarrow \quad \underline{f(x) = O(g(x))}$

For $x > K$, $\exists\, C > 0$

$|f(x)| \leq C|g(x)|$, $x > K$.

Example 2:

$$x\sin\left(\frac{1}{x}\right)=O(x). \quad x \in \mathbb{R}$$

Sol.

Let $f(x) = x\sin\frac{1}{x}$, $g(x) = x$

$$|f(x)| = \left|x\sin\frac{1}{x}\right|$$

$$= |x|\left|\sin\frac{1}{x}\right| \text{ —— (1)}$$

If $x>1$, then $\left|\sin\frac{1}{x}\right| \leq 1$

$$\boxed{f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))}$$

Using in (1),

$$|f(x)| \leq |x|, \quad x>1$$

$$= |g(x)|$$

$$\Rightarrow f(x) = O(g(x)),$$

Here $C=1$, $K=1$ w.r.t definition of Big-O-notation.

Example of Little o Notation:

$$f(x) = o(g(x)) \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

Example 3:

$$2x = o(x^2), \quad x \in \mathbb{R}.$$

Sol.

$$f(x) = 2x, \qquad g(x) = x^2$$

Now $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{2x}{x^2}$$

$$= \lim_{x\to\infty} \frac{2}{x}$$

$$= 0$$

$$\Rightarrow 2x = o(x^2).$$

Example 4:

$$x\sin\left(\frac{1}{x}\right)=o(x).$$

Sol. Assume $f(x) = x\sin\frac{1}{x}$, $g(x) = x$.

Now

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x\sin\frac{1}{x}}{x}$$

$$= \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{1/x}\cdot\lim_{x\to\infty}\left(\frac{1}{x}\right)$$

If we take $\frac{1}{x} = y$,

then $y\to 0$ as $x\to\infty$.

So

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{y\to 0}\frac{\sin y}{y}\cdot\lim_{y\to 0} y$$

$$= 1\cdot 0 = 0$$

$$\Rightarrow \quad f(x) = o(g(x))$$

$\square$

Examples of Little-Omega Notation

$f(x) = \omega(g(x)) \Rightarrow \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$

Example 5:

$$3x^2 - 2x + 1 = \omega(x+1).$$

Sol. Let $f(x) = 3x^2 - 2x + 1$, $g(x) = x + 1$.

Now $\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{x+1}{3x^2 - 2x + 1}$

$= \lim_{x\to\infty} \frac{x\left(1 + \frac{1}{x}\right)}{x^2\left(3 - \frac{2}{x} + \frac{1}{x^2}\right)}$

$= \lim_{x\to\infty} \frac{1}{x} \frac{\left(1 + \frac{1}{x}\right)}{\left(3 - \frac{2}{x} + \frac{1}{x^2}\right)}$

$= 0 \cdot \frac{(1+0)}{(3-0+0)} = 0$

$\Rightarrow 3x^2 - 2x + 1 = \omega(x+1)$ $\square$

Example 6:

$$x^5 - 3x^2 + 4x - 1 = \omega(x^2+5).$$

Sol. Let $f(x) = x^5 - 3x^2 + 4x - 1$,
$g(x) = x^2 + 5$.

Now
$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{x^2+5}{x^5-3x^2+4x-1}$$

$$= \lim_{x\to\infty} \frac{x^2(1+5/x^2)}{x^5\left(1-\frac{3}{x^3}+\frac{4}{x^4}-\frac{1}{x^5}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^3} \frac{\left(1+\frac{5}{x^2}\right)}{\left(1-\frac{3}{x^3}+\frac{4}{x^4}-\frac{1}{x^5}\right)}$$

$$= 0 \,\frac{(1+0)}{(1-0+0-0)} = 0$$

$$\Rightarrow f(x) = \omega(g(x)). \quad \square$$

$$\underbrace{t^5 - 3t^2 + 4t - 1}_{f(t)} = \omega(\underbrace{t^2+5}_{g(t)}) \quad t \in \mathbb{R}.$$

$$\lim_{t\to\infty} \frac{g(t)}{f(t)}$$

Thanks for your attention.