Lecture 05: Discrete Mathematics

Course Title: Discrete Mathematics **Course Code:** MTH211 **Class:** BSM-II

Objectives

The main aim of the lecture is to discuss about

- *Propositional Function*
- *Universal Quantifier*
- *Existential Quantifier*
- *Negation of Quantified Statement*

References:

- S. Lipschutz and M. Lipson, Schaum's Outlines Discrete Mathematics, Third Edition, McGraw-Hil, 2007.
- K.H. Rosen, Discrete Mathematics and its Application, MeGraw-Hill, 6th edition. 2007.
- K.A. Ross, C.R.B. Wright, Discrete Mathematics, Prentice Hall. New Jersey, 2003.

Propositional Functions

Let A be a given set. *A propositional function* (or an *open sentence* or *condition*) defined on A is an expression $p(x)$, which has the property that $p(a)$ is true or false for each $a \in A$.

The set A is called the *domain* of $p(x)$, and the set T_p of all elements of A for which $p(a)$ is true is called the *truth set* of $p(x)$. In other words,

$$T_p = \{x \mid x \in A, p(x) \text{ is true}\} \text{ or } T_p = \{x \mid p(x)\}$$

Example: Consider propositional function $p(x)$ defined on the set **N** of positive integers.

(a) Let $p(x)$ be "$x + 2 > 7$." Its truth set is $\{6, 7, 8, \ldots\}$ consisting of all integers greater than 5.

(b) Let $p(x)$ be "$x + 5 < 3$." Its truth set is the empty set. That is, $p(x)$ is not true for any integer in **N**.

(c) Let $p(x)$ be "$x + 5 > 1$." Its truth set is **N**. That is, $p(x)$ is true for every element in **N**.

Universal Quantifier

Let *p(x)* be a propositional function defined on a set *A*. Consider the expression

$$(\forall x \in A)\ p(x) \qquad \text{or} \qquad \forall x\ p(x)$$

which reads "For every *x* in *A*, *p*(*x*) is a true statement" or, simply, "For all *x*, *p*(*x*)."

The symbol ∀, which reads "for all" or "for every" is called the *universal quantifier*.

The above statement is equivalent to the statement

$$T_{\mathrm{p}} = \{x \mid x \in A, p(x)\} = A$$

that is, that the truth set of *p(x)* is the entire set *A*. So, we have a conclusion:

If $\{x \mid x \in A, p(x)\} = A$ then $\forall x\ p(x)$ is true; otherwise, $\forall x\ p(x)$ is false.

Examples

(a) The proposition $(\forall n \in \mathbf{N})(n + 4 > 3)$ is true since $\{n \mid n + 4 > 3\} = \{1, 2, 3, \ldots\} = \mathbf{N}$.

(b) The proposition $(\forall n \in \mathbf{N})(n + 2 > 8)$ is false since $\{n \mid n + 2 > 8\} = \{7, 8, \ldots\} \neq \mathbf{N}$.

(c) The symbol ∀ can be used to define the intersection of an indexed collection $\{A_i \mid i \in I\}$ of sets A_i as follows:

$$\cap(A_i \mid i \in I) = \{x \mid \forall i \in I, x \in A_i\}.$$

Existential Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)\ p(x) \qquad \text{or} \qquad \exists x,\ p(x),$$

which reads "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$." The symbol ∃, which reads "there exists" or "for some" or "for at least one" is called the *existential quantifier*. Above statement is equivalent to the statement

$$T_p = \{x \mid x \in A, p(x)\} \neq \phi$$

i.e., that the truth set of $p(x)$ is not empty. Accordingly, $\exists x\ p(x)$, that is, $p(x)$ preceded by the quantifier ∃, does have a truth value. Specifically:

If $\{x \mid p(x)\} \neq \phi$ then $\exists x\ p(x)$ is true; otherwise, $\exists x\ p(x)$ is false.

Examples

(a) The proposition $(\exists n \in \mathbf{N})(n + 4 < 7)$ is true since $\{n \mid n + 4 < 7\} = \{1, 2\} \neq \phi$.

(b) The proposition $(\exists n \in \mathbf{N})(n + 6 < 4)$ is false since $\{n \mid n + 6 < 4\} = \phi$.

(c) The symbol ∃ can be used to define the union of an indexed collection $\{A_i \mid i \in I\}$ of sets Ai as follows:

$$\cup(A_i \mid i \in I) = \{x \mid \exists\ i \in I, x \mid \in A_i\}.$$

Negation of Quantified Statements

Consider the statement: “All math majors are male.” Its negation reads:
“It is not the case that all math majors are male” or, equivalently, “There exists at least one math major who is a female (not male)”

Symbolically, using M to denote the set of math majors, the above can be written as

$$\neg(\forall x \in M)\,(x \text{ is male}) \equiv (\exists\, x \in M)\,(x \text{ is not male})$$

or, when $p(x)$ denotes “x is male,”

$$\neg(\forall x \in M)\,p(x) \equiv (\exists\, x \in M)\neg p(x) \quad \text{or} \quad \neg\forall x p(x) \equiv \exists x \neg p(x).$$

The above is true for any proposition $p(x)$. That is:

Theorem (DeMorgan):

(a) $\neg(\forall x \in A)p(x) \equiv (\exists\, x \in A)\neg p(x)$ (b) $\neg(\exists x \in A)p(x) \equiv (\forall x \in A)\neg p(x)$.

Example

(a) The following statements are negatives of each other:

"For all positive integers n we have $n + 2 > 8$"

"There exists a positive integer n such that $n + 2 \not> 8$"

(b) The following statements are also negatives of each other:

"There exists a (living) person who is 150 years old"

"Every living person is not 150 years old"

Remark: The expression $\neg\, p(x)$ has the obvious meaning:

"The statement $\neg\, p(a)$ is true when $p(a)$ is false, and vice versa"

Previously, $\neg$ was used as an operation on statements; here $\neg$ is used as an operation on propositional functions.

Similarly, $p(x) \wedge q(x)$, read "$p(x)$ and $q(x)$," is defined by:

"The statement $p(a) \wedge q(a)$ is true when $p(a)$ and $q(a)$ are true"

Similarly, $p(x) \vee q(x)$, read "$p(x)$ or $q(x)$," is defined by:

"The statement $p(a) \vee q(a)$ is true when $p(a)$ or $q(a)$ is true"

Thus, in terms of truth sets:

(i) $\neg\, p(x)$ is the complement of $p(x)$.

(ii) $p(x) \wedge q(x)$ is the intersection of $p(x)$ and $q(x)$.

(iii) $p(x) \vee q(x)$ is the union of $p(x)$ and $q(x)$.

THANKS FOR YOUR ATTENTION