Lecture 02: Discrete Mathematics

Course Title: Discrete Mathematics **Course Code:** MTH211 **Class:** BSM-II

Objectives

The main aim of the lecture is to discuss about

- *proposition and its truth table*
- *define the notion of tautologies and contradiction.*
- *define and discuss logical equivalence.*
- *discuss algebra of propositions.*
- *define conditional and biconditional statements.*

References:

- S. Lipschutz and M. Lipson, Schaum's Outlines Discrete Mathematics, Third Edition, McGraw-Hil, 2007.
- K.H. Rosen, Discrete Mathematics and its Application, MeGraw-Hill, 6th edition. 2007.
- K.A. Ross, C.R.B. Wright, Discrete Mathematics, Prentice Hall. New Jersey, 2003.

Propositions & Truth Tables

Let *P(p, q, . . .)* denote an expression constructed from logical variables *p, q, . . .*, which take on the value TRUE (T) or FALSE (F), and the logical connectives ∧, ∨, and ¬ (and others discussed subsequently). Such an expression *P(p, q, . . .)* will be called a *proposition*.

For example: $\sim(p \wedge q) \vee r$ and $(\neg p \wedge q) \vee (r \vee \neg s)$.

The main property of a proposition *P(p, q, . . .)* is that its truth value depends exclusively upon the truth values of its variables, that is, the truth value of a proposition is known once the truth value of each of its variables is known. A simple concise way to show this relationship is through a *truth table*. We describe a way to obtain such a truth table below.

For example truth tables of $p \wedge q$, $p \vee q$ and $\neg p$ are as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	$\neg p$
T	F
F	T

As another example, consider proposition $\neg(p \wedge \neg q)$. Its truth table is as follows:

p	q	$\neg(p \wedge \neg q)$
T	T	T
T	F	F
F	T	T
F	F	T

But how we come to the last column in above table for $\neg(p \wedge \neg q)$. One method is as follows:

Alternate Method for Constructing a Truth Table

Another way to construct the truth table of $\neg(p \wedge \neg q)$ *as* follows:

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Tautologies and Contradictions

Some propositions $P(p, q, \ldots)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called *tautologies*. Analogously, a proposition $P(p, q, \ldots)$ is called a *contradiction* if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables.

For example: From the truth table, $p \vee \neg p$ is tautology and $p \wedge \neg p$ is contradiction.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Logical Equivalence

Two propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are said to be *logically equivalent*, or simply *equivalent* or *equal*, denoted by

$$P(p, q, \ldots) \equiv Q(p, q, \ldots)$$

if they have identical truth tables.

For example, Consider the truth tables of $\neg(p \wedge q)$ and $\neg p \vee \neg q$:

Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

Accordingly, we can write: $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Remark: Let p be "Roses are red" and q be "Violets are blue." Let S be the statement:

"It is not true that roses are red and violets are blue."

Then S can be written in the form $\neg(p \wedge q)$. However, as noted above, $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Accordingly, S has the same meaning as the statement:

"Roses are not red, or violets are not blue."

Algebra of Propositions

Propositions satisfy various laws which are listed in the table below. (In this table, *T* and *F* are restricted to the truth values "True" and "False," respectively.)

Laws of the algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Conditional and Biconditional Statements

A statement of the form "If p then q" is called *conditional* statement and is denoted by $p \rightarrow q$. The conditional $p \rightarrow q$ is frequently read "p implies q" or "p only if q."

A statement of the form "p if and only if q" is called *biconditional* statement and is denoted by

$$p \leftrightarrow q.$$

The truth values of $p \rightarrow q$ and $p \leftrightarrow q$ are defined by the the following tables:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(*a*) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(*b*) $p \leftrightarrow q$

THANKS FOR YOUR ATTENTION