Metric Spaces: An Introduction

❖ Metric Spaces

Let X be a non-empty set and $\mathbb{R}$ denotes the set of real numbers. A function $d: X \times X \to \mathbb{R}$ is said to be metric if it satisfies the following axioms $\forall \ x, y, z \in X$.

[M$_1$] $d(x,y) \geq 0$ i.e. d is finite and non-negative real valued function.

[M$_2$] $d(x,y) = 0$ if and only if $x = y$.

[M$_3$] $d(x,y) = d(y,x)$ (Symmetric property)

[M$_4$] $d(x,z) \leq d(x,y) + d(y,z)$ (Triangular inequality)

The pair (X, d) is then called *metric space*, d is called *distance function* and $d(x, y)$ is the distance from x to y.

Note: If (X, d) be a metric space then X is called *underlying set*.

❖ Examples:

i) Let X be a non-empty set. Then $d: X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1 & if \ \ x \neq y \\ 0 & if \ \ x = y \end{cases}$$

is a metric on X and is called *trivial metric* or *discrete metric*.

ii) Let $\mathbb{R}$ be the set of real number. Then $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x,y) = |x - y| \text{ is a metric on } \mathbb{R}.$$

The space $(\mathbb{R}, d)$ is called *real line* and d is called *usual metric on* $\mathbb{R}$.

iii) Let X be a non-empty set and $d: X \times X \to \mathbb{R}$ be a metric on X. Then $d': X \times X \to \mathbb{R}$ defined by $d'(x,y) = \min(1, d(x,y))$ is also a metric on X.

iv) Let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x,y) = \sqrt{|x - y|}.$$

Then d is a metric space on $\mathbb{R}$.

v) Let $x = (x_1, y_1)$, $y = (x_2, y_2)$. We define

$$d(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \text{ is a metric on } \mathbb{R}$$

and called *Euclidean metric on* $\mathbb{R}^2$ or *usual metric on* $\mathbb{R}^2$.

vi) A $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is not a metric, where $\mathbb{R}$ is the set of real number and d defined by

$$d(x,y) = (x - y)^2$$

vii) Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. We define

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

is a metric on $\mathbb{R}^2$, called *Taxi-Cab metric* on $\mathbb{R}^2$.

viii) Let $\mathbb{R}^n$ be the set of all real *n*-tuples. For

$x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)$ in $\mathbb{R}^n$.

we define $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+...+(x_n-y_n)^2}$

then *d* is metric on $\mathbb{R}^n$, called *Euclidean metric on* $\mathbb{R}^n$ or *usual metric on* $\mathbb{R}^n$.

ix) The space l^∞. As points we take bounded sequence $x=(x_1,x_2,...)$, also written as $x=(x_i)$, of complex numbers such that

$$|x_i|\leq C_x \quad \forall\ i=1,2,3,...$$

where C_x is fixed real number. The metric is defined as

$$d(x,y)=\sup_{i\in\mathbb{N}}|x_i-y_i| \qquad \text{where } y=(y_i)$$

x) The space l^p, $p\geq 1$ is a real number, we take as member of l^p, all sequence

$x=(\xi_j)$ of complex number such that $\sum_{j=1}^{\infty}|\xi_j|^p<\infty$.

The metric is defined by $d(x,y)=\left(\sum_{j=1}^{\infty}|\xi_j-\eta_j|^p\right)^{\frac{1}{p}}$,

where $y=(\eta_j)$ such that $\sum_{j=1}^{\infty}|\eta_j|^p<\infty$

❖ Open Ball

Let (X,d) be a metric space. An open ball in (X,d) is denoted by

$$B(x_0;r)=\{x\in X\mid d(x_0,x)<r\}$$

x_0 is called centre of the ball and *r* is called radius of ball and $r\geq 0$.

❖ Closed Ball

The set $\overline{B}(x_0;r)=\{x\in X\mid d(x_0,x)\leq r\}$ is called closed ball in (X,d).

❖ Sphere

The set $S(x_0;r)=\{x\in X\mid d(x_0,x)=r\}$ is called sphere in (X,d).

❖ Examples

Consider the set of real numbers with usual metric $d=|x-y|\ \forall\ x,y\in\mathbb{R}$

then $B(x_\circ;r)=\{x\in\mathbb{R}\mid d(x_\circ,x)<r\}$

i.e. $B(x_\circ;r)=\{x\in\mathbb{R}:|x-x_\circ|<r\}$

i.e. $x_0-r<x<x+r=(x_0-r,x_0+r)$

i.e. open ball is the real line with usual metric is an open interval.

And $\overline{B}(x_\circ;r)=\left\{x\in\mathbb{R}:\left|x-x_0\right|\le r\right\}$
i.e. $x_0-r\le x\le x_0+r=[x_0-r,x_0+r]$
i.e. closed ball in a real line is a closed interval.
And $S(x_\circ;r)=\left\{x\in\mathbb{R}:\left|x-x_0\right|=r\right\}=\left\{x_0-r,x_0+r\right\}$
i.e. two point x_0-r and x_0+r only.

❖ Open Set

Let (X,d) be a metric space. A set G is called open in X if for every $x\in G$, there exists an open ball $B(x;\, r)\subset G$.

❖ Theorem

An open ball in metric space X is open.

Proof.

Let $B(x_0;\, r)$ be an open ball in (X,d).
Let $y\in B(x_0;\, r)$. Then $d(x_0,y)=r_1<r$
Let $r_2<r-r_1$. Then $B(y;r_2)\subset B(x_0;r)$
Hence $B(x_0;\, r)$ is an open set.

Note: Let $\left(X,d\right)$ be a metric space. Then
i) X and φ are open sets.
ii) union of any number of open sets is open.
iii) intersection of a finite number of open sets is open.

❖ Limit point of a set

Let (X,d) be a metric space and $A\subset X$. Then $x\in X$ is called a *limit point* or *accumulation point* of *A* if for every open ball $B(x;r)$ with centre *x*,

$$B(x;r)\cap\left\{A-\{x\}\right\}\neq\varphi,$$

i.e. every open ball contains a point of *A* other than *x*.

❖ Closed Set

A subset *A* of metric space *X* is *closed* if it contains every limit point of itself. The set of all limit points of *A* is called the *derived set of A* and denoted by A'.

❖ Theorem

A subset *A* of a metric space is closed if and only if its complement A^c is open.

❖ Theorem

A closed ball is a closed set.

❖ Theorem

Let (X,d) be a metric space and $A\subset X$. If $x\in X$ is a limit point of *A*, then every open ball $B(x;r)$ with centre *x* contain an infinite numbers of point of *A*.

❖ Closure of a Set

Let (X,d) be a metric space and $M\subset X$. Then *closure of M* is denoted by $\overline{M}=M\cup M'$, where M' is the set of all limit points of *M*. It is the smallest closed superset of *M*.

❖ Dense Set

Let (*X*, *d*) be a metric space. Then a set $M\subset X$ is called dense in *X* if $\overline{M}=X$.

❖ Countable Set

A set *A* is *countable* if it is finite or there exists a function $f:A\to\mathbb{N}$ which is one-one and onto, where $\mathbb{N}$ is the set of natural numbers.

e.g. $\mathbb{N},\mathbb{Q}$ and $\mathbb{Z}$ are countable sets. The set of real numbers, the set of irrational numbers and any interval are not countable sets.

❖ Separable Space

A space *X* is said to be *separable* if it contains a countable dense subsets.
e.g. the real line $\mathbb{R}$ is separable since it contain the set $\mathbb{Q}$ of rational numbers, which is dense is $\mathbb{R}$.

❖ Theorem

Let (*X*, *d*) be a metric space. A set $A\subset X$ is dense if and only if *A* has non-empty intersection with any open subset of *X*.

❖ Neighbourhood of a Point

Let (*X*, *d*) be a metric space and $x_0\in X$. A set $N\subset X$ is called a *neighbourhood of* x_0 if there exists an open ball $B(x_0;\varepsilon)$ with centre x_0 such that $B(x_0;\varepsilon)\subset N$.

Shortly "*neighbourhood*" is written as "*nhood*".

❖ Interior Point

Let (*X*, *d*) be a metric space and $A\subset X$. A point $x_0\in X$ is called an *interior point* of *A* if there is an open ball $B(x_0;r)$ with centre x_0 such that $B(x_0;r)\subset A$.
The set of all interior points of *A* is called *interior of A* and is denoted by *int(A)* or A°.
It is the largest open set contain in A. i.e. $A^\circ\subset A$.

❖ Continuity

A function $f:\left(X,d\right)\to\left(Y,d'\right)$ is called continuous at a point $x_0\in X$ if for any $\varepsilon>0$ there is a $\delta>0$ such that $d'\left(f(x),f(x_0)\right)<\varepsilon$ for all x satisfying $d(x,x_0)<\delta$.

Alternative:

A function $f:X\to Y$ is continuous at $x_0\in X$ if for any $\varepsilon>0$, there is a $\delta>0$ such that

$$x\in B(x_0;\delta)\qquad\Rightarrow f(x)\in B\left(f(x_0);\varepsilon\right).$$

❖ Theorem

A function $f:(X,d)\to(Y,d')$ is continuous at $x_0\in X$ if and only if $f^{-1}(G)$ is open is X wherever G is open in Y.

❖ Convergence of Sequence:

Let $(x_n)=(x_1,x_2,...)$ be a sequence in a metric space (X,d). We say (x_n) converges to $x\in X$ if $\lim\limits_{n\to\infty} d(x_n,x)=0$.

We write $\lim\limits_{n\to\infty} x_n=x$ or simply $x_n\to x$ as $n\to\infty$.

Alternatively, we say $x_n\to x$ if for every $\varepsilon>0$ there is an $n_0\in\mathbb{N}$, such that

$$\forall\ n>n_0,\quad d(x_n,x)<\varepsilon.$$

❖ Theorem

i) A convergent sequence is bounded.

ii) If $x_n\to x$ and $y_n\to y$ then $d(x_n,y_n)\to d(x,y)$.

❖ Cauchy Sequence

A sequence (x_n) in a metric space (X,d) is called *Cauchy* if for any $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that $\forall\ m,n>n_0,\quad d(x_m,x_n)<\varepsilon$.

❖ Theorem

A convergent sequence in a metric space (X,d) is Cauchy.

Proof.

Let $x_n\to x\in X$, therefore any $\varepsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\forall\ m,n>n_0,\quad d(x_n,x)<\frac{\varepsilon}{2}\quad\text{and}\quad d(x_m,x)<\frac{\varepsilon}{2}.$$

Then by using triangular inequality

$$\begin{aligned} d\left(x_m,x_n\right) &\le d\left(x_m,x\right)+d\left(x,x_n\right)\\ &\le d\left(x_m,x\right)+d\left(x_n,x\right) && \because\ d(x,y)=d(y,x)\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

Thus every convergent sequence in a metric space is Cauchy.

❖ Example

Let (x_n) be a sequence in the discrete space (X,d). If (x_n) be a Cauchy sequence, then for $\varepsilon = \frac{1}{2}$, there is a natural number n_0 depending on ε such that

$$d(x_m,x_n) < \tfrac{1}{2} \qquad \forall\ m,n \geq n_0$$

Since in discrete space d is either 0 or 1 therefore $d(x_m,x_n)=0 \Rightarrow x_m = x_n = x$ (say)

Thus a Cauchy sequence in (X,d) become constant after a finite number of terms,

i.e. $(x_n) = (x_1, x_2, \ldots, x_{n_0}, x, x, x, \ldots)$

❖ Subsequence

Let $(a_1,a_2,a_3,\ldots)$ be a sequence in (X,d) and let $(i_1,i_2,i_3,\ldots)$ be a sequence of positive integers such that $i_1 < i_2 < i_3 < \ldots$ then $(a_{i_1}, a_{i_2}, a_{i_3}, \ldots)$ is called *subsequence* of $(a_n : n \in \mathbb{N})$.

❖ Complete Space

A metric space (X,d) is called *complete* if every Cauchy sequence in X converges to a point of X.

❖ Example

Let $X = (0,1)$ then $(x_n) = (x_1, x_2, x_3, \ldots) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ is a sequence in X. Then $x_n \to 0$ but 0 is not a point of X.

❖ Subspace

Let (X,d) be a metric space and $Y \subset X$ then Y is called *subspace* if Y is itself a metric space under the metric d.

❖ Theorem

A subspace of a complete metric space (X,d) is complete if and only if Y is closed in X.

❖ Nested Sequence:

A sequence sets $A_1, A_2, A_3, \ldots$ is called *nested* if $A_1 \supset A_2 \supset A_3 \supset \ldots$

❖ Theorem (Cantor's Intersection Theorem)

A metric space (X,d) is complete if and only if every nested sequence of non-empty closed subset of X, whose diameter tends to zero, has a non-empty intersection.

❖ Complete Space (Examples)

(*i*) The discrete space is complete.

Since in discrete space a Cauchy sequence becomes constant after finite terms i.e. (x_n) is Cauchy in discrete space if it is of the form

$$(x_1, x_2, x_3, ..., x_n = b, b, b, ...)$$

(ii) The set $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ of integers with usual metric is complete.

(iii) The set of rational numbers with usual metric is not complete.
Since $(1.1, 1.41, 1.412, ...)$ is a Cauchy sequence of rational numbers but its limit is $\sqrt{2}$, which is not rational.

(iv) The space of irrational number with usual metric is not complete.
We take $(-1,1), (-\tfrac{1}{2}, \tfrac{1}{2}), (-\tfrac{1}{3}, \tfrac{1}{3}), ..., (-\tfrac{1}{n}, \tfrac{1}{n})$
We choose one irrational number from each interval and these irrational tends to zero as we goes toward infinity, as zero is a rational so space of irrational is not complete.

❖ **Theorem**

The real line is complete.
The Euclidean space $\mathbb{R}^n$ is complete.
The space l^∞ is complete.
The space **C** of all convergent sequence of complex number is complete.
The space l^p, $p \geq 1$ is a real number, is complete.
The space **C**[a, b] is complete.

❖ **Theorem**

If (X, d_1) and (Y, d_2) are complete then $X \times Y$ is complete.
Note: The metric d (say) on $X \times Y$ is defined as
$d(x, y) = \max\left(d_1(\xi_1, \xi_2), d_2(\eta_1, \eta_2)\right)$
where $x = (\xi_1, \eta_1)$, $y = (\xi_2, \eta_2)$ and $\xi_1, \xi_2 \in X$, $\eta_1, \eta_2 \in Y$.
