Chapter 5 – Differentiation

Course Title: Real Analysis 1 Course Code: MTH321
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Differentiation allows us to find rates of change. For example, it allows us to find the rate of change of velocity with respect to time (which is acceleration). Calculus courses succeed in conveying an idea of what a derivative is, and the students develop many technical skills in computations of derivatives or applications of them. We shall return to the subject of derivatives but with a different objective.
Now we wish to see a little deeper and to understand the basis on which that theory develops.

❖ *Derivative of a function:*
Let f be defined and real valued on (a,b). For any point $c\in(a,b)$, form the quotient

$$\frac{f(x)-f(c)}{x-c}.$$

We fix point c and study the behaviour of this quotient as $x\to c$.

❖ *Definition*
Let f be defined on an open interval (a,b), and assume that $c\in(a,b)$. Then f is said to be differentiable at c whenever the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists. This limit is denoted by $f'(c)$ and is called the derivative of f at point c.

❖ *Remarks*
- There are so many notations to represents the derivative of the function in the literature.
- If $x-c=h$, then we have
$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}.$$

❖ *Definition*
If f is differentiable at each point of (a,b), then we say f is differentiable on (a,b).

❖ *Example*
(i) A function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ; \ x\neq 0 \\ 0 & ; \ x=0 \end{cases}$$

This function is differentiable at $x=0$ because

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}$$

$$= \lim_{x\to 0} x\sin\tfrac{1}{x} = 0.$$

(ii) Let $f(x) = x^n$; $n \ge 0$ (n is integer), $x \in \mathbb{R}$.

Then

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{x^n - c^n}{x-c}$$

$$= \lim_{x\to c}\frac{(x-c)(x^{n-1}+cx^{n-2}+\dots+c^{n-2}x+c^{n-1})}{x-c}$$

$$= \lim_{x\to c}(x^{n-1}+cx^{n-2}+\dots+c^{n-2}x+c^{n-1})$$

$$= nc^{n-1}$$

implies that f is differentiable every where and $f'(x) = nx^{n-1}$.

❖ ***Theorem*** (Differentiability implies continuity)

Let f be defined on (a,b), if f is differentiable at a point $x \in (a,b)$, then f is continuous at x.

Proof

We know that

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x} = f'(x) \qquad \text{where } t \ne x \quad \text{and} \quad a<t<b$$

Now

$$\lim_{t\to x}\big(f(t)-f(x)\big) = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}(t-x)$$

$$= f'(x)\cdot 0$$

$$= 0$$

$$\Rightarrow \lim_{t\to x} f(t) = f(x).$$

This show that f is continuous at x.

❖ ***Remarks***

(i) The converse of the above theorem does not hold.

Consider $f(x) = |x| = \begin{cases} x & if \;\; x \ge 0 \\ -x & if \;\; x < 0 \end{cases}$

$f'(0)$ does not exists but $f(x)$ is continuous at $x = 0$

(ii) If f is discontinuous at some point c of the domain of the function then $f'(c)$ does not exist. e.g.

$$f(x) = \begin{cases} 1 & if \;\; x > 0 \\ 0 & if \;\; x \le 0 \end{cases}$$

is discontinuous at $x = 0$ therefore it is not differentiable at $x = 0$.

❖ ***Question***

Prove that a differentiable function is continuous, but the converse is not true.

❖ ***Theorem***
Suppose f and g are defined on (a,b) and are differentiable at a point $x\in(a,b)$, then $f+g$, fg and $\dfrac{f}{g}$ are differentiable at x and

(i) $(f+g)'(x)=f'(x)+g'(x)$

(ii) $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$

(iii) $\left(\dfrac{f}{g}\right)'(x)=\dfrac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$, proved $g(x)\neq 0$.

The proof of this theorem can be get from any F.Sc or B.Sc textbook.

❖ ***Remark***
As we know $\lim\limits_{t\to x}\dfrac{f(t)-f(x)}{t-x}=f'(x)$, this gives

$$\frac{f(t)-f(x)}{t-x}=f'(x)+u(t),$$

where $u(t)$ is a function such that $u(t)\to 0$ as $t\to x$.

This gives us $f(t)-f(x)=(t-x)\left[f'(x)+u(t)\right]$, where $u(t)\to 0$ as $t\to x$, as an alternative definition of derivative.

❖ ***Theorem (Chain Rule)***
Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x\in(a,b)$. A function g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t)=g\left(f(t)\right)$; $a\le t\le b$, then h is differentiable at x and

$$h'(x)=g'\left(f(x)\right)\cdot f'(x).$$

Proof
Let $y=f(x)$
By the definition of the derivative we have

$$f(t)-f(x)=(t-x)\left[f'(x)+u(t)\right] \quad \ldots\ldots\ldots\ldots (i)$$

and $$g(s)-g(y)=(s-y)\left[g'(y)+v(s)\right] \quad \ldots\ldots\ldots\ldots (ii)$$

where $t\in\left[a,b\right]$, $s\in I$ and $u(t)\to 0$ as $t\to x$ and $v(s)\to 0$ as $s\to y$.
Let us suppose $s=f(t)$ then

$$\begin{aligned} h(t)-h(x) &= g\left(f(t)\right)-g\left(f(x)\right)=g(s)-g(y)\\ &=\left[s-y\right]\left[g'(y)+v(s)\right] && \text{by } (ii)\\ &=\left[f(t)-f(x)\right]\left[g'(y)+v(s)\right]\\ &=(t-x)\left[f'(x)+u(t)\right]\left[g'(y)+v(s)\right] && \text{by } (i)\end{aligned}$$

or if $t\neq x$

$$\frac{h(t)-h(x)}{t-x}=\left[f'(x)+u(t)\right]\left[g'(y)+v(s)\right]$$

taking the limit as $t\to x$ we have

$$h'(x) = [f'(x)+0][g'(y)+0]$$
$$= g'(f(x))\cdot f'(x), \qquad \because \; y = f(x)$$

This is the required result.

❖ *Example*

Let us find the derivative of $\sin(2x)$, One way to do that is through some trigonometric identities. Indeed, we have

$$\sin(2x) = 2\sin(x)\cos(x)\cdot$$

So we will use the product formula to get

$$\left(\sin(2x)\right)' = 2\left(\sin'(x)\cos(x) + \sin(x)\cos'(x)\right)$$

which implies

$$\left(\sin(2x)\right)' = 2\left(\cos^2(x) - \sin^2(x)\right)\cdot$$

Using the trigonometric formula $\cos(2x) = \cos^2(x) - \sin^2(x)$, we have

$$\left(\sin(2x)\right)' = 2\cos(2x)\cdot$$

Once this is done, you may ask about the derivative of $\sin(5x)$? The answer can be found using similar trigonometric identities, but the calculations are not as easy as before. We will see how the Chain Rule formula will answer this question in an elegant way.

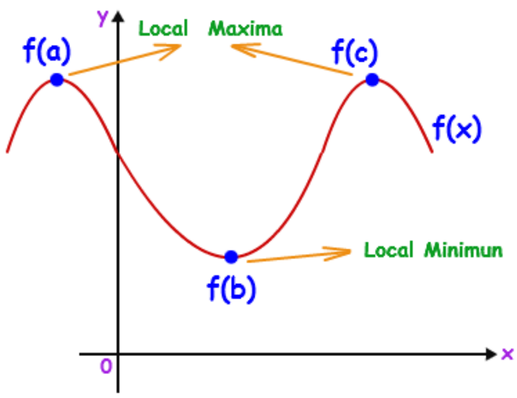
Let us find the derivative of $\sin(5x)\cdot$

We have $h(x) = f(g(x))$, where $g(x) = 5x$ and $f(x) = \sin x$. Then the Chain rule implies that $h'(x)$ exists and

$$h'(x) = 5\cdot[\cos(5x)] = 5\cos(5x)\cdot$$

❖ *Local Maximum*

Let f be a real valued function defined on a set $E \subseteq \mathbb{R}$, we say that f has a local maximum at a point $p \in E$ if there exist $\delta > 0$ such that $f(x) \le f(p) \;\; \forall \;\; x \in E$ with $|x - p| < \delta$.



Local minimum is defined likewise.

❖ *Theorem*

Let f be defined on $[a,b]$ and it is differentiable on (a,b). If f has a local maximum at a point $x \in [a,b]$ and if $f'(x)$ exist then $f'(x) = 0$.

Proof

Choose a $\delta > 0$ such that

$$a < x - \delta < x < x + \delta < b$$

Now if $x - \delta < t < x$ then

$$\frac{f(t)-f(x)}{t-x} \geq 0$$

Taking limit as $t \to x$ we get

$$f'(x) \geq 0 \ \ldots\ldots\ldots\ldots (i)$$

If $x < t < x + \delta$

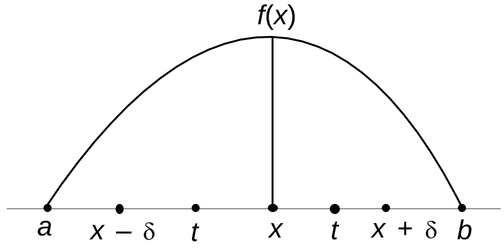
Then

$$\frac{f(t)-f(x)}{t-x} \leq 0$$

Again taking limit when $t \to x$ we get

$$f'(x) \leq 0 \ \ldots\ldots\ldots\ldots\ldots (ii)$$

Combining (i) and (ii) we have $f'(x) = 0$.



❖ ***Theorem***

Let f be defined on $[a,b]$ and it is differentiable on (a,b). If f has a local minimum at a point $x \in [a,b]$ and if $f'(x)$ exist then $f'(x) = 0$.

The proof of this theorem is like the proof of above theorem.

❖ ***Lagrange's Mean Value Theorem.***

Let f be

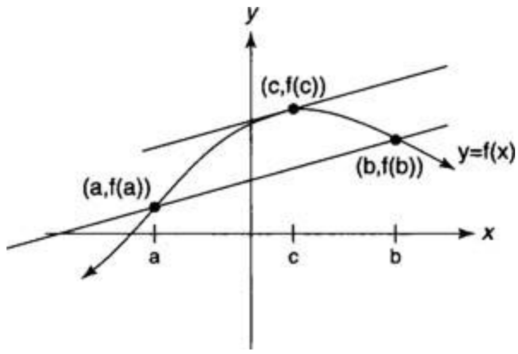
i) continuous on $[a,b]$

ii) differentiable on (a,b).

Then there exists a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

We are skipping the proof as it is included in BSc calculus book.

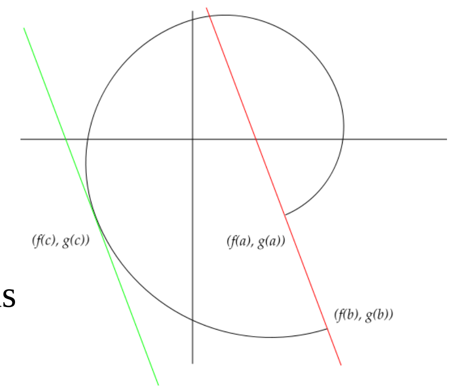


❖ ***Generalized Mean Value Theorem***

If f and g are continuous real valued functions on closed interval $[a,b]$ and f and g are differentiable on (a,b), then there is a point $c \in (a,b)$ at which

$$[f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c)$$

We are skipping the proof as it is included in BSc calculus book.



❖ ***Theorem (Intermediate Value Theorem or Darboux's Theorem)***

Suppose f is a real differentiable function on some interval I with $a,b \in I, a < b$ and suppose $f'(a) < \lambda < f'(b)$ then there exist a point $x \in (a,b)$ such that $f'(x) = \lambda$.

A similar result hold if $f'(a) > f'(b)$.

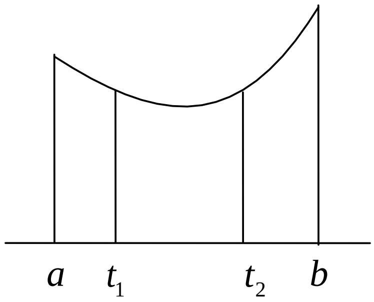
Proof

Put $g(t) = f(t) - \lambda t$

Then $g'(t) = f'(t) - \lambda$

If $t = a$ we have

$$g'(a) = f'(a) - \lambda$$



Since $f'(a) - \lambda < 0$, therefore $g'(a) < 0$.
This implies that g is monotonically decreasing at a.
So there exists a point $t_1 \in (a,b)$ such that $g(a) > g(t_1)$.
Similarly,

$$g'(b) = f'(b) - \lambda$$

Since $f'(b) - \lambda > 0$, therefore $g'(b) > 0$.
This implies that g is monotonically increasing at b.
So there exists a point $t_2 \in (a,b)$ such that $g(t_2) < g(b)$
This implies the function attain its minimum on (a,b) at a point x (say)
such that $g'(x) = 0 \quad \Rightarrow \quad f'(x) - \lambda = 0$

$$\Rightarrow \quad f'(x) = \lambda.$$

❖ ***Question***

Prove that the derivative of constant function is zero.

❖ ***Question***

Let f be defined for all real x and suppose that
$|\, f(x) - f(y) \,| \leq (x-y)^2 \quad \forall$ real x & y. Prove that f is constant.

Solution

Since $\quad |\, f(x) - f(y) \,| \leq (x-y)^2$

Therefore

$$-(x-y)^2 \leq f(x) - f(y) \leq (x-y)^2$$

Dividing throughout by $x - y$, we get

$$-(x-y) \leq \frac{f(x) - f(y)}{x-y} \leq (x-y) \qquad \text{when} \quad x > y$$

and

$$-(x-y) \geq \frac{f(x) - f(y)}{x-y} \geq (x-y) \qquad \text{when} \quad x < y$$

Taking limit as $x \to y$, we get

$$\left.\begin{array}{l} 0 \leq f'(y) \leq 0 \\ 0 \geq f'(y) \geq 0 \end{array}\right] \quad \Rightarrow \quad f'(y) = 0$$

which shows that function is constant.

❖ *Question (L'Hospital Rule)*

Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$ and $f(x) = g(x) = 0$.

Prove that $\lim\limits_{t \to x} \dfrac{f(t)}{g(t)} = \dfrac{f'(x)}{g'(x)}$

Proof

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t) - 0}{g(t) - 0} = \lim_{t \to x} \frac{f(t) - f(x)}{g(t) - (x)} \qquad \because \; f(x) = g(x) = 0$$

$$= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot \frac{t - x}{g(t) - (x)}$$

$$= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot \lim_{t \to x} \frac{1}{\dfrac{g(t) - (x)}{t - x}}$$

$$= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot \frac{1}{\lim\limits_{t \to x} \dfrac{g(t) - (x)}{t - x}} = f'(x) \cdot \frac{1}{g'(x)} = \frac{f'(x)}{g'(x)}.$$

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