Summary: Riemann-Stieltjes Integral

Course Title: Real Analysis II | Course Code: MTH322
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➢ *Partition*

Let $[a,b]$ be a given interval. A finite set $P=\{a=x_0,x_1,x_2,\ldots,x_k,\ldots,x_n=b\}$ is said to be a partition of $[a,b]$ which divides it into n such intervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n].$$

Each sub-interval is called a *component* of the partition.

Obviously, corresponding to different choices of the points x_i we shall have different partition.

The maximum of the length of the components is defined as the *norm* of the partition and it is denoted by $\|P\|$.

➢ *Refinement of a Partition*

Let P and P^* be two partitions of an interval $[a,b]$ such that $P\subset P^*$ i.e. P^* contains all the points of P and possibly some other points as well. Then P^* is said to be a *refinement* of P.

➢ *Common Refinement*

Let P_1 and P_2 be two partitions of $[a,b]$. Then a partition P^* is said to be their *common refinement* if $P^*=P_1\cup P_2$.

➢ *Examples*

Consider an interval $[1,10]$ and following partitions of this interval.

$P_1=\{1,2,3,10\}$,

$P_2=\{1,2,3,6,9,10\}$,

$$P_3=\left\{1,1+\frac{9}{100},1+2\left(\frac{9}{100}\right),1+3\left(\frac{9}{100}\right),\ldots,1+99\left(\frac{9}{100}\right),10\right\}$$

and more generally for any positive integer n, we can write

$$P_4=\left\{1,1+\frac{9}{n},1+2\left(\frac{9}{n}\right),1+3\left(\frac{9}{n}\right),\ldots,1+(n-1)\left(\frac{9}{n}\right),1+n\left(\frac{9}{n}\right)=10\right\}.$$

One can note that P_2 is refinement of P_1.

Also note that $\|P_1\|=7$, $\|P_2\|=3$, $\|P_3\|=\dfrac{9}{100}$, $\|P_4\|=\dfrac{9}{n}$.

➢ *Remark*

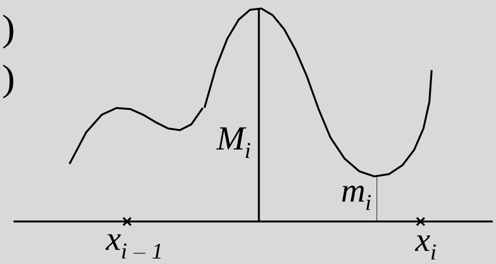
Note that if $P\subseteq P'$ implies $\|P'\|\le\|P\|$. That is, refinement of a partition decreases its norm but the convers does not necessarily hold.

Also note that $P_1\subseteq P_2$ and $\|P_2\|\le\|P_1\|$.

➢ *Riemann Integral*

Let f be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we put

$$M_i = \sup f(x) \qquad (x_{i-1} \le x \le x_i)$$
$$m_i = \inf f(x) \qquad (x_{i-1} \le x \le x_i)$$



We define upper and lower sums as

$$U(P,f) = \sum_{i=1}^{n} M_i \,\Delta x_i$$

and $$L(P,f) = \sum_{i=1}^{n} m_i \,\Delta x_i .$$

where $\Delta x_i = x_i - x_{i-1} \quad (i = 1,2,\dots,n)$

and finally $$\overline{\int_a^b} f\,dx = \inf U(P,f) \;\dots\dots\dots\dots\dots\; (i)$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f) \;\dots\dots\dots\dots\dots (ii)$$

Where the infimum and the supremum are taken over all partitions P of $[a,b]$. Then $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called the upper and lower Riemann Integrals of f over $[a,b]$ respectively.

In case the upper and lower integrals are equal, we say that f is Riemann-Integrable on $[a,b]$ and we write $f \in \mathcal{R}$, where $\mathcal{R}$ denotes the set of Riemann integrable functions.

The common value of (i) and (ii) is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Which is known as the Riemann integral of f over $[a,b]$.

➢ *Theorem*

The upper and lower integrals are defined for every bounded function f.

Proof

Take M and m to be the upper and lower bounds of $f(x)$ in $[a,b]$.

$$\Rightarrow \quad m \le f(x) \le M \qquad (a \le x \le b)$$

Then $M_i \le M$ and $m_i \ge m \qquad (i = 1,2,\dots\dots,n)$

Where M_i and m_i denote the supremum and infimum of $f(x)$ in (x_{i-1}, x_i) for certain partition P of $[a,b]$.

$$\Rightarrow L(P,f) = \sum_{i=1}^{n} m_i \,\Delta x_i \ge \sum_{i=1}^{n} m\,\Delta x_i \qquad (\Delta x_i = x_i - x_{i-1})$$

$$\Rightarrow L(P,f) \ge m \sum_{i=1}^{n} \Delta x_i$$

But $\sum_{i=1}^{n}\Delta x_i = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + ... + (x_n - x_{n-1})$

$$= x_n - x_0 = b - a$$

$\Rightarrow\ L(P, f) \geq m(b-a)$

Similarity $U(P, f) \leq M(b-a)$

$\Rightarrow\ m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$

Which shows that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set.

$\Rightarrow$ The upper and lower integrals are defined for every bounded function f. ⊙

➢ *Riemann-Stieltjes Integral*

It is a generalization of the Riemann Integral. Let $\alpha(x)$ be a monotonically increasing function on $[a,b]$. $\alpha(a)$ and $\alpha(b)$ being finite, it follows that $\alpha(x)$ is bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

(Difference of values of α at x_i & x_{i-1})

$\because\ \alpha(x)$ is monotonically increasing.

$\therefore\ \Delta\alpha_i \geq 0$

Let f be a real function which is bounded on $[a,b]$.

Put $U(P, f, \alpha) = \sum_{i=1}^{n} M_i\, \Delta\alpha_i$

$L(P, f, \alpha) = \sum_{i=1}^{n} m_i\, \Delta\alpha_i$,

where M_i and m_i have their usual meanings.

Define

$$\overline{\int_a^b} f\, d\alpha = \inf U(P, f, \alpha) \ \ldots\ldots\ldots\ldots\ldots\ (i)$$

$$\underline{\int_a^b} f\, d\alpha = \sup L(P, f, \alpha) \ \ldots\ldots\ldots\ldots\ldots\ (ii)$$

Where the infimum and supremum are taken over all partitions of $[a,b]$.

If $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$, we denote their common value by $\int_a^b f\, d\alpha$ or $\int_a^b f(x)\, d\alpha(x)$.

This is the Riemann-Stieltjes integral or simply the Stieltjes Integral of f w.r.t. α over $[a,b]$.

If $\int_a^b f\, d\alpha$ exists, we say that f is integrable w.r.t. α, in the Riemann sense, and write $f \in R(\alpha)$.

➢ *Note*

The Riemann-integral is a special case of the Riemann-Stieltjes integral when we take $\alpha(x)=x$.

$\because$ The integral depends upon f,α,a and b but not on the variable of integration.

$\therefore$ We can omit the variable and prefer to write $\int_a^b f\,d\alpha$ instead of $\int_a^b f(x)\,d\alpha(x)$.

In the following discussion f will be assume to be real and bounded, and α monotonically increasing on $[a,b]$.

➢ *Theorem*

If P^* is a refinement of P, then following holds:

(i) $L(P,f,\alpha) \le L(P^*,f,\alpha)$,

(ii) $U(P,f,\alpha) \ge U(P^*,f,\alpha)$.

➢ *Theorem*

Let f be a real valued function defined on $[a,b]$ and α be a monotonically increasing function on $[a,b]$. Then

$$\sup L(P,f,\alpha) \le \inf U(P,f,\alpha)$$

$$\text{i.e.}\quad \int_{\underline{a}}^{b} f\,d\alpha \le \int_a^{\bar{b}} f\,d\alpha\,.$$

➢ *Theorem* *(Condition of Integrability or Cauchy's Criterion for Integrability.)*

A function $f\in R(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon>0$ there exists a partition P such that $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$.

➢ *Theorem*

If $f\in R(\alpha)$ on $[a,b]$, then $|f|\in R(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

➢ *Theorem (Fundamental Theorem of Calculus)*

If $f\in R$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F'=f$, then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$
