Ch 01: Improper Integrals of 1st and 2nd Kinds

Course Title: Real Analysis II — Course Code: MTH322
Course instructor: Dr. Atiq ur Rehman — Class: MSc-III
Course URL: *www.mathcity.org/atiq/sp17-mth322*

"The objective of this chapter is to learn about different kind of improper integral. To learn the meaning of convergence and divergence of improper integrals. Learn the theory to develop different tests and techniques to find convergence or divergence of improper integrals"

We discussed Riemann-Stieltjes's integrals of the form $\int_a^b f\,d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. The integral of the form $\int_a^b f\,d\alpha$ are called definite integrals. To extend the concept, we shall relax some condition on definite integral like f on finite interval or boundedness of f on finite interval.

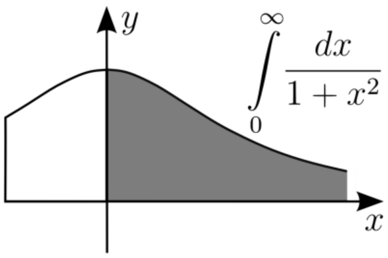
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

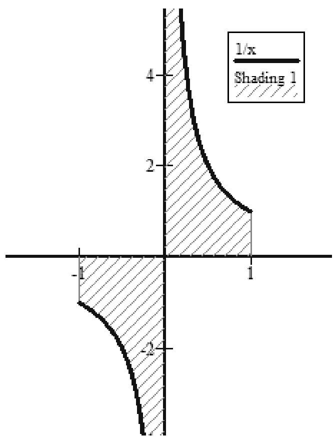
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded with infinite discontinuity at one or more points of $a\le x\le b$.

➢ *Examples*



- $\int\limits_0^\infty \frac{1}{1+x^2}dx$, $\int\limits_{-\infty}^1 \frac{1}{x-2}dx$ and $\int\limits_{-\infty}^\infty (x^2+1)dx$ are examples of improper integrals of first kind.



- $\int\limits_{-1}^1 \frac{1}{x}dx$ and $\int\limits_0^1 \frac{1}{2x-1}dx$ are examples of improper integrals of second kind.

➢ *Notations*

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

➢ *MCQs*

(i) Which of the following is an improper integral of 1st kind.

(I) $\int_1^2 \frac{1}{x}dx$ (II) $\int_1^\infty \frac{1}{x^2}dx$ (III) $\int_{-\infty}^\infty (2t+1)dt$

A. I and III only B. III only
C. II only D. II and III only

(ii) Which of the following is an improper integral of 2nd kind.

(I) $\int_{-1}^0 \frac{1}{x}dx$ (II) $\int_2^3 \frac{1}{x^2-1}dx$ (III) $\int_0^1 \tan\frac{\pi t}{2}dt$

A. I and III only B. III only
C. I only D. II and III only

(ii) The integral $\int_0^1 \frac{\sin\theta}{\theta}d\theta$ is ………………..

A. improper integral of 1st kind.
B. Improper integral of 2nd kind.
C. None of these.

IMPROPER INTEGRAL OF THE FIRST KIND

➢ *Definition*

Assume that $f\in R(\alpha;a,b)$ for every $b\geq a$. Keep a,α and f fixed and define a function I on $[a,\infty)$ as follows:

$$I(b)=\int_a^b f(x)\,d\alpha(x) \quad \text{if} \quad b\geq a.$$

The integral $\int_a^\infty f(x)\,d\alpha(x)$ is said to converge if the $\lim_{b\to\infty} I(b)$ exists (finite).

Otherwise, $\int_a^\infty f\,d\alpha$ is said to diverge.

If the $\lim_{b\to\infty} I(b)$ exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,d\alpha = A$.

➢ *Remark*

If $\int_a^\infty f\,d\alpha$ is convergent(divergent), then $\int_c^\infty f\,d\alpha$ is convergent(divergent) for $c>a$.

If $\int_c^\infty f\,d\alpha$ is convergent (divergent), then $\int_a^\infty f\,d\alpha$ is convergent (divergent) for $a<c$ if f in bounded in $[a,c]$.

➢ *Example*

Consider and integral $\int_1^\infty x^{-p}\,dx$, where p is any real number. Discuss its convergence or divergence.

Solution

Let $I(b)=\int_1^b x^{-p}\,dx$ where $b\geq 1$.

Then $I(b)=\int_1^b x^{-p}\,dx=\left.\frac{x^{1-p}}{1-p}\right|_1^b=\frac{1-b^{1-p}}{p-1}$ if $p\neq 1$.

If $b\to\infty$, then $b^{1-p}\to 0$ for $p>1$ and $b^{1-p}\to\infty$ for $p<1$.

Therefore we have

$$\lim_{b\to\infty} I(b)=\lim_{b\to\infty}\frac{1-b^{1-p}}{p-1}=\begin{cases}\infty & \text{if } p<1,\\ \dfrac{1}{p-1} & \text{if } p>1.\end{cases}$$

Now if $p=1$, we get $\int_1^b x^{-1}\,dx=\log b\to\infty$ as $b\to\infty$.

Hence we concluded: $\int_1^\infty x^{-p}\,dx=\begin{cases}\text{diverges} & \text{if } p\leq 1,\\ \dfrac{1}{p-1} & \text{if } p>1.\end{cases}$

➢ *Example*

Is the integral $\int_0^\infty \sin 2\pi x\,dx$ converges or diverges?

Solution:

Consider $I(b)=\int_0^b \sin 2\pi x\,dx$, where $b\geq 0$.

We have $\int_0^b \sin 2\pi x\,dx=\left.\frac{-\cos 2\pi x}{2\pi}\right|_0^b=\frac{1-\cos 2\pi b}{2\pi}$.

Also $\cos 2\pi b\to l$ as $b\to\infty$, where l has values between -1 and 1, that is, limit is not unique.

Therefore the integral $\int_0^\infty \sin 2\pi x\,dx$ diverges.

➢ *Exercises*

- Show that $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p>1$.
- Evaluate: (i) $\int_{-\infty}^0 \sin x\,dx$ (ii) $\int_{-\infty}^0 e^x\,dx$

➢ *Note*

If $\int_{-\infty}^{a} f\,d\alpha$ and $\int_{a}^{\infty} f\,d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\,d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\,d\alpha = \int_{-\infty}^{a} f\,d\alpha + \int_{a}^{\infty} f\,d\alpha .$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} fd\alpha$ converges, its value is equal to the limit: $\lim_{b\to+\infty}\int_{-\infty}^{\infty} fd\alpha$.

For improper integral of first kind we will discuss the results for integral of the type $\int_{a}^{\infty} f\,d\alpha$. The results for other cases can be derived in a similar manner.

➢ *Exercises*

Evaluate the improper integral $\int_{-\infty}^{\infty} e^{x}\,dx$.

➢ *MCQ*

(i) For what value of m the integral $\int_{1}^{\infty} \frac{dx}{x^{m+1}}$ is convergent.

A. $m>1$ B. $m\le 1$ C. $m>0$ D. $m\ge 0$

(ii) Which of the following integrals is divergent.

A. $\int_{2}^{\infty} \frac{dx}{x^{2}}$ B. $\int_{1}^{\infty} \frac{dt}{t^{\alpha+1}},\ \alpha>0$ C. $\int_{1}^{\infty} z^{-\frac{3}{2}}dz$ D. $\int_{1}^{\infty} x^{\frac{3}{2}}dx$

(iii) If $\int_{2}^{\infty} fdx$ is convergent then ……………… is convergent.

A. $\int_{0}^{\infty} fdx$ B. $\int_{1}^{\infty} fdx$ C. $\int_{3}^{\infty} fdx$ D. $\int_{-2}^{\infty} fdx$

➢ *Review:*

- A function f is said to be increasing, if for all $x_1, x_2 \in D_f$ (domain of f) and $x_1 \le x_2$ implies $f(x_1)\le f(x_2)$.
- A function f is said to be bounded if there exist some positive number μ such that $|f(t)|\le \mu$ for all $t\in D_f$.
- If f is define on $[a,+\infty)$ and $\lim_{x\to\infty} f(x)$ exists then f is bounded on $[a,+\infty)$.
- If $f\in R(\alpha;a,b)$ and $c\in[a,b]$, then $\int_{a}^{b} fd\alpha = \int_{a}^{c} fd\alpha + \int_{c}^{b} fd\alpha$.
- If $f\in R(\alpha;a,b)$ and $f(x)\ge 0$ for all $x\in[a,b]$, then $\int_{a}^{b} fd\alpha \ge 0$.
- If f is monotonically increasing and bounded on $[a,+\infty)$, then $\lim_{x\to\infty} f(x) = \sup_{x\in[a,\infty)} f(x)$.

- If $f, g \in R(\alpha; a,b)$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f d\alpha \le \int_a^b g d\alpha$.

> ➢ ***Theorem***
>
> Assume that α is monotonically increasing on $[a,+\infty)$ and suppose that $f \in R(\alpha; a,b)$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^\infty f\, d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that
>
> $$\int_a^b f\, d\alpha \le M \quad \text{for every } b \ge a.$$

Proof

Let $I(b) = \int_a^b f\, d\alpha$ for $b \ge a$.

First suppose that $\int_a^\infty f\, d\alpha$ is convergent, then $\lim\limits_{b\to+\infty} I(b)$ exists, that is, $I(b)$ is bounded on $[a,+\infty)$.

So there exists a constant $M > 0$ such that

$$|I(b)| < M \text{ for every } b \ge a.$$

As $f(x) \ge 0$ for each $x \ge a$, therefore $\int_a^b f\, d\alpha \ge 0$.

This gives $I(b) = \int_a^b f\, d\alpha \le M$ for every $b \ge a$.

Conversely, suppose that there exists a constant $M > 0$ such that $\int_a^b f\, d\alpha \le M$ for every $b \ge a$. This give $|I(b)| \le M$ for every $b \ge a$.

That is, I is bounded on $[a,+\infty)$.

Now for $b_2 \ge b_1 > a$, we have

$$I(b_2) = \int_a^{b_2} f\, d\alpha = \int_a^{b_1} f\, d\alpha + \int_{b_1}^{b_2} f\, d\alpha$$

$$\ge \int_a^{b_1} f\, d\alpha = I(b_1), \qquad \because \int_{b_1}^{b_2} f\, d\alpha \ge 0 \text{ as } f(x) \ge 0 \text{ for all } x \ge a.$$

This gives I is monotonically increasing on $[a,+\infty)$.

As I is monotonically increasing and bounded on $[a,+\infty)$, therefore $\lim\limits_{b\to\infty} I(b)$ exists, that is, $\int_a^\infty f\, d\alpha$ converges.

➢ *Theorem: (Comparison Test)*

Assume that α is monotonically increasing on $[a,+\infty)$ and $f \in R(\alpha;a,b)$ for every $b \ge a$. If $0 \le f(x) \le g(x)$ for every $x \ge a$ and $\int_a^\infty g d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ converges and we have $\int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$.

Proof

Let $I_1(b) = \int_a^b f\,d\alpha$ and $I_2(b) = \int_a^b g\,d\alpha$, $b \ge a$.

Since $0 \le f(x) \le g(x)$ for every $x \ge a$, therefore

$$\int_a^b f\,d\alpha \le \int_a^b g\,d\alpha ,$$

that is, $I_1(b) \le I_2(b)$ …………………. (i)

Since $\int_a^\infty g\,d\alpha$ converges, there exists a constant $M > 0$ such that

$$\int_a^b g\,d\alpha \le M \quad , \quad b \ge a \ \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

From (i) and (ii) we have $I_1(b) \le M$ for every $b \ge a$.

This implies $\int_a^\infty f d\alpha$ converges, that is, $\lim\limits_{b\to\infty} I_1(b)$ exists and is finite.

Also $\lim\limits_{b\to\infty} I_1(b) \le \lim\limits_{b\to\infty} I_2(b) \le M$,

this gives $\int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$.

➢ *Example*

Is the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ convergent or divergent?

Solution:

Since $\sin^2 x \le 1$ for all $x \in [1,+\infty)$, therefore $\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ for all $x \in [1,+\infty)$.

This gives $\int_1^\infty \frac{\sin^2 x}{x^2} dx \le \int_1^\infty \frac{1}{x^2} dx$.

Now $\int_1^\infty \frac{1}{x^2} dx$ is convergent, therefore $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent.

➢ MCQs

(i) A function f is said to be bounded if there exist a positive number α such that ………….. for all $t \in D_f$ (domain of f)

A. $f(t) \le \alpha$ B. $|f(t)| \le \alpha$ C. $|f(t)| > \alpha$ D. $f(t) > \alpha$

(i) If $f:[a,b] \to (0,\infty)$ is a bounded function then

A. $\int_a^b f(t)dt \ge 0$ B. $\int_a^\infty f(t)dt \ge 0$ C. $f(t) \ge \mu$ for $\mu \in \mathbb{R}$ D. None of these

➢ Review:

- For all $a,b,c \in \mathbb{R}$, $|a-b| < c \iff c-b < a < c+b$ or $c-a < b < c+a$.
- If $\lim\limits_{x\to\infty} f(x) = m$, then for all real $\varepsilon > 0$, there exists $N > 0$ such that
$$|f(x)-m| < \varepsilon \text{ whenever } |x| > N.$$
- If $\int_a^\infty f\,d\alpha$ converges(diverges), then $\int_N^\infty f\,d\alpha$ converges(diverges) if $N > a$.
- If $\int\limits_N^\infty f\,d\alpha$ is convergent (divergent), then $\int\limits_a^\infty f\,d\alpha$ is convergent (divergent) for $a < N$ if f is bounded in $[a,N]$.

➢ Theorem (Limit Comparison Test)

Assume that α is monotonically increasing on $[a,+\infty)$. Suppose that $f \in R(\alpha;a,b)$ and that $g \in R(\alpha;a,b)$ for every $b \ge a$, where $f(x) \ge 0$ and $g(x) \ge 0$ for $x \ge a$. If
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$
then $\int\limits_a^\infty f\,d\alpha$ and $\int\limits_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

Suppose $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$, then for all real $\varepsilon > 0$, we can find some $N > 0$, such that
$$\left|\frac{f(x)}{g(x)} - 1\right| < \varepsilon \qquad \forall \ x > N \ge a.$$
$$\Rightarrow\ 1-\varepsilon < \frac{f(x)}{g(x)} < 1+\varepsilon \qquad \forall \ x > N \ge a.$$
If we choose $\varepsilon = \frac{1}{2}$, then we have
$$\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2} \qquad \forall \ x > N \ge a.$$

This implies $g(x) < 2f(x)$ ……….(i) and $2f(x) < 3g(x)$ ……....(ii)

From (i) $\int_N^\infty g\,d\alpha < 2\int_N^\infty f\,d\alpha$,

so if $\int_a^\infty f\,d\alpha$ converges, then $\int_N^\infty f\,d\alpha$ converges and hence by comparison test we get $\int_N^\infty g\,d\alpha$ is convergent, which implies $\int_a^\infty g\,d\alpha$ is convergent.

Now if $\int_a^\infty g\,d\alpha$ diverges, then $\int_N^\infty g\,d\alpha$ diverges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is divergent, which implies $\int_a^\infty f\,d\alpha$ is divergent.

From (ii), we have $2\int_N^\infty f\,d\alpha < 3\int_N^\infty g\,d\alpha$,

so if $\int_a^\infty g\,d\alpha$ converges, then $\int_N^\infty g\,d\alpha$ converges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is convergent, which implies $\int_a^\infty f\,d\alpha$ is convergent.

Now if $\int_a^\infty f\,d\alpha$ diverges, then $\int_N^\infty f\,d\alpha$ diverges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is divergent, which implies $\int_a^\infty f\,d\alpha$ is divergent.

$\Rightarrow$ The integrals $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ converge or diverge together.

➢ *Note*

The above theorem also holds if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = c$, provided that $c \neq 0$. If $c = 0$, we can only conclude that convergence of $\int_a^\infty g\,d\alpha$ implies convergence of $\int_a^\infty f\,d\alpha$.

➢ *Example*

For every real p, the integral $\int_1^\infty e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int_1^\infty \frac{1}{x^2}dx$.

Let $f(x)=e^{-x}x^p$ and $g(x)=\frac{1}{x^2}$.

Now $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{e^{-x}x^p}{1/x^2}$

$\Rightarrow \lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty} e^{-x}x^{p+2}=\lim\limits_{x\to\infty}\frac{x^{p+2}}{e^x}=0$. (find this limit yourself)

Since $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore the given integral $\int_1^\infty e^{-x}x^p\,dx$ is also convergent.

➢ *Remark*

It is easy to show that if $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ are convergent, then

- $\int_a^\infty (f\pm g)\,d\alpha$ is convergent.
- $\int_a^\infty cf\,d\alpha$, where c is some constant, is convergent.

➢ *Absolutely & conditionally convergent*

An improper integral $\int_a^\infty f\,d\alpha$ is said to converge absolutely if $\int_a^\infty |f|\,d\alpha$ converges.

It is said to be convergent conditionally if $\int_a^\infty f\,d\alpha$ converges but $\int_a^\infty |f|\,d\alpha$.

➢ ***Note:*** The definition of absolutely and conditionally convergence is the same as above for other type of improper integrals.

➢ *Theorem*

Assume α is monotonically increasing on $[a,+\infty)$. If $f\in R(\alpha;a,b)$ for every $b\geq a$ and if $\int_a^\infty |f|\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ also converges.

Or: An absolutely convergent integral is convergent.

Proof

Let $\int_a^\infty f d\alpha$ be absolutely convergent, i.e., $\int_a^\infty |f| d\alpha$ is convergent.

Since $0 \le | f(x)| - f(x) \le 2| f(x)|$ for all $x \ge a$,

therefore by comparison test, we have $\int_a^\infty (| f | - f) d\alpha$ converges.

Now difference of $\int_a^\infty | f | d\alpha$ and $\int_a^\infty (| f | - f) d\alpha$ is convergent,

that is, $\int_a^\infty | f | d\alpha - \int_a^\infty (| f | - f) d\alpha = \int_a^\infty f \, d\alpha$ is convergent.

➢ ***Note:*** The converse of the above theorem doesn't hold in general.

For example: The integral $\int_1^\infty \frac{\sin x}{x} dx$ is convergent (prove yourself) but $\int_1^\infty \left| \frac{\sin x}{x} \right| dx$ is divergent (it is hard to prove).

➢ ***Review***

- If $\lim_{x\to\infty} f(x) = m$, then for all real $\varepsilon > 0$, there exists real $N > 0$ such that
$$|f(x) - m| < \varepsilon \text{ whenever } |x| > N.$$
- A sequence $\{a_n\}$ is said to be convergent if there exist a number l such that for all $\varepsilon > 0$, there exists a positive integer n_0 (depending on ε) such that
$$|a_n - l| < \varepsilon \text{ whenever } n > n_0.$$
The number l is called limit of the sequence and we write $\lim_{n\to\infty} a_n = l$.
- A sequence $\{a_n\}$ is said to be Cauchy if for all $\varepsilon > 0$, there exists a positive integer n_0 such that
$$|a_n - a_m| < \varepsilon \text{ whenever } n, m > n_0.$$
- A sequence of real numbers is Cauchy if and only if it is convergent.

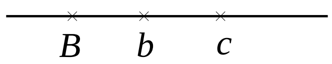
➢ *Theorem (Cauchy condition for infinite integrals)*

Assume that $f \in R(\alpha; a, b)$ for every $b \geq a$. Then the integral $\int_a^\infty f\, d\alpha$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $c > b > B$ implies

$$\left| \int_b^c f\ d\alpha \right| < \varepsilon .$$

Proof

Let $\int_a^\infty f\, d\alpha$ be convergent, that is $\lim\limits_{b\to\infty} \int_a^b f\, d\alpha = \int_a^\infty f\, d\alpha$.



Then for all $\varepsilon > 0$, there exists $B > a$ such that

$$\left| \int_a^b f\, d\alpha - \int_a^\infty f\, d\alpha \right| < \frac{\varepsilon}{2} \text{ for every } b > B. \ \ldots\ldots\ldots..(i)$$

Also for $c > b > B$,

$$\left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha \right| < \frac{\varepsilon}{2} \ \ldots\ldots\ldots\ldots\ldots.. (ii)$$

As we know $\int_a^c f\, d\alpha = \int_a^b f\, d\alpha + \int_b^c f\, d\alpha$, this gives

$$\left| \int_b^c f\, d\alpha \right| = \left| \int_a^c f\, d\alpha - \int_a^b f\, d\alpha \right|$$

$$= \left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha + \int_a^\infty f\, d\alpha - \int_a^b f\, d\alpha \right|$$

$$\leq \left| \int_a^c f\, d\alpha - \int_a^\infty f\, d\alpha \right| + \left| \int_a^\infty f\, d\alpha - \int_a^b f\, d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \left| \int_b^c f\, d\alpha \right| < \varepsilon \quad \text{when} \quad c > b > B.$$

Conversely, assume that the Cauchy condition holds.

Define $a_n = \int_a^{a+n} f\, d\alpha$ if $n = 1, 2, \ldots\ldots$

Consider n, m such that $a+n, a+m > b > B$, then

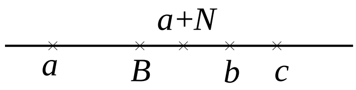
$$|a_n - a_m| = \left| \int_a^{a+n} f\, d\alpha - \int_a^{a+m} f\, d\alpha \right| = \left| \int_a^b f\, d\alpha + \int_b^{a+n} f\, d\alpha - \int_a^b f\, d\alpha - \int_b^{a+m} f\, d\alpha \right|$$

$$=\left|\int_b^{a+n} f\,d\alpha-\int_b^{a+m} f\,d\alpha\right|\le\left|\int_b^{a+n} f\,d\alpha\right|+\left|\int_b^{a+m} f\,d\alpha\right|<\varepsilon+\varepsilon=2\varepsilon$$

This gives, the sequence $\{a_n\}$ is a Cauchy sequence $\Rightarrow$ it is convergent.

Let $\lim\limits_{n\to\infty} a_n = A$. Then for given $\varepsilon>0$, choose B so that

$$|a_n - A|<\frac{\varepsilon}{2}\quad\text{whenever}\quad a+n\ge B.$$



Also for $\varepsilon>0$, we can have (by Cauchy condition)

$$\left|\int_b^{c} f\,d\alpha\right|<\frac{\varepsilon}{2}\quad\text{if}\quad c>b>B.$$

Choose an integer N such that $a+N>B$.

Then, if $b>a+N$, we have

$$\left|\int_a^{b} f\,d\alpha-A\right|=\left|\int_a^{a+N} f\,d\alpha-A+\int_{a+N}^{b} f\,d\alpha\right|$$

$$\le|a_N-A|+\left|\int_{a+N}^{b} f\,d\alpha\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

$$\Rightarrow\int_a^{\infty} f\,d\alpha=A$$

This completes the proof.

➢ *Review*

- A function $f(x)$ is bounded for $x\ge a$ if there exist some positive number K such that $|f(x)|\le K$ for all $x\ge a$.
- An integral $\int_a^{\infty} f(x)dx$ is said to be absolutely convergent if $\int_a^{\infty}|f(x)|dx$ is convergent.
- Let $f\in R(a,b)$ for each $b\ge a$. An integral $\int_a^{\infty} f\,dx$ converges if, and only if, there exists a constant $M>0$ such that $\int_a^b f dx\le M$ for every $b\ge a$.

➢ *Theorem*

If $f(x)$ is bounded for all $x\ge a$, integrable on every closed subinterval of $[a,\infty)$ (i.e. $f\in R(a,b)$ for each $b\ge a$) and $\int_a^{\infty} g(x)dx$ is absolutely convergent, then $\int_a^{\infty} f(x)g(x)dx$ is absolutely convergent.

Proof

Since $f(x)$ is bounded for all $x\ge a$, there exists $K>0$ such that

$$|f(x)| \le K \text{ for all } x \ge a. \quad \ldots\ldots \text{(i)}$$

Since $\int_a^\infty g(x)dx$ is absolutely convergent, that is, $\int_a^\infty |g(x)|dx$ is convergent, there exists $M > 0$ such that

$$\int_a^b |g(x)|dx \le M \text{ for all } b \ge a. \quad \ldots\ldots\ldots \text{(ii)}$$

Now
$$\int_a^b |f(x)g(x)|\, dx = \int_a^b |f(x)||g(x)|\, dx$$

$$\le K\int_a^b |g(x)|\, dx \qquad \text{from (i)}$$

$$\le KM \quad \text{for all } b \ge a \text{ by using (ii).}$$

Hence $\int_a^b |f(x)g(x)|\, dx$ is convergent, this implies $\int_a^b f(x)g(x)\, dx$ is absolutely convergent.

> ### *Theorem (Abel)*
> If $f(x)$ is bounded and monotone for all $x \ge a$ and $\int_a^\infty g(x)dx$ is convergent, then $\int_a^\infty f(x)g(x)dx$ is convergent.

> ### *Theorem (Dirichlet)*
> If $f(x)$ is bounded, monotone for all $x \ge a$ and $\lim_{x\to\infty} f(x) = 0$. Also $\int_a^X g(x)dx$ is is bounded for all $X \ge a$, then $\int_a^\infty f(x)g(x)dx$ is convergent.

One can see the proof of above theorem in [4, page 495-497] or in any other book of the same subject.

Example

Prove that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent;

Solution:

Since $\frac{\sin x}{x} \to 1$ as $x \to 0$, therefore 0 is not a point of infinite discontinuity.

We write $\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \int_1^\infty \frac{\sin x}{x} dx$ and note that $\int_0^1 \frac{\sin x}{x} dx$ is a proper integral. Therefore it is enough to test the convergence of $\int_1^\infty \frac{\sin x}{x} dx$.

Denote $f(x) = \frac{1}{x}$ and $g(x) = \sin x$.

Note that $|f(x)| \le 1$ and $f(x)$ is decreasing for all for all $x \ge 1$, it gives $f(x)$ is bounded and monotone for all $x \ge 1$. Also $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$.

Now $$\left|\int_1^X g(x)\,dx\right| = \left|\int_1^X \sin x\,dx\right|$$
$$= |-\cos X + \cos(1)| \le |\cos X| + |\cos(1)| < 2$$

This gives $\int_1^X g(x)\, dx$ is bounded for every $X \ge 1$.

Hence by Dirichlet theorem $\int_1^\infty f(x)g(x)\, dx = \int_1^\infty \frac{\sin x}{x}\, dx$ is convergent.

➢ *Example*

Discuss the convergence of $\int_1^\infty \sin x^2\, dx$.

Solution: We write $\sin x^2 = \frac{1}{2x} \cdot 2x \cdot \sin x^2$, i.e., $\int_1^\infty \sin x^2\, dx = \int_1^\infty \frac{1}{2x} \cdot 2x \cdot \sin x^2\, dx$

Take $f(x) = \frac{1}{2x}$ and $g(x) = 2x \sin x^2$.

Note that $|f(x)| \le \frac{1}{2}$ and $f(x)$ is decreasing for all for all $x \ge 1$, it gives $f(x)$ is bounded and monotone for all $x \ge 1$. Also $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$.

Now $$\left|\int_1^X g(x)\,dx\right| = \left|\int_1^X 2x \sin x^2\, dx\right| = |-\cos X^2 + \cos(1)| < 2.$$

This gives $\int_1^X g(x)\, dx$ is bounded for every $X \ge 1$.

Hence by Dirichlet's theorem $\int_1^\infty f(x)g(x)\,dx = \int_1^\infty \frac{1}{2x}\cdot 2x\sin x^2\,dx$ i.e. $\int_1^\infty \sin x^2\,dx$ is convergent.

➢ *Example*

Discus the convergence of $\int_0^\infty e^{-x}\frac{\sin x}{x}\,dx$.

Solution:

Let $f(x)=e^{-x}$ and $g(x)=\frac{\sin x}{x}$.

Since $|f(x)|\le 1$ for all $x\ge 0$ and $f'(x)=-e^{-x}<0$ for all $x\ge 0$, therefore $f(x)$ is bounded and monotonically decreasing for all $x\ge 0$.

Also $\int_0^\infty g(x)\,dx = \int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

Hence by Abel's theorem $\int_0^\infty f(x)g(x)dx = \int_0^\infty e^{-x}\frac{\sin x}{x}\,dx$ is convergent.

➢ *Question*

Show that $\int_0^\infty \frac{\sin x}{(1+x)^\alpha}\,dx$ converges for $\alpha>0$.

Solution

$\int_0^\infty \sin x\,dx$ is bounded because $\int_0^X \sin x\,dx \le 2 \quad \forall\ x>0$.

Furthermore the function $\frac{1}{(1+x)^\alpha}$, $\alpha>0$ is monotonic on $[0,+\infty)$.

$\Rightarrow$ the integral $\int_0^\infty \frac{\sin x}{(1+x)^\alpha}\,dx$ is convergent.

➢ *Question*

Show that $\int_0^\infty e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because\ \left|e^{-x}\cos x\right| < e^{-x}$ and $\int_0^\infty e^{-x}\,dx = 1$

$\therefore$ the given integral is absolutely convergent. (comparison test).

IMPROPER INTEGRAL OF THE SECOND KIND

➢ Definition

Let f be defined on the half open interval $(a,b]$ (having point of infinite discontinuity at a) and assume that $f \in R(\alpha; x, b)$ for every $x \in (a,b]$. Define a function I on $(a,b]$ as follows:

$$I(x) = \int_x^b f\,d\alpha \quad \text{if} \quad x \in (a,b]$$

If $\lim\limits_{x\to a+} I(x)$ exists then the integral $\int\limits_{a+}^{b} f\,d\alpha$ is said to be convergent. Otherwise, $\int\limits_{a+}^{b} f\,d\alpha$ is said to be divergent. If $\lim\limits_{x\to a+} I(x) = A$, the number A is called the value of the integral and we write $\int\limits_{a+}^{b} f\,d\alpha = A.$

Similarly, if f is defined on $[a,b)$ (having point of infinite discontinuity at b) and $f \in R(\alpha; a, x) \;\; \forall \; x \in [a,b)$ then define $I(x) = \int\limits_a^x f\,d\alpha$ if $x \in [a,b)$. If $\lim\limits_{x\to b-} I(x)$ exists (finite) then we say $\int\limits_a^{b-} f\,d\alpha$ is convergent.

➢ Note

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

➢ Example

$f(x) = x^{-p}$ is defined on $(0,b]$ and $f \in R(x,b)$ for every $x \in (0,b]$.

$$I(x) = \int_x^b u^{-p}\,du \qquad \text{if} \quad x \in (0,b]$$

$$\int_{0+}^{b} u^{-p}\,du = \lim_{\varepsilon\to 0} \int_{0+\varepsilon}^{b} u^{-p}\,du = \lim_{\varepsilon\to 0} \left| \frac{u^{1-p}}{1-p} \right|_{\varepsilon}^{b} = \lim_{\varepsilon\to 0} \frac{b^{1-p} - \varepsilon^{1-p}}{1-p} \qquad , \qquad (p \neq 1)$$

$$= \left[\begin{array}{ll} \textit{finite} \;, & p < 1 \\ \textit{infinite} \;, & p > 1 \end{array} \right.$$

When $p = 1$, we get $\int\limits_{\varepsilon}^{b} \frac{1}{x}\,dx = \log b - \log \varepsilon \to \infty \quad$ as $\;\varepsilon \to 0$.

$\Rightarrow \int_{0+}^{b} x^{-1}\,dx$ also diverges.

Hence the integral converges when $p<1$ and diverges when $p \ge 1$.

➢ *Note*

If the two integrals $\int_{a+}^{c} f\,d\alpha$ and $\int_{c}^{b-} f\,d\alpha$ both converge, we write

$$\int_{a+}^{b-} f\,d\alpha = \int_{a+}^{c} f\,d\alpha + \int_{c}^{b-} f\,d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^{b} f\,d\alpha + \int_{b}^{\infty} f\,d\alpha \quad \text{which can be written as} \quad \int_{a+}^{\infty} f\,d\alpha .$$

➢ *Question:*

Prove that $\int_{a}^{b} \frac{dx}{(x-a)^n}$ and $\int_{a}^{b} \frac{dx}{(b-x)^n}$ converges if $n<1$. (see [4, page 490])

➢ *Question*

Examine the convergence of

(*i*) $\int_{0}^{1} \frac{dx}{x^{1/3}\left(1+x^2\right)}$ (*ii*) $\int_{0}^{1} \frac{dx}{x^2(1+x)^2}$ (*iii*) $\int_{0}^{1} \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Solution: (*i*) $\int_{0}^{1} \frac{dx}{x^{1/3}\left(1+x^2\right)}$

Here ‘0’ is the only point of infinite discontinuity of the integrand.

Let $f(x) = \frac{1}{x^{1/3}\left(1+x^2\right)}$ and take $g(x) = \frac{1}{x^{1/3}}$.

Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{1+x^2} = 1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \int_{0}^{1} \frac{dx}{x^{1/3}}$ converges $\therefore \int_{0}^{1} \frac{dx}{x^{1/3}\left(1+x^2\right)}$ also converges.

(*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x) = \frac{1}{x^2(1+x)^2}$$

Take $g(x) = \dfrac{1}{x^2}$

Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{1}{(1+x)^2} = 1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n = 2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge. Hence the given integral also does not converge.

(*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.
We have

$$f(x) = \frac{1}{x^{1/2}(1-x)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\displaystyle\int_0^{1/2} f(x)\,dx$ and $\displaystyle\int_{1/2}^{1} f(x)\,dx$.

To examine the convergence of $\displaystyle\int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}}\,dx$, we take $g(x) = \dfrac{1}{x^{1/2}}$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{1}{(1-x)^{1/3}} = 1$$

$\because \displaystyle\int_0^{1/2} \frac{1}{x^{1/2}}\,dx$ converges $\therefore \displaystyle\int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}}\,dx$ is convergent.

To examine the convergence of $\displaystyle\int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{1/3}}\,dx$, we take $g(x) = \dfrac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{1}{x^{1/2}}=1$$

$\because\ \int\limits_{1/2}^{1}\frac{1}{(1-x)^{1/3}}dx$ converges $\quad\because\ \int\limits_{1/2}^{1}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

➢ *Question*

Show that the following improper integrals are convergent.

$(i)\ \int\limits_1^\infty \sin^2\frac{1}{x}\,dx \quad (ii)\ \int\limits_1^\infty \frac{\sin^2 x}{x^2}\,dx \quad (iii)\ \int\limits_0^1 \frac{x\log x}{(1+x)^2}\,dx \quad (iv)\ \int\limits_0^1 \log x\cdot\log(1+x)\,dx$

Solution: (*i*) Let $f(x)=\sin^2\frac{1}{x}$ and $g(x)=\frac{1}{x^2}$

then $\displaystyle \lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2=1$

$\Rightarrow\ \int\limits_1^\infty f(x)\,dx$ and $\int\limits_1^\infty \frac{1}{x^2}\,dx$ behave alike.

$\because\ \int\limits_1^\infty \frac{1}{x^2}\,dx$ is convergent $\ \therefore\ \int\limits_1^\infty \sin^2\frac{1}{x}\,dx$ is also convergent.

(*ii*) $\int\limits_1^\infty \frac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\frac{\sin^2 x}{x^2}$ and $g(x)=\frac{1}{x^2}$

$\sin^2 x\le 1 \quad \Rightarrow\ \frac{\sin^2 x}{x^2}\le\frac{1}{x^2} \quad \forall\ x\in(1,\infty)$

and $\int\limits_1^\infty \frac{1}{x^2}\,dx$ converges $\ \therefore\ \int\limits_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges.

➢ *Note*

$\int\limits_0^1 \frac{\sin^2 x}{x^2}\,dx$ is a proper integral because $\lim\limits_{x\to 0}\frac{\sin^2 x}{x^2}=1$ so that ‘0’ is not a point of infinite discontinuity. Therefore $\int\limits_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

(*iii*) $\displaystyle\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$

$\because\ \log x < x\ ,\quad x\in(0,1)\quad \therefore\ x\log x < x^2$

$\Rightarrow\ \dfrac{x\log x}{(1+x)^2} < \dfrac{x^2}{(1+x)^2}$

Now $\displaystyle\int_0^1 \frac{x^2}{(1+x)^2}\,dx$ is a proper integral.

$\therefore\ \displaystyle\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$ is convergent.

(*iv*) $\displaystyle\int_0^1 \log x\cdot\log(1+x)\,dx$

$\because\ \log x < x\quad \therefore\ \log(x+1) < x+1$

$\Rightarrow\ \log x\cdot\log(1+x) < x(x+1)$

$\because\ \displaystyle\int_0^1 x(x+1)\,dx$ is a proper integral $\therefore\ \displaystyle\int_0^1 \log x\cdot\log(1+x)\,dx$ is convergent.

➢ *Note*

(*i*) $\displaystyle\int_0^a \frac{1}{x^p}dx$ diverges when $p\ge 1$ and converges when $p<1$.

(*ii*) $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges iff $p>1$.

➢ *Questions*

Examine the convergence of

(*i*) $\displaystyle\int_1^\infty \frac{x}{(1+x)^3}dx$ (*ii*) $\displaystyle\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ (*iii*) $\displaystyle\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$

Solution: (***i***) Let $f(x)=\dfrac{x}{(1+x)^3}$ and take $g(x)=\dfrac{1}{x^2}$.

As $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^3}{(1+x)^3}=1$

Therefore the two integrals $\displaystyle\int_1^\infty \frac{x}{(1+x)^3}dx$ and $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ have identical behaviour for convergence at ∞.

$\because \int_1^\infty \frac{1}{x^2}dx$ is convergent $\therefore \int_1^\infty \frac{x}{(1+x)^3}dx$ is convergent.

(***ii***) Let $f(x)=\frac{1}{(1+x)\sqrt{x}}$ and take $g(x)=\frac{1}{x\sqrt{x}}=\frac{1}{x^{3/2}}$

We have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{x}{1+x}=1$

and $\int_1^\infty \frac{1}{x^{3/2}}dx$ is convergent. Thus $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ is convergent.

(***iii***) Let $f(x)=\frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x)=\frac{1}{x^{1/3}\cdot x^{1/2}}=\frac{1}{x^{5/6}}$

We have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=1$ and $\int_1^\infty \frac{1}{x^{5/6}}dx$ is divergent $\therefore \int_1^\infty f(x)\,dx$ is divergent.

➢ *Question*

Show that $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$ is convergent.

Solution: We have

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{a\to\infty}\left[\int_{-a}^{0}\frac{1}{1+x^2}dx+\int_0^a\frac{1}{1+x^2}dx\right]$$

$$=\lim_{a\to\infty}\left[\int_0^a\frac{1}{1+x^2}dx+\int_0^a\frac{1}{1+x^2}dx\right]=2\lim_{a\to\infty}\left[\int_0^a\frac{1}{1+x^2}dx\right]$$

$$=2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a=2\left(\frac{\pi}{2}\right)=\pi$$

therefore the integral is convergent.

➢ *Question*

Show that $\int_0^\infty \frac{\tan^{-1}x}{1+x^2}dx$ is convergent.

Solution: $\because (1+x^2)\cdot\frac{\tan^{-1}x}{(1+x^2)}=\tan^{-1}x\to\frac{\pi}{2}$ as $x\to\infty$

$\int_0^\infty \frac{\tan^{-1}x}{1+x^2}dx$ & $\int_0^\infty \frac{1}{1+x^2}dx$ behave alike.

Here $f(x)=\frac{\tan^{-1}x}{1+x^2}$

and $g(x)=\frac{1}{1+x^2}$

$\because \int_0^\infty \frac{1}{1+x^2}dx$ is convergent $\therefore$ A given integral is convergent.

➢ *Question*

Show that $\int_0^1 \frac{e^{-x}}{\sqrt{1-x^4}}dx$ is convergent.

Solution: $\because$ $e^{-x} < 1$ and $1+x^2 > 1$ for all $x \in (0,1)$.

$$\therefore \frac{e^{-x}}{\sqrt{1-x^4}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1-x^2}}$$

Also $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \lim_{\varepsilon\to 0} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

$\Rightarrow \displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{1-x^4}}dx$ is convergent. (by comparison test)

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