

## Chapter 2 – Sequences

Course Title: Real Analysis 1

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### Sequence

A sequence is a function whose domain of definition is the set of natural numbers.

Or it can also be defined as an ordered set.

Notation:

An infinite sequence is denoted as

$$\{s_n\}_{n=1}^{\infty} \text{ or } \{s_n : n \in \mathbb{N}\} \text{ or } \{s_1, s_2, s_3, \dots\} \text{ or simply as } \{s_n\},$$

e.g. i)  $\{n\} = \{1, 2, 3, \dots\}$ .

ii)  $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ .

iii)  $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \dots\}$ .

iv)  $\{2, 3, 5, 7, 11, \dots\}$ , a sequence of positive prime numbers.

### Subsequence

It is a sequence whose terms are contained in given sequence.

A subsequence of  $\{s_n\}_{n=1}^{\infty}$  is usually written as  $\{s_{n_k}\}_{k=1}^{\infty}$ .

### Increasing Sequence

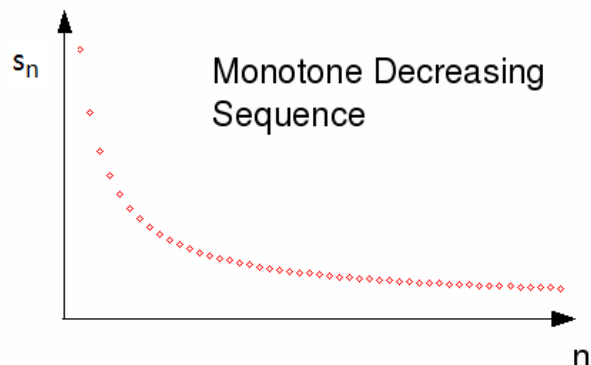
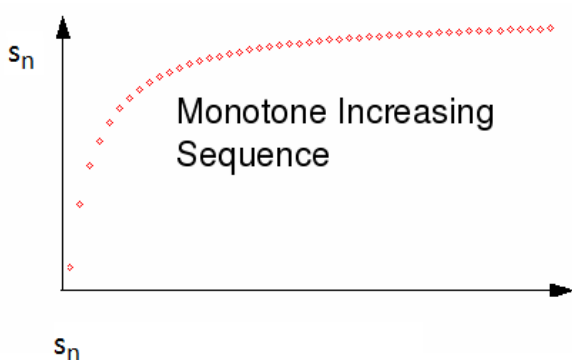
A sequence  $\{s_n\}$  is said to be an increasing sequence if  $s_{n+1} \geq s_n \quad \forall n \geq 1$ .

### Decreasing Sequence

A sequence  $\{s_n\}$  is said to be a decreasing sequence if  $s_{n+1} \leq s_n \quad \forall n \geq 1$ .

### Monotonic Sequence

A sequence  $\{s_n\}$  is said to be a monotonic sequence if it is either increasing or decreasing.



**Remarks:**

- A sequence  $\{s_n\}$  is monotonically increasing if  $s_{n+1} - s_n \geq 0$ .
- A positive term sequence  $\{s_n\}$  is monotonically increasing if  $\frac{s_{n+1}}{s_n} \geq 1, \forall n \geq 1$ .
- A sequence  $\{s_n\}$  is monotonically decreasing if  $s_n - s_{n+1} \geq 0$ .
- A positive term sequence  $\{s_n\}$  is monotonically decreasing if  $\frac{s_n}{s_{n+1}} \geq 1, \forall n \geq 1$ .

**Strictly Increasing or Decreasing**

A sequence  $\{s_n\}$  is called strictly increasing or decreasing according as

$$s_{n+1} > s_n \text{ or } s_{n+1} < s_n \quad \forall n \geq 1.$$

**Bernoulli's Inequality**

Let  $p \in \mathbb{R}, p \geq -1$  and  $p \neq 0$  then for  $n \geq 2$  we have

$$(1+p)^n > 1+np.$$

*Proof:*

We shall use mathematical induction to prove this inequality.

If  $n = 2$

$$L.H.S = (1+p)^2 = 1+2p+p^2,$$

$$R.H.S = 1+2p,$$

$$\Rightarrow L.H.S > R.H.S,$$

i.e. condition I of mathematical induction is satisfied.

Suppose  $(1+p)^k > 1+kp$  ..... (i) where  $k \geq 2$

$$\begin{aligned} \text{Now } (1+p)^{k+1} &= (1+p)(1+p)^k \\ &> (1+p)(1+kp) && \text{using (i)} \\ &= 1+kp+p+kp^2 \\ &= 1+(k+1)p+kp^2 \\ &\geq 1+(k+1)p && \text{ignoring } kp^2 \geq 0, \end{aligned}$$

$$\Rightarrow (1+p)^{k+1} > 1+(k+1)p.$$

Since the truth for  $n = k$  implies the truth for  $n = k + 1$  therefore condition II of mathematical induction is satisfied. Hence we conclude that  $(1+p)^n > 1+np$ .  $\square$

**Example:**

Prove that  $\left\{ \left( 1 + \frac{1}{n} \right)^n \right\}$  is an increasing sequence.

Let  $s_n = \left(1 + \frac{1}{n}\right)^n$  where  $n \geq 1$ .

To prove that this sequence is an increasing sequence, we use  $p = \frac{-1}{n^2}$ ,  $n \geq 2$  in

Bernoulli's inequality to have

$$\begin{aligned} & \left(1 - \frac{1}{n^2}\right)^n > 1 - \frac{n}{n^2} \\ \Rightarrow & \left(\left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)\right)^n > 1 - \frac{1}{n} \\ \Rightarrow & \left(1 + \frac{1}{n}\right)^n > \left(1 - \frac{1}{n}\right)^{1-n} = \left(\frac{n-1}{n}\right)^{1-n} = \left(\frac{n}{n-1}\right)^{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} \\ \Rightarrow & s_n > s_{n-1} \quad \forall n \geq 1. \end{aligned}$$

This shows that  $\{s_n\}$  is increasing sequence. □

**Example:**

Prove that a sequence  $\left\{\left(1 + \frac{1}{n}\right)^{n+1}\right\}$  is a decreasing sequence.

Let  $t_n = \left(1 + \frac{1}{n}\right)^{n+1}$  ;  $n \geq 1$ .

We use  $p = \frac{1}{n^2 - 1}$  in Bernoulli's inequality.

$$\left(1 + \frac{1}{n^2 - 1}\right)^n > 1 + \frac{n}{n^2 - 1} \dots\dots\dots (i)$$

where

$$\begin{aligned} 1 + \frac{1}{n^2 - 1} &= \frac{n^2}{n^2 - 1} = \left(\frac{n}{n-1}\right)\left(\frac{n}{n+1}\right) \\ \Rightarrow & \left(1 + \frac{1}{n^2 - 1}\right)\left(\frac{n+1}{n}\right) = \left(\frac{n}{n-1}\right) \dots\dots\dots (ii) \end{aligned}$$

$$\begin{aligned} \text{Now } t_{n-1} &= \left(1 + \frac{1}{n-1}\right)^n = \left(\frac{n}{n-1}\right)^n \\ &= \left(\left(1 + \frac{1}{n^2 - 1}\right)\left(\frac{n+1}{n}\right)\right)^n && \text{from (ii)} \\ &= \left(1 + \frac{1}{n^2 - 1}\right)^n \left(\frac{n+1}{n}\right)^n \\ &> \left(1 + \frac{n}{n^2 - 1}\right)\left(\frac{n+1}{n}\right)^n && \text{from (i)} \end{aligned}$$

$$\begin{aligned}
 &> \left(1 + \frac{1}{n}\right) \left(\frac{n+1}{n}\right)^n && \because \frac{n}{n^2-1} > \frac{n}{n^2} = \frac{1}{n} \\
 &= \left(\frac{n+1}{n}\right)^{n+1} = t_n,
 \end{aligned}$$

i.e.  $t_{n-1} > t_n$ .

Hence the given sequence is decreasing sequence.  $\square$

### Bounded Sequence

A sequence  $\{s_n\}$  is said to be bounded if there exists a positive real number  $\lambda$  such that  $|s_n| < \lambda \quad \forall n \in \mathbb{N}$ .

If  $S$  and  $s$  are the supremum and infimum of elements forming the bounded sequence  $\{s_n\}$  we write  $S = \sup s_n$  and  $s = \inf s_n$ .

All the elements of the sequence  $s_n$  such that  $|s_n| < \lambda \quad \forall n \in \mathbb{N}$  lie within the strip  $\{y : -\lambda < y < \lambda\}$ . But the elements of the unbounded sequence can not be contained in any strip of a finite width.

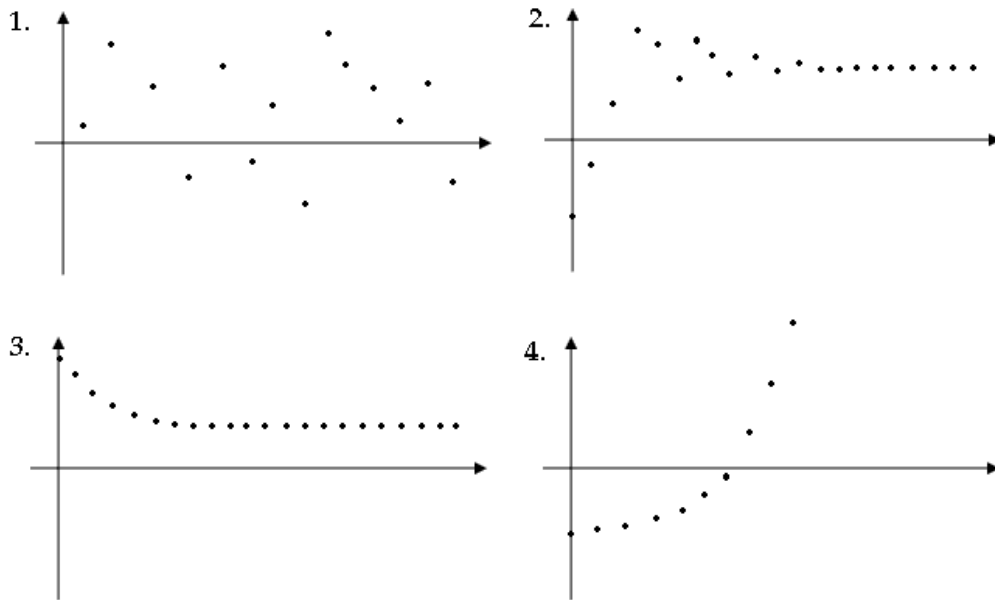
### Examples

- (i)  $\{u_n\} = \left\{\frac{(-1)^n}{n}\right\}$  is a bounded sequence
- (ii)  $\{v_n\} = \{\sin nx\}$  is also bounded sequence. Its supremum is 1 and infimum is  $-1$ .
- (iii) The geometric sequence  $\{ar^{n-1}\}$ ,  $r > 1$  is an unbounded above sequence. It is bounded below by  $a$ .
- (iv)  $\left\{\tan \frac{n\pi}{2}\right\}$  is an unbounded sequence.

### Convergence of the Sequence

A sequence  $\{s_n\}$  of real numbers is said to be convergent to limit 's' as  $n \rightarrow \infty$ , if for every positive real number  $\varepsilon > 0$ , there exists a positive integer  $n_0$ , depending upon  $\varepsilon$ , such that  $|s_n - s| < \varepsilon \quad \forall n > n_0$ .

We will try to understand it by graph of some sequence. Graph of any four sequences is drawn in the picture below.



### Theorem

A convergent sequence of real number has one and only one limit (i.e. limit of the sequence is unique.)

*Proof:*

Suppose  $\{s_n\}$  converges to two limits  $s$  and  $t$ , where  $s \neq t$ .

Put  $\varepsilon = \frac{|s-t|}{2}$  then there exists two positive integers  $n_1$  and  $n_2$  such that

$$|s_n - s| < \varepsilon \quad \forall n > n_1$$

$$\text{and } |s_n - t| < \varepsilon \quad \forall n > n_2.$$

$$\Rightarrow |s_n - s| < \varepsilon \text{ and } |s_n - t| < \varepsilon \text{ hold simultaneously } \forall n > \max(n_1, n_2).$$

Thus for all  $n > \max(n_1, n_2)$  we have

$$|s - t| = |s - s_n + s_n - t|$$

$$\leq |s_n - s| + |s_n - t|$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

$$\Rightarrow |s - t| < 2\left(\frac{|s-t|}{2}\right)$$

$$\Rightarrow |s - t| < |s - t|$$

Which is impossible, therefore the limit of the sequence is unique.  $\square$

**Note:** If  $\{s_n\}$  converges to  $s$  then all of its infinite subsequence converge to  $s$ .

### Cauchy Sequence

A sequence  $\{x_n\}$  of real number is said to be a *Cauchy sequence* if for given positive real number  $\varepsilon$ , there exists a positive integer  $n_0(\varepsilon)$  such that

$$|x_n - x_m| < \varepsilon \quad \forall m, n > n_0$$

**Theorem**

A Cauchy sequence of real numbers is bounded.

*Proof:*

Let  $\{s_n\}$  be a Cauchy sequence.

Take  $\varepsilon = 1$ , then there exists a positive integers  $n_0$  such that

$$|s_n - s_m| < 1 \quad \forall m, n > n_0.$$

Fix  $m = n_0 + 1$  then

$$\begin{aligned} |s_n| &= |s_n - s_{n_0+1} + s_{n_0+1}| \\ &\leq |s_n - s_{n_0+1}| + |s_{n_0+1}| \\ &< 1 + |s_{n_0+1}| \quad \forall n > n_0 \\ &< \lambda \quad \forall n > 1, \text{ and } \lambda = 1 + |s_{n_0+1}| \quad (n_0 \text{ changes as } \varepsilon \text{ changes}) \end{aligned}$$

Hence we conclude that  $\{s_n\}$  is a Cauchy sequence, which is bounded one.  $\square$

**Note:**

(i) Convergent sequence is bounded.

(ii) The converse of the above theorem does not hold.

i.e. every bounded sequence is not Cauchy.

Consider the sequence  $\{s_n\}$  where  $s_n = (-1)^n$ ,  $n \geq 1$ . It is bounded sequence because

$$|(-1)^n| = 1 < 2 \quad \forall n \geq 1.$$

But it is not a Cauchy sequence if it is then for  $\varepsilon = 1$  we should be able to find a positive integer  $n_0$  such that  $|s_n - s_m| < 1$  for all  $m, n > n_0$ .

But with  $m = 2k + 1$ ,  $n = 2k + 2$  when  $2k + 1 > n_0$ , we arrive at

$$\begin{aligned} |s_n - s_m| &= |(-1)^{2n+2} - (-1)^{2k+1}| \\ &= |1 + 1| = 2 < 1 \quad \text{is absurd.} \end{aligned}$$

Hence  $\{s_n\}$  is not a Cauchy sequence. Also this sequence is not a convergent sequence. (it is an oscillatory sequence).

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**Divergent Sequence**

A  $\{s_n\}$  is said to be divergent if it is not convergent or it is unbounded.

e.g.  $\{n^2\}$  is divergent, it is unbounded.

(ii)  $\{(-1)^n\}$  tends to 1 or -1 according as  $n$  is even or odd. It oscillates finitely.

(iii)  $\{(-1)^n n\}$  is a divergent sequence. It oscillates infinitely.

**Note:** If two subsequence of a sequence converges to two different limits then the sequence itself is a divergent.

**Theorem**

If  $s_n < u_n < t_n \quad \forall n \geq n_0$  and if both the  $\{s_n\}$  and  $\{t_n\}$  converge to same limits as  $s$ , then the sequence  $\{u_n\}$  also converges to  $s$ .

*Proof:*

Since the sequence  $\{s_n\}$  and  $\{t_n\}$  converge to the same limit  $s$ , therefore, for given  $\varepsilon > 0$  there exists two positive integers  $n_1, n_2 > n_0$  such that

$$\begin{aligned} |s_n - s| < \varepsilon & \quad \forall n > n_1 \\ |t_n - s| < \varepsilon & \quad \forall n > n_2 \\ \text{i.e. } s - \varepsilon < s_n < s + \varepsilon & \quad \forall n > n_1 \\ s - \varepsilon < t_n < s + \varepsilon & \quad \forall n > n_2 \end{aligned}$$

Since we have given

$$\begin{aligned} s_n < u_n < t_n & \quad \forall n > n_0 \\ \therefore s - \varepsilon < s_n < u_n < t_n < s + \varepsilon & \quad \forall n > \max(n_0, n_1, n_2) \\ \Rightarrow s - \varepsilon < u_n < s + \varepsilon & \quad \forall n > \max(n_0, n_1, n_2) \\ \text{i.e. } |u_n - s| < \varepsilon & \quad \forall n > \max(n_0, n_1, n_2) \\ \text{i.e. } \lim_{n \rightarrow \infty} u_n = s. & \quad \square \end{aligned}$$

**Example**

Show that  $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$

*Solution*

Using Bernoulli's Inequality

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \frac{n}{\sqrt{n}} \geq \sqrt{n} \geq 1 \quad \forall n.$$

Also

$$\begin{aligned} \left(1 + \frac{1}{\sqrt{n}}\right)^2 &= \left[\left(1 + \frac{1}{\sqrt{n}}\right)^n\right]^{\frac{2}{n}} > (\sqrt{n})^{\frac{2}{n}} > n^{\frac{1}{n}} \geq 1, \\ \Rightarrow 1 \leq n^{\frac{1}{n}} < \left(1 + \frac{1}{\sqrt{n}}\right)^2, \\ \Rightarrow \lim_{n \rightarrow \infty} 1 \leq \lim_{n \rightarrow \infty} n^{\frac{1}{n}} < \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{n}}\right)^2, \\ \Rightarrow 1 \leq \lim_{n \rightarrow \infty} n^{\frac{1}{n}} < 1. \\ \text{i.e. } \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1. & \quad \square \end{aligned}$$

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**Example**

Show that  $\lim_{n \rightarrow \infty} \left( \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2} \right) = 0$

*Solution*

Consider

$$s_n = \left( \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2} \right)$$

and

$$\frac{n}{(2n)^2} < s_n < \frac{n}{n^2}$$

$$\Rightarrow \frac{1}{4n} < s_n < \frac{1}{n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{4n} < \lim_{n \rightarrow \infty} s_n < \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$\Rightarrow 0 < \lim_{n \rightarrow \infty} s_n < 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} s_n = 0$$

□

**Theorem**

If the sequence  $\{s_n\}$  converges to  $s$  then  $\exists$  a positive integer  $n$  such that  $|s_n| > \frac{1}{2}s$ .

*Proof:*

$$\text{We fix } \varepsilon = \frac{1}{2}|s| > 0$$

$\Rightarrow \exists$  a positive integer  $n_1$  such that

$$|s_n - s| < \varepsilon \quad \text{for } n > n_1$$

$$\Rightarrow |s_n - s| < \frac{1}{2}|s|$$

Now

$$\frac{1}{2}|s| = |s| - \frac{1}{2}|s|$$

$$< |s| - |s_n - s| \leq |s + (s_n - s)|$$

$$\Rightarrow \frac{1}{2}|s| < |s_n|$$

□



**Theorem**

Let  $a$  and  $b$  be fixed real numbers if  $\{s_n\}$  and  $\{t_n\}$  converge to  $s$  and  $t$  respectively, then

- (i)  $\{as_n + bt_n\}$  converges to  $as + bt$ .
- (ii)  $\{s_n t_n\}$  converges to  $st$ .
- (iii)  $\left\{\frac{s_n}{t_n}\right\}$  converges to  $\frac{s}{t}$ , provided  $t_n \neq 0 \quad \forall n$  and  $t \neq 0$ .

*Proof:*

Since  $\{s_n\}$  and  $\{t_n\}$  converge to  $s$  and  $t$  respectively,

$$\begin{aligned} \therefore |s_n - s| < \varepsilon & \quad \forall n > n_1 \in \mathbb{N} \\ |t_n - t| < \varepsilon & \quad \forall n > n_2 \in \mathbb{N} \end{aligned}$$

Also  $\exists \lambda > 0$  such that  $|s_n| < \lambda \quad \forall n > 1$  ( $\because \{s_n\}$  is bounded)

(i) We have

$$\begin{aligned} |(as_n + bt_n) - (as + bt)| &= |a(s_n - s) + b(t_n - t)| \\ &\leq |a(s_n - s)| + |b(t_n - t)| \\ &< |a|\varepsilon + |b|\varepsilon & \quad \forall n > \max(n_1, n_2) \\ &= \varepsilon_1, \end{aligned}$$

where  $\varepsilon_1 = |a|\varepsilon + |b|\varepsilon$  a certain number.

This implies  $\{as_n + bt_n\}$  converges to  $as + bt$ .

$$\begin{aligned} \text{(ii)} \quad |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &= |s_n(t_n - t) + t(s_n - s)| \leq |s_n| \cdot |(t_n - t)| + |t| \cdot |(s_n - s)| \\ &< \lambda \varepsilon + |t|\varepsilon & \quad \forall n > \max(n_1, n_2) \\ &= \varepsilon_2, & \quad \text{where } \varepsilon_2 = \lambda \varepsilon + |t|\varepsilon \text{ a certain number.} \end{aligned}$$

This implies  $\{s_n t_n\}$  converges to  $st$ .

$$\begin{aligned} \text{(iii)} \quad \left| \frac{1}{t_n} - \frac{1}{t} \right| &= \left| \frac{t - t_n}{t_n t} \right| \\ &= \frac{|t_n - t|}{|t_n| |t|} < \frac{\varepsilon}{\frac{1}{2}|t| |t|} & \quad \forall n > \max(n_1, n_2) \quad \because |t_n| > \frac{1}{2}t \\ &= \frac{\varepsilon}{\frac{1}{2}|t|^2} = \varepsilon_3, & \quad \text{where } \varepsilon_3 = \frac{\varepsilon}{\frac{1}{2}|t|^2} \text{ a certain number.} \end{aligned}$$

This implies  $\left\{\frac{1}{t_n}\right\}$  converges to  $\frac{1}{t}$ .

Hence  $\left\{\frac{s_n}{t_n}\right\} = \left\{s_n \cdot \frac{1}{t_n}\right\}$  converges to  $s \cdot \frac{1}{t} = \frac{s}{t}$ . (from (ii))  $\square$

**Theorem**

For each irrational number  $x$ , there exists a sequence  $\{r_n\}$  of distinct rational numbers such that  $\lim_{n \rightarrow \infty} r_n = x$ .

*Proof:*

Since  $x$  and  $x + 1$  are two different real numbers

$\therefore \exists$  a rational number  $r_1$  such that

$$x < r_1 < x + 1$$

Similarly  $\exists$  a rational number  $r_2 \neq r_1$  such that

$$x < r_2 < \min\left(r_1, x + \frac{1}{2}\right) < x + 1$$

Continuing in this manner we have

$$x < r_3 < \min\left(r_2, x + \frac{1}{3}\right) < x + 1$$

$$x < r_4 < \min\left(r_3, x + \frac{1}{4}\right) < x + 1$$

.....  
 .....  
 .....

$$x < r_n < \min\left(r_{n-1}, x + \frac{1}{n}\right) < x + 1$$

This implies that  $\exists$  a sequence  $\{r_n\}$  of the distinct rational number such that

$$x - \frac{1}{n} < x < r_n < x + \frac{1}{n}.$$

Since

$$\lim_{n \rightarrow \infty} \left(x - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \left(x + \frac{1}{n}\right) = x.$$

Therefore

$$\lim_{n \rightarrow \infty} r_n = x. \quad \square$$

**Theorem**

Let a sequence  $\{s_n\}$  be a bounded sequence.

(i) If  $\{s_n\}$  is monotonically increasing then it converges to its supremum.

(ii) If  $\{s_n\}$  is monotonically decreasing then it converges to its infimum.

*Proof*

Let  $S = \sup s_n$  and  $s = \inf s_n$

Take  $\varepsilon > 0$

(i) Since  $S = \sup s_n$

$\therefore \exists s_{n_0}$  such that  $S - \varepsilon < s_{n_0}$

Since  $\{s_n\}$  is  $\uparrow$  ( $\uparrow$  stands for monotonically increasing)

$$\therefore S - \varepsilon < s_{n_0} < s_n < S < S + \varepsilon \quad \text{for } n > n_0$$

$$\Rightarrow S - \varepsilon < s_n < S + \varepsilon \quad \text{for } n > n_0$$

$$\Rightarrow |s_n - S| < \varepsilon \quad \text{for } n > n_0$$

$$\Rightarrow \lim_{n \rightarrow \infty} s_n = S$$

(ii) Since  $s = \inf S_n$

$$\therefore \exists s_{n_1} \text{ such that } s_{n_1} < s + \varepsilon$$

Since  $\{s_n\}$  is  $\downarrow$ . ( $\downarrow$  stands for monotonically decreasing)

$$\therefore s - \varepsilon < s < s_n < s_{n_1} < s + \varepsilon \quad \text{for } n > n_1$$

$$\Rightarrow s - \varepsilon < s_n < s + \varepsilon \quad \text{for } n > n_1$$

$$\Rightarrow |s_n - s| < \varepsilon \quad \text{for } n > n_1$$

$$\text{Thus } \lim_{n \rightarrow \infty} s_n = s$$

□

### Note

A monotonic sequence can not oscillate infinitely.

### Example:

Show that  $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}$  is bounded sequence.

$$\text{Consider } \{s_n\} = \left\{ \left(1 + \frac{1}{n}\right)^n \right\}$$

As shown earlier it is an increasing sequence

$$\text{Take } s_{2n} = \left(1 + \frac{1}{2n}\right)^{2n}, \text{ then } \sqrt{s_{2n}} = \left(1 + \frac{1}{2n}\right)^n,$$

$$\Rightarrow \frac{1}{\sqrt{s_{2n}}} = \left(\frac{2n}{2n+1}\right)^n \Rightarrow \frac{1}{\sqrt{s_{2n}}} = \left(1 - \frac{1}{2n+1}\right)^n$$

Using Bernoulli's Inequality we have

$$\Rightarrow \frac{1}{\sqrt{s_{2n}}} \geq 1 - \frac{n}{2n+1} > 1 - \frac{n}{2n} = \frac{1}{2}$$

$$\therefore \left(1 - \frac{1}{2n+1}\right)^n \geq 1 - \frac{n}{2n+1}$$

$$\Rightarrow \sqrt{s_{2n}} < 2 \quad \forall n = 1, 2, 3, \dots$$

$$\Rightarrow s_{2n} < 4 \quad \forall n = 1, 2, 3, \dots$$

$$\Rightarrow s_n < s_{2n} < 4 \quad \forall n = 1, 2, 3, \dots$$

Which show that the sequence  $\{s_n\}$  is bounded one.

Hence  $\{s_n\}$  is a convergent sequence the number to which it converges is its supremum, which is denoted by 'e' and  $2 < e < 3$ .

□

**Recurrence Relation**

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

**Example:**

Let  $t_1 > 1$  and let  $\{t_n\}$  be defined by  $t_{n+1} = 2 - \frac{1}{t_n}$  for  $n \geq 1$ .

(i) Show that  $\{t_n\}$  is decreasing sequence.

(ii) It is bounded below.

(iii) Find the limit of the sequence.

Since  $t_1 > 1$  and  $\{t_n\}$  is defined by  $t_{n+1} = 2 - \frac{1}{t_n}$  ;  $n \geq 1$

$$\Rightarrow t_n > 0 \quad \forall n \geq 1$$

$$\begin{aligned} \text{Also } t_n - t_{n+1} &= t_n - 2 + \frac{1}{t_n} \\ &= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n - 1)^2}{t_n} > 0. \end{aligned}$$

$$\Rightarrow t_n > t_{n+1} \quad \forall n \geq 1.$$

This implies that  $t_n$  is monotonically decreasing.

Since  $t_n > 1 \quad \forall n \geq 1$ ,

$\Rightarrow t_n$  is bounded below.

Since  $t_n$  is decreasing and bounded below therefore  $t_n$  is convergent.

Let us suppose  $\lim_{n \rightarrow \infty} t_n = t$ .

$$\text{Then } \lim_{n \rightarrow \infty} t_{n+1} = \lim_{n \rightarrow \infty} t_n \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \left( 2 - \frac{1}{t_n} \right) = \lim_{n \rightarrow \infty} t_n$$

$$\Rightarrow 2 - \frac{1}{t} = t \quad \Rightarrow \quad \frac{2t-1}{t} = t \quad \Rightarrow \quad 2t-1 = t^2 \quad \Rightarrow \quad t^2 - 2t + 1 = 0$$

$$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow \quad t = 1. \quad \square$$

**Theorem**

Every Cauchy sequence of real numbers has a convergent subsequence.

*Proof:*

Suppose  $\{s_n\}$  is a Cauchy sequence.

Let  $\varepsilon > 0$  then  $\exists$  a positive integer  $n_0 \geq 1$  such that

$$\left| s_{n_k} - s_{n_{k-1}} \right| < \frac{\varepsilon}{2^k} \quad \forall n_k, n_{k-1}, k = 1, 2, 3, \dots$$

$$\text{Put } b_k = (s_{n_1} - s_{n_0}) + (s_{n_2} - s_{n_1}) + \dots + (s_{n_k} - s_{n_{k-1}})$$

$$\begin{aligned} \Rightarrow |b_k| &= \left| (s_{n_1} - s_{n_0}) + (s_{n_2} - s_{n_1}) + \dots + (s_{n_k} - s_{n_{k-1}}) \right| \\ &\leq \left| (s_{n_1} - s_{n_0}) \right| + \left| (s_{n_2} - s_{n_1}) \right| + \dots + \left| (s_{n_k} - s_{n_{k-1}}) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots + \frac{\varepsilon}{2^k} \\ &= \varepsilon \left( \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} \right) = \varepsilon \left( \frac{\frac{1}{2} \left( 1 - \frac{1}{2^k} \right)}{1 - \frac{1}{2}} \right) = \varepsilon \left( 1 - \frac{1}{2^k} \right) \end{aligned}$$

$$\Rightarrow |b_k| < \varepsilon \quad \forall k \geq 1$$

$$\Rightarrow \{b_k\} \text{ is convergent}$$

$$\because b_k = s_{n_k} - s_{n_0} \quad \therefore s_{n_k} = b_k + s_{n_0},$$

where  $s_{n_0}$  is a certain fix number therefore  $\{s_{n_k}\}$  which is a subsequence of  $\{s_n\}$  is convergent.  $\square$

### Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

*Proof:*

#### Necessary Condition

Let  $\{s_n\}$  be a convergent sequence, which converges to  $s$ .

Then for given  $\varepsilon > 0 \exists$  a positive integer  $n_0$ , such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \forall n > n_0$$

Now for  $n > m > n_0$

$$\begin{aligned} |s_n - s_m| &= |s_n - s + s - s_m| \\ &\leq |s_n - s| + |s - s_m| = |s_n - s| + |s_m - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Which shows that  $\{s_n\}$  is a Cauchy sequence.

#### Sufficient Condition

Let us suppose that  $\{s_n\}$  is a Cauchy sequence then for  $\varepsilon > 0$ ,  $\exists$  a positive integer  $m_1$  such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \quad \forall n, m > m_1 \dots\dots\dots (i)$$

Since  $\{s_n\}$  is a Cauchy sequence

therefore it has a subsequence  $\{s_{n_k}\}$  converging to  $s$  (say).

$\Rightarrow \exists$  a positive integer  $m_2$  such that

$$|s_{n_k} - s| < \frac{\varepsilon}{2} \quad \forall n > m_2 \dots\dots\dots (ii)$$

Now

$$\begin{aligned} |s_n - s| &= |s_n - s_{n_k} + s_{n_k} - s| \\ &\leq |s_n - s_{n_k}| + |s_{n_k} - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n > \max(m_1, m_2), \end{aligned}$$

this shows that  $\{s_n\}$  is a convergent sequence.  $\square$

### Example

Prove that  $\left\{1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots + \frac{1}{n}\right\}$  is divergent sequence.

Let  $\{t_n\}$  be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots + \frac{1}{n}.$$

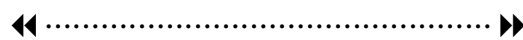
For  $m, n \in \mathbb{N}$ ,  $n > m$  we have

$$\begin{aligned} |t_n - t_m| &= \frac{1}{m+1} + \frac{1}{m+2} + \dots\dots\dots + \frac{1}{n} \\ &> (n-m) \frac{1}{n} = 1 - \frac{m}{n}. \end{aligned}$$

In particular if  $n = 2m$  then

$$|t_n - t_m| > \frac{1}{2}.$$

This implies that  $\{t_n\}$  is not a Cauchy sequence therefore it is divergent.  $\square$



### Theorem (nested intervals)

Suppose that  $\{I_n\}$  is a sequence of the closed interval such that  $I_n = [a_n, b_n]$ ,  $I_{n+1} \subset I_n \quad \forall n \geq 1$ , and  $(b_n - a_n) \rightarrow 0$  as  $n \rightarrow \infty$  then  $\bigcap I_n$  contains one and only one point.

*Proof:*

Since  $I_{n+1} \subset I_n$

$$\therefore a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n < b_n < b_{n-1} < \dots < b_3 < b_2 < b_1$$

$\{a_n\}$  is increasing sequence, bounded above by  $b_1$  and bounded below by  $a_1$ .

And  $\{b_n\}$  is decreasing sequence bounded below by  $a_1$  and bounded above by  $b_1$ .

$\Rightarrow \{a_n\}$  and  $\{b_n\}$  both are convergent.

Suppose  $\{a_n\}$  converges to  $a$  and  $\{b_n\}$  converges to  $b$ .

$$\text{But } |a - b| = |a - a_n + a_n - b_n + b_n - b|$$

$$\leq |a_n - a| + |a_n - b_n| + |b_n - b| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\Rightarrow a = b$$

and  $a_n < a < b_n \quad \forall n \geq 1.$  □

### Theorem (Bolzano-Weierstrass theorem)

Every bounded sequence has a convergent subsequence.

*Proof:*

Let  $\{s_n\}$  be a bounded sequence.

Take  $a_1 = \inf s_n$  and  $b_1 = \sup s_n$

Then  $a_1 < s_n < b_1 \quad \forall n \geq 1.$

Now bisect interval  $[a_1, b_1]$  such that at least one of the two sub-intervals contains infinite numbers of terms of the sequence.

Denote this sub-interval by  $[a_2, b_2]$ .

If both the sub-intervals contain infinite number of terms of the sequence then choose the one on the right hand.

Then clearly  $a_1 \leq a_2 < b_2 \leq b_1.$

Suppose there exist a subinterval  $[a_k, b_k]$  such that

$$a_1 \leq a_2 \leq \dots \leq a_k < b_k \leq \dots \leq b_2 \leq b_1$$

$$\Rightarrow (b_k - a_k) = \frac{1}{2^k} (b_1 - a_1)$$

Bisect the interval  $[a_k, b_k]$  in the same manner and choose  $[a_{k+1}, b_{k+1}]$  to have

$$a_1 \leq a_2 \leq \dots \leq a_k \leq a_{k+1} < b_{k+1} \leq b_k \leq \dots \leq b_2 \leq b_1$$

and 
$$b_{k+1} - a_{k+1} = \frac{1}{2^{k+1}} (b_1 - a_1)$$

This implies that we obtain a sequence of interval  $[a_n, b_n]$  such that

$$b_n - a_n = \frac{1}{2^n} (b_1 - a_1) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$\Rightarrow$  we have a unique point  $s$  such that

$$s = \bigcap [a_n, b_n]$$

there are infinitely many terms of the sequence whose length is  $\varepsilon > 0$  that contain  $s$ .

For  $\varepsilon = 1$  there are infinitely many values of  $n$  such that

$$|s_n - s| < 1$$

Let  $n_1$  be one of such value then

$$|s_{n_1} - s| < 1$$

Again choose  $n_2 > n_1$  such that

$$|s_{n_2} - s| < \frac{1}{2}$$

Continuing in this manner we find a sequence  $\{s_{n_k}\}$  for each positive integer  $k$  such that  $n_k < n_{k+1}$  and

$$|s_{n_k} - s| < \frac{1}{k} \quad \forall k = 1, 2, 3, \dots$$

Hence there is a subsequence  $\{s_{n_k}\}$  which converges to  $s$ .  $\square$

### Limit Inferior of the sequence

Suppose  $\{s_n\}$  is bounded below then we define limit inferior of  $\{s_n\}$  as follow

$$\lim_{n \rightarrow \infty} (\inf s_n) = \lim_{n \rightarrow \infty} u_k, \quad \text{where } u_k = \inf \{s_n : n \geq k\}$$

If  $s_n$  is not bounded below then

$$\lim_{n \rightarrow \infty} (\inf s_n) = -\infty.$$

### Limit Superior of the sequence

Suppose  $\{s_n\}$  is bounded above then we define limit superior of  $\{s_n\}$  as follow

$$\lim_{n \rightarrow \infty} (\sup s_n) = \lim_{n \rightarrow \infty} v_k, \quad \text{where } v_k = \sup \{s_n : n \geq k\}$$

If  $s_n$  is not bounded above then we have

$$\lim_{n \rightarrow \infty} (\sup s_n) = +\infty.$$

### Note:

(i) A bounded sequence has unique limit inferior and superior

(ii) Let  $\{s_n\}$  contains all the rational numbers, then every real number is a subsequential limit then limit superior of  $s_n$  is  $+\infty$  and limit inferior of  $s_n$  is  $-\infty$

(iii) Let  $\{s_n\} = (-1)^n \left(1 + \frac{1}{n}\right)$

then limit superior of  $s_n$  is 1 and limit inferior of  $s_n$  is  $-1$ .

(iv) Let  $s_n = \left(1 + \frac{1}{n}\right) \cos n\pi$ .

Then  $u_k = \inf \{s_n : n \geq k\}$

$$= \inf \left\{ \left(1 + \frac{1}{k}\right) \cos k\pi, \left(1 + \frac{1}{k+1}\right) \cos(k+1)\pi, \left(1 + \frac{1}{k+2}\right) \cos(k+2)\pi, \dots \right\}$$

$$= \begin{cases} \left(1 + \frac{1}{k}\right) \cos k\pi & \text{if } k \text{ is odd} \\ \left(1 + \frac{1}{k+1}\right) \cos(k+1)\pi & \text{if } k \text{ is even} \end{cases}$$

$$\Rightarrow \lim_{n \rightarrow \infty} (\inf s_n) = \lim_{n \rightarrow \infty} u_k = -1$$

Also  $v_k = \sup \{s_n : n \geq k\}$



$$= \begin{cases} \left(1 + \frac{1}{k+1}\right) \cos(k+1)\pi & \text{if } k \text{ is odd} \\ \left(1 + \frac{1}{k}\right) \cos k\pi & \text{if } k \text{ is even} \end{cases}$$

$$\Rightarrow \lim_{n \rightarrow \infty} (\sup s_n) = \lim_{n \rightarrow \infty} v_k = 1$$

□

⋮

**Theorem**

If  $\{s_n\}$  is a convergent sequence then

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (\inf s_n) = \lim_{n \rightarrow \infty} (\sup s_n)$$

*Proof:*

Let  $\lim_{n \rightarrow \infty} s_n = s$  then for a real number  $\varepsilon > 0$ ,  $\exists$  a positive integer  $n_0$  such that

$$|s_n - s| < \varepsilon \quad \forall n \geq n_0 \dots\dots\dots (i)$$

i.e.  $s - \varepsilon < s_n < s + \varepsilon \quad \forall n \geq n_0$

If  $v_k = \sup \{s_n : n \geq k\}$

Then  $s - \varepsilon < v_n < s + \varepsilon \quad \forall k \geq n_0$

$$\Rightarrow s - \varepsilon < \lim_{k \rightarrow \infty} v_n < s + \varepsilon \quad \forall k \geq n_0 \dots\dots\dots (ii)$$

from (i) and (ii) we have

$$s = \lim_{k \rightarrow \infty} \sup \{s_n\}$$

We can have the same result for limit inferior of  $\{s_n\}$  by taking

$$u_k = \inf \{s_n : n \geq k\}.$$

□

⋮

