Chapter 1 – Real Number System

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Course instructor: Dr. Atiq ur Rehman Class: MSc-II
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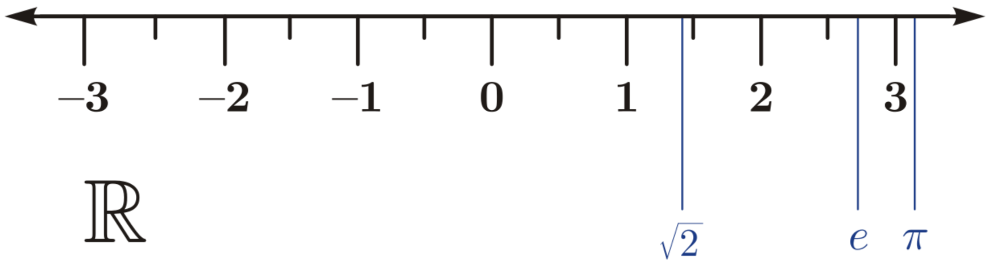
In mathematics, a real number is a value that represents a quantity along a continuous line. The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as $\sqrt{2}$ (1.41421356…, the square root of two, an irrational algebraic number) and π (3.14159265…, a transcendental number). Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one.



The real number system can be describe as a "complete ordered field". Therefore let's discusses and understand these notions first.

❖ **Order**

Let S be a non-empty set. An *order* on a set S is a relation denoted by "$<$" with the following two properties

(i) If $x, y \in S$,

then one and only one of the statement $x < y$, $x = y$, $y < x$ is true.

(ii) If $x, y, z \in S$ and if $x < y$, $y < z$ then $x < z$.

❖ **Ordered Set**

A set is said to be *ordered set* if an order is defined on S.

❖ **Bound**

Upper Bound

Let S be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \le \beta \ \ \forall \ x \in E$, then we say that E is bounded above. And β is known as upper bound of E.

Lower Bound

Let S be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \geq \beta \ \forall \ x \in E$, then we say that E is bounded below. And β is known as lower bound of E.

❖ **Example**

Consider $S = \{1,2,3,...,50\}$ and $E = \{5,10,15,20\}$.

Set of all lower bound of $E = \{1,2,3,4,5\}$.

Set of all upper bound of $E = \{20,21,22,23,...,50\}$.

❖ **Least Upper Bound (Supremum)**

Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose there exists an $\alpha \in S$ such that

(*i*) α is an upper bound of E.

(*ii*) If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called *least upper bound* of E or *supremum* of E and written as $\sup E = \alpha$.

In other words α is the least member of the set of upper bound of E.

❖ **Example**

Consider $S = \{1,2,3,...,50\}$ and $E = \{5,10,15,20\}$.

(i) It is clear that 20 is upper bound of E.

(ii) If $\gamma < 20$ then clearly γ is not an upper bound of E. Hence $\sup E = 20$.

❖ **Greatest Lower Bound (Infimum)**

Suppose S is an ordered set, $E \subset S$ and E is bounded below. Suppose there exists a $\beta \in S$ such that

(*i*) β is a lower bound of E.

(*ii*) If $\beta < \gamma$, then γ is not a lower bound of E.

Then β is called *greatest lower bound* of E or *infimum* of E and written as $\inf E = \beta$.

In other words β is the greatest member of the set of lower bound of E.

❖ **Example**

Consider the sets

$$A = \{p : p \in \mathbb{Q} \ \wedge \ p^2 < 2\} \text{ and}$$

$$B = \{p : p \in \mathbb{Q} \ \wedge \ p^2 > 2\},$$

where $\mathbb{Q}$ is set of rational numbers. Then the set A is bounded above. The upper bound of A are the exactly the members of B. Since B contain no smallest member therefore A has no supremum in $\mathbb{Q}$. Similarly B is bounded below. The set of all lower bounds of B consists of A and $r \in \mathbb{Q}$ with $r \leq 0$. Since A has no largest member therefore B has no infimum in $\mathbb{Q}$.

❖ **Example**

If α is supremum of E then α may or may not belong to E.

Let $E_1 = \{r : r \in \mathbb{Q} \wedge r < 0\}$ and $E_2 = \{r : r \in \mathbb{Q} \wedge r \geq 0\}$.

Then $\sup E_1 = \inf E_2 = 0$ and $0 \notin E_1$ and $0 \in E_2$.

❖ **Example**

Let E be the set of all numbers of the form $1/n$, where n is the natural numbers, that is,

$$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}.$$

Then $\sup E = 1$ which is in E, but $\inf E = 0$ which is not in E.

❖ **Least Upper Bound Property**

A set S is said to have the *least upper bound property* if the followings is true

(i) S is non-empty and ordered.

(ii) If $E \subset S$ and E is non-empty and bounded above then sup$E$ exists in S.

Greatest lower bound property can be defined in a similar manner.

❖ **Example**

Let S be set of rational numbers and

$$E = \{p : p \in \mathbb{Q} \wedge p^2 < 2\}$$

then $E \subset \mathbb{Q}$, E is non-empty and also bounded above but supremum of E is not in S, this implies that $\mathbb{Q}$ the set of rational number does not posses the least upper bound property.

❖ **Theorem**

Suppose S is an ordered set with least upper bound property. $B \subset S$, B is non-empty and is bounded below. Let L be set of all lower bound of B then $\alpha = \sup L$ exists in S and also $\alpha = \inf B$.

In particular infimum of B exists in S.

OR

An ordered set which has the least upper bound property has also the greatest lower bound property.

Proof

Since B is bounded below therefore L is non-empty.

Since L consists of exactly those $y \in S$ which satisfy the inequality.

$$y \leq x \qquad \forall\ x \in B$$

We see that every $x \in B$ is an upper bound of L.

$\Rightarrow$ L is bounded above.

Since S is ordered and non-empty therefore L has a supremum in S. Let us call it α.
If $\gamma < \alpha$, then γ is not upper bound of L.

$\Rightarrow \quad \gamma \notin B,$

$\Rightarrow \quad \alpha \le x \qquad \forall\, x \in B \quad \Rightarrow \alpha \in L.$

Now if $\alpha < \beta$ then $\beta \notin L$ because $\alpha = \sup L$.

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$.

In other words, if $\alpha < \beta$, then α is a lower bound of B, but β is not, this means that $\alpha = \inf B$.

❖ Field

A set F with two operations called addition and multiplication satisfying the following axioms is known to be field.

Axioms for Addition:

(i) If $x,\, y \in F$ then $x + y \in F$. *Closure Law*

(ii) $x + y = y + x, \quad \forall\, x, y \in F$. *Commutative Law*

(iii) $x + (y + z) = (x + y) + z \quad \forall\, x, y, z \in F$. *Associative Law*

(iv) For any $x \in F$, $\exists\; 0 \in F$ such that $x + 0 = 0 + x = x$ *Additive Identity*

(v) For any $x \in F$, $\exists\; -x \in F$ such that $x + (-x) = (-x) + x = 0$ *+tive Inverse*

Axioms for Multiplication:

(i) If $x,\, y \in F$ then $x\,y \in F$. *Closure Law*

(ii) $x\,y = y\,x, \quad \forall\, x, y \in F$ *Commutative Law*

(iii) $x(y\,z) = (x\,y)z \quad \forall\, x, y, z \in F$

(iv) For any $x \in F$, $\exists\; 1 \in F$ such that $x \cdot 1 = 1 \cdot x = x$ *Multiplicative Identity*

(v) For any $x \in F$, $x \neq 0$, $\exists\; \frac{1}{x} \in F$, such that $x\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)x = 1$ *×tive Inverse.*

Distributive Law

For any $x, y, z \in F$, $\quad (i)\; x(y + z) = xy + xz$

$(ii)\; (x + y)z = xz + yz$

❖ Ordered Field

An ordered field is a field F which is also an ordered set such that

$i)$ $x + y < x + z$ if $x, y, z \in F$ and $y < z$.

$ii)$ $xy > 0$ if $x, y \in F$, $x > 0$ and $y > 0$.

e.g the set $\mathbb{Q}$ of rational number is an ordered field.

❖ Existance of Real Field

There exists an ordered field $\mathbb{R}$ (set of real) which has the least upper bound property and it contain $\mathbb{Q}$ (set of rationales) as a subfield.

> *There are many other ways to construct a set of real numbers. We are not interested to do so therefore we leave it on the reader if they are interested then following page is useful:*
> *http://en.wikipedia.org/wiki/Construction_of_the_real_numbers*

❖ Theorem

Let $x, y, z \in \mathbb{R}$. Then axioms for addition imply the following.

(***a***) If $x+y=x+z$ then $y=z$

(***b***) If $x+y=x$ then $y=0$

(***c***) If $x+y=0$ then $y=-x$.

(***d***) $-(-x)=x$

Proof

(a) Suppose $x+y=x+z$.

Since

$$\begin{aligned} y &= 0+y \\ &= (-x+x)+y && \because\ -x+x=0 \\ &= -x+(x+y) && \text{by Associative law} \\ &= -x+(x+z) && \text{by supposition} \\ &= (-x+x)+z && \text{by Associative law} \\ &= (0)+z && \because\ -x+x=0 \\ &= z \end{aligned}$$

(b) Take $z=0$ in (a)

$$x+y=x+0$$

$$\Rightarrow\ y=0$$

(c) Take $z=-x$ in (a)

$$x+y=x+(-x)$$

$$\Rightarrow\ y=-x$$

(d) Since $(-x)+x=0$

then (c) gives $x=-(-x)$

❖ Theorem

Let $x, y, z \in \mathbb{R}$. Then axioms of multiplication imply the following.

(***a***) If $x \neq 0$ and $xy=xz$ then $y=z$.

(***b***) If $x \neq 0$ and $xy = x$ then $y = 1$.

(***c***) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$.

(***d***) If $x \neq 0$, then $\frac{1}{1/x} = x$.

Proof

(a) Suppose $xy = xz$

$$
\begin{aligned}
\text{Since } y &= 1 \cdot y \\
&= \left(\frac{1}{x} \cdot x\right) y && \because \frac{1}{x} \cdot x = 1 \\
&= \frac{1}{x}(xy) && \text{by associative law} \\
&= \frac{1}{x}(xz) && \because xy = xz \\
&= \left(\frac{1}{x} \cdot x\right) z && \text{by associative law} \\
&= 1 \cdot z = z
\end{aligned}
$$

(b) Take $z = 1$ in (a)

$$xy = x \cdot 1 \quad \Rightarrow y = 1$$

(c) Take $z = \frac{1}{x}$ in (a)

$$xy = x \cdot \frac{1}{x} \quad \text{i.e. } xy = 1$$

$$\Rightarrow \; y = \frac{1}{x}$$

(d) Since $\frac{1}{x} \cdot x = 1$

then (c) give

$$x = \frac{1}{1/x}$$

❖ Theorem

Let $x, y, z \in \mathbb{R}$. Then field axioms imply the following.

(***i***) $0 \cdot x = x$

(***ii***) if $x \neq 0$, $y \neq 0$ then $xy \neq 0$.

(***iii***) $(-x)y = -(xy) = x(-y)$

(***iv***) $(-x)(-y) = xy$

Proof

(***i***) Since $0x + 0x = (0+0)x$

$\Rightarrow\ 0x + 0x = 0x$

$\Rightarrow\ 0x = 0$ $\qquad \because x + y = x \Rightarrow y = 0$

(***ii***) Suppose $x \neq 0,\ y \neq 0$ but $x\,y = 0$

Since $1 = \dfrac{1}{x\,y} \cdot x\,y$

$\Rightarrow 1 = \dfrac{1}{x\,y}(0)$ $\qquad \because\ xy = 0$

$\Rightarrow\ 1 = 0$ $\qquad$ from (*i*) $\quad \because\ x0 = 0$

a contradiction, thus (*ii*) is true.

(***iii***) Since $(-x)y + xy = (-x + x)y = 0y = 0$ …….. (1)

Also $x(-y) + xy = x(-y + y) = x0 = 0$ ……… (2)

Also $-(xy) + xy = 0$ …………. (3)

Combining (1) and (2)

$(-x)y + xy = x(-y) + xy$

$\Rightarrow\ (-x)y = x(-y)$ ………… (4)

Combining (2) and (3)

$x(-y) + xy = -(xy) + xy$

$\Rightarrow\ x(-y) = -xy$ …………. (5)

From (4) and (5)

$(-x)y = x(-y) = -xy$

(***iv***) $(-x)(-y) = -\left[x(-y)\right] = -\left[-xy\right] = xy$ $\qquad$ using (*iii*)

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then the following statements are true in every ordered field.

i) If $x > 0$ then $-x < 0$ and vice versa.

ii) If $x > 0$ and $y < z$ then $xy < xz$.

iii) If $x < 0$ and $y < z$ then $xy > xz$.

iv) If $x \neq 0$ then $x^2 > 0$ in particular $1 > 0$.

v) If $0 < x < y$ then $0 < \dfrac{1}{y} < \dfrac{1}{x}$.

Proof

$i)$ If $x>0$ then $0=-x+x>-x+0$ so that $-x<0$.

If $x<0$ then $0=-x+x<-x+0$ so that $-x>0$.

$ii)$ Since $z>y$ we have $z-y>y-y=0$

which means that $z-y>0$ also $x>0$

$$\therefore \quad x(z-y)>0$$
$$\Rightarrow \quad xz-xy>0$$
$$\Rightarrow \quad xz-xy+xy>0+xy$$
$$\Rightarrow \quad xz+0>0+xy$$
$$\Rightarrow \quad xz>xy$$

$iii)$ Since $y<z \quad \Rightarrow \quad -y+y<-y+z$

$$\Rightarrow \quad z-y>0$$

Also $x<0 \quad \Rightarrow -x>0$

Therefore $-x(z-y)>0$

$$\Rightarrow -xz+xy>0 \qquad \Rightarrow -xz+xy+xz>0+xz$$
$$\Rightarrow \quad xy>xz$$

$iv)$ If $x>0$ then $x\cdot x>0 \quad \Rightarrow x^2>0$

If $x<0$ then $-x>0 \Rightarrow (-x)(-x)>0 \Rightarrow (-x)^2>0 \Rightarrow x^2>0$

i.e. if $x>0$ then $x^2>0$, since $1^2=1$ then $1>0$.

$v)$ If $y>0$ and $v\leq 0$ then $yv\leq 0$, But $y\left(\frac{1}{y}\right)=1>0 \quad \Rightarrow \frac{1}{y}>0$

Likewise $\frac{1}{x}>0$ as $x>0$

If we multiply both sides of the inequality $x<y$ by the positive quantity

$\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$ we obtain $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x<\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y$

i.e. $\quad \frac{1}{y}<\frac{1}{x}$

finally $\quad 0<\frac{1}{y}<\frac{1}{x}$.

❖ Theorem (Archimedean Property)

If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$ then there exists a positive integer n such that $nx > y$.

Proof

Let $A = \{nx : n \in \mathbb{Z}^+ \wedge x > 0, x \in \mathbb{R}\}$

Suppose the given statement is false i.e. $nx \le y$.

$\Rightarrow$ y is an upper bound of A.

Since we are dealing with a set of real therefore it has the least upper bound property.

Let $\alpha = \sup A$

$\Rightarrow$ $\alpha - x$ is not an upper bound of A.

$\Rightarrow$ $\alpha - x < mx$ where $mx \in A$ for some positive integer m.

$\Rightarrow$ $\alpha < (m+1)x$ where $m + 1$ is integer, therefore $(m+1)x \in A$

Which is impossible because α is least upper bound of A i.e. $\alpha = \sup A$.

Hence we conclude that the given statement is true i.e. $nx > y$.

❖ The Density Theorem

If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{Q}$ such that $x < p < y$.

i.e. between any two real numbers there is a rational number *or* $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof

Since $x < y$, therefore $y - x > 0$

$\Rightarrow$ $\exists$ a +ive integer n such that

$$n(y - x) > 1 \qquad (\textit{by Archimedean Property})$$

$$\Rightarrow ny > 1 + nx \quad \ldots\ldots\ldots\ldots\ldots (i)$$

We apply (a) part of the theorem again to obtain two +ive integers m_1 and m_2 such that $m_1 \cdot 1 > nx$ and $m_2 \cdot 1 > -nx$

$$\Rightarrow -m_2 < nx < m_1$$

then there exists and integers $m(-m_2 \le m \le m_1)$ such that

$$m - 1 \le nx < m$$

$$\Rightarrow nx < m \quad \text{and} \quad m \le 1 + nx$$

$$\Rightarrow nx < m < 1 + nx$$

$$\Rightarrow nx < m < ny \qquad \text{from } (i)$$

$$\Rightarrow x < \frac{m}{n} < y$$

$$\Rightarrow \quad x < p < y \quad \text{where} \quad p = \frac{m}{n} \text{ is a rational.}$$

❖ Theorem

Given two real numbers x and y, $x < y$ there is an irrational number u such that $x < u < y$.

Proof

Take $x > 0,\ y > 0$

Then $\exists$ a rational number q such that

$$0 < \frac{x}{\alpha} < q < \frac{y}{\alpha} \qquad \text{where } \alpha \text{ is an irrational.}$$

$$\Rightarrow \quad x < \alpha q < y$$

$$\Rightarrow \quad x < u < y,$$

where $u = \alpha q$ is an irrational as product of rational and irrational is irrational.

❖ Theorem

For every real number x there is a set E of rational number such that $x = \sup E$.

Proof

Take $E = \{q \in \mathbb{Q} : q < x\}$ where x is a real.

Then E is bounded above. Since $E \subset \mathbb{R}$ therefore supremum of E exists in $\mathbb{R}$.

Suppose $\sup E = \lambda$.

It is clear that $\lambda \leq x$.

If $\lambda = x$ then there is nothing to prove.

If $\lambda < x$ then $\exists\ q \in \mathbb{Q}$ such that $\lambda < q < x$,

which can not happened hence we conclude that real x is sup$E$.

❖ The Extended Real Numbers

The extended real number system consists of real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$, We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty \quad \forall\ x \in \mathbb{R}.$$

The extended real number system does not form a field. Mostly we write $+\infty = \infty$.

We make following conventions:

i) If x is real the $x + \infty = \infty,\ x - \infty = -\infty,\ \dfrac{x}{\infty} = \dfrac{x}{-\infty} = 0$.

ii) If $x > 0$ then $x(\infty) = \infty,\ x(-\infty) = -\infty$.

iii) If $x < 0$ then $x(\infty) = -\infty,\ x(-\infty) = \infty$.

❖ Euclidean Space

For each positive integer *k*, let $\mathbb{R}^k$ be the set of all ordered *k*-tuples

$$\underline{x} = (x_1, x_2, ..., x_k)$$

where $x_1, x_2, ..., x_k$ are real numbers, called the *coordinates* of $\underline{x}$. The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$.

If $\underline{y} = (y_1, y_2, ..., y_n)$ and α is a real number, put

$$\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, ..., x_k + y_k)$$

and $$\alpha\underline{x} = (\alpha x_1, \alpha x_2, ..., \alpha x_k)$$

So that $\underline{x} + \underline{y} \in \mathbb{R}^k$ and $\alpha\underline{x} \in \mathbb{R}^k$. These operations make $\mathbb{R}^k$ into a vector space over the real field.

The inner product or scalar product of $\underline{x}$ and $\underline{y}$ is defined as

$$\underline{x}.\underline{y} = \sum_{i=1}^{k} x_i y_i = (x_1 y_1 + x_2 y_2 + ... + x_k y_k)$$

And the norm of $\underline{x}$ is defined by

$$\| \underline{x} \| = (x \cdot x)^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}$$

The vector space $\mathbb{R}^k$ with the above inner product and norm is called *Euclidean k-space*.

❖ Theorem

Let $\underline{x}, \underline{y} \in \mathbb{R}^n$ then

i) $\| \underline{x} \|^2 = x \cdot x$

ii) $\left| \underline{x} \cdot \underline{y} \right| \leq \| \underline{x} \| \| \underline{y} \|$ (*Cauchy-Schwarz's inequality*)

Proof

i) Since $\| \underline{x} \| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}}$ therefore $\| \underline{x} \|^2 = \underline{x} \cdot \underline{x}$

ii) If $\underline{x} = 0$ or $\underline{y} = 0$, then Cauchy-Schwarz's inequality holds with equality.

If $\underline{x} \neq 0$ and $\underline{y} \neq 0$, then for $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} 0 \leq \| \underline{x} - \lambda\underline{y} \|^2 &= \left(\underline{x} - \lambda\underline{y}\right) \cdot \left(\underline{x} - \lambda\underline{y}\right) \\ &= \underline{x} \cdot \left(\underline{x} - \lambda\underline{y}\right) + \left(-\lambda\underline{y}\right) \cdot \left(\underline{x} - \lambda\underline{y}\right) \\ &= \underline{x} \cdot \underline{x} + \underline{x} \cdot (-\lambda\underline{y}) + (-\lambda\underline{y}) \cdot \underline{x} + (-\lambda\underline{y}) \cdot (-\lambda\underline{y}) \\ &= \| \underline{x} \|^2 - 2\lambda(\underline{x} \cdot \underline{y}) + \lambda^2 \| \underline{y} \|^2 \end{aligned}$$

Now put $\lambda=\dfrac{\underline{x}\cdot\underline{y}}{\|\underline{y}\|^2}$ (*certain real number*)

$$\Rightarrow\ 0\le\|\underline{x}\|^2-2\frac{(\underline{x}\cdot\underline{y})(\underline{x}\cdot\underline{y})}{\|\underline{y}\|^2}+\frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{y}\|^4}\|\underline{y}\|^2\quad\Rightarrow\ 0\le\|\underline{x}\|^2-\frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{y}\|^2}$$

$$\Rightarrow\ 0\le\|\underline{x}\|^2\|\underline{y}\|^2-\left|\underline{x}\cdot\underline{y}\right|^2\qquad\because\ a^2=|a|^2\ \forall\ a\in\mathbb{R},$$

$$\Rightarrow\ 0\le\left(\|\underline{x}\|\|\underline{y}\|+\left|\underline{x}\cdot\underline{y}\right|\right)\left(\|\underline{x}\|\|\underline{y}\|-\left|\underline{x}\cdot\underline{y}\right|\right).$$

Which hold if and only if

$$0\le\|\underline{x}\|\|\underline{y}\|-\left|\underline{x}\cdot\underline{y}\right|$$

i.e. $\left|\underline{x}\cdot\underline{y}\right|\le\|\underline{x}\|\|\underline{y}\|$.

❖ Question

Suppose $\underline{x},\underline{y},\underline{z}\in\mathbb{R}^n$ the prove that

a) $\|\underline{x}+\underline{y}\|\le\|\underline{x}\|+\|\underline{y}\|$

b) $\|\underline{x}-\underline{z}\|\le\|\underline{x}-\underline{y}\|+\|\underline{y}-\underline{z}\|$

Proof

a) Consider

$$\begin{aligned}\|\underline{x}+\underline{y}\|^2&=(\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y})\\&=\underline{x}\cdot\underline{x}+\underline{x}\cdot\underline{y}+\underline{y}\cdot\underline{x}+\underline{y}\cdot\underline{y}\\&=\|\underline{x}\|^2+2(\underline{x}\cdot\underline{y})+\|\underline{y}\|^2\\&\le\|\underline{x}\|^2+2\left|\underline{x}\cdot\underline{y}\right|+\|\underline{y}\|^2&&\because\ |a|\ge a\ \forall\ a\in\mathbb{R}.\\&\le\|\underline{x}\|^2+2\|\underline{x}\|\|\underline{y}\|+\|\underline{y}\|^2&&\because\ \|\underline{x}\|\|\underline{y}\|\ge\left|\underline{x}\cdot\underline{y}\right|\\&=\left(\|\underline{x}\|+\|\underline{y}\|\right)^2\end{aligned}$$

$$\Rightarrow\ \|\underline{x}+\underline{y}\|\le\|\underline{x}\|+\|\underline{y}\|\ \ \ldots\ldots\ldots\ldots\ (i)$$

b) We have

$$\begin{aligned}\|\underline{x}-\underline{z}\|&=\|\underline{x}-\underline{y}+\underline{y}-\underline{z}\|\\&\le\|\underline{x}-\underline{y}\|+\|\underline{y}-\underline{z}\|&&\text{from }(i)\end{aligned}$$

❖ Relatively Prime

Let $a,b \in \mathbb{Z}$. Then a and b are said to be relatively prime or co-prime if a and b don't have common factor other than 1. If a and b are relatively prime then we write $(a,b)=1$.

❖ Question

If r is non-zero rational and x is irrational then prove that $r+x$ and rx are irrational.

Proof

Let $r+x$ be rational.

$$\Rightarrow \; r+x=\frac{a}{b} \qquad \text{where } a,b \in \mathbb{Z}\,,\; b \neq 0 \text{ such that } {}^*(a,b)=1$$

$$\Rightarrow \; x=\frac{a}{b}-r$$

Since r is rational therefore $r=\dfrac{c}{d}$ where $c,d \in \mathbb{Z}$, $d \neq 0$ such that $(c,d)=1$

$$\Rightarrow \; x=\frac{a}{b}-\frac{c}{d} \quad \Rightarrow \; x=\frac{ad-bc}{bd}$$

Which is rational, which can not happened because x is given to be irrational.

Similarly let us suppose that rx is rational then

$$rx=\frac{a}{b} \qquad \text{for some } a,b \in \mathbb{Z},\; b \neq 0 \text{ such that } (a,b)=1$$

$$\Rightarrow \; x=\frac{a}{b}\cdot\frac{1}{r}$$

Since r is rational therefore $r=\dfrac{c}{d}$ where $c,d \in \mathbb{Z}$, $d \neq 0$ such that $(c,d)=1$

$$\Rightarrow \; x=\frac{a}{b}\cdot\frac{1}{c/d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Which shows that x is rational, which is again contradiction; hence we conclude that $r+x$ and rx are irrational. ⊙

❖ Question

If n is a positive integer which is not perfect square then prove that $\sqrt{n}$ is irrational number.

Solution

There will be two cases

Case I. When n contain no square factor greater then 1.

Let us suppose that $\sqrt{n}$ is a rational number.

$$\Rightarrow \sqrt{n} = \frac{p}{q} \qquad \text{where } p,q \in \mathbb{Z},\ q \neq 0 \text{ and } (p,q) = 1$$

$$\Rightarrow n = \frac{p^2}{q^2} \quad \Rightarrow p^2 = nq^2 \ ....(i)$$

$$\Rightarrow q^2 = \frac{p^2}{n}$$

$$\Rightarrow n \mid p^2 \qquad\qquad (n \mid p \text{ means “} n \text{ divides } p\text{” })$$

Since n has no square factor greater than 1, therefore

$$n \mid p \ ....(ii)$$

So there exists $c \in \mathbb{Z}$, such that

$$p = nc \quad \Rightarrow p^2 = n^2c^2$$

Putting this value of p^2 in equation (i)

$$n^2c^2 = nq^2$$

$$\Rightarrow nc^2 = q^2 \quad \Rightarrow c^2 = \frac{q^2}{n}$$

$$\Rightarrow n \mid q^2 \ ....(iii)$$

From (ii) and (iii) we get p and q both have common factor n i.e. $(p,q) = n$

Which is a contradiction.

Hence our supposition is wrong.

Case II When n contain a square factor greater then 1.

Let us suppose $n = k^2m > 1$

$$\Rightarrow \sqrt{n} = k\sqrt{m}$$

Where k is rational and $\sqrt{m}$ is irrational because m has no square factor greater than one, this implies $\sqrt{n}$, the product of rational and irrational, is irrational.

❖ **Question**

Prove that $\sqrt{12}$ is irrational.

Proof

Since $\sqrt{12} = 2\sqrt{3}$ and $\sqrt{3}$ is an irrational number, therefore $2\sqrt{3}$ is irrational.

❖ **Question**

Let E be a non-empty subset of an ordered set, suppose α is a lower bound of E and β is an upper bound then prove that $\alpha \le \beta$.

Proof

Since E is a subset of an ordered set S i.e. $E \subseteq S$.

Also α is a lower bound of E therefore by definition of lower bound

$$\alpha \le x \quad \forall \quad x \in E \quad \ldots\ldots\ldots\ldots\ldots \ (i)$$

Since β is an upper bound of E therefore by the definition of upper bound

$$x \le \beta \quad \forall \quad x \in E \quad \ldots\ldots\ldots\ldots\ldots \ (ii)$$

Combining (i) and (ii)

$$\alpha \le x \le \beta$$

$\Rightarrow \ \alpha \le \beta$ as required.

References:

(1) Principles of Mathematical Analysis
Walter Rudin (McGraw-Hill, Inc.)

(2) Introduction to Real Analysis
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(3) Mathematical Analysis,
Tom M. Apostol, (Pearson; 2nd edition.)

A password protected "zip" archive of above resources can be downloaded from the following URL: www.bit.ly/mth321
