Simple Random Walk

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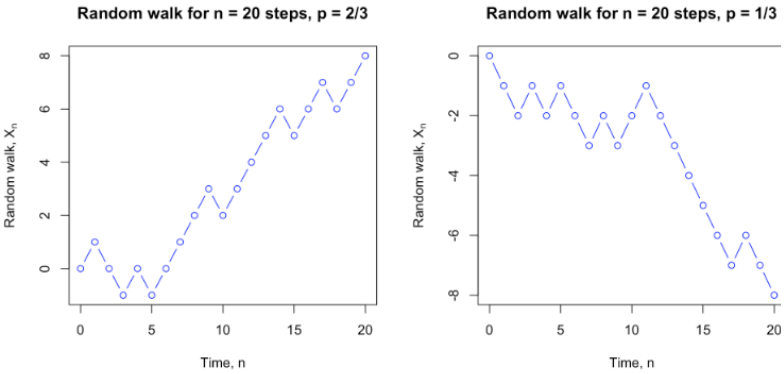
We can write this as a stochastic process (X_n) with discrete time $n = \{0,1,2,\ldots\}$ and discrete state space $S = \mathbb{Z}$, where $X_0 = 0$ and, for $n \geq 0$, we have



$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \\ X_n - 1 & \text{with probability } q \end{cases}$$

It's clear from this definition that X_{n+1} (the future) depends on X_n (the present), but, given X_n, does not depend on $X_{n-1}, \dots, X_1, X_0$ (the past). Thus the Markov property holds, and the ***simple random walk*** is a **discrete time Markov process** or **Markov chain**.

In other words:

A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, \dots$ is said to be a simple random walk if, for some number $0 < p < 1$,

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 0, \pm 1, \pm 2, \dots$$

The simple random walk is called ***simple symmetric random walk*** if $p = q = \frac{1}{2}$, that is,

$$X_0 = 0,\ X_n = X_{n-1} + Y_n,\ \text{where } Y_n = \begin{cases} +1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2} \end{cases}$$

Unbiased Simple Random Walk on {0, 1, 2, 3}

Let's consider a random walk on a finite set of states $\{0,1,2,3\}$. We need to define what happens at the boundaries. A common choice is to make them **absorbing boundaries**, meaning once the walker reaches 0 or 3, they cannot leave.

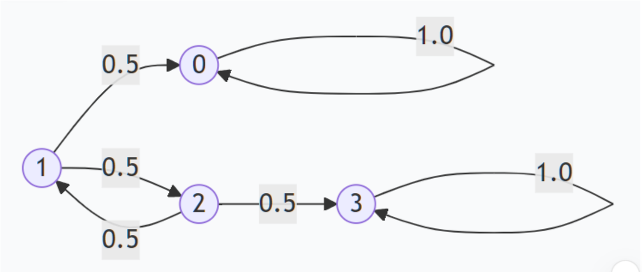
Here are the rules:

- From state **1**, you move to state **0** with probability p and to state **2** with probability $1-p$.
- From state **2**, you move to state **1** with probability p and to state **3** with probability $1-p$.
- States **0** and **3** are absorbing. If you are in state 0 or 3, you stay there with probability 1.

For a **simple, unbiased** walk, the probability of moving left or right is equal.

Let's set $p=0.5$.

State Transition Diagram



Transition Probability Matrix, P

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hopefully, you know the answer if one asks:

What is the probability of being in state 3 after 2 steps, starting from state 1?

Thank you so much