DEPARTMENT OF MATHEMATICS
COMSATS University Islamabad, Attock Campus

Exponential and Logarithmic Functions
Sample Questions: Set 01

1. Consider a sequence of functions $E_n : \mathbb{R} \to \mathbb{R}$ defined as follows:
$$E_1(x) = 1 + x \text{ and } E_{n+1}(x) = 1 + \int_0^x E_n(t)dt,$$
for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Prove that E_n is well-defined.
2. Consider a sequence of functions $E_n : \mathbb{R} \to \mathbb{R}$ defined by $E_1(x) = 1 + x$ and $E_{n+1}(x) = 1 + \int_0^x E_n(t)dt$, for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Prove that for all $n \in \mathbb{N}$, we have
$$E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}.$$
3. Prove that $\lim_{n\to\infty} \frac{A^n}{n!} = 0$ for $A > 0$.
4. Prove that if $\{s_n\}$ is convergent then $\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} s_n$.
5. Consider a sequence of function $\{E_n(x)\}$ define by
$$E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}.$$
Prove that $\{E_n\}$ converges uniformly on the interval $[-A, A]$, where $A > 0$.
6. Only sketch the proof that there exists a function $E : \mathbb{R} \to \mathbb{R}$ such that
$$E'(x) = E(x) \text{ for all } x \in \mathbb{R} \quad \text{and} \quad E(0) = 1.$$
7. State Taylor's and Maclaurin's theorem with remainder after n terms.
8. Consider a function $E : \mathbb{R} \to \mathbb{R}$ defined by $E'(x) = E(x)$ for all $x \in \mathbb{R}$ and $E(0) = 1$. Prove that such a function E is unique.
9. Define an exponential function.
10. Define the Euler's number e.
11. Prove that the exponential function E is non-zero for all real, that is, $E(x) \neq 0$ for all $x \in \mathbb{R}$.
12. Prove that exponential function E is strictly increasing on $\mathbb{R}$.
13. Prove that the exponential function satisfies that
$$E(x + y) = E(x)E(y) \text{ for all } x, y \in \mathbb{R}.$$
14. Prove that $E(r) = e^r$, where $E(1) = e$ and $r \in \mathbb{Q}$.
15. Define logarithm function.

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