DEPARTMENT OF MATHEMATICS
COMSATS Institute of Information Technology, Attock

Exponential, Logarithmic and Trigonometric Functions
Sample Questions

1. Only sketch the proof that there exists a function $E : \mathbb{R} \to \mathbb{R}$ such that
$$E'(x) = E(x) \text{ for all } x \in \mathbb{R} \quad \text{and} \quad E(0) = 1.$$

2. Consider a sequence of functions $E_n : \mathbb{R} \to \mathbb{R}$ defined as follows:
$$E_1(x) := 1 + x \text{ and } E_{n+1}(x) = 1 + \int_0^x E_n(t)dt,$$
for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Prove that E_n is well-defined.

3. Consider a sequence of functions $E_n : \mathbb{R} \to \mathbb{R}$ defined by $E_1(x) = 1 + x$ and $E_{n+1}(x) = 1 + \int_0^x E_n(t)dt$, for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Prove that for all $n \in \mathbb{N}$, we have
$$E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}.$$

4. Prove that $\lim\limits_{n\to\infty} \frac{A^n}{n!} = 0$ for $A > 0$.

5. Consider a sequence of function $\{E_n(x)\}$ define by
$$E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}.$$
Prove that $\{E_n\}$ converges uniformly on the interval $[-A, A]$, where $A > 0$.

6. Consider a function $E : \mathbb{R} \to \mathbb{R}$ defined by $E'(x) = E(x)$ for all $x \in \mathbb{R}$ and $E(0) = 1$. Prove that such a function E is unique.

7. Define an exponential function.

8. Prove that exponential function satisfies the following properties
   a. $E(x) \neq 0$ for all $x \in \mathbb{R}$.
   b. $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.

9. Define logarithm function.

10. Only sketch the proof that there exist functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that
    (i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$
    (ii) $C(0) = 1$, $C'(0) = 0$ and $S(0) = 0$, $S'(0) = 1$.

11. Consider a sequence of functions $\{C_n(x)\}$ define by
$$C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x \in \mathbb{R}.$$
Prove that $\{C_n\}$ converges uniformly on the interval $[-A, A]$, where $A > 0$.

12. Consider a sequence of functions $\{S_n(x)\}$ define by

$$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \in \mathbb{R}.$$

Prove that $\{S_n\}$ converges uniformly on the interval $[-A, A]$, where $A > 0$.

13. Consider the functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that

    (i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$

    (ii) $C(0) = 1$, $C'(0) = 0$ and $S(0) = 0$, $S'(0) = 1$.

    Prove that $C^2(x) + S^2(x) = 1$ for $x \in \mathbb{R}$.

14. Consider the functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that

    (i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$

    (ii) $C(0) = 1$, $C'(0) = 0$ and $S(0) = 0$, $S'(0) = 1$.

    Prove that the functions C and S defined in this way are unique.

15. Define cosine and sine functions.

16. If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f''(x) = -f(x) \quad \text{for} \quad x \in \mathbb{R},$$

then there exist real numbers α and β such that

$$f(x) = \alpha C(x) + \beta S(x) \quad \text{for} \quad x \in \mathbb{R},$$

where C and S represents cosine and sine function.

Above question can be rephrases as follows:

17. If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f''(x) = -f(x) \quad \text{for} \quad x \in \mathbb{R},$$

then there exist real numbers α and β such that

$$f(x) = \alpha \cos x + \beta \sin x \quad \text{for} \quad x \in \mathbb{R},$$

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