Improper Integrals

Course Title: Real Analysis II Course Code: MTH322
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We discussed (in MTH321: Real Analysis I) Riemann-Stieltjes's integrals of the form $\int_a^b f\,d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. To extend the concept, we shall relax these restrictions on f and α.

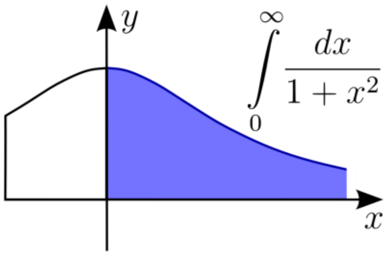
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

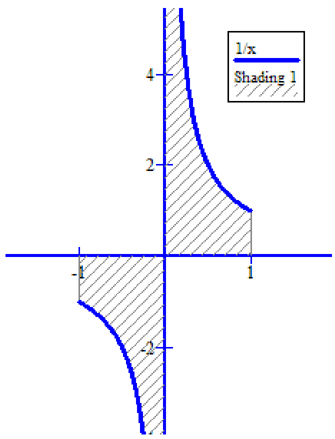
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded at one or more points of $a\le x\le b$. Such points are called singularities of $f(x)$.

➢ *Examples*



- $\int_0^\infty \frac{1}{1+x^2}dx$, $\int_{-\infty}^1 \frac{1}{x-2}dx$ and $\int_{-\infty}^\infty (x^2+1)dx$ are examples of improper integrals of first kind.



- $\int_{-1}^1 \frac{1}{x}dx$ and $\int_0^1 \frac{1}{2x-1}dx$ are examples of improper integrals of second kind.

➢ *Notations*

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

IMPROPER INTEGRAL OF THE FIRST KIND

➢ Definition

Assume that $f \in R(\alpha;a,b)$ for every $b \ge a$. Keep a,α and f fixed and define a function I on $[a,\infty)$ as follows:

$$I(b)=\int_a^b f(x)\,d\alpha(x) \quad \text{if} \quad b \ge a \;\ldots\ldots\ldots\ldots (i)$$

The function I so defined is called an infinite (or an improper) integral of first kind and is denoted by the symbol $\int_a^\infty f(x)\,d\alpha(x)$ or by $\int_a^\infty f\,d\alpha$.

The integral $\int_a^\infty f\,d\alpha$ is said to converge if the limit

$$\lim_{b\to\infty} I(b) \;\ldots\ldots\ldots\ldots (ii)$$

exists (finite). Otherwise, $\int_a^\infty f\,d\alpha$ is said to diverge.

If the limit in (ii) exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,d\alpha = A$

➢ Example

Consider and integral $\int_1^\infty x^{-p}\,dx$, where p is any real number.

Now $I(b)=\int_1^b x^{-p}\,dx = \left.\dfrac{x^{1-p}}{1-p}\right|_1^b = \dfrac{1-b^{1-p}}{p-1}$ if $p \ne 1$.

As we know

$$\lim_{b\to\infty} I(b)=\lim_{b\to\infty}\frac{1-b^{1-p}}{p-1}=\begin{cases}\infty & if \quad p<1,\\ \dfrac{1}{p-1} & if \quad p>1.\end{cases}$$

Thus integral $\int_1^\infty x^{-p}\,dx$ diverges if $p<1$ and converges if $p>1$ and has the value $\dfrac{1}{p-1}$.

If $p=1$, we get $\int_1^b x^{-1}\,dx = \log b \to \infty$ as $b\to\infty$. $\Rightarrow \int_1^\infty x^{-1}\,dx$ diverges.

Hence we concluded: $\displaystyle\int_1^\infty x^{-p}\,dx = \begin{cases} diverges & if \quad p \le 1,\\ \dfrac{1}{p-1} & if \quad p>1.\end{cases}$

➢ *Example*

Consider $\int_0^\infty \sin 2\pi x\, dx$

Since $\int_0^b \sin 2\pi x\, dx = \frac{1-\cos 2\pi b}{2\pi} \to l$ as $b \to \infty$, where l has values between 0 and $\frac{1}{\pi}$, that is, limit is not unique.

Therefore the integral $\int_0^\infty \sin 2\pi x\, dx$ diverges.

➢ *Note*

If $\int_{-\infty}^{a} f\, d\alpha$ and $\int_{a}^{\infty} f\, d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\, d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\, d\alpha = \int_{-\infty}^{a} f\, d\alpha + \int_{a}^{\infty} f\, d\alpha$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} f\, d\alpha$ converges, its value is equal to the limit: $\lim_{b\to+\infty} \int_{-b}^{b} f\, d\alpha$.

➢ *Theorem*

Assume that α is monotonically increasing on $[a,+\infty)$ and suppose that $f \in R(\alpha;a,b)$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^\infty f\, d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that

$$\int_a^b f\, d\alpha \le M \quad \text{for every } b \ge a.$$

Proof

Let $I(b) = \int_a^b f\, d\alpha$ and suppose that $\int_a^\infty f\, d\alpha$ is convergent, then $\lim_{b\to+\infty} I(b)$ exists, that is, $I(b)$ is bounded.

So there exists a constant $M > 0$ such that

$$|I(b)| < M \text{ for every } b \ge a.$$

As $f(x) \ge 0$ for each $x \ge a$, therefore $\int_a^b f\, d\alpha \ge 0$.

This gives $I(b)=\int_a^b f\,d\alpha \;\le\; M$ for every $b \ge a$.

Conversely, suppose that there exists a constant $M>0$ such that $\int_a^b f\,d\alpha \;\le\; M$ for every $b \ge a$. This give $\left|I(b)\right| \le M$ for every $b \ge a$.

That is, I is bounded on $[a,+\infty)$.

Now for $b_2 \ge b_1 > a$, we have

$$I(b_2)=\int_a^{b_2} f(x)\,d\alpha(x)=\int_a^{b_1} f(x)\,d\alpha(x)+\int_{b_1}^{b_2} f(x)\,d\alpha(x)$$

$$\ge \int_a^{b_1} f(x)\,d\alpha(x)=I(b_1) \qquad \because \int_{b_1}^{b_2} f(x)\,d\alpha(x)\ge 0 \text{ as } f(x)\ge 0.$$

This gives I is monotonically increasing on $[a,+\infty)$.

As I is monotonically increasing and bounded on $[a,+\infty)$, therefore it is convergent, that is $\int_a^\infty f\,d\alpha$ converges.

> **➢ *Theorem: (Comparison Test)***
>
> Assume that α is monotonically increasing on $[a,+\infty)$. If $f \in R(\alpha;a,b)$ for every $b \ge a$, if $0 \le f(x) \le g(x)$ for every $x \ge a$, and if $\int_a^\infty g\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ converges and we have
>
> $$\int_a^\infty f\,d\alpha \;\le\; \int_a^\infty g\,d\alpha$$

Proof

Let $I_1(b)=\int_a^b f\,d\alpha$ and $I_2(b)=\int_a^b g\,d\alpha$, $b \ge a$

$\because \quad 0 \le f(x) \le g(x)$ for every $x \ge a$

$\therefore \quad I_1(b) \le I_2(b)$ …………………. (i)

$\because \int_a^\infty g\,d\alpha$ converges $\therefore$ $\exists$ a constant $M>0$ such that

$$\int_a^b g\,d\alpha \le M \quad , \quad b \ge a \text{ …………………}(ii)$$

From (i) and (ii) we have $I_1(b) \le M$ for every $b \ge a$.

$\Rightarrow \lim\limits_{b\to\infty} I_1(b)$ exists and is finite.

$$\Rightarrow \int_a^\infty f\,d\alpha \quad \text{converges.}$$

Also $\lim\limits_{b\to\infty} I_1(b) \le \lim\limits_{b\to\infty} I_2(b) \le M$

$$\Rightarrow \int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha\,.$$

➢ *Theorem (Limit Comparison Test)*

Assume that α is monotonically increasing on $[a,+\infty)$. Suppose that $f \in R(\alpha;a,b)$ and that $g \in R(\alpha;a,b)$ for every $b \ge a$, where $f(x) \ge 0$ and $g(x) \ge 0$ if $x \ge a$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

then $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

For all $b \ge a$, we can find some $N > 0$ such that

$$\left| \frac{f(x)}{g(x)} - 1 \right| < \varepsilon \qquad \forall \;\; x \ge N \quad \text{for every} \quad \varepsilon > 0\,.$$

$$\Rightarrow \; 1-\varepsilon < \frac{f(x)}{g(x)} < 1+\varepsilon$$

Let $\varepsilon = \frac{1}{2}$. Then we have

$$\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}\,.$$

$$\Rightarrow \; g(x) < 2f(x) \;\; \ldots\ldots\ldots(i) \qquad \text{and} \qquad 2f(x) < 3g(x) \;\; \ldots\ldots\ldots(ii)$$

From (i) $\quad \int_a^\infty g\,d\alpha < 2\int_a^\infty f\,d\alpha\,,$

$\Rightarrow \int_a^\infty g\,d\alpha$ converges if $\int_a^\infty f\,d\alpha$ converges and $\int_a^\infty g\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

From (ii) $\quad 2\int_a^\infty f\,d\alpha < 3\int_a^\infty g\,d\alpha\,,$

$\Rightarrow \int_a^\infty f\,d\alpha$ converges if $\int_a^\infty g\,d\alpha$ converges and $\int_a^\infty g\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

$\Rightarrow$ The integrals $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ converge or diverge together.

➢ *Note*

The above theorem also holds if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=c$, provided that $c\neq 0$. If $c=0$, we can only conclude that convergence of $\int_a^\infty g\,d\alpha$ implies convergence of $\int_a^\infty f\,d\alpha$.

➢ *Example*

For every real p, the integral $\int_1^\infty e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int_1^\infty \frac{1}{x^2}\,dx$.

Let $f(x)=e^{-x}x^p$ and $g(x)=\frac{1}{x^2}$.

Now $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{e^{-x}x^p}{1/x^2}$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}e^{-x}x^{p+2}=\lim_{x\to\infty}\frac{x^{p+2}}{e^x}=0.$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent, therefore the given integral $\int_1^\infty e^{-x}x^p\,dx$ is also convergent.

➢ *Remark*

It is easy to show that if $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ are convergent, then

- $\int_a^\infty (f\pm g)\,d\alpha$ is convergent.
- $\int_a^\infty cf\,d\alpha$, where c is some constant, is convergent.

➢ *Theorem*

Assume $\alpha\uparrow$ on $[a,+\infty)$. If $f\in R(\alpha;a,b)$ for every $b\geq a$ and if $\int_a^\infty |f|\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ also converges.

Or: An absolutely convergent integral is convergent.

Proof

If $x \ge a$, $\pm f(x) \le |f(x)|$

$\Rightarrow |f(x)| - f(x) \ge 0$

$\Rightarrow 0 \le |f(x)| - f(x) \le 2|f(x)|$

$\Rightarrow \int\limits_a^\infty (|f| - f)\,d\alpha$ converges.

Now difference of $\int\limits_a^\infty |f|\,d\alpha$ and $\int\limits_a^\infty (|f| - f)\,d\alpha$ is convergent,

that is, $\int\limits_a^\infty f\,d\alpha$ is convergent.

➢ *Note*

$\int\limits_a^\infty f\,d\alpha$ is said to converge absolutely if $\int\limits_a^\infty |f|\,d\alpha$ converges. It is said to be convergent conditionally if $\int\limits_a^\infty f\,d\alpha$ converges but $\int\limits_a^\infty |f|\,d\alpha$ diverges.

➢ *Remark*

Every absolutely convergent integral is convergent.

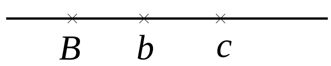
➢ *Theorem (Cauchy condition for infinite integrals)*

Assume that $f \in R(\alpha; a, b)$ for every $b \ge a$. Then the integral $\int\limits_a^\infty f\,d\alpha$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $c > b > B$ implies

$$\left| \int_b^c f(x)\,d\alpha(x) \right| < \varepsilon$$

Proof

Let $\int\limits_a^\infty f\,d\alpha$ be convergent. Then $\exists$ $B > 0$ such that



$$\left| \int_a^b f\,d\alpha - \int_a^\infty f\,d\alpha \right| < \frac{\varepsilon}{2} \quad \text{for every } b \ge B \quad \ldots\ldots\ldots..(i)$$

Also for $c > b > B$,

$$\left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha \right| < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots..(ii)$$

Consider

$$\left| \int_b^c f\,d\alpha \right| = \left| \int_a^c f\,d\alpha - \int_a^b f\,d\alpha \right|$$

$$= \left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha + \int_a^\infty f\,d\alpha - \int_a^b f\,d\alpha \right|$$

$$\leq \left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha \right| + \left| \int_a^\infty f\,d\alpha - \int_a^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \left| \int_b^c f\,d\alpha \right| < \varepsilon \quad \text{when} \quad c > b > B.$$

Conversely, assume that the Cauchy condition holds.

Define $a_n = \int_a^{a+n} f\,d\alpha$ if $n = 1, 2, \ldots\ldots$

Consider n, m such that $a+n, a+m > b > B$, then

$$|a_n - a_m| = \left| \int_a^{a+n} f\,d\alpha - \int_a^{a+m} f\,d\alpha \right| = \left| \int_a^b f\,d\alpha + \int_b^{a+n} f\,d\alpha - \int_a^b f\,d\alpha - \int_b^{a+m} f\,d\alpha \right|$$

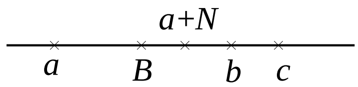
$$= \left| \int_b^{a+n} f\,d\alpha - \int_b^{a+m} f\,d\alpha \right| \leq \left| \int_b^{a+n} f\,d\alpha \right| + \left| \int_b^{a+m} f\,d\alpha \right| < \varepsilon + \varepsilon = 2\varepsilon$$

This gives, the sequence $\{a_n\}$ is a Cauchy sequence $\Rightarrow$ it converges.

Let $\lim\limits_{n\to\infty} a_n = A$

Given $\varepsilon > 0$, choose B so that $\left| \int_b^c f\,d\alpha \right| < \frac{\varepsilon}{2}$ if $c > b > B$.

and also that $|a_n - A| < \frac{\varepsilon}{2}$ whenever $a + n \geq B$.



Choose an integer N such that $a + N > B$ i.e. $N > B - a$.
Then, if $b > a + N$, we have

$$\left| \int_a^b f\,d\alpha - A \right| = \left| \int_a^{a+N} f\,d\alpha - A + \int_{a+N}^b f\,d\alpha \right|$$

$$\leq |a_N - A| + \left| \int_{a+N}^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \int_a^\infty f\,d\alpha = A$$

This completes the proof.

➢ *Remarks*

It follows from the above theorem that convergence of $\int_a^\infty f\,d\alpha$ implies

$\lim\limits_{b\to\infty} \int_b^{b+\varepsilon} f\,d\alpha = 0$ for every fixed $\varepsilon > 0$.

However, this does not imply that $f(x) \to 0$ as $x \to \infty$.

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IMPROPER INTEGRAL OF THE SECOND KIND

➢ Definition

Let f be defined on the half open interval $(a,b]$ and assume that $f \in R(\alpha; x,b)$ for every $x \in (a,b]$. Define a function I on $(a,b]$ as follows:

$$I(x) = \int_x^b f\, d\alpha \quad \text{if} \quad x \in (a,b] \;\ldots\ldots\ldots\ldots (i)$$

The function I so defined is called an improper integral of the second kind and is denoted by the symbol $\int_{a+}^{b} f(t)\, d\alpha(t)$ or $\int_{a+}^{b} f\, d\alpha$.

The integral $\int_{a+}^{b} f\, d\alpha$ is said to converge if the limit

$$\lim_{x \to a+} I(x) \;\ldots\ldots\ldots (ii) \quad \text{exists (finite).}$$

Otherwise, $\int_{a+}^{b} f\, d\alpha$ is said to diverge. If the limit in (*ii*) exists and equals A, the number A is called the value of the integral and we write $\int_{a+}^{b} f\, d\alpha = A$.

Similarly, if f is defined on $[a,b)$ and $f \in R(\alpha; a,x)$ $\forall$ $x \in [a,b)$ then $I(x) = \int_a^x f\, d\alpha$ if $x \in [a,b)$ is also an improper integral of the second kind and is denoted as $\int_a^{b-} f\, d\alpha$ and is convergent if $\lim_{x \to b-} I(x)$ exists (finite).

➢ Example

$f(x) = x^{-p}$ is defined on $(0,b]$ and $f \in R(x,b)$ for every $x \in (0,b]$.

$$I(x) = \int_x^b x^{-p}\, dx \qquad \text{if} \quad x \in (0,b]$$

$$= \int_{0+}^{b} x^{-p}\, dx = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{b} x^{-p}\, dx$$

$$= \lim_{\varepsilon \to 0} \left| \frac{x^{1-p}}{1-p} \right|_{\varepsilon}^{b} = \lim_{\varepsilon \to 0} \frac{b^{1-p} - \varepsilon^{1-p}}{1-p} \qquad , \qquad (p \neq 1)$$

$$= \left[\begin{array}{ll} \text{finite} \;, & p < 1 \\ \text{infinite} \;, & p > 1 \end{array} \right.$$

When $p = 1$, we get $\int_{\varepsilon}^{b} \frac{1}{x}\, dx = \log b - \log \varepsilon \to \infty$ as $\varepsilon \to 0$.

$$\Rightarrow \int_{0+}^{b} x^{-1}\,dx \text{ also diverges.}$$

Hence the integral converges when $p<1$ and diverges when $p \ge 1$.

➢ *Note*

If the two integrals $\int_{a+}^{c} f\,d\alpha$ and $\int_{c}^{b-} f\,d\alpha$ both converge, we write

$$\int_{a+}^{b-} f\,d\alpha = \int_{a+}^{c} f\,d\alpha + \int_{c}^{b-} f\,d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^{b} f\,d\alpha + \int_{b}^{\infty} f\,d\alpha \quad \text{which can be written as} \quad \int_{a+}^{\infty} f\,d\alpha .$$

➢ *Example*

Consider $\int_{0+}^{\infty} e^{-x} x^{p-1}\,dx \quad , \quad (p>0)$

This integral must be interpreted as a sum as

$$\int_{0+}^{\infty} e^{-x} x^{p-1}\,dx = \int_{0+}^{1} e^{-x} x^{p-1}\,dx + \int_{1}^{\infty} e^{-x} x^{p-1}\,dx$$

$$= I_1 + I_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (i)$$

I_2, the second integral, converges for every real p as proved earlier.

To test I_1, put $t=\dfrac{1}{x} \quad \Rightarrow \quad dx = -\dfrac{1}{t^2}\,dt$

$$\Rightarrow \; I_1 = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} e^{-x}x^{p-1}\,dx \;=\lim_{\varepsilon\to 0}\int_{1/\varepsilon}^{1} e^{-\frac{1}{t}}\, t^{1-p}\left(-\frac{1}{t^2}dt\right) \;=\lim_{\varepsilon\to 0}\int_{1}^{1/\varepsilon} e^{-\frac{1}{t}}\, t^{-p-1}\,dt$$

Take $\quad f(t)=e^{-\frac{1}{t}}\, t^{-p-1} \quad$ and $\quad g(t)=t^{-p-1}$

Then $\displaystyle \lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{t\to\infty}\frac{e^{-\frac{1}{t}}\cdot t^{-p-1}}{t^{-p-1}} = 1$ and since $\displaystyle\int_{1}^{\infty} t^{-p-1}\,dt$ converges when $p>0$

$\therefore \displaystyle\int_{1}^{\infty} e^{-\frac{1}{t}}\, t^{-p-1}\,dt$ converges when $p>0$

Thus $\displaystyle\int_{0+}^{\infty} e^{-x} x^{p-1}\,dx$ converges when $p>0$.

When $p>0$, the value of the sum in (i) is denoted by $\Gamma(p)$. The function so defined is called the Gamma function.

➢ Note

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

➢ A Useful Comparison Integral

$$\int_a^b \frac{dx}{(x-a)^n}$$

We have, if $n \neq 1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \left| \frac{1}{(1-n)(x-a)^{n-1}} \right|_{a+\varepsilon}^{b}$$

$$= \frac{1}{(1-n)} \left(\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}} \right)$$

Which tends to $\dfrac{1}{(1-n)(b-a)^{n-1}}$ or $+\infty$ according as $n<1$ or $n>1$, as $\varepsilon \to 0$.

Again, if $n=1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a) - \log\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

Hence the improper integral $\displaystyle\int_a^b \frac{dx}{(x-a)^n}$ converges iff $n<1$.

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➢ Question

Examine the convergence of

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ (*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$ (*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Solution

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

Here ‘0’ is the only point of infinite discontinuity of the integrand.

We have

$$f(x) = \frac{1}{x^{1/3}(1+x^2)}$$

Take $g(x) = \dfrac{1}{x^{1/3}}$

Then $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to 0}\dfrac{1}{1+x^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \int\limits_0^1 \dfrac{dx}{x^{1/3}}$ converges $\therefore \int\limits_0^1 \dfrac{dx}{x^{1/3}\left(1+x^2\right)}$ also converges.

(*ii*) $\int\limits_0^1 \dfrac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x)=\frac{1}{x^2(1+x)^2}$$

Take $g(x)=\dfrac{1}{x^2}$

Then $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to 0}\dfrac{1}{(1+x)^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge. Hence the given integral also does not converge.

(*iii*) $\int\limits_0^1 \dfrac{dx}{x^{1/2}(1-x)^{1/3}}$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/2}(1-x)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\int\limits_0^{1/2} f(x)\,dx$ and $\int\limits_{1/2}^{1} f(x)\,dx$.

To examine the convergence of $\int\limits_0^{1/2} \dfrac{1}{x^{1/2}(1-x)^{1/3}}\,dx$, we take $g(x)=\dfrac{1}{x^{1/2}}$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1-x)^{1/3}}=1$$

$\because \int_0^{1/2} \frac{1}{x^{1/2}}dx$ converges $\therefore \int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

To examine the convergence of $\int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to1}\frac{f(x)}{g(x)}=\lim_{x\to1}\frac{1}{x^{1/2}}=1$$

$\because \int_{1/2}^{1} \frac{1}{(1-x)^{1/3}}dx$ converges $\because \int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

➢ *Question*

Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists iff m, n are both positive.

Solution

The integral is proper if $m \geq 1$ and $n \geq 1$.

The number '0' is a point of infinite discontinuity if $m<1$ and the number '1' is a point of infinite discontinuity if $n<1$.

Let $m<1$ and $n<1$.

We take any number, say $1/2$, between 0 & 1 and examine the convergence of the improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ and $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ at '0' and '1' respectively.

Convergence at 0:

We write

$$f(x)=x^{m-1}(1-x)^{n-1}=\frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and take} \quad g(x)=\frac{1}{x^{1-m}}$$

Then $\frac{f(x)}{g(x)} \to 1$ as $x \to 0$

As $\int_0^{1/2} \frac{1}{x^{1-m}}dx$ is convergent at 0 iff $1-m<1$ i.e. $m>0$

We deduce that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ is convergent at 0, iff m is +ive.

Convergence at 1:

We write $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{x^{m-1}}{(1-x)^{1-n}}$ and take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

Then $\dfrac{f(x)}{g(x)}\to 1$ as $x\to 1$

As $\displaystyle\int_{1/2}^{1}\frac{1}{(1-x)^{1-n}}dx$ is convergent, iff $1-n<1$ i.e. $n>0$.

We deduce that the integral $\displaystyle\int_{1/2}^{1}x^{m-1}\left(1-x\right)^{n-1}dx$ converges iff $n>0$.

Thus $\displaystyle\int_{0}^{1}x^{m-1}\left(1-x\right)^{n-1}dx$ exists for positive values of m, n only.

It is a function which depends upon m & n and is defined for all positive values of m & n. It is called Beta function.

➢ *Question*

Show that the following improper integrals are convergent.

(i) $\displaystyle\int_{1}^{\infty}\sin^2\frac{1}{x}\,dx$ (ii) $\displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$ (iii) $\displaystyle\int_{0}^{1}\frac{x\log x}{(1+x)^2}\,dx$ (iv) $\displaystyle\int_{0}^{1}\log x\cdot\log(1+x)\,dx$

Solution

(*i*) Let $f(x)=\sin^2\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x^2}$

then $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2=1$

$\Rightarrow \displaystyle\int_{1}^{\infty}f(x)\,dx$ and $\displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ behave alike.

$\because \displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ is convergent $\therefore \displaystyle\int_{1}^{\infty}\sin^2\frac{1}{x}\,dx$ is also convergent.

(*ii*) $\displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\dfrac{\sin^2 x}{x^2}$ and $g(x)=\dfrac{1}{x^2}$

$\sin^2 x\le 1 \quad\Rightarrow\quad \dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2} \quad \forall\ x\in\left(1,\infty\right)$

and $\displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ converges $\therefore \displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$ converges.

➢ *Note*

$\int_0^1 \frac{\sin^2 x}{x^2}\,dx$ is a proper integral because $\lim_{x\to 0}\frac{\sin^2 x}{x^2}=1$ so that '0' is not a point of infinite discontinuity. Therefore $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

(*iii*) $\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$

$\because\ \log x < x\ ,\quad x\in(0,1)$

$\therefore\ x\log x < x^2$

$\Rightarrow\ \frac{x\log x}{(1+x)^2} < \frac{x^2}{(1+x)^2}$

Now $\int_0^1 \frac{x^2}{(1+x)^2}\,dx$ is a proper integral.

$\therefore\ \int_0^1 \frac{x\log x}{(1+x)^2}\,dx$ is convergent.

(*iv*) $\int_0^1 \log x\cdot\log(1+x)\,dx$

$\because\ \log x < x \qquad \therefore\ \log(x+1) < x+1$

$\Rightarrow\ \log x\cdot\log(1+x) < x(x+1)$

$\because\ \int_0^1 x(x+1)\,dx$ is a proper integral $\quad\therefore\ \int_0^1 \log x\cdot\log(1+x)\,dx$ is convergent.

➢ *Note*

(i) $\int_0^a \frac{1}{x^p}\,dx$ diverges when $p\ge 1$ and converges when $p<1$.

(ii) $\int_a^\infty \frac{1}{x^p}\,dx$ converges iff $p>1$.

UNIFORM CONVERGENCE OF IMPROPER INTEGRALS

➢ ***Definition***

Let f be a real valued function of two variables x & y, $x \in [a,+\infty)$, $y \in S$ where $S \subset \mathbb{R}$. Suppose further that, for each y in S, the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ is convergent. If F denotes the function defined by the equation

$$F(y) = \int_a^\infty f(x,y)\,d\alpha(x) \qquad \text{if} \quad y \in S$$

the integral is said to converge *pointwise* to F on S

➢ ***Definiton***

Assume that the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges pointwise to F on S. The integral is said to converge Uniformly on S if, for every $\varepsilon > 0$ there exists a $B > 0$ (depending only on ε) such that $b > B$ implies

$$\left| F(y) - \int_a^b f(x,y)\,d\alpha(x) \right| < \varepsilon \qquad \forall \;\; y \in S.$$

(Pointwise convergence means convergence when y is fixed but uniform convergence is for every $y \in S$).

➢ ***Theorem (Cauchy condition for uniform convergence.)***

The integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S, iff, for every $\varepsilon > 0$ there exists a $B > 0$ (depending on ε) such that $c > b > B$ implies

$$\left| \int_b^c f(x,y)\,d\alpha(x) \right| < \varepsilon \qquad \forall \;\; y \in S.$$

Proof

Proceed as in the proof for Cauchy condition for infinite integral $\int_a^\infty f\,d\alpha$.

➢ ***Theorem (Weierstrass M-test)***

Assume that $\alpha \uparrow$ on $[a,+\infty)$ and suppose that the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b \geq a$ and for every y in S. If there is a positive function M defined on $[a,+\infty)$ such that the integral $\int_a^\infty M(x)\,d\alpha(x)$ converges and $|f(x,y)| \leq M(x)$ for each $x \geq a$ and every y in S, then the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

$\because \ |f(x,y)| \le M(x)$ for each $x \ge a$ and every y in S.

$\therefore$ For every $c \ge b$, we have

$$\left|\int_b^c f(x,y)\,d\alpha(x)\right| \le \int_b^c |f(x,y)\,d\alpha(x)| \le \int_b^c M\,d\alpha \quad \ldots\ldots\ldots\ldots (i)$$

$\because \ I = \int_a^\infty M\,d\alpha$ is convergent

$\therefore$ given $\varepsilon > 0$, $\exists\ B > 0$ such that $b > B$ implies

$$\left|\int_a^b M\,d\alpha - I\right| < \varepsilon/2 \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Also if $c > b > B$, then

$$\left|\int_a^c M\,d\alpha - I\right| < \varepsilon/2 \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

Then $\left|\int_b^c M\,d\alpha\right| = \left|\int_a^c M\,d\alpha - \int_a^b M\,d\alpha\right|$

$$= \left|\int_a^c M\,d\alpha - I + I - \int_a^b M\,d\alpha\right|$$

$$\le \left|\int_a^c M\,d\alpha - I\right| + \left|\int_a^b M\,d\alpha - I\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{(By } ii \text{ \& } iii\text{)}$$

$$\Rightarrow \left|\int_b^c f(x,y)\,d\alpha(x)\right| < \varepsilon, \quad c > b > B \text{ \& for each } y \in S$$

Cauchy condition for convergence (uniform) being satisfied.

Therefore the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

➢ ***Example***

Consider $\int_0^\infty e^{-xy} \sin x\,dx$

$$|e^{-xy}\sin x| \le |e^{-xy}| = e^{-xy} \qquad (\because |\sin x| \le 1)$$

and $\quad e^{-xy} \le e^{-xc} \quad$ if $\quad c \le y$

Now take $\quad M(x) = e^{-cx}$

The integral $\int_0^\infty M(x)\,dx = \int_0^\infty e^{-cx}\,dx$ is convergent & converging to $\dfrac{1}{c}$.

$\therefore$ The conditions of M-test are satisfied and $\int_0^\infty e^{-xy}\sin x\,dx$ converges uniformly on $[c,+\infty)$ for every $c>0$.

> **➢ *Theorem (Dirichlet's test for uniform convergence)***
>
> Assume that α is bounded on $[a,+\infty)$ and suppose the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b\ge a$ and for every y in S. For each fixed y in S, assume that $f(x,y)\le f(x',y)$ if $a\le x'<x<+\infty$. Furthermore, suppose there exists a positive function g, defined on $[a,+\infty)$, such that $g(x)\to 0$ as $x\to+\infty$ and such that $x\ge a$ implies
>
> $$|f(x,y)|\le g(x) \quad \text{for every } y \text{ in } S.$$
>
> Then the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

Let $M>0$ be an upper bound for $|\alpha|$ on $[a,+\infty)$.

Given $\varepsilon>0$, choose $B>a$ such that $x\ge B$ implies

$$g(x)<\frac{\varepsilon}{4M}$$

($\because$ $g(x)$ is +ive and $\to 0$ as $x\to\infty$ $\therefore$ $|g(x)-0|<\frac{\varepsilon}{4M}$ for $x\ge B$)

If $c>b$, integration by parts yields

$$\int_b^c f\,d\alpha = \left| f(x,y)\cdot\alpha(x)\right|_b^c - \int_b^c \alpha\,df$$

$$= f(c,y)\alpha(c)-f(b,y)\alpha(b)+\int_b^c \alpha\,d(-f) \quad \ldots\ldots\ldots\ldots (i)$$

But, since $-f$ is increasing (for each fixed y), we have

$$\left|\int_b^c \alpha\,d(-f)\right| \le M\int_b^c d(-f) \qquad (\because \text{ upper bound of } |\alpha| \text{ is } M)$$

$$= M\,f(b,y)-M\,f(c,y) \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Now if $c>b>B$, we have from (i) and (ii)

$$\left|\int_b^c f\,d\alpha\right| \le \left|f(c,y)\alpha(c)-f(b,y)\alpha(b)\right| + \left|\int_b^c \alpha\,d(-f)\right|$$

$$\le |\alpha(c)||f(c,y)|+|f(b,y)||\alpha(b)|+M\,|f(b,y)-f(c,y)|$$

$$\le |\alpha(c)||f(c,y)|+|\alpha(b)||f(b,y)|+M\,|f(b,y)|+M\,|f(c,y)|$$

$$\le M\,g(c)+M\,g(b)+M\,g(b)+M\,g(c)$$

$$=2M\left[g(b)+g(c)\right]$$

$$< 2M\left[\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\right] = \varepsilon$$

$$\Rightarrow \left|\int_b^c f\,d\alpha\right| < \varepsilon \quad \text{for every } y \text{ in } S.$$

Therefore the Cauchy condition is satisfied and $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

➢ *Example*

Consider $\int_0^\infty \frac{e^{-xy}}{x}\sin x\,dx$

Take $\alpha(x) = \cos x$ and $f(x,y) = \frac{e^{-xy}}{x}$ if $x > 0$, $y \geq 0$.

If $S = [0,+\infty)$ and $g(x) = \frac{1}{x}$ on $[\varepsilon,+\infty)$ for every $\varepsilon > 0$ then

i) $f(x,y) \leq f(x',y)$ if $x' \leq x$ and $\alpha(x)$ is bounded on $[\varepsilon,+\infty)$.

ii) $g(x) \to 0$ as $x \to +\infty$

iii) $|f(x,y)| = \left|\frac{e^{-xy}}{x}\right| \leq \frac{1}{x} = g(x) \quad \forall\ y \in S$.

So that the conditions of Dirichlet's theorem are satisfied.
Hence

$$\int_\varepsilon^\infty \frac{e^{-xy}}{x}\sin x\,dx = +\int_\varepsilon^\infty \frac{e^{-xy}}{x}\,d(-\cos x) \quad \text{converges uniformly on } [\varepsilon,+\infty) \text{ if } \varepsilon > 0.$$

$\because \lim\limits_{x\to 0}\frac{\sin x}{x} = 1 \qquad \therefore \int_0^\varepsilon e^{-xy}\frac{\sin x}{x}dx$ converges being a proper integral.

$\Rightarrow \int_0^\infty e^{-xy}\frac{\sin x}{x}dx$ also converges uniformly on $[0,+\infty)$.

➢ *Remarks*

Dirichlet's test can be applied to test the convergence of the integral of a product. For this purpose the test can be modified and restated as follows:

Let $\phi(x)$ be bounded and monotonic in $[a,+\infty)$ and let $\phi(x) \to 0$, when $x \to \infty$.

Also let $\int_a^X f(x)\,dx$ be bounded when $X \geq a$.

Then $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

➢ *Example*

Consider $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$

$\because\ \dfrac{\sin x}{x} \to 1$ as $x \to 0$.

$\therefore\ 0$ is not a point of infinite discontinuity.

Now consider the improper integral $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$.

The factor $\dfrac{1}{x}$ of the integrand is monotonic and $\to 0$ as $x \to \infty$.

Also $\left|\displaystyle\int_1^X \sin x\,dx\right| = \left|-\cos X + \cos(1)\right| \le \left|\cos X\right| + \left|\cos(1)\right| < 2$

So that $\displaystyle\int_1^X \sin x\,dx$ is bounded above for every $X \ge 1$.

$\Rightarrow \displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent. Now since $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral, we see that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

➢ *Example*

Consider $\displaystyle\int_0^\infty \sin x^2\,dx$.

We write $\sin x^2 = \dfrac{1}{2x}\cdot 2x\cdot \sin x^2$

Now $\displaystyle\int_1^\infty \sin x^2\,dx = \int_1^\infty \frac{1}{2x}\cdot 2x\cdot \sin x^2\,dx$

$\dfrac{1}{2x}$ is monotonic and $\to 0$ as $x \to \infty$.

Also $\left|\displaystyle\int_1^X 2x\sin x^2\,dx\right| = \left|-\cos X^2 + \cos(1)\right| < 2$

So that $\displaystyle\int_1^X 2x\sin x^2\,dx$ is bounded for $X \ge 1$.

Hence $\displaystyle\int_1^\infty \frac{1}{2x}\cdot 2x\cdot \sin x^2\,dx$ i.e. $\displaystyle\int_1^\infty \sin x^2\,dx$ is convergent.

Since $\int_0^1 \sin x^2\, dx$ is only a proper integral, we see that the given integral is convergent.

➢ *Example*

Consider $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx \quad , \quad a > 0$

Here e^{-ax} is monotonic and bounded and $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

Hence $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ is convergent.

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➢ *Example*

Show that $\int_0^\infty \frac{\sin x}{x} dx$ is not absolutely convergent.

Solution

Consider the proper integral $\int_0^{n\pi} \frac{|\sin x|}{x} dx$

where n is a positive integer. We have

We need not take $|x|$ because $x \geq 0$.

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx$$

Put $x = (r-1)\pi + y$ so that y varies in $[0, \pi]$.

We have $|\sin[(r-1)\pi + y]| = |(-1)^{r-1} \sin y| = \sin y$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx = \int_0^\pi \frac{\sin y}{(r-1)\pi + y} dy$$

$\because$ $r\pi$ is the max. value of $[(r-1)\pi + y]$ in $[0, \pi]$

$$\therefore \int_0^\pi \frac{\sin y}{(r-1)\pi + y} dy \geq \frac{1}{r\pi} \int_0^\pi \sin y\, dy = \frac{2}{r\pi}$$

[$\because$ Division by max. value will lessen the value

$$\Rightarrow \int_0^{n\pi} \frac{|\sin x|}{x} dx \geq \sum_1^n \frac{2}{r\pi} = \frac{2}{\pi} \sum_1^n \frac{1}{r}$$

$\because$ $\sum_1^n \frac{1}{r} \to \infty$ as $n \to \infty$, we see that

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx \to \infty \quad \text{as} \quad n \to \infty.$$

Let, now, X be any real number.
There exists a +tive integer n such that $n\pi \le X < (n+1)\pi$.

We have $\int_0^X \frac{|\sin x|}{x}\,dx \ge \int_0^{n\pi} \frac{|\sin x|}{x}\,dx$

Let $X \to \infty$ so that n also $\to \infty$. Then we see that $\int_0^X \frac{|\sin x|}{x}\,dx \to \infty$

So that $\int_0^\infty \frac{|\sin x|}{x}\,dx$ does not converge.

➢ *Questions*

Examine the convergence of

$$(i)\ \int_1^\infty \frac{x}{(1+x)^3}\,dx \qquad (ii)\ \int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx \qquad (iii)\ \int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$$

Solution

(***i***) Let $f(x) = \frac{x}{(1+x)^3}$ and take $g(x) = \frac{x}{x^3} = \frac{1}{x^2}$

As $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3}{(1+x)^3} = 1$

Therefore the two integrals $\int_1^\infty \frac{x}{(1+x)^3}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$ have identical behaviour for convergence at ∞.

$\because \int_1^\infty \frac{1}{x^2}\,dx$ is convergent $\therefore \int_1^\infty \frac{x}{(1+x)^3}\,dx$ is convergent.

(***ii***) Let $f(x) = \frac{1}{(1+x)\sqrt{x}}$ and take $g(x) = \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$

We have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{1+x} = 1$

and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is convergent. Thus $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$ is convergent.

(***iii***) Let $f(x) = \frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x) = \frac{1}{x^{1/3}\cdot x^{1/2}} = \frac{1}{x^{5/6}}$

We have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^\infty \frac{1}{x^{5/6}}\,dx$ is convergent $\therefore \int_1^\infty f(x)\,dx$ is convergent.

➢ *Question*

Show that $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$ is convergent.

Solution

We have

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx=\lim_{a\to\infty}\left[\int_{-a}^{0}\frac{1}{1+x^2}\,dx+\int_{0}^{a}\frac{1}{1+x^2}\,dx\right]$$

$$=\lim_{a\to\infty}\left[\int_{0}^{a}\frac{1}{1+x^2}\,dx+\int_{0}^{a}\frac{1}{1+x^2}\,dx\right]=2\lim_{a\to\infty}\left[\int_{0}^{a}\frac{1}{1+x^2}\,dx\right]$$

$$=2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a=2\left(\frac{\pi}{2}\right)=\pi$$

therefore the integral is convergent.

➢ *Question*

Show that $\displaystyle\int_{0}^{\infty}\frac{\tan^{-1}x}{1+x^2}\,dx$ is convergent.

Solution

$$\because\ (1+x^2)\cdot\frac{\tan^{-1}x}{(1+x^2)}=\tan^{-1}x\to\frac{\pi}{2}\quad\text{as}\quad x\to\infty$$

Here $f(x)=\dfrac{\tan^{-1}x}{1+x^2}$ and $g(x)=1+x^2$

$\displaystyle\int_{0}^{\infty}\frac{\tan^{-1}x}{1+x^2}\,dx$ & $\displaystyle\int_{0}^{\infty}\frac{1}{1+x^2}\,dx$ behave alike.

$\because\ \displaystyle\int_{0}^{\infty}\frac{1}{1+x^2}\,dx$ is convergent $\therefore$ A given integral is convergent.

➢ *Question*

Show that $\displaystyle\int_{0}^{\infty}\frac{\sin x}{(1+x)^{\alpha}}\,dx$ converges for $\alpha>0$.

Solution

$\displaystyle\int_{0}^{\infty}\sin x\,dx$ is bounded because $\displaystyle\int_{0}^{X}\sin x\,dx\le 2\quad\forall\ x>0$.

Furthermore the function $\dfrac{1}{(1+x)^{\alpha}}$, $\alpha>0$ is monotonic on $[0,+\infty)$.

$\Rightarrow$ the integral $\displaystyle\int_{0}^{\infty}\frac{\sin x}{(1+x)^{\alpha}}\,dx$ is convergent.

➢ *Question*

Show that $\int_0^\infty e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because\ \left|e^{-x}\cos x\right| < e^{-x}$ and $\int_0^\infty e^{-x}\,dx = 1$

$\therefore$ the given integral is absolutely convergent. (comparison test)

➢ *Question*

Show that $\int_0^1 \dfrac{e^{-x}}{\sqrt{1-x^4}}\,dx$ is convergent.

Solution

$\because\ e^{-x} < 1$ and $1+x^2 > 1$

$$\therefore\ \frac{e^{-x}}{\sqrt{1-x^4}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1-x^2}}$$

Also $$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{\varepsilon\to 0}\sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

$\Rightarrow \int_0^1 \dfrac{e^{-x}}{\sqrt{1-x^4}}\,dx$ is convergent. (by comparison test)

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